第百二十號

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抄錄

翼の揚力に對する地面効果に就いての續研究

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航空研究所報告第 97 號 (1933) に於いて著者は一つの無限平面壁の近くに置かれた平板に働らく揚力を計算し，色々な數値計算を遂行して，揚力が迎へ角及び平板の壁からの距離によつて如何に變るかを研究した．そしてその理論的結果は，少くとも定性的には，從來しられてゐる實驗結果とかなりよく合ふことを示した．ここに提出する短い報文は上の報告の續きで，更に詳細な數値計算の結果を示すものである．斯樣な追加計算を遂行した理由は，前報告發表後に知れた實驗結果と理論とを詳しく比較したい爲である．

前報告發表後間もなく著者は瑞西の DÄTWYLER 博士から一書を受け，氏もまた地面効果に就いての或る理論的研究と實驗的研究とを遂行したことを知つた．そして，氏の實驗結果によると，迎へ角が實用的範圍にある場合，平板（翼）の後端が壁に近づくに從つて揚力が非常に增大するが，著者の理論的結果は果して同樣な結果を與へるや否やといふ興味ある問題が起つた．しかし，殘念ながら，前報告で遂行した數値計算の範圍では揚力增大の傾向は認められるも，この問題に對して，はつきりした理論的解答を與へることは不可能である．そこで，更に詳しい計算を遂行し，平板の後端がかなり壁に近い場合の揚力を計算し，その結果を著者が着英後に入手した DÄTWYLER の報文に於ける實驗結果と比較した次第である．計算の結果は第 2 圖，第 3 圖，第 4 圖に示す通りであるが，DÄTWYLER の實驗結果とかなりよく合ふことが認められる．

附錄は二つの平行な平面壁の間に置かれた平板に働らく揚力に就いての補遺的計算の結果を示す．即ち，平板の中點が平面壁の丁度中間になくて，それから或る有限な距離だけ離れてゐる場合の揚力の近似式を，平板の幅が壁間距離に比べて小さいといふ假定のもとに導出した結果を示すものである．この樣な計算は既に ROSENHEAD も行つたが，報告第 101 號 (1934) に示した樣に揚力に對する ROSENHEAD の一般式は正しくないから，それから導出された該近似式も亦正しくないのである．

No. 120.

(Published April, 1935.)

Further Studies on the Effect of the Ground upon the Lift of a Monoplane Aerofoil.[1]

By

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Formerly the Member of the Institute.

I. Introduction.

§ 1. In a previous paper[2], I have investigated, with Messrs. T. NAGAMIYA and Y. TAKENOUTI, the two-dimensional continuous flow of an incompressible perfect fluid past a flat plate near an infinite plane wall, and I have calculated the lift acting on the plate, with the intention of examining how the boundary wall affects the lifting force on the plate. Performing various numerical calculations and applying the results to the case of a thin monoplane wing while its take-off run or its flight near the ground, I have discussed the effect of the ground upon the lift of a thin aerofoil, which may be replaced by a plate, and arrived at some interesting results which are in full agreement with the known experimental results. Namely, I have shown theoretically that the lift acting on a monoplane aerofoil near the ground is generally

(1) Communicated by Prof. K. TERAZAWA, The Member of the Institute.

(2) S. TOMOTIKA, T. NAGAMIYA and Y. TAKENOUTI, The Lift on a Flat Plate placed near a Plane Wall, with Special Reference to the Effect of the Ground upon the Lift of a Monoplane Aerofoil. Report Aeron. Res. Inst., Tokyo Imp. Univ., No. 97 (1933).

more or less greater than that which the same aerofoil would experience in an unlimited stream, so far as the angle of attack is not too large; in other words, the lift on the aerofoil is more or less increased due to the presence of the ground, when the angle of attack is smaller than a certain critical angle, and further that when the distance of the aerofoil from the ground is fixed, the larger the angle of attack becomes, the smaller becomes the amount of increase of the lift due to the effect of the ground, and also when the angle of attack is greater than the critical angle just mentioned, the lift on the aerofoil is rather decreased, on the contrary, due to the presence of the ground.

Quite recently Dr. G. DÄTWYLER[1] has made thorough experimental investigations concerning the problem of the ground-effect, together with some theoretical discussions for special cases. He obtained not only the results hitherto known that the lift on the aerofoil is increased due to the presence of the ground, when the angle of attack is small as in practically important cases, but also an interesting result that the lift is remarkably increased when the distance of the trailing edge of the aerofoil from the ground becomes very small.

Although my theoretical results agree, for the most part, tolerably well with the experimental ones obtained by Dätwyler, it is desirable to carry out further numerical calculations, since in the previous paper I have not calculated the lifting force when the distance of the trailing edge of the plate from the wall is sufficiently small.

The chief object of the present short note is to show the results of further numerical calculations for the value of the lift on the plate near the bounding plane wall and then to discuss, applying the results to the case of an aerofoil near the ground, the effect of the ground upon the lift of the aerofoil, especially when the distance of the trailing edge of the aerofoil from the ground is sufficiently small.

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(1) G. DÄTWYLER, Untersuchungen über das Verhalten von Tragflügelprofilen sehr nahe am Boden. Mitteilung aus dem Institut für Aerodynamik, E. T. H., Zürich. (1934)

II. The Expressions for the Lift and Other Quantities.

§ 2. In the previous paper cited in the above, I have developed, in detail, the conformal transformations necessary for the determination of the two-dimensional continuous flow of an incompressible perfect fluid past a flat plate placed in the vicinity of an infinite plane wall and I have obtained the general expressions for various quantities. Also the general expression for the lift acting on the plate has been obtained. For the sake of reference, we shall here write down various expressions that are going to be of use in the present discussion.

