第百四十六號

(昭和十二年二月發行)

抄　　錄

水上飛行機の揚力に及ぼす海の表面の影響

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水上飛行機が海面上を滑走してゐる場合，或は海面の極く近くを飛行してゐる際に，海面が飛行機の揚力に相當大きい影響を及ぼすことが知れてゐる．實際，數年前英國の Felixstowe に於いてなされた實物飛行機による試驗の結果は，水上機の最大揚力が海面の影響によつて約 10% も增加することを示してゐるのである．

この様な揚力に對する海面の影響を理論的に研究することは興味あることであるが，この現象を滿足に說明し得る理論は今までに提出されてゐない樣である．普通の所謂渦理論の如きは到底うまくこれを說明し得べしとも思へない．

本論文では，一つの試みとして，純粹な流體力學的の一問題を研究し，その結果を水上機の揚力の問題に應用することを試みてゐる．先づ，一つの自由表面を持ち且つ半無限大に擴がつてゐる二次元的の流れの中に一つの平板を置く場合，その平板の受ける揚力は如何といふ流體力學的問題を考へ，その揚力を嚴密に計算した．但し，この場合，自由表面は平板の下方に於いて流れを限界してゐるものとした．次に，平板の迎へ角及び平板と自由表面との間の距離を色々に變へて，平板の揚力が自由表面のために如何に影響されるかを詳しく數値的に研究したものである．

この様な問題は一つの理論的の問題としても興味があると信ずるものであるが，又冒頭に述べた水上飛行機の揚力に對する海面の影響といふ實際的の問題にも密接に關聯してゐると考へられる．即ち，若しも海水が靜止してゐるものと假定し且つ重力を無視するならば海水中の壓力は到る處一定であり，從つてその場合には海水とその上の空氣との境界である海の表面は一つの自由表面を形成すると考へることが出來る故に，吾々の研究した理論的の問題は，かゝる假定の下に於いては，水上飛行機が海面上を滑走する場合，或は海面の近くを飛行する場合の條件とよく似て居て，平板は即ち飛行機の翼に對應し，自由表面は海面に對應することになる．從つて，吾々の問題から得られた結果を，大きい誤なしに，實際問題に應用することが出來ると思はれるのである．

本論文に於ける色々な詳しい數値計算の結果を適用すると，水上機が海面上を滑走するか，又はそれの極く近くを飛行してゐて，翼と海面との間の距離が翼の幅と同程度の大いさの場合には，翼の最大揚力は海面のために約 6% 増加することが理論的に豫想される．この理論的結果は前述の實物試驗の結果と比較さるべきものであつて，吾々の理論的問題では流れが二次元的であるに反し實際の場合には三次元的である故に完全な一致を期待し得ないのは當然であることを考慮すると，理論の結果と實際とはかなりよく合ふと云つてもよいと思はれる．

No. 146.

(Published February, 1937)

The Interference Effect of the Surface of the Sea on the Lift of a Seaplane.

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I. Introduction.

§ 1. The interference effect of the ground upon the lift of a monoplane aerofoil has been completely discussed theoretically by one of us in the previous papers(1), and it has been found that the theoretical results are in good agreement with experiments. In a recent paper(2) Dr. P. DE HALLER has also re-investigated the problem theoretically by using JACOBI's elliptic functions instead of WEIERSTRASS's functions and has confirmed our results.

(1) S. TOMOTIKA, T. NAGAMIYA and Y. TAKENOUTI, The Lift on a Flat Plate placed near a Plane Wall, with Special Reference to the Effect of the Ground upon the Lift of a Monoplane Aerofoil. Report Aeron. Res. Inst., Tokyo Imp. Univ., No. 97 (1933); S. TOMOTIKA, Further Studies on the Effect of the Ground upon the Lift of a Monoplane Aerofoil. ibid., No. 120 (1935).

(2) P. DE HALLER, La portance et la traînée induite minimum d'une aile au voisinage du sol. Mitteilungen aus dem Institut für Aerodynamik, E. T. H., Zürich. Nos. 4/5 (1936), 99–131.

A similar case in which the semi-infinite flow of fluid past a flat plate or an aerofoil is bounded, on the lower side of the plate or aerofoil, by a free surface, instead of a rigid plane wall, has not yet been subjected to any rigorous mathematical analysis, although a simple approximate treatment has already been made by replacing an aerofoil by a bound rectilinear vortex(1). The theoretical discussion of the interference effect of such a free surface upon the lift of the plate or aerofoil placed in the vicinity of the free surface is however not only interesting from the theoretical point of view, but also important from the practical standpoint.

In effect, if the sea water is assumed to be at rest and the gravity is not taken into consideration, then the fluid pressure in the sea is everywhere constant, in accordance with the fundamental hydrodynamical equation for pressure, and therefore the boundary surface between the sea water and the air may be considered as a free surface along which the pressure is constant.

Thus, the present hydrodynamical problem has an intimate connection with the practically important case of a seaplane flying near the surface of the sea.

The interference of the sea upon the lift and pitching moment of a seaplane while taxi-ing over the sea has a large effect upon the take-off run, because the resulting change in angle of attack and immersion of the hull will give rise to a change in water resistance. Full scale tests at Felixstowe, England, have shown that the maximum lift of a seaplane is increased by about 10 per cent due to the interference effect of the sea(2).

It seems that the approximate vortex theory is incapable of explaining satisfactorily this interference effect, and so far as we are aware, no exact theory has yet been proposed in order to explain the phenomenon.

In view of the practical importance of the problem, a mathematical analysis has been worked out, the results of which will be described in the present paper.

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(1) See, e.g., W. F. DURAND, Aerodynamic Theory, II (1935), 241.

(2) Report for the year 1933, National Physical Laboratory, 202.

In this paper, the lifting force acting on a flat plate is first calculated in the case when the semi-infinite two-dimensional flow of fluid past the plate is bounded, on the lower side of the plate, by a free surface, and the interference effect of such a free surface upon the lift of the plate is discussed in detail for various values of the angle of attack as well as of the distance of the plate from the bounding free surface. The theoretical results are then applied to the case of a seaplane flying near the surface of the sea and the interference effect of the sea upon the lift of a seaplane while taxi-ing over the sea is discussed.

II. The Conformal Transformations.

§ 2. Taking the z-plane as the plane of fluid motion, we consider a steady irrotational continuous two-dimensional flow of an incompressible perfect fluid past a flat plate AA′ placed near an infinite free surface which bounds the fluid on the lower side of the plate. What we are concerned with in the present paper is the investigation of the interference effect of such a free surface upon the lift experienced by the plate.

We shall begin with the conformal transformations necessary for the problem.

We assume for the present that the circulation round the plate is zero. Also we assume that at infinity upstream the fluid flows, with a constant velocity U, from left to right parallel to the boundary free surface.

A part of a particular stream line coincides with the surface of the plate. Let this stream line be defined by $\psi = 0$, where ψ is the stream function. If we suppose that the axis of x is drawn parallel to the direction of flow at infinity upstream, the flow pattern in the z-plane may become as shown in Fig. 1. We denote the value of ψ on the bounding free stream line by $-\psi_0$, and β is the angle of attack of the plate. Since the pressure on the free surface is everywhere constant, the velocity of flow along this surface is also constant and is equal to U.

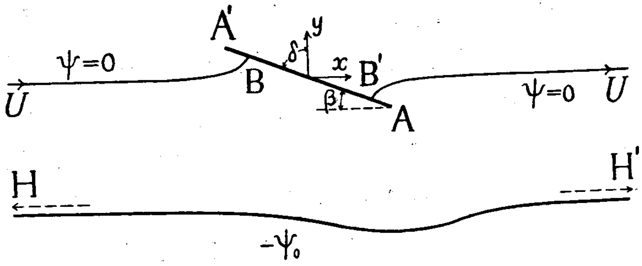


Fig. 1. z-plane.

If then we denote the complex velocity potential for the irrotational continuous flow under consideration by $f=\phi+i\psi$, where ϕ is the velocity potential, the f-plane becomes as shown in Fig. 2.

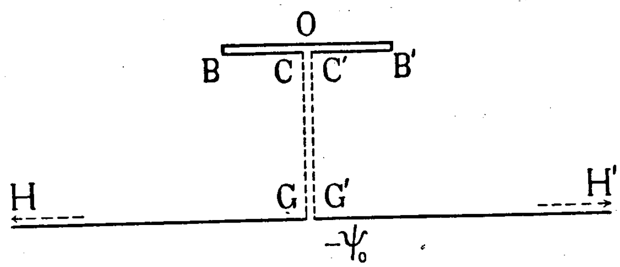


Fig. 2. f-plane.

By making a cut along C G G′ C′ as shown in the figure, we transform the f-plane on to the upper half of a t-plane by SCHWARZ-CHRISTOFFEL'S method. The transformation equation is

$$\frac{df}{dt}=M\frac{t^2-b^2}{\sqrt{(t^2-c^2)(t^2-g^2)}}, \tag{1}$$

where b corresponds to B′, $-b$ to B, c to C′, $-c$ to C, g to G′ and $-g$ to G respectively. The constant M will be determined presently.

The t-plane is shown in Fig. 3.

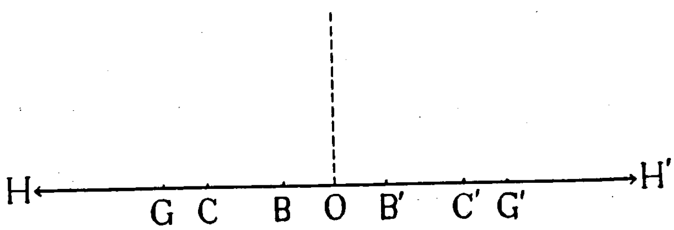


Fig. 3. t-plane.

Next, by employing $\wp$ function with periods $2\omega_1$, $2\omega_3$, where ω_1 is real and ω_3 is purely imaginary such that $\omega_1 > 0$, $\omega_3/i > 0$, we transform conformally the upper half of the t-plane into a rectangle of sides $2\omega_1$ and ω_3/i in an s-plane by the relation:

$$t^2 = \wp(s) - e_3 . \tag{2}$$

Then, the points B, B′, C, C′, G, G′, H, H′, i. e., $t = -b,\ b, -c,\ c, -g,\ g, -\infty,\ \infty$ correspond to $s = \mu,\ -\bar{\mu},\ \omega_1 + \omega_3,\ -\omega_1 + \omega_3,\ \omega_1,\ -\omega_1,\ 0,\ 0$ respectively[1], and the s-plane becomes as illustrated in Fig. 4.

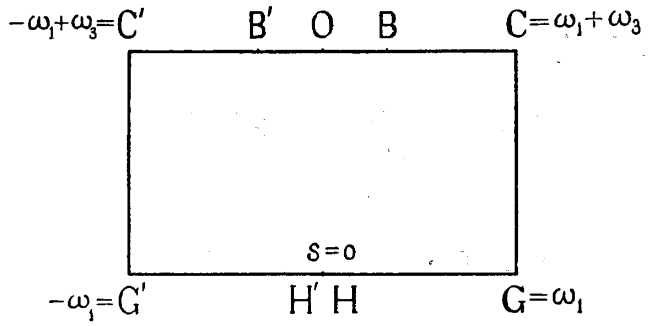


Fig. 4. s-plane.

We have from (1) and (2)

$$\frac{df}{ds} = \frac{df}{dt}\frac{dt}{ds} = M\Big[\wp(s) - \wp(\mu)\Big] . \tag{3}$$

This differential equation can be integrated immediately and we get

$$f = -M\Big[\zeta(s) + \wp(\mu)s\Big] . \tag{4}$$

To this expression must be added an arbitrary constant, which can however be neglected.

Since we assume that the circulation round the plate is zero in the flow defined by f, this function f has a period $2\omega_1$, and this condition gives a relation as follows:

(1) In the Report No. 97, $-\mu$ on page 5 should be read as $-\bar{\mu}$, where $\bar{\mu}$ is the conjugate complex of μ.

$$\wp(\mu) = -\frac{\eta_1}{\omega_1}\,. \tag{5}$$

Also, from the condition that f differs by $i\psi_0$ at $s=\omega_1$ and $s=\omega_1+\omega_3$, we have

$$f_C - f_G = i\psi_0 = -M\left[\eta_3 + \wp(\mu)\omega_3\right], \tag{6}$$

which, in conjunction with (5) and the well-known LEGENDRE's relation, gives

$$M = \frac{2\psi_0\omega_1}{\pi}\,. \tag{7}$$

Thus, we get finally

$$f = -\frac{2\psi_0\omega_1}{\pi}\left[\zeta(s) + \wp(\mu)s\right], \tag{8}$$

and

$$\begin{aligned} \frac{df}{ds} &= \frac{2\psi_0\omega_1}{\pi}\left[\wp(s) - \wp(\mu)\right] \\ &= -\frac{2\psi_0\omega_1}{\pi}\,\frac{\sigma(s+\mu)\,\sigma(s-\mu)}{[\sigma(s)\sigma(\mu)]^2}\,. \end{aligned} \tag{9}$$

Further, the inside of the rectangle in the s-plane is transformed conformally into a ring region in a Z-plane bounded by two concentric circles of radii 1 and $q\left[=\exp\left(\frac{\omega_3}{\omega_1}\pi i\right)<1\right]$, by the relation:

$$s = \omega_1 + \omega_3 - \frac{\omega_1}{i\pi}\log Z\,. \tag{10}$$

Then, the face of the plate corresponds to the outer circle and the bounding free surface to the inner circle.

§ 3. Now, for the flow defined by f the conjugate complex velocity $v_1\,(=v_x - iv_y)$ at any point in the z-plane is given by[1]

(1) As in our former papers, we define the velocity potential ϕ as $\boldsymbol{v} = \text{grad}\ \phi$, where $\boldsymbol{v}$ is the velocity vector of the fluid element at any point.

$$v_1 = \frac{df}{dz}. \tag{11}$$

If we denote by $|v_1|$ the absolute magnitude of the fluid velocity at any point and by Θ an angle which the direction of velocity at that point makes with the positive direction of the x-axis, we have $v_1 = |v_1| e^{-i\Theta}$. Therefore, writing

$$\Omega = \Theta + i \log |v_1|, \tag{12}$$

we have

$$v_1 = \frac{df}{dz} = e^{-i\Omega}. \tag{13}$$

The direction of flow at every point on the surface of the flat plate is known, so that the real part Θ of the function Ω is given from the outset. On the bounding free surface, however, the value of Θ is not known from the beginning, but the fluid velocity along it is constant and equal to U, since the fluid pressure on the free surface is everywhere constant and is equal to the pressure in the fluid at infinity. Thus, the imaginary part of the function Ω on the free surface is known and is equal to $\log U$.

We assume further that there exist neither sources, sinks nor vortices, so that the function Ω is everywhere regular in the field of fluid motion.

As mentioned in the preceding paragraph, the region of fluid motion in the z-plane is transformed conformally into the ring region in the Z-plane, and the face of the plate corresponds to the outer circle of radius 1, while the free surface corresponds to the inner circle of radius q. Thus, in the Z-plane the function Ω, expressed as a function of Z, must be such that it is everywhere regular in the said ring region and its real part on the outer circle assumes the prescribed value, $\Phi(\theta)$ say, expressed as a function of the central angle θ, while its imaginary part on the inner circle is constant. Such a function can however be determined if we make use of the formula given below, which has been established rigorously in our previous papers(1).

In general, let $F(Z)$ be an analytic function satisfying the conditions that it is everywhere regular in a ring region bounded by two concentric circles of radii 1 and $q\left[=\exp\left(\frac{\omega_3}{\omega_1}\pi i\right)<1\right]$ in the Z-plane and its real part on the outer circle takes the form $\Phi(\theta)$, the known function of the central angle θ, while its imaginary part on the inner circle is equal to a constant, k say. Then, the expression for $F(Z)$ is

$$F(Z)=\frac{i\omega_1}{\pi^2}\int_0^{2\pi}\Phi(\theta)\left\{\zeta\left[\left(\frac{\omega_1}{i\pi}\log Z-\frac{\omega_1}{\pi}\theta\right)\middle|\,\omega_1,\,2\omega_3\right]\right.$$

$$\left.-\zeta_3\left[\left(\frac{\omega_1}{i\pi}\log Z-\frac{\omega_1}{\pi}\theta\right)\middle|\,\omega_1,\,2\omega_3\right]\right\}d\theta+ik\,, \tag{14}$$

the half-periods of ζ functions here used being ω_1, $2\omega_3$ as indicated in the formula.

Thus, if we substitute the known value for $\Phi(\theta)$ and put $k=\log U$, we can obtain by this formula the expression for our function $\Omega(Z)$.

Since, however, we are not interested, from the practical point of view, in an irrotational flow with no circulatory motion round the plate, we shall not enter into the detailed calculations of Ω for this case, and only the results will be described briefly. On calculating Ω by the above formula and then taking account of the obvious relations:

$$\frac{dz}{ds}=\frac{dz}{df}\frac{df}{ds}=e^{i\Omega}\frac{df}{ds}\,,$$

(1) S. TOMOTIKA, On Certain Problems of DIRICHLET for an Annular Region, with Special Reference to Hydrodynamical Applications. Proc. Phys.-Math. Soc., Japan. [3] 14 (1932), 197–213. S. TOMOTIKA, A New Derivation of the Formula solving a Kind of DIRICHLET's Problem for a Ring Region. ibid., [3] 18 (1936), 427-435. In these papers, the imaginary part on the inner circle was assumed to be zero. However, when it has a constant value, k say, the required formula can be obtained simply by adding ik.

we express dz/ds as a function of s. Then, if we denote by s_1 and s_2 those points in the s-plane which correspond to the edges A and A′ of the plate respectively, it is easily proved that

$$\left(\frac{dz}{ds}\right)_{s=s_1}=0\,,\qquad \left(\frac{dz}{ds}\right)_{s=s_2}=0\,.$$

Consequently, from the relation:

$$v_1=\frac{df}{dz}=-\frac{i\pi}{\omega_1}Z\frac{df}{dZ}\bigg/\frac{dz}{ds}\,,$$

we see that the fluid velocity at A and A′ becomes infinite in the flow under consideration where the circulation round the plate is zero.