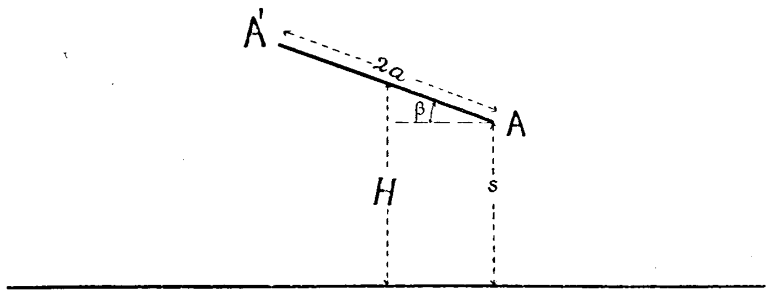

Fig. 1.

Referring to Fig. 1, we denote the breadth and the angle of attack of the plate AA′ by $2a$ and β respectively, and also we denote the distances of the mid-point and trailing edge of the plate from the wall by H and s respectively.

The expression for the ratio $2a/H$ was found to be[1]

$$\frac{2a}{H}=\frac{2}{\sin\beta}\,\frac{\left[\vartheta_3\left(\frac{\theta_4}{2\pi}\right)\right]^2-\left[\vartheta_3\left(\frac{\theta_3}{2\pi}\right)\right]^2}{\left[\vartheta_3\left(\frac{\theta_4}{2\pi}\right)\right]^2+\left[\vartheta_3\left(\frac{\theta_3}{2\pi}\right)\right]^2}\,, \qquad (1)$$

where θ_3 and θ_4 are real quantities satisfying the conditions that $\pi>\theta_3>0$, $0>\theta_4>-\pi$, and are determined by the following two equations :

(1) S. TOMOTIKA, T. NAGAMIYA and Y. TAKENOUTI, loc. cit. p. 23.

$$\theta_3+\theta_4 = 2\beta\ , \tag{2}$$

$$\frac{\vartheta_3'\left(\frac{\theta_3}{2\pi}\right)}{\vartheta_3\left(\frac{\theta_3}{2\pi}\right)}+\frac{\vartheta_3'\left(\frac{\theta_4}{2\pi}\right)}{\vartheta_3\left(\frac{\theta_4}{2\pi}\right)} = 0\ . \tag{3}$$

Also, determining, with JOUKOWSKI, the value of the circulation round the plate uniquely such that the flow leaves the trailing edge smoothly, I have calculated, in the previous paper, the lifting force acting on the plate with the aid of the well-known BLASIUS' formula. If we denote the said lift by L, we have

$$\frac{L}{L_0} = \frac{1}{2\sin^2\beta}\frac{\left[\vartheta_1\left(\frac{\beta}{\pi}\right)\right]^3\vartheta_1\left(\frac{\theta_3-\theta_4}{2\pi}\right)}{\left[\vartheta_3\left(\frac{\theta_4}{2\pi}\right)\right]^2\left\{\left[\vartheta_3\left(\frac{\theta_4}{2\pi}\right)\right]^2-\left[\vartheta_2\left(\frac{\theta_3}{2\pi}\right)\right]^2\right\}}\ , \tag{4}$$

where L_0 is the lift which the same plate would experience in an unlimited stream and, as is well known, it is given by

$$L_0 = 2\pi a U^2\rho\sin\beta\ ,$$

ρ being the density of the fluid concerned and U being the velocity of the flow at infinity.

Thus, if we determine the values of θ_3 and θ_4 for various values of q, by solving the simultaneous equations (2) and (3), we can calculate the values of $2a/H$ and L/L_0 by means of (1) and (4) respectively.

III. Numerical Discussions.

§ 3. Although the equations (2) and (3) are fairly simple in their forms, the determination of the values of θ_3 and θ_4 by these equations is rather troublesome. In the previous paper, I have first obtained, by solving (3), the q-expansion formula for $\cos\frac{1}{2}(\theta_3-\theta_4)$ up to the fifteenth power of q, i. e., q^{15}, and that formula was conveniently used for the determination of the value of $\theta_3-\theta_4$, especially when q is smaller than 0·2. But, since the said q-series converges rather slowly when q

becomes larger, it can not be used for the accurate determination of the values of θ_3, θ_4 for larger values of q. Consequently, this time I have determined, directly by means of (2) and (3), the values of θ_3, θ_4 accurately for the larger values of q, by employing the method of trial and error.

Then, I have calculated the values of $2a/H$ and L/L_0 by the formulae (1) and (4), the results of which, together with some previous results, are shown in the following tables.