§ 4. As mentioned just in the above, in the continuous flow with no circulation round the plate the fluid velocity at both edges of the plate is infinite and the stream line does not leave the trailing edge smoothly. In order to avoid this we superpose, as usual, a circulatory flow in the clockwise sense round the plate and following JOUKOWSKI's hypothesis, we determine the constant of circulation κ such that the velocity at the trailing edge A becomes finite.

Now, if this circulatory flow is transformed into the Z-plane, we may obtain a circulatory flow with the same circulation κ occurring in the counter-clockwise sense round the inner circle.

Thus, denoting by f' the complex velocity potential for the superposed circulatory motion, we have

$$f'=-\frac{i\kappa}{2\pi}\log Z\,, \tag{15}$$

and since the outer and inner circles in the Z-plane are stream lines of this flow, f' satisfies the boundary conditions in the z-plane.

If we express f' in terms of s, by the aid of (10), we get

$$f'=-\frac{\kappa}{2\omega_1}(s-\omega_1-\omega_3)\,. \tag{16}$$

Consequently, the complex velocity potential χ for an irrotational continuous flow around the flat plate is, in the most general case, given by

$$\chi = f + f',$$

that is,

$$\chi = -\frac{2\psi_0\omega_1}{\pi}\Big[\zeta(s) + \wp(\mu)s\Big] - \frac{\kappa}{2\omega_1}(s-\omega_1-\omega_3). \tag{17}$$

Differentiating this with respect to s, we have

$$\frac{d\chi}{ds} = \frac{2\psi_0\omega_1}{\pi}\Big[\wp(s) - \wp(\mu)\Big] - \frac{\kappa}{2\omega_1}. \tag{18}$$

We now determine, with JOUKOWSKI, the circulation κ such that the flow leaves the trailing edge A of the plate smoothly. If, as before, we denote by s_1 the point in the s-plane which corresponds to A in the z-plane, the condition for determining κ is proved without difficulty to be

$$\left(\frac{d\chi}{ds}\right)_{s=s_1} = 0. \tag{19}$$

Putting (18) in (19), κ can be determined as:

$$\frac{\kappa}{2\omega_1} = \frac{2\psi_0\omega_1}{\pi}\Big[\wp(s_1) - \wp(\mu)\Big], \tag{20}$$

and substituting this in (18), we have

$$\frac{d\chi}{ds} = \frac{2\psi_0\omega_1}{\pi}\Big[\wp(s) - \wp(s_1)\Big], \tag{21}$$

or

$$\frac{d\chi}{ds} = -\frac{2\psi_0\omega_1}{\pi}\frac{\sigma(s+s_1)\sigma(s-s_1)}{[\sigma(s)\sigma(s_1)]^2}. \tag{22}$$

The flow pattern in the z-plane becomes as shown in Fig. 5.

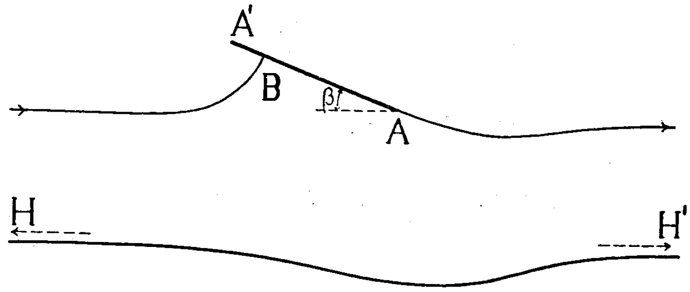
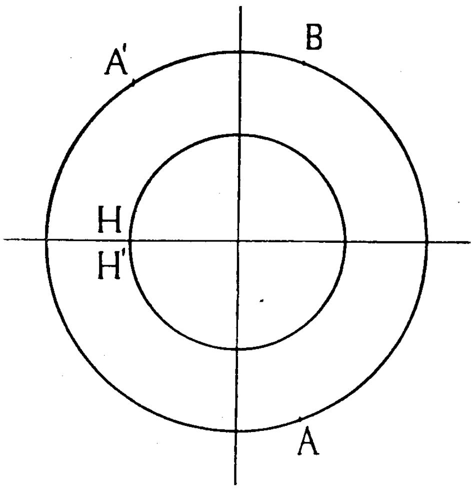


Fig. 5. z-plane.

It is easily proved from (22) that dX/ds vanishes also when $s=-s_1+2\omega_3$. This point in the s-plane corresponds to the point B on the plate shown in the above figure, where the fluid velocity is nil.

The region of fluid motion in the z-plane is transformed, as before, conformally into a ring region in the Z-plane bounded by two concentric circles of radii 1 and $q\left[=\exp\left(\frac{\omega_3}{\omega_1}\pi i\right)<1\right]$, and the face of the plate corresponds to the outer circle, whilst the free surface to the inner circle. The various points are transformed as shown in Fig. 6, where we assume that $\theta_1>0$ and consequently $\theta_1>\theta_2$.



$$A=e^{i\theta_1},$$
$$B=e^{i(2\pi-\theta_1)},$$
$$A'=e^{i\theta_2}.$$

Fig. 6. Z-plane.

Further, since the points A, B and A′ correspond to the points s_1, $-s_1+2\omega_3$ and s_2 in the s-plane respectively, we have

$$\left.\begin{aligned} \frac{\omega_1}{\pi}\theta_1 &= \omega_1+\omega_3-s_1\,, \\ \frac{\omega_1}{\pi}\theta_2 &= \omega_1+\omega_3-s_2\,. \end{aligned}\right\} \tag{23}$$

§ 5. In the flow defined by χ, the conjugate complex velocity in the z-plane is given by $d\chi/dz$ and this can be written in the form:

$$\frac{d\chi}{dz} = e^{-i\Omega}\,, \tag{24}$$

where Ω stands for $\Theta + i\log|v_1|$, as in § 3, Θ being the angle between the direction of flow and the positive direction of the x-axis. We shall now calculate the function Ω.

As mentioned previously, the function Ω must be, when expressed as a function of Z, of such properties that it is everywhere regular in the ring region in the Z-plane, and its real part Θ on the outer circle assumes the given form $\Phi(\theta)$, say, whilst its imaginary part on the inner circle is constant and equal to $\log U$.

Since these properties are the same as those for the function $F(Z)$ in § 3, the function $\Omega(Z)$ can be determined by the aid of the formula (14). Thus,

$$\Omega(Z) = \frac{i\omega_1}{\pi^2}\int_0^{2\pi}\Phi(\theta)\left\{\zeta\left[\left(\frac{\omega_1}{i\pi}\log Z-\frac{\omega_1}{\pi}\theta\right)\middle|\,\omega_1,\,2\omega_3\right]\right.$$
$$\left.-\zeta_3\left[\left(\frac{\omega_1}{i\pi}\log Z-\frac{\omega_1}{\pi}\theta\right)\middle|\,\omega_1,\,2\omega_3\right]\right\}d\theta + i\log U\,. \tag{25}$$

The function $\Phi(\theta)$ for our case is defined, as can easily be seen from Figs. 5 and 6, as follows:

$$\left.\begin{aligned} \Phi(\theta) &= \frac{\pi}{2}+\delta\,, && (2\pi-\theta_1<\theta<\theta_2)\,; \\ &= -\frac{\pi}{2}+\delta\,, && (\theta_2<\theta<\theta_1)\,; \\ &= -\frac{\pi}{2}+\delta\,, && (\theta_1<\theta<4\pi-\theta_1)\,. \end{aligned}\right\} \tag{26}$$

Putting this in the integrand in the formula (25) and carrying out the integration, we get

$$\Omega(Z) = i \log U - \frac{i}{\pi}\left\{\left(-\frac{\pi}{2}+\delta\right)\log(-1)\right.$$

$$\left. + \pi \log \frac{\xi_{30}\left[\left(\frac{\omega_1}{i\pi}\log Z + \frac{\omega_1}{\pi}\theta_1 - 2\omega_1\right)\Big|\, \omega_1,\, 2\omega_3\right]}{\xi_{30}\left[\left(\frac{\omega_1}{i\pi}\log Z - \frac{\omega_1}{\pi}\theta_2\right)\Big|\, \omega_1,\, 2\omega_3\right]}\right\}, \tag{27}$$

where, in general, $\xi_{30}(u) = \sigma_3(u)/\sigma(u)$.