TABLE I. ($\beta = 5^\circ$)

q	$\frac{2a}{H}$	$\frac{L}{L_0}$
0·02	0·1601	0·9946
0·05	0·4020	0·9925
0·10	0·8156	1·0033
0·15	1·2527	1·0296
0·20	1·7251	1·0686
0·25	2·245	1.118
0·30	2·827	1·180
0·40	4·245	1·315
0·50	6·157	1.485
0·60	8·854	1·677

TABLE II. ($\beta = 10^\circ$)

q	$\frac{2a}{H}$	$\frac{L}{L_0}$
0·02	0·1601	0·9878
0·05	0·4018	0·9756
0·10	0·8145	0·9700
0·15	1·2486	0·9782
0·20	1·7055	1·0014
0·25	2·222	1·023
0·30	2·782	1·053
0·40	4·098	1·122
0·50	5·732	1·195
0·60	7·690	1·260

TABLE III. ($\beta = 18^\circ$)

q	$\frac{2a}{H}$	$\frac{L}{L_0}$
0·02	0·1601	0·9773
0·05	0·4014	0·9501
0·10	0·8112	0·9199
0·15	1·2370	0·9041
0·20	1·6849	0·8987
0.25	2·159	0·897
0·30	2·661	0·900(1)
0·40	3·735	0·907
0·50	4·826	0·911

TABLE IV. ($\beta = 36^\circ$)

q	$\frac{2a}{H}$	$\frac{L}{L_0}$
0·02	0·1600	0·9561
0·10	0·7990	0·8301
0·20	1·5825	0·7422
0·30	2·299	0·692
0·40	2·863	0·661

§ 4. Since aerofoils are usually used, in experimental work as well as in practice, at small angles of attack, it may not be usuless to discuss the lift of the flat plate when the angle of attack is very small. In the following calculations the important terms of finite magnitude only will be retained and terms of infinitesimal magnitudes will be neglected.

Now, if we eliminate θ_4 between (2) and (3), we have

$$\frac{\vartheta_3'\left(\frac{\theta_3}{2\pi}\right)}{\vartheta_3\left(\frac{\theta_3}{2\pi}\right)}+\frac{\vartheta_3'\left(\frac{2\beta-\theta_3}{2\pi}\right)}{\vartheta_3\left(\frac{2\beta-\theta_3}{2\pi}\right)}=0\,.$$

(1) There is unfortunately a misprint in the previous paper. The value 0·902 in the lowest row in the third column of Table III, p. 42 should be read as 0·900.

Then, expanding the second term in ascending powers of β, which is assumed to be very small, and retaining only the most important term, we get

$$\frac{\vartheta_3''\left(\frac{\theta_3}{2\pi}\right)}{\vartheta_3\left(\frac{\theta_3}{2\pi}\right)}-\left\{\frac{\vartheta_3'\left(\frac{\theta_3}{2\pi}\right)}{\vartheta_3\left(\frac{\theta_3}{2\pi}\right)}\right\}^2=0\,. \qquad (5)$$

By this equation we can determine the single quantity θ_3.

Under these assumptions we have, from (1),

$$\frac{2a}{H}=-\frac{2}{\pi}\frac{\vartheta_3'\left(\frac{\theta_3}{2\pi}\right)}{\vartheta_3\left(\frac{\theta_3}{2\pi}\right)}\,. \qquad (6)$$

Also, the expression for the ratio L/L_0 reduces to the form:

$$\frac{L}{L_0}=-\frac{1}{4\pi^2}\frac{\left[\vartheta_1'(0)\right]^3\vartheta_1\left(\frac{\theta_3}{\pi}\right)}{\left[\vartheta_3\left(\frac{\theta_3}{2\pi}\right)\right]^3\vartheta_3'\left(\frac{\theta_3}{2\pi}\right)}\,. \qquad (7)$$

Combining (6) and (7), the ratio L/L_0 can also be put in the form:

$$\frac{L}{L_0}=\frac{H}{4\pi^3a}\frac{\left[\vartheta_1'(0)\right]^3\vartheta_1\left(\frac{\theta_3}{\pi}\right)}{\left[\vartheta_3\left(\frac{\theta_3}{2\pi}\right)\right]^4}\,. \qquad (8)$$

I have determined the value of θ_3 by (5), by employing the method of trial and error, for a number of values of q, and then the values of $2a/H$ and L/L_0 have been calculated. The results are embodied in the following table.

TABLE V. ($\beta \rightarrow 0°$)

q	$\frac{2a}{H}$	$\frac{L}{L_0}$
0·01	0·0800	1·0004
0·05	0·4020	1·0100
0·10	0·8160	1·0393
0·20	1·7287	1·1505
0·30	2·8430	1·3256
0·40	4·2972	1·5716
0·50	6·3192	1·9187
0·60	9·3539	2·4374

Fig. 2 shows approximate curves of L/L_0 drawn against the ratio $2a/H$ for various values of the angle of attack.

From the preceding tables and the figure we see clearly the manner of variation of the ratio with respect to $2a/H$. It is easily seen that the theoretical results are in good agreement, at least qualitatively, with experimental ones which have been fully described in the previous paper.

§ 5. In this paper already referred to, DÄTWYLER shows some interesting curves, by plotting experimental values of the lift coefficient of the aerofoil model against the distance of the trailing edge of the aerofoil from the bounding wall.

In order to compare the theoretical results obtained in this paper with his experimental results, we shall now draw similar curves showing the manner of variation of the lift coefficient of the flat plate with respect to the distance of its trailing edge from the wall.