If, further, we express Ω as a function of s by making use of the relations (10) and (23), we have

$$\Omega(s) = i \log U - \frac{i}{\pi}\left[\left(-\frac{\pi}{2}+\delta\right)\log(-1)\right.$$

$$\left. + \pi \log \frac{\xi_{30}[(s+s_1-2\omega_3)\,|\,\omega_1,\, 2\omega_3]}{\xi_{30}[(s-s_2)\,|\,\omega_1,\, 2\omega_3]}\right]. \tag{28}$$

Thus, putting $\log(-1) = i\pi$, $e^{i\Omega(s)}$ can be written in the form:

$$e^{i\Omega(s)} = \frac{1}{U} e^{i\left(-\frac{\pi}{2}+\delta\right)} \frac{\xi_{30}[(s+s_1-2\omega_3)\,|\,\omega_1,\, 2\omega_3]}{\xi_{30}[(s-s_2)\,|\,\omega_1,\, 2\omega_3]}. \tag{29}$$

Since, however, this is expressed in terms of the elliptic functions with half-periods ω_1 , $2\omega_3$, we have now to express it in terms of elliptic functions with half-periods ω_1 , ω_3 .

We put in general

$$\wp(\omega_1\,|\,\omega_1,\, 2\omega_3) = E_1, \quad \wp(\omega_2\,|\,\omega_1,\, 2\omega_3) = E_2, \quad \wp(2\omega_3\,|\,\omega_1,\, 2\omega_3) = E_3,$$

where $\omega_2 = -(\omega_1 + 2\omega_3)$. Then, we have the following relations[1]:

$$\left.\begin{aligned}
\sigma(u \mid \omega_1, \omega_3) &= e^{\frac{E_3 u^2}{2}} \sigma(u \mid \omega_1, 2\omega_3)\sigma_3(u \mid \omega_1, 2\omega_3), \\
\sigma_2(u \mid \omega_1, \omega_3) &= e^{\frac{E_3 u^2}{2}} \Big\{[\sigma_3(u \mid \omega_1, 2\omega_3)]^2 \\
&\qquad + \sqrt{E_1 - E_3}\sqrt{E_2 - E_3}[\sigma(u \mid \omega_1, 2\omega_3)]^2\Big\}, \\
\sigma_3(u \mid \omega_1, \omega_3) &= e^{\frac{E_3 u^2}{2}} \Big\{[\sigma_3(u \mid \omega_1, 2\omega_3)]^2 \\
&\qquad - \sqrt{E_1 - E_3}\sqrt{E_2 - E_3}[\sigma(u \mid \omega_1, 2\omega_3)]^2\Big\}, \\
e_2 - e_3 &= -4\sqrt{E_1 - E_3}\sqrt{E_2 - E_3},
\end{aligned}\right\} \qquad (30)$$

in which $\wp(\omega_1 \mid \omega_1, \omega_3) = e_1$, $\wp(\omega_2 \mid \omega_1, \omega_3) = e_2$, $\wp(\omega_3 \mid \omega_1, \omega_3) = e_3$. Further we have[2], for the functions with periods $2\omega_1$, $2\omega_3$,

$$\left.\begin{aligned}
\xi_{\beta 0}(u - 2\omega_\alpha) &= -\xi_{\beta 0}(u), \\
\xi_{\alpha 0}(u - 2\omega_\alpha) &= \xi_{\alpha 0}(u),
\end{aligned}\right\} \quad (\alpha, \beta = 1, 2, 3) \qquad (31)$$

where in general $\xi_{\alpha 0}(u) = \sigma_\alpha(u)/\sigma(u)$.

By the aid of these formulae we can express $e^{i\Omega(s)}$ in terms of elliptic functions with periods $2\omega_1$, $2\omega_3$. In the following calculations, as in the formulae (31), the half-periods ω_1, ω_3 will not be indicated explicitly for the elliptic functions with periods $2\omega_1$, $2\omega_3$.

We have from (30), by dividing respectively the second and third formulae by the first and taking the last formula into account,

$$\left.\begin{aligned}
\xi_{20}(u) &= \xi_{30}(u \mid \omega_1, 2\omega_3) - \frac{1}{4}(e_2 - e_3)\xi_{03}(u \mid \omega_1, 2\omega_3), \\
\xi_{30}(u) &= \xi_{30}(u \mid \omega_1, 2\omega_3) + \frac{1}{4}(e_2 - e_3)\xi_{03}(u \mid \omega_1, 2\omega_3),
\end{aligned}\right\} \qquad (32)$$

(1) J. Tannery et J. Molk, Éléments de la théorie des fonctions elliptiques, 2 (1896), 244-245.

(2) J. Tannery et J. Molk, loc. cit., 2 (1896), 280.

where $\xi_{0\alpha}(u) = 1/\xi_{\alpha 0}(u)$, $(\alpha = 1, 2, 3)$, and from these formulae we get easily

$$\frac{1}{2}(e_2 - e_3)\xi_{03}(u \mid \omega_1, 2\omega_3) = -\Big[\xi_{20}(u) - \xi_{30}(u)\Big]. \tag{33}$$

Thus, writing $(s - s_2)$ and $(s + s_1 - 2\omega_3)$ for u respectively we have

$$\frac{1}{2}(e_2 - e_3)\xi_{03}\Big[(s - s_2) \mid \omega_1, 2\omega_3\Big] = -\Big[\xi_{20}(s - s_2) - \xi_{30}(s - s_2)\Big], \tag{34}$$

and

$$\frac{1}{2}(e_2 - e_3)\xi_{03}\Big[(s + s_1 - 2\omega_3) \mid \omega_1, 2\omega_3\Big]$$
$$= -\Big[\xi_{20}(s + s_1 - 2\omega_3) - \xi_{30}(s + s_1 - 2\omega_3)\Big]. \tag{35}$$

By (31), equation (35) can also be written as:

$$\frac{1}{2}(e_2 - e_3)\xi_{03}\Big[(s + s_1 - 2\omega_3) \mid \omega_1, 2\omega_3\Big] = \xi_{20}(s + s_1) + \xi_{30}(s + s_1). \tag{36}$$

Dividing (34) by (36) side by side we get

$$\frac{\xi_{03}[(s - s_2) \mid \omega_1, 2\omega_3]}{\xi_{03}[(s + s_1 - 2\omega_3) \mid \omega_1, 2\omega_3]} = -\frac{\xi_{20}(s - s_2) - \xi_{30}(s - s_2)}{\xi_{20}(s + s_1) + \xi_{30}(s + s_1)}, \tag{37}$$

or

$$\frac{\xi_{30}[(s + s_1 - 2\omega_3) \mid \omega_1, 2\omega_3]}{\xi_{30}[(s - s_2) \mid \omega_1, 2\omega_3]} = -\frac{\xi_{20}(s - s_2) - \xi_{30}(s - s_2)}{\xi_{20}(s + s_1) + \xi_{30}(s + s_1)}. \tag{38}$$

However we have, in general,

$$\Big[\xi_{\alpha 0}(u)\Big]^2 - \Big[\xi_{\beta 0}(u)\Big]^2 = (e_\beta - e_\alpha), \ (\alpha, \beta = 1, 2, 3). \tag{39}$$

Therefore

$$\frac{\xi_{30}[(s + s_1 - 2\omega_3) \mid \omega_1, 2\omega_3]}{\xi_{30}[(s - s_2) \mid \omega_1, 2\omega_3]}$$
$$= \frac{1}{e_2 - e_3}\Big[\xi_{20}(s + s_1) - \xi_{30}(s + s_1)\Big]\Big[\xi_{20}(s - s_2) - \xi_{30}(s - s_2)\Big]. \tag{40}$$

Thus, combining this with (29) and introducing the angle of attack β by $\frac{1}{2}\pi-\delta=\beta$, we have $e^{i\Omega(s)}$ in terms of elliptic functions with periods $2\omega_1$, $2\omega_3$ in the form:

$$e^{i\Omega(s)}=\frac{e^{-i\beta}}{U(e_2-e_3)}\Big[\xi_{20}(s+s_1)-\xi_{30}(s+s_1)\Big]\Big[\xi_{20}(s-s_2)-\xi_{30}(s-s_2)\Big]. \qquad (41)$$

§ 6. We have assumed that at infinity H upstream the fluid flows, with the constant velocity U, from left to right parallel to the free surface, and we have taken the x-axis parallel to the direction of flow at infinity upstream. Therefore, since the point H at infinity upstream corresponds to the point $s=0$ in the s-plane, the condition at H is given by

$$\Big[e^{i\Omega(s)}\Big]_{s=0}=\frac{1}{U}. \qquad (42)$$

The point H′ at infinity downstream corresponds however to the same point $s=0$ in the s-plane, so that the condition at H′ is the same as (42).