We define, with DÄTWYLER, the lift coefficient c_a as follows:

$$c_a = \frac{L}{a\rho U^2}.$$

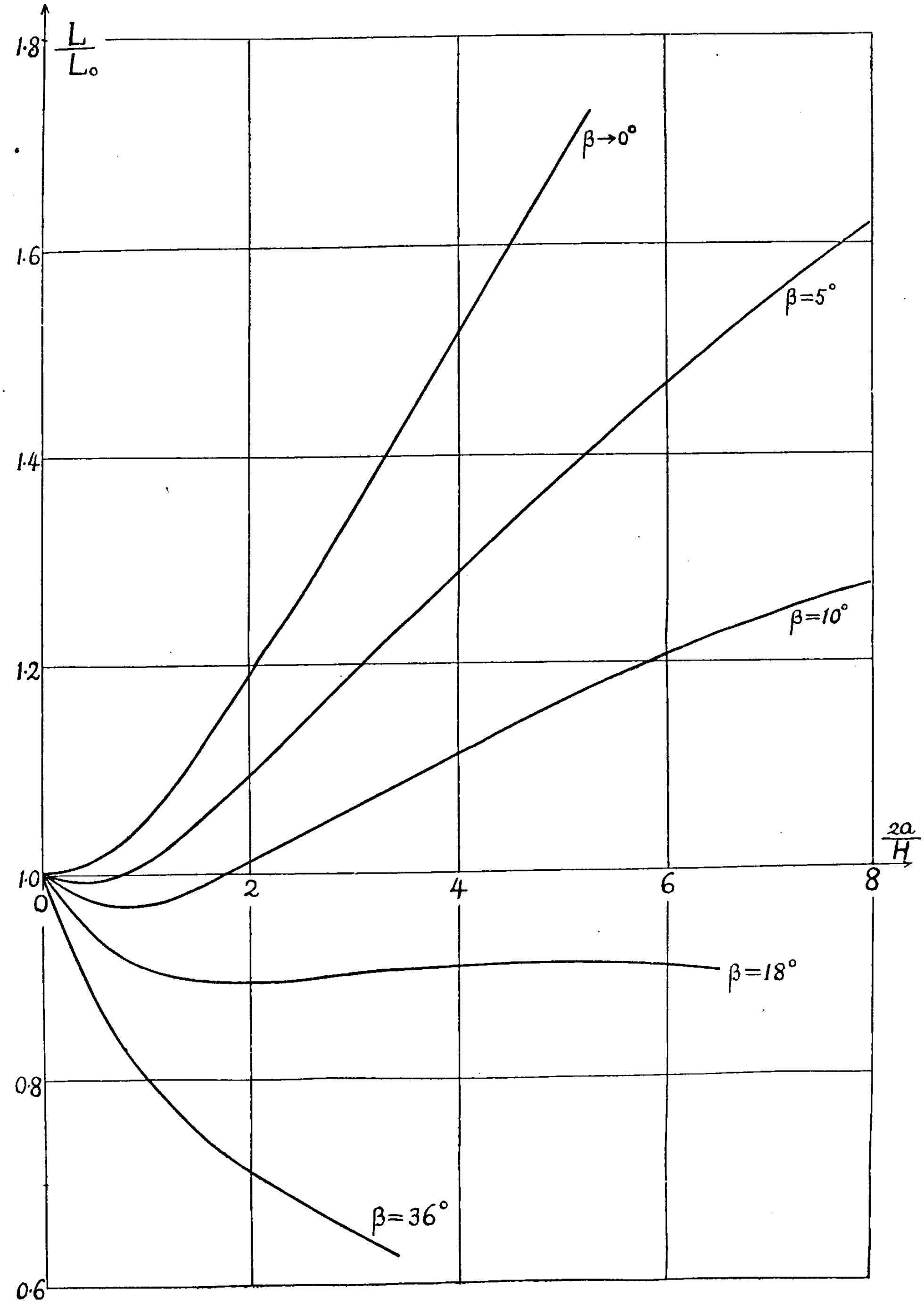

Fig. 2.

Then, we have

$$c_a = c_{a0}\frac{L}{L_0}$$

where $c_{a0} = 2\pi \sin\beta$.

Also we denote, as mentioned already, by s the distance of the trailing edge of the plate from the wall. Then we evidently have the following relation:

$$\frac{s}{2a} = \frac{H}{2a} - \frac{\sin\beta}{2} .$$

Since the values of L/L_0 and $2a/H$ have been calculated for some cases, we can calculate, by the above formulae, the values of c_a and $s/2a$, and I have calculated their values for three cases in which β is equal to 5°, 10°, and 18° respectively. The results of calculation are shown in the following tables.

TABLE VI. ($\beta = 5°$)

q	$\frac{s}{2a}$	c_a
0·02	6·2025	0·5447
0·05	2·4440	0·5435
0·10	1·1824	0·5495
0·15	0·7547	0·5639
0·20	0·5361	0·5852
0·25	0·402	0·612
0·30	0·310	0·646
0·40	0·192	0·720
0·50	0·119	0·813
0·60	0·069	0·919

TABLE VII. ($\beta = 10^\circ$)

q	$\frac{s}{2a}$	c_a
0·02	6·1593	1·0778
0·05	2·4020	1·0645
0·10	1·1409	1·0583
0·15	0·7141	1·0673
0·20	0·4995	1·0926
0·25	0·363	1·116
0·30	0·273	1·148
0·40	0·157	1·224
0·50	0·088	1·304
0·60	0·043	1·375

TABLE VIII. ($\beta = 18^\circ$)

q	$\frac{s}{2a}$	c_a
0·02	6·0916	1·8976
0·05	2·3367	1·8447
0·10	1·0782	1·7861
0·15	0·6539	1·7555
0·20	0·4390	1·7449
0·25	0·309	1·742
0·30	0·221	1·747
0·40	0·113	1·761
0·50	0·053	1·769

The curves of c_a plotted against the ratio $s/2a$ are shown in Figs. 3 and 4, where the values of c_a for the case $s = 0$ have been calculated by DÄTWYLER's theoretical formula. Fig. 4 is evidently a part of Fig. 3 and gives the curves of c_a for small values of $s/2a$ ranging from 0 to 1 on a larger scale.

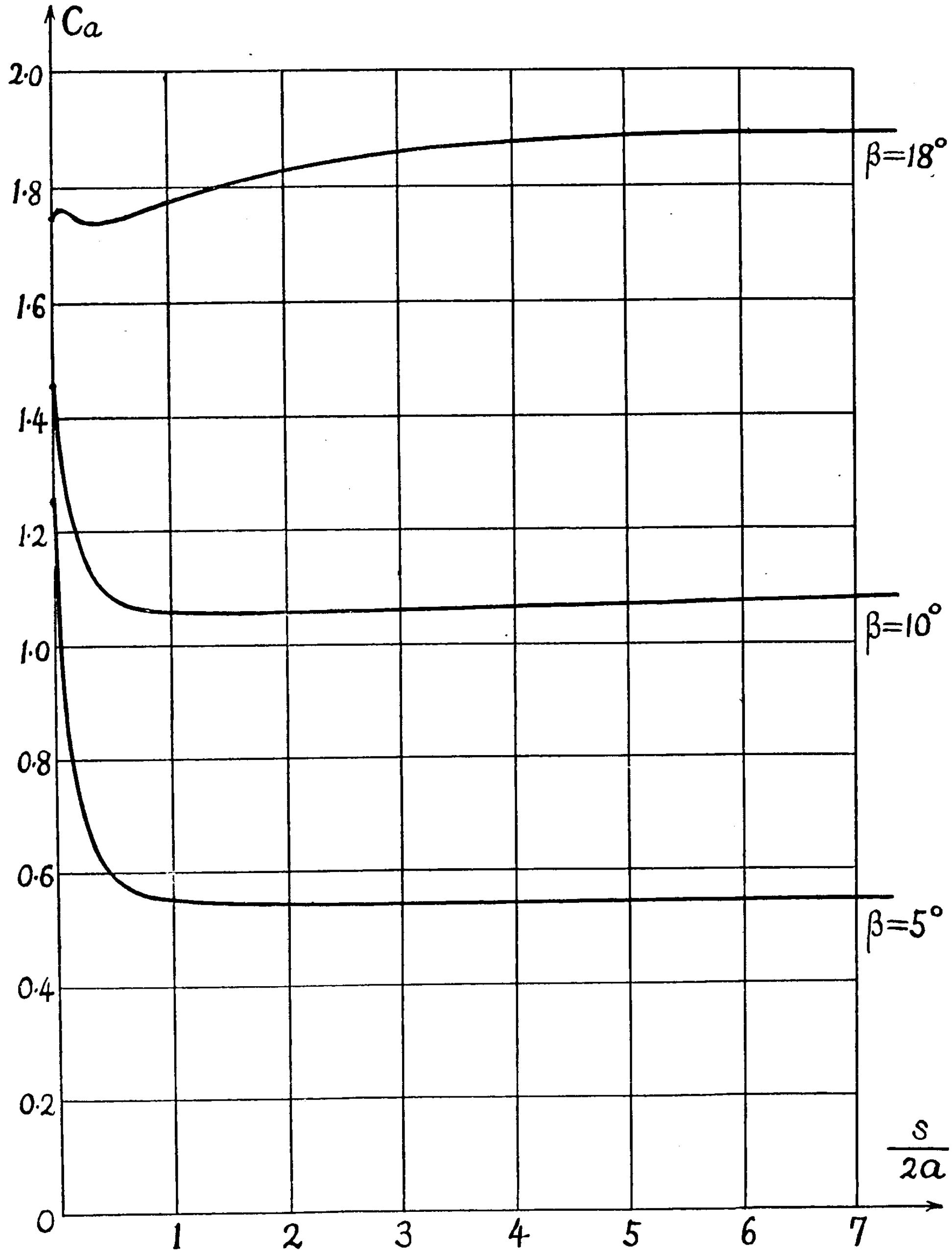
C_a
2·0
1·8
1·6
1·4
1·2
1·0
0·8
0·6
0·4
0·2
0
1
2
3
4
5
6
7
$\frac{s}{2a}$
$\beta=18°$
$\beta=10°$
$\beta=5°$

Fig. 3.