Inserting (41) in (42) we have

$$-\frac{e^{-i\beta}}{U(e_2-e_3)}\Big[\xi_{20}(s_1)-\xi_{30}(s_1)\Big]\Big[\xi_{20}(s_2)-\xi_{30}(s_2)\Big]=\frac{1}{U},$$

i. e.,

$$\Big[\xi_{20}(s_1)-\xi_{30}(s_1)\Big]\Big[\xi_{20}(s_2)-\xi_{30}(s_2)\Big]=-(e_2-e_3)e^{i\beta}. \qquad (43)$$

By using the formulae (39), equation (43) can also be written in the form:

$$\Big[\xi_{20}(s_1)+\xi_{30}(s_1)\Big]\Big[\xi_{20}(s_2)+\xi_{30}(s_2)\Big]=-(e_2-e_3)e^{-i\beta}. \qquad (44)$$

From (43) and (44) we have, by adding and subtracting respectively,

$$\left.\begin{aligned}\xi_{20}(s_1)\xi_{20}(s_2)+\xi_{30}(s_1)\xi_{30}(s_2)&=-(e_2-e_3)\cos\beta,\\ \xi_{20}(s_1)\xi_{30}(s_2)+\xi_{30}(s_1)\xi_{20}(s_2)&=i(e_2-e_3)\sin\beta.\end{aligned}\right\} \qquad (45)$$

Now, we put

$$x_1 = \omega_1 - \frac{\omega_1}{\pi}\theta_1\,, \quad x_2 = \omega_1 - \frac{\omega_1}{\pi}\theta_2\,. \tag{46}$$

Then, x_1 and x_2 are evidently both real, and by (23) s_1 and s_2 are written in the forms:

$$s_1 = x_1 + \omega_3\,, \qquad s_2 = x_2 + \omega_3\,. \tag{47}$$

We have

$$\left.\begin{aligned} \xi_{30}(s_j) &= \xi_{30}(x_j+\omega_3) = -\sqrt{e_1-e_3}\sqrt{e_2-e_3}\,\xi_{03}(x_j)\,, \\ \xi_{20}(s_j) &= \xi_{20}(x_j+\omega_3) = i\sqrt{e_2-e_3}\,\xi_{13}(x_j)\,, \end{aligned}\right\} \quad (j = 1, 2)$$

where $\xi_{\alpha\beta}(u) = \sigma_\alpha(u)/\sigma_\beta(u)$, $(\alpha, \beta = 1, 2, 3)$ in general.

Substituting these in (45) we have

$$\left.\begin{aligned} &\xi_{13}(x_1)\xi_{13}(x_2) - (e_1-e_3)\xi_{03}(x_1)\xi_{03}(x_2) = \cos\beta\,, \\ &\sqrt{e_1-e_3}\Big[\xi_{03}(x_1)\xi_{13}(x_2) + \xi_{13}(x_1)\xi_{03}(x_2)\Big] = -\sin\beta\,, \end{aligned}\right\} \tag{48}$$

from which we get the following equation:

$$\tan^{-1}\Big[\sqrt{e_1-e_3}\,\xi_{01}(x_1)\Big] + \tan^{-1}\Big[\sqrt{e_1-e_3}\,\xi_{01}(x_2)\Big] + \beta = 0\,, \tag{49}$$

or, in terms of ϑ functions,

$$\tan^{-1}\left[\frac{\vartheta_3(0)}{\vartheta_4(0)}\,\frac{\vartheta_1\!\left(\dfrac{x_1}{2\omega_1}\right)}{\vartheta_2\!\left(\dfrac{x_1}{2\omega_1}\right)}\right] + \tan^{-1}\left[\frac{\vartheta_3(0)}{\vartheta_4(0)}\,\frac{\vartheta_1\!\left(\dfrac{x_2}{2\omega_1}\right)}{\vartheta_2\!\left(\dfrac{x_2}{2\omega_1}\right)}\right] + \beta = 0\,. \tag{50}$$

This is the first equation for determining the two real quantities x_1 and x_2.

III. Development of the Transformations.

§ 7. In the next place, we shall express z in terms of s. From (24) we have

$$\frac{dz}{ds} = e^{i\Omega}\frac{d\chi}{ds}\,, \tag{51}$$

and by (22) and (41) this can be written in the form:

$$\frac{dz}{ds} = -\frac{2\psi_0\omega_1 e^{-i\beta}}{\pi U(e_2-e_3)}$$

$$\times \frac{\sigma(s-s_1)}{\sigma(s-s_2)} \frac{[\sigma_2(s+s_1)-\sigma_3(s+s_1)][\sigma_2(s-s_2)-\sigma_3(s-s_2)]}{[\sigma(s_1)\sigma(s)]^2}. \quad (52)$$

We put

$$\frac{dz}{ds} = -\frac{2\psi_0\omega_1 e^{-i\beta}}{\pi U(e_2-e_3)} F(s)\,, \quad (53)$$

where

$$F(s) = \frac{\sigma(s-s_1)}{\sigma(s-s_2)} \frac{[\sigma_2(s+s_1)-\sigma_3(s+s_1)][\sigma_2(s-s_2)-\sigma_3(s-s_2)]}{[\sigma(s_1)\sigma(s)]^2}. \quad (54)$$

Also, we put

$$\left.\begin{aligned} F_1(s) &= \frac{\sigma(s-s_1)}{\sigma(s-s_2)} \frac{\sigma_2(s+s_1)\sigma_2(s-s_2)+\sigma_3(s+s_1)\sigma_3(s-s_2)}{[\sigma(s_1)\sigma(s)]^2}\,, \\ F_2(s) &= \frac{\sigma(s-s_1)}{\sigma(s-s_2)} \frac{\sigma_3(s+s_1)\sigma_2(s-s_2)+\sigma_2(s+s_1)\sigma_3(s-s_2)}{[\sigma(s_1)\sigma(s)]^2}\,. \end{aligned}\right\} \quad (55)$$

Then, we evidently have

$$F(s) = F_1(s) - F_2(s)\,, \quad (56)$$

and it will be proved that

$$\left.\begin{aligned} F_1(s+2\omega_1) &= F_1(s)\,, \\ F_1(s+2\omega_3) &= F_1(s)\,; \end{aligned}\right\} \quad (57)$$

and

$$\left.\begin{aligned} F_2(s+2\omega_1) &= F_2(s)\,, \\ F_2(s+2\omega_3) &= -F_2(s)\,. \end{aligned}\right\} \quad (58)$$

Thus, $F_1(s)$ is an elliptic function of the first kind, whilst $F_2(s)$ is an elliptic function of the second kind, and both functions have a double pole at $s = 0$ and a simple pole at $s = s_2$. Hence, remembering their respective periodicity-properties given in (57) and (58), they can be split up into simple elements in the forms:

$$F_1(s) = C_0 + C_1\zeta(s) + C_2\zeta'(s) + C_3\zeta(s-s_2)\,, \tag{59}$$

$$F_2(s) = B_1\xi_{10}(s) + B_2\xi'_{10}(s) + B_3\xi_{10}(s-s_2)\,, \tag{60}$$

where C_0, C_1, C_2, C_3, B_1, B_2 and B_3 are constants which will be determined in the following lines.

These constants are, however, by no means all independent of each other, and there exist some relations between them. In the first place, the perfect periodicity-property of the function $F_1(s)$ requires a relation that

$$C_1 + C_3 = 0\,, \tag{61}$$

because the periodicity-property for $\zeta(s)$ is $\zeta(s+2\omega_\alpha) = \zeta(s) + 2\eta_\alpha$, $(\alpha = 1, 2, 3)$, though $\zeta'(s) = -\wp(s)$ has the perfect periodicity-property. Secondly, since it is evident from the definitions of $F_1(s)$ and $F_2(s)$ that

$$\Big[F_1(s)\Big]_{s=s_1} = 0\,, \qquad \Big[F_2(s)\Big]_{s=s_1} = 0\,,$$

we get two relations, namely:

$$C_0 + C_1\zeta(s_1) + C_2\zeta'(s_1) + C_3\zeta(s_1-s_2) = 0\,, \tag{62}$$

$$B_1\xi_{10}(s_1) + B_2\xi'_{10}(s_1) + B_3\xi_{10}(s_1-s_2) = 0\,. \tag{63}$$

Thus, we have three relations between seven constants C_0, C_1, C_2, C_3, B_1, B_2, B_3, so that only four of them are independent of each other.