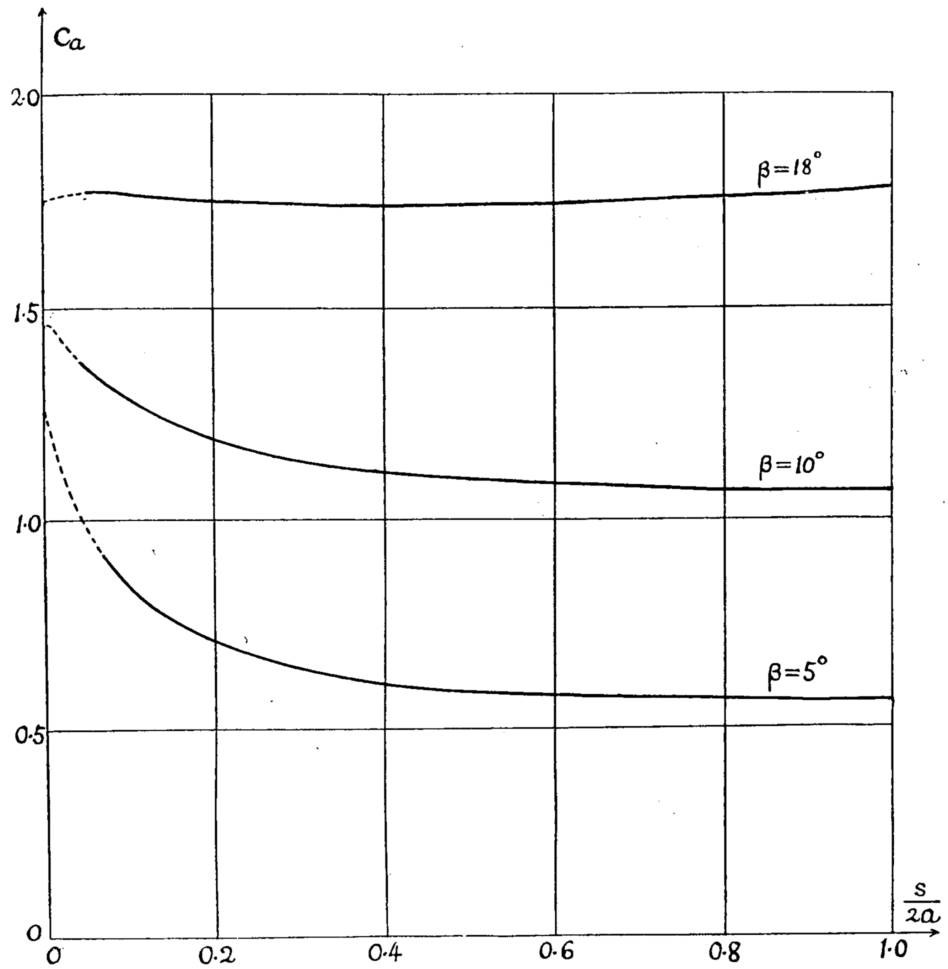

Fig. 4.

We see that in practically important cases where the angle of attack is small, the lift coefficient of the plate is considerably increased by the influence of the boundary wall when the distance of the trailing edge of the plate from the wall becomes very small. These theoretical results for the plate may duly be applied to a thin symmetrical aerofoil near the ground and thus we see that our theoretical results agree satisfactorily well with DÄTWYLER's experimental results(1).

(1) G. DÄTWYLER, loc. cit., p. 87.

IV. Summary.

§6. In this short note I have carried out further numerical calculations of the lift acting on a flat plate which is placed in an incompressible perfect fluid in the neighbourhood of an infinite plane wall, with the special intention of examining how the said lift changes when the distance of the trailing edge of the plate from the wall becomes fairly small, since for such cases interesting experimental results have been obtained by G. DÄTWYLER. It was found that the theoretical results obtained in the present paper are in full agreement with experimental results obtained by DÄTWYLER and others.

August, 1934.

Cambridge, England.

Appendix.

Supplementary Note on the Lift on a Flat Plate in a Stream between Two Parallel Walls.

§ 1. In No. 101 of the Reports of this Institute[(1)], I have investigated the problem of finding the effect of two parallel walls upon the lift on a flat plate placed in a stream between the said walls, which has been discussed previously by Dr. T. SASAKI and Dr. L. ROSENHEAD. I have pointed out that the relation $(df/ds)_{s=s_3} = -\kappa/2\omega_1$ used by ROSENHEAD in the course of calculation of the lift does not hold for the general case and in consequence of that ROSENHEAD's expression for the lift for the general case is incorrect, although his expression gives the correct result for a special case in which the mid-point of the plate lies on the centre line of the channel, since in such a special case the relation $(df/ds)_{s=s_3} = -\kappa/2\omega_1$ holds good.

I have obtained the correct general expression for the lift on the plate in the channel and carried out some numerical calculations, and also I have shown clearly that my general expression gives, as its limiting forms, the already known expressions for the lift on a plate placed in a semi-infinite stream bounded by a single infinite plane wall below or above the plate.

Since, as mentioned just in the above, ROSENHEAD's expression for the lift does not hold for the general case, his approximate formula[(2)] for $\Delta k_L/k_L$, which is equal to $(L-L\ )/L_0$ in our notations, for the case where the mid-point of the plate is not on the central line of the channel can not be said to be correct.

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(1) S. TOMOTIKA, The Lift on a Flat Plate placed in a Stream between Two Parallel Walls and Some Allied Problems. January, 1934. There are unfortunately some small misprints in this paper. The term $[\sin(n\theta-2\beta)-q^{2n}\sin\theta]$ on pages 177 and 178 should be corrected as $[\sin(n\theta-2\beta)-q^{2n}\sin n\theta]$.

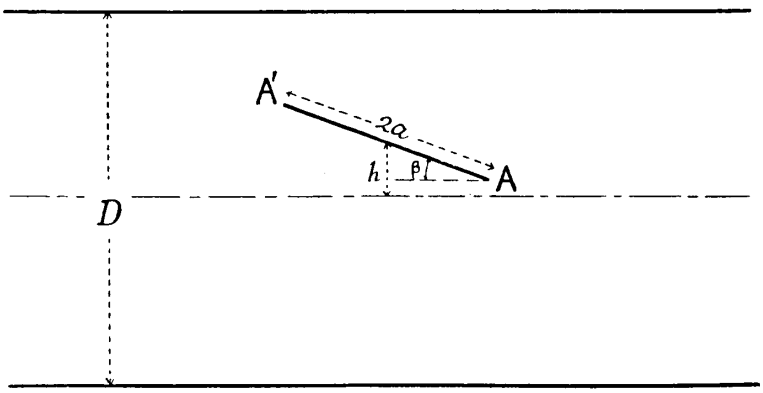
(2) L. ROSENHEAD, The Lift on a Flat Plate between Parallel Walls. Proc. Roy. Soc., London. **132** (1931), p. 129 and p. 146.