We now calculate the values of these constants. By virtue of the formulae:

$$\left.\begin{aligned} \sigma(u+a)\sigma(u-a) &= \Big[\sigma(u)\sigma_\alpha(a)\Big]^2 - \Big[\sigma_\alpha(u)\sigma(a)\Big]^2, \\ \sigma_\alpha(u+a)\sigma(u-a) &= \sigma(u)\sigma_\alpha(u)\sigma_\beta(a)\sigma_\gamma(a) \\ &\qquad -\sigma(a)\sigma_\alpha(a)\sigma_\beta(u)\sigma_\gamma(u)\,, \\ &(\alpha, \beta, \gamma = 1, 2, 3) \end{aligned}\right\} \tag{64}$$

we have, for small values of s, the following expansions:

$$\sigma_3(s+s_1)\sigma(s-s_1) = -\sigma(s_1)\sigma_3(s_1)+\sigma_1(s_1)\sigma_2(s_1)s+\cdots\cdot$$
$$= -\left[\sigma(s_1)\right]^2\left\{\xi_{30}(s_1)-\xi_{10}(s_1)\,\xi_{20}(s_1)s+\cdots\cdot\right\},$$

$$\sigma_2(s+s_1)\sigma(s-s_1) = -\sigma(s_1)\sigma_2(s_1)+\sigma_1(s_1)\sigma_3(s_1)s+\cdots\cdot$$
$$= -\left[\sigma(s_1)\right]^2\left\{\xi_{20}(s_1)-\xi_{10}(s_1)\,\xi_{30}(s_1)s+\cdots\cdot\right\},$$

$$\frac{\sigma_3(s-s_2)}{\sigma(s-s_2)} = \frac{\sigma(s+s_2)\sigma_3(s-s_2)}{\sigma(s+s_2)\sigma(s-s_2)} = -\left\{\frac{\sigma_3(s_2)}{\sigma(s_2)}+\frac{\sigma_1(s_2)\sigma_2(s_2)}{[\sigma(s_2)]^2}s+\cdots\cdot\right\}$$
$$= -\left\{\xi_{30}(s_2)+\xi_{10}(s_2)\xi_{20}(s_2)s+\cdots\cdot\right\},$$

$$\frac{\sigma_2(s-s_2)}{\sigma(s-s_2)} = \frac{\sigma(s+s_2)\,\sigma_2(s-s_2)}{\sigma(s+s_2)\sigma(s-s_2)} = -\left\{\frac{\sigma_2(s_2)}{\sigma(s_2)}+\frac{\sigma_1(s_2)\sigma_3(s_2)}{[\sigma(s_2)]^2}s+\cdots\cdot\right\}$$
$$= -\left\{\xi_{20}(s_2)+\xi_{10}(s_2)\xi_{30}(s_2)s+\cdots\cdot\right\}.$$

Thus, the function $F_1(s)$ is expanded in a power series of s in the form:

$$F_1(s) = \left\{\xi_{20}(s_1)\xi_{20}(s_2)+\xi_{30}(s_1)\xi_{30}(s_2)\right\}\frac{1}{s^2}$$
$$-\left\{\xi_{10}(s_1)-\xi_{10}(s_2)\right\}\left\{\xi_{20}(s_1)\xi_{30}(s_2)+\xi_{30}(s_1)\xi_{20}(s_2)\right\}\frac{1}{s}+\cdots\cdot. \quad (65)$$

On the other hand, we have, from (59),

$$F_1(s) = -\frac{C_2}{s^2}+\frac{C_1}{s}+C_0+\cdots\cdot\,. \quad (66)$$

Thus, we have

$$\left.\begin{aligned} C_1 &= -\left\{\xi_{10}(s_1)-\xi_{10}(s_2)\right\}\left\{\xi_{20}(s_1)\xi_{30}(s_2)+\xi_{30}(s_1)\xi_{20}(s_2)\right\}, \\ C_2 &= -\left\{\xi_{20}(s_1)\xi_{20}(s_2)+\xi_{30}(s_1)\xi_{30}(s_2)\right\}, \end{aligned}\right\} \quad (67)$$

or, taking (45) into account,

$$\left.\begin{aligned} C_1 &= -i(e_2-e_3)\Big\{\xi_{10}(s_1)-\xi_{10}(s_2)\Big\}\sin\beta\,, \\ C_2 &= (e_2-e_3)\cos\beta\,. \end{aligned}\right\} \tag{68}$$

The constant C_3 can be determined by the relation (61). Alternatively, it is given by

$$C_3 = \lim_{s\to s_2}\Big[(s-s_2)F_1(s)\Big]\,.$$

By performing the calculations we get

$$\begin{aligned} C_3 &= -\frac{\sigma(s_1-s_2)}{[\sigma(s_1)\sigma(s_2)]^2}\Big\{\sigma_2(s_1+s_2)+\sigma_3(s_1+s_2)\Big\} \\ &= \Big\{\xi_{10}(s_1)-\xi_{10}(s_2)\Big\}\Big\{\xi_{20}(s_1)\xi_{30}(s_2)+\xi_{30}(s_1)\xi_{20}(s_2)\Big\} \\ &= -C_1\,, \end{aligned}$$

as we should have expected. Thus,

$$C_3 = i(e_2-e_3)\Big\{\xi_{10}(s_1)-\xi_{10}(s_2)\Big\}\sin\beta\,. \tag{69}$$

The constant C_0 is determined by (62), namely:

$$C_0 = -C_1\zeta(s_1)-C_2\zeta'(s_1)-C_3\zeta(s_1-s_2)\,. \tag{70}$$

In like manner, the function $F_2(s)$ can be expanded in the form:

$$\begin{aligned} F_2(s) = {} & \Big\{\xi_{30}(s_1)\xi_{20}(s_2)+\xi_{20}(s_1)\xi_{30}(s_2)\Big\}\frac{1}{s^2} \\ & -\Big\{\xi_{10}(s_1)-\xi_{10}(s_2)\Big\}\Big\{\xi_{20}(s_1)\xi_{20}(s_2)+\xi_{30}(s_1)\xi_{30}(s_2)\Big\}\frac{1}{s}+\cdots\cdots, \end{aligned} \tag{71}$$

and also we have, by (60),

$$F_2(s) = -\frac{B_2}{s^2}+\frac{B_1}{s}+\cdots\cdots\,. \tag{72}$$

Thus,

$$\left.\begin{aligned} B_1 &= -\left\{\xi_{10}(s_1)-\xi_{10}(s_2)\right\}\left\{\xi_{20}(s_1)\xi_{20}(s_2)+\xi_{30}(s_1)\xi_{30}(s_2)\right\}, \\ B_2 &= -\left\{\xi_{30}(s_1)\xi_{20}(s_2)+\xi_{20}(s_1)\xi_{30}(s_2)\right\}, \end{aligned}\right\} \tag{73}$$

or, taking the relations (45) into account, we have

$$\left.\begin{aligned} B_1 &= (e_2-e_3)\left\{\xi_{10}(s_1)-\xi_{10}(s_2)\right\}\cos\beta, \\ B_2 &= -i(e_2-e_3)\sin\beta. \end{aligned}\right\} \tag{74}$$

Finally, the constant B_3 can be determined by

$$B_3 = \lim_{s\to s_2}\left[(s-s_2)F_2(s)\right],$$

and we have

$$\begin{aligned} B_3 &= -\frac{\sigma(s_1-s_2)}{[\sigma(s_1)\sigma(s_2)]^2}\left\{\sigma_2(s_1+s_2)+\sigma_3(s_1+s_2)\right\} \\ &= C_3. \end{aligned}$$

Therefore

$$B_3 = i(e_2-e_3)\left\{\xi_{10}(s_1)-\xi_{10}(s_2)\right\}\sin\beta. \tag{75}$$

As shown in the above, the three constants B_1, B_2, B_3 must satisfy the condition (63) identically. This can however be verified without difficulty by making use of a formula[1]:

$$\begin{aligned} &\sigma_\gamma(u+a)\sigma_\beta(u-a)\sigma_\alpha(b+c)\sigma(b-c) \\ &+\sigma_\gamma(u+b)\sigma_\beta(u-b)\sigma_\alpha(c+a)\sigma(c-a) \\ &+\sigma_\gamma(u+c)\sigma_\beta(u-c)\sigma_\alpha(a+b)\sigma(a-b) = 0, \\ &\qquad (\alpha, \beta, \gamma = 1, 2, 3). \end{aligned}$$

(1) J. Tannery et J. Molk, loc. cit., **1** (1893), 195.

Lastly, we can express z as a function of s. We have

$$\frac{dz}{ds}=-\frac{2\psi_0\omega_1 e^{-i\beta}}{\pi U(e_2-e_3)}\left\{F_1(s)-F_2(s)\right\}$$

$$=-\frac{2\psi_0\omega_1 e^{-i\beta}}{\pi U(e_2-e_3)}\Big\{C_0+C_1\zeta(s)+C_2\zeta'(s)+C_3\zeta(s-s_2)$$
$$-B_1\xi_{10}(s)-B_2\xi'_{10}(s)-B_3\xi_{10}(s-s_2)\Big\}, \qquad (76)$$

and therefore on integrating we get

$$z=-\frac{2\psi_0\omega_1 e^{-i\beta}}{\pi U(e_2-e_3)}\Big\{C_0 s+C_1\log\sigma(s)+C_2\zeta(s)+C_3\log\sigma(s-s_2)$$
$$-B_1\log\Big[\xi_{30}(s)-\xi_{20}(s)\Big]-B_2\xi_{10}(s)$$
$$-B_3\log\Big[\xi_{30}(s-s_2)-\xi_{20}(s-s_2)\Big]\Big\}, \qquad (77)$$

or, remembering the relation $C_3=B_3$, we have

$$z=-\frac{2\psi_0\omega_1 e^{-i\beta}}{\pi U(e_2-e_3)}\Big\{C_0 s+C_1\log\sigma(s)+C_2\zeta(s)$$
$$-B_1\log\Big[\xi_{30}(s)-\xi_{20}(s)\Big]-B_2\xi_{10}(s)$$
$$+B_3\log\Big[\sigma_2(s-s_2)+\sigma_3(s-s_2)\Big]\Big\}, \qquad (78)$$

where an arbitrary constant of integration has been neglected.

§ 8. When we start from a point in the z-plane and arrive, after encircling once round the plate, at that point, z must return to its original value; in other words, z must be a one-valued function. This condition requires that

$$\oint dz=0, \qquad (79)$$

where the integral is taken round a closed contour surrounding the plate.

Since, however, such a closed contour corresponds to a contour Σ in the ring region in the Z-plane surrounding the inner circle, (79) can be replaced by

$$\oint_{\Sigma} \left(\frac{dz}{ds}\right)\frac{dZ}{Z} = 0,$$

which further takes the following form, if we substitute the expression for dz/ds given by (53),

$$\oint_{\Sigma} F(s)\frac{dZ}{Z} = 0. \tag{80}$$

To this integral, only a constant term in the LAURENT expansion of the function $F(s)$ makes a contribution, and since $F(s) = F_1(s) - F_2(s)$, we have next to obtain the constant terms in the LAURENT expansions of the functions $F_1(s)$ and $F_2(s)$:

$$F_1(s) = C_0 + C_1\zeta(s) + C_2\zeta'(s) + C_3\zeta(s-s_2),$$

$$F_2(s) = B_1\xi_{10}(s) + B_2\xi'_{10}(s) + B_3\xi_{10}(s-s_2).$$

We have the expansion formula for ζ function:

$$\zeta(u) = \frac{\eta_1 u}{\omega_1} + \frac{\pi}{2\omega_1}\cot\frac{\pi u}{2\omega_1} + \frac{2\pi}{\omega_1}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi u}{\omega_1}. \tag{81}$$

Putting in this $u = s = \omega_1 + \omega_3 - \frac{\omega_1}{i\pi}\log Z$ and then remembering that $\left|\exp\left(\frac{i\pi}{\omega_1}s\right)\right| = |qZ^{-1}| < 1$, we get

$$\zeta(s) = \frac{\eta_1}{\omega_1}\left(\omega_1 + \omega_3 - \frac{\omega_1}{i\pi}\log Z\right) - \frac{i\pi}{2\omega_1}$$
$$+ (\text{positive and negative integral powers of } Z). \tag{82}$$

Using this and taking account of the obvious relations:

$$\zeta'(s) = \frac{d\zeta}{ds} = \frac{d\zeta}{dZ}\frac{dZ}{ds} = -\frac{i\pi}{\omega_1}Z\frac{d\zeta}{dZ},$$

we get the expansion for $\zeta'(s)$ in the form:

$$\zeta'(s) = \frac{\eta_1}{\omega_1} + (\text{positive and negative integral powers of } Z) . \quad (83)$$

Next, we put in (81) $u = s - s_2 = \frac{\omega_1}{\pi}\theta_2 - \frac{\omega_1}{i\pi}\log Z$. Then, remembering that $\left|\exp\left[\frac{i\pi}{\omega_1}(s-s_2)\right]\right| = |e^{i\theta_2}Z^{-1}| = |Z^{-1}| > 1$, we have

$$\zeta(s-s_2) = \frac{\eta_1}{\omega_1}\left(\omega_1 + \omega_3 - \frac{\omega_1}{i\pi}\log Z - s_2\right) + \frac{i\pi}{2\omega_1}$$
$$+ (\text{positive and negative integral powers of } Z) . \quad (84)$$

Thus, the expansion for $F_1(s)$ becomes:

$$F_1(s) = C_0 + (C_1 + C_3)\frac{\eta_1}{\omega_1}\left(\omega_1 + \omega_3 - \frac{\omega_1}{i\pi}\log Z\right)$$
$$- \frac{i\pi}{2\omega_1}(C_1 - C_3) + \frac{\eta_1}{\omega_1}(C_2 - C_3 s_2)$$
$$+ (\text{positive and negative integral powers of } Z) ,$$

or, since $C_1 + C_3 = 0$,

$$F_1(s) = C_0 + \frac{i\pi}{\omega_1}C_3 + \frac{\eta_1}{\omega_1}(C_2 - C_3 s_2)$$
$$+ (\text{positive and negative integral powers of } Z) . \quad (85)$$

We have next to expand the function $F_2(s)$ in powers of Z. For that purpose we shall expand the functions $\xi_{10}(s)$, $\xi'_{10}(s)$ and $\xi_{10}(s-s_2)$ in powers of Z respectively.

Now, we have

$$\xi_{10}(u) = \frac{1}{2\omega_1}\frac{\vartheta_1'(0)\vartheta_2\left(\frac{u}{2\omega_1}\right)}{\vartheta_2(0)\vartheta_1\left(\frac{u}{2\omega_1}\right)} , \quad (86)$$

and this can be put in the following two forms, namely:

$$\xi_{10}(u) = \frac{1}{2\omega_1 i} \frac{\vartheta_1'(0)\vartheta_4\left(\frac{u}{2\omega_1} - \frac{1+\tau}{2}\right)}{\vartheta_2(0)\vartheta_3\left(\frac{u}{2\omega_1} - \frac{1+\tau}{2}\right)}, \tag{87}$$

and

$$\xi_{10}(u) = \frac{i}{2\omega_1} \frac{\vartheta_1'(0)\vartheta_4\left(\frac{u}{2\omega_1} + \frac{1+\tau}{2}\right)}{\vartheta_2(0)\vartheta_3\left(\frac{u}{2\omega_1} + \frac{1+\tau}{2}\right)}. \tag{88}$$

In general, when a complex quantity v satisfies the inequality:

$$-\Re\left(\frac{\tau}{i}\right) < 2\Re\left(\frac{v}{i}\right) < \Re\left(\frac{\tau}{i}\right),$$

where $\tau = \omega_3/\omega_1$ and $\Re(z)$ means "the real part of z", we have the expansion formula[1]:

$$\frac{1}{4\pi} \frac{\vartheta_1'(0)\vartheta_4(v)}{\vartheta_2(0)\vartheta_3(v)} = \frac{1}{4} + \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1+q^{2n}} \cos 2n\pi v. \tag{89}$$

Therefore, if

$$-\Re\left(\frac{\tau}{i}\right) < 2\Re\left(\frac{1}{i}\left[\frac{u}{2\omega_1} - \frac{\tau}{2}\right]\right) < \Re\left(\frac{\tau}{i}\right),$$

we have, by (87) and (89),

$$\begin{aligned}\xi_{10}(u) &= \frac{1}{2\omega_1 i} \frac{\vartheta_1'(0)\vartheta_4\left(\frac{u}{2\omega_1} - \frac{1+\tau}{2}\right)}{\vartheta_2(0)\vartheta_3\left(\frac{u}{2\omega_1} - \frac{1+\tau}{2}\right)} \\ &= \frac{\pi}{2\omega_1 i} + \frac{2\pi}{\omega_1 i} \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1+q^{2n}} \cos 2n\pi\left(\frac{u}{2\omega_1} - \frac{1+\tau}{2}\right).\end{aligned} \tag{90}$$

---

(1) J. Tannery et J. Molk, loc. cit., 4 (1902), 105.

In the present problem, when s varies from 0 to ω_3, the quantity $\left(\frac{s}{2\omega_1}-\frac{\tau}{2}\right)$ varies from $-\tau/2$ to 0, so that s satisfies the inequality:

$$-\Re\left(\frac{\tau}{i}\right)<2\Re\left(\frac{1}{i}\left[\frac{s}{2\omega_1}-\frac{\tau}{2}\right]\right)<\Re\left(\frac{\tau}{i}\right),$$

which is the same as the above inequality satisfied by u. Thus, we get

$$\xi_{10}(s)=\frac{\pi}{2\omega_1 i}$$
$$+\text{(positive and negative integral powers of } Z\text{)}. \quad (91)$$

By the aid of the obvious relations:

$$\xi_{10}'(s)=\frac{d\xi_{10}}{ds}=\frac{d\xi_{10}}{dZ}\frac{dZ}{ds}=-\frac{i\pi}{\omega_1}Z\frac{d\xi_{10}}{dZ},$$

it follows from (91) that the expansion in terms of Z for $\xi_{10}'(s)$ has no constant term.