Thus, it is not quite useless to obtain the correct approximate expression for L/L_0 in the case where the mid-point of the plate is at a finite distance h from the central line of the channel, although, in mounting aerofoils in wind tunnels, it is not usual to put the aerofoil at a great distance from the middle.

The object of this supplementary note is to derive the said approximate formula for the lift on the plate, assuming that the breadth of the plate is fairly small compared with the width of the channel.

The method of images has already been applied to this case by the late H. GLAUERT and the approximate formula for the lift has been obtained, under the supposition that the dimension of the aerofoil is sufficiently small in comparison with that of wind tunnel and the angle of attack is small.

§ 2. Referring to the annexed figure, we denote the breadth and the angle of attack of the plate AA′ by $2a$ and β respectively. Also we denote by h the distance of the mid-point of the plate from the central line of the channel and by D the breadth of the channel.



Then, we have[1]

$$\frac{2\pi a}{D}=8\sum_{n=1}^{\infty}\frac{q^n\sin n\theta_2\sin\frac{1}{2}n(\theta_3-\theta_4)}{n(1-2q^{2n}\cos 2\beta+q^{4n})}\left\{\cos(n-1)\beta-q^{2n}\cos(n+1)\beta\right\}, \tag{1}$$

(1) S. TOMOTIKA, loc. cit., p. 174, and p. 180.

and

$$\frac{\pi h}{D}=\left(\frac{\pi}{2}-\theta_2\right)+4\sin\beta\sum_{n=1}^{\infty}\frac{q^n\sin n\theta_2\cos\frac{1}{2}n(\theta_3-\theta_4)}{n(1-2q^{2n}\cos2\beta+q^{4n})}$$
$$\times\left\{\sin(n-1)\beta-q^{2n}\sin(n+1)\beta\right\}, \qquad (2)$$

where θ_2, θ_3 and θ_4 are real quantities and they must satisfy the following two equations:

$$\theta_3+\theta_4=2\beta, \qquad (3)$$

and

$$\vartheta_4\left(\frac{\theta_3-\theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4-\theta_2}{2\pi}\right)=\vartheta_4\left(\frac{\theta_3+\theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4+\theta_2}{2\pi}\right). \qquad (4)$$

Next, we denote by L the lift acting on the plate in the channel and hy L_0 the corresponding quantity in the case of an unlimited stream, which is given by $L_0=2\pi aU^2\rho\sin\beta$, U being the velocity of flow at infinity and ρ the density of the fluid. Then, the most general expression for the ratio L/L_0 is

$$\frac{L}{L_0}=\frac{\pi(\kappa-\kappa')^2}{4aDU^2\sin^2\beta}\,\frac{\vartheta_4\left(\frac{\theta_3+\theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4-\theta_2}{2\pi}\right)\left[\vartheta_4\left(\frac{\theta_3-\theta_2}{2\pi}\right)\right]^2}{\left[\vartheta_1'(0)\right]^2\vartheta_1\left(\frac{\theta_2}{\pi}\right)\vartheta_1\left(\frac{\theta_3-\theta_4}{2\pi}\right)}, \qquad (5)$$

where κ is the circulation round the plate, the value of which has been determined by JOUKOWSKI's hypothesis such that the flow leaves the trailing edge of the plate smoothly, and is given by

$$\kappa=\frac{UD}{\pi}\left\{\frac{\vartheta_4'\left(\frac{\theta_4+\theta_2}{2\pi}\right)}{\vartheta_4\left(\frac{\theta_4+\theta_2}{2\pi}\right)}-\frac{\vartheta_4'\left(\frac{\theta_4-\theta_2}{2\pi}\right)}{\vartheta_4\left(\frac{\theta_4-\theta_2}{2\pi}\right)}\right\}$$
$$=8UD\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\cos n\theta_4\sin n\theta_2, \qquad (6)$$

and also κ' is the constant defined in the form:

$$\kappa' = \frac{UD}{\pi}\left\{\frac{\vartheta_4'\left(\frac{\theta_3+\theta_2}{2\pi}\right)}{\vartheta_4\left(\frac{\theta_3+\theta_2}{2\pi}\right)} - \frac{\vartheta_4'\left(\frac{\theta_3-\theta_2}{2\pi}\right)}{\vartheta_4\left(\frac{\theta_3-\theta_2}{2\pi}\right)}\right\}$$

$$= 8\,UD\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\cos n\theta_3 \sin n\theta_2 . \qquad (7)$$

§ 3. We shall now obtain approximate expressions for various quantities, under the assumption that the ratio $2a/D$ is fairly small; in other words, the value of q is fairly small.

We put

$$\theta_2 = \frac{1}{2}\pi - \lambda .$$

When q tends to zero, $\theta_3 \to \frac{1}{2}\pi + \beta$ and $\theta_4 \to -\frac{1}{2}\pi + \beta$, as shown in the previous paper[1]. Therefore, since $\theta_3 + \theta_4 = 2\beta$, we may assume that

$$\left.\begin{aligned}\theta_3 &= \frac{1}{2}\pi + \beta + a_1 q + a_2 q^2 + a\ q^3 ,\\ \theta_4 &= -\frac{1}{2}\pi + \beta - a_1 q - a_2 q^2 - a_3 q^3 .\end{aligned}\right\}$$

Putting these in (4) and determining the coefficients a_1, a_2 and a_3, we get

$$\theta_3 = \frac{1}{2}\pi + \beta + 2q\left\{1 - 2q^2\left(1 - \frac{1}{3}\cos^2\beta \sin^2\lambda\right)\right\}\cos\beta \sin\lambda ,$$

$$\theta_4 = -\frac{1}{2}\pi + \beta - 2q\left\{1 - 2q^2\left(1 - \frac{1}{3}\cos^2\beta \sin^2\lambda\right)\right\}\cos\beta \sin\lambda .$$

(1) S. Tomotika, loc. cit., p. 194.

Thus we have

$$\theta_3-\theta_4 = \pi-4q\left\{1-2q^2\left(1-\frac{1}{3}\cos^2\beta\sin^2\lambda\right)\right\}\cos\beta\sin\lambda\,. \qquad (8)$$

In like manner, we get, from the general expressions (1), (2), (6 and (7), the approximate expressions for $2\pi a/D$, $\pi h/D$, κ/UD and κ'/UD respectively. The results are

$$\left.\begin{aligned}
2\pi a/D &= 8q\cos\lambda\left[1+q^2\left(\frac{4}{3}\cos 2\beta\cos^2\lambda+2\cos^2\beta\sin^2\lambda\right)\right],\\
\pi h/D &= \lambda-2q^2\sin^2\beta\sin 2\lambda\,,\\
\kappa/UD &= 8q\sin\beta\cos\lambda\Big[1+2q\sin\beta\sin\lambda\\
&\qquad +2q^2\left(2\cos^2\beta+\sin^2\lambda+\sin^2\beta\sin^2\lambda\right)\Big],\\
\kappa'/UD &= 8q\sin\beta\cos\lambda\Big[-1+2q\sin\beta\sin\lambda\\
&\qquad -2q^2\left(2\cos^2\beta+\sin^2\lambda+\sin^2\beta\sin^2\lambda\right)\Big].
\end{aligned}\right\} \quad (9)$$

Also, taking the above several results into account, we obtain, from (5), the approximate expression for L/L_0 in the form:

$$\frac{L}{L_0} = 1+4q\sin\beta\sin\lambda+\frac{8}{3}q^2\left[(1+\sin^2\beta)+\frac{1}{2}\sin^2\lambda(1+7\sin^2\beta)\right], \qquad (10)$$

Solving λ from the second equation in (9), we have

$$\lambda = \frac{\pi h}{D}+2q^2\sin^2\beta\sin\left(\frac{2\pi h}{D}\right), \qquad (11)$$

and putting this in the first equation in (9) we get

$$\frac{2\pi a}{D} = 8q\cos\left(\frac{\pi h}{D}\right)\left[1+\frac{4}{3}q^2\left\{(1-2\sin^2\beta)+\frac{1}{2}(1-5\sin^2\beta)\sin^2\left(\frac{\pi h}{D}\right)\right\}\right],$$

from which we can solve q as a power series of $2a/D$. The result is

$$q = \frac{\pi}{8}\sec\left(\frac{\pi h}{D}\right)\left(\frac{2a}{D}\right) - \frac{\pi^3}{384}\sec^3\left(\frac{\pi h}{D}\right)\left\{(1-2\sin^2\beta) + \frac{1}{2}(1-5\sin^2\beta)\sin^2\left(\frac{\pi h}{D}\right)\right\}\left(\frac{2a}{D}\right)^3. \quad (12)$$

Finally, substituting (11) and (12) in (10) we have

$$\frac{L}{L_0} = 1 + \left\{\frac{\pi}{2}\sin\beta\tan\left(\frac{\pi h}{D}\right)\right\}\left(\frac{2a}{D}\right) + \frac{\pi^2}{24}\sec^2\left(\frac{\pi h}{D}\right)\left\{(1+\sin^2\beta) + \frac{1}{2}\sin^2\left(\frac{\pi h}{D}\right)(1+7\sin^2\beta)\right\}\left(\frac{2a}{D}\right)^2 \quad (13)$$

and at small angles of attack this becomes

$$\frac{L}{L_0} = 1 + \frac{\pi^2}{16}\left\{\sec^2\left(\frac{\pi h}{D}\right) - \frac{1}{3}\right\}\left(\frac{2a}{D}\right)^2 + \left\{\frac{\pi}{2}\frac{2a}{D}\tan\left(\frac{\pi h}{D}\right)\right\}\beta. \quad (14)$$

These are the required correct approximate formulae for the lift and they should be used when necessary instead of ROSENHEAD's erroneous formulae.

Further, if we suppose β to be very small and neglect the second term which is proportional to β, we get

$$\frac{L}{L_0} = 1 + \frac{\pi^2}{16}\left\{\sec^3\left(\frac{\pi h}{D}\right) - \frac{1}{3}\right\}\left(\frac{2a}{D}\right)^2. \quad (15)$$

This is the approximate formula deduced by H. GLAUERT by the method of images.