Also, when

$$-\Re\left(\frac{\tau}{i}\right)<2\Re\left(\frac{1}{i}\left[\frac{u}{2\omega_1}+\frac{\tau}{2}\right]\right)<\Re\left(\frac{\tau}{i}\right),$$

we have, by (88) and (89), the following expansion:

$$\xi_{10}(u)=\frac{i}{2\omega_1}\frac{\vartheta_1'(0)\vartheta_4\left(\frac{u}{2\omega_1}+\frac{1+\tau}{2}\right)}{\vartheta_2(0)\vartheta_3\left(\frac{u}{2\omega_1}+\frac{1+\tau}{2}\right)}$$
$$=\frac{i\pi}{2\omega_1}+\frac{2i\pi}{\omega_1}\sum_{n=1}^{\infty}(-1)^n\frac{q^n}{1+q^{2n}}\cos 2n\pi\left(\frac{u}{2\omega_1}+\frac{1+\tau}{2}\right). \quad (92)$$

Since $s_2=x_2+\omega_3$, it will easily be seen that $s-s_2$ satisfies the inequality:

$$-\Re\left(\frac{\tau}{i}\right)<2\Re\left(\frac{1}{i}\left[\frac{s-s_2}{2\omega_1}+\frac{\tau}{2}\right]\right)<\Re\left(\frac{\tau}{i}\right),$$

and this is the same as the above inequality satisfied by u. Therefore, we get

$$\xi_{10}(s-s_2) = \frac{i\pi}{2\omega_1}$$
$$+(\text{positive and negative integral powers of } Z). \quad (93)$$

Thus, we have

$$F_2(s) = \frac{\pi}{2\omega_1 i}(B_1-B_3)$$
$$+(\text{positive and negative integral powers of } Z). \quad (94)$$

Finally we have, by (85) and (94),

$$F(s) = F_1(s)-F_2(s)$$
$$= C_0+\frac{i\pi}{\omega_1}C_3+\frac{\eta_1}{\omega_1}(C_2-C_3 s_2)-\frac{\pi}{2\omega_1 i}(B_1-B_3)$$
$$+(\text{positive and negative integral powers of } Z), \quad (95)$$

and, as mentioned already, only the constant term in this expansion makes a contribution to the integral on the left-hand side of (80). Therefore, the condition that z must be a one-valued function requires that

$$C_0+\frac{i\pi}{\omega_1}C_3+\frac{\eta_1}{\omega_1}(C_2-C_3 s_2)-\frac{\pi}{2\omega_1 i}(B_1-B_3) = 0. \quad (96)$$

This equation can be transformed as follows. By (70), we have

$$C_0 = -C_1\zeta(s_1)-C_2\zeta'(s_1)-C_3\zeta(s_1-s_2),$$

and therefore since $s_1 = x_1+\omega_3$, $s_2 = x_2+\omega_3$,

$$C_0 = -C_1\left\{\zeta_3(x_1)+\eta_3\right\}-C_2\zeta_3'(x_1)-C_3\zeta(x_1-x_2).$$

Inserting this in (96) and remembering that $C_1 = -C_3$, $C_3 = B_3$, we have

$$C_1\zeta_3(x_1)+C_2\zeta_3'(x_1)+C_3\zeta(x_1-x_2)-\frac{\eta_1}{\omega_1}C_2+\frac{\eta_1}{\omega_1}x_2C_3-\frac{i\pi}{2\omega_1}B_1$$

$$+C_3\frac{1}{\omega_1}\left(\eta_1\omega_3-\eta_3\omega_1-\frac{i\pi}{2}\right)=0\,.$$

However, the well-known LEGENDRE's relation gives

$$\eta_1\omega_3-\eta_3\omega_1-\frac{i\pi}{2}=0\,.$$

Thus, using again the relation $C_1=-C_3$, we get

$$C_1\left[\zeta_3(x_1)-\frac{\eta_1x_1}{\omega_1}\right]+C_2\left[\zeta_3'(x_1)-\frac{\eta_1}{\omega_1}\right]$$

$$+C_3\left[\zeta(x_1-x_2)-\frac{\eta_1}{\omega_1}(x_1-x_2)\right]=\frac{i\pi}{2\omega_1}B_1\,. \qquad (97)$$

§9. For the sake of later use, we shall here summarize, in revised forms, the values of the constants C_1, C_2, C_3, B_1, B_2 and B_3 given by (68), (69), (74) and (75). Using the obvious result that

$$\xi_{10}(s_1)-\xi_{10}(s_2)=-i\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\,,$$

which is purely imaginary, we have

$$\left.\begin{aligned}C_1&=-(e_2-e_3)\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\sin\beta\,,\\ C_2&=(e_2-e_3)\cos\beta\,,\\ C_3&=(e_2-e_3)\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\sin\beta\,;\end{aligned}\right\} \qquad (98)$$

$$\left.\begin{aligned}B_1&=-i(e_2-e_3)\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\cos\beta\,,\\ B_2&=-i(e_2-e_3)\sin\beta\,,\\ B_3&=(e_2-e_3)\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\sin\beta\,.\end{aligned}\right\} \qquad (99)$$

Also, C_0 becomes:

$$C_0 = (e_2-e_3)\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\left\{\zeta_3(x_1)-\zeta(x_1-x_2)+\eta_3\right\}\sin\beta$$
$$-(e_2-e_3)\zeta_3'(x_1)\cos\beta\,. \qquad (100)$$

Substituting these values in (97) and simplifying, we get

$$\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\left[\left\{\zeta_3(x_1)-\frac{\eta_1 x_1}{\omega_1}\right\}-\left\{\zeta(x_1-x_2)-\frac{\eta_1}{\omega_1}(x_1-x_2)\right\}\right]\sin\beta$$
$$=\left[\zeta_3'(x_1)-\frac{\eta_1}{\omega_1}-\frac{\pi}{2\omega_1}\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\right]\cos\beta\,. \qquad (101)$$

This can be expressed in terms of ϑ functions in the following manner. We have

$$\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\} = \frac{\pi}{2\omega_1}\vartheta_3(0)\vartheta_4(0)\left\{\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}-\frac{\vartheta_3\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}\right\}; \qquad (102)$$

and

$$\left.\begin{aligned}\zeta_3(x_1)-\frac{\eta_1 x_1}{\omega_1} &= \frac{1}{2\omega_1}\frac{\vartheta_4'\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)},\\ \zeta(x_1-x_2)-\frac{\eta_1}{\omega_1}(x_1-x_2) &= \frac{1}{2\omega_1}\frac{\vartheta_1'\left(\frac{x_1-x_2}{2\omega_1}\right)}{\vartheta_1\left(\frac{x_1-x_2}{2\omega_1}\right)}.\end{aligned}\right\} \qquad (103)$$

Also, with the help of the formula[1]:

$$\zeta_3'(u) = (e_1-e_3)\left[\xi_{23}(u)\right]^2-e_1\,,$$

(1) J. Tannery et J. Molk, loc. cit., 2 (1896), 281.

we have

$$\zeta_3'(x_1)-\frac{\eta_1}{\omega_1}=\frac{\pi^2}{4\omega_1^2}\left[\vartheta_3(0)\vartheta_4(0)\right]^2\left[\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}\right]^2-\left(\frac{\eta_1}{\omega_1}+e_1\right)$$

$$=\frac{\pi^2}{4\omega_1^2}\left[\vartheta_3(0)\vartheta_4(0)\right]^2\left[\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}\right]^2+\frac{1}{4\omega_1^2}\frac{\vartheta_2''(0)}{\vartheta_2(0)}. \quad (104)$$

Therefore, substituting all these results in (101) we get

$$\vartheta_3(0)\vartheta_4(0)\left\{\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}-\frac{\vartheta_3\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}\right\}\frac{1}{\pi}\left\{\frac{\vartheta_4'\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}-\frac{\vartheta_1'\left(\frac{x_1-x_2}{2\omega_1}\right)}{\vartheta_1\left(\frac{x_1-x_2}{2\omega_1}\right)}\right\}\sin\beta$$

$$=\left\{\left[\vartheta_3(0)\vartheta_4(0)\right]^2\left[\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}\right]^2\right.$$

$$\left.+\frac{1}{\pi^2}\frac{\vartheta_2''(0)}{\vartheta_2(0)}-\vartheta_3(0)\vartheta_4(0)\left[\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}-\frac{\vartheta_3\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}\right]\right\}\cos\beta. \quad (105)$$

Being the second equation, this determines, in conjunction with the first equation (50), the values of the two constants x_1 and x_2.

§ 10. Next, we shall obtain the expression for the breadth of the plate, which will be denoted by $2a$. We have evidently

$$z_{A'}-z_A=2ae^{i\left(\frac{\pi}{2}+\delta\right)}$$

$$=-2ae^{-i\beta}. \quad (106)$$

Since, however, the points A′ and A in the z-plane correspond to $s=s_2$ and $s=s_1$ respectively, we have by (78),

$$z_{A'}-z_A = -\frac{2\psi_0\omega_1 e^{-i\beta}}{\pi U(e_2-e_3)}\Big\{C_0(s_2-s_1)+C_1\log\frac{\sigma(s_2)}{\sigma(s_1)}+C_2\Big[\zeta(s_2)-\zeta(s_1)\Big]$$
$$-B_1\log\frac{\xi_{30}(s_2)-\xi_{20}(s_2)}{\xi_{30}(s_1)-\xi_{20}(s_1)}-B_2\Big[\xi_{10}(s_2)-\xi_{10}(s_1)\Big]$$
$$-B_3\log\frac{1}{2}\Big[\sigma_2(s_1-s_2)+\sigma_3(s_1-s_2)\Big]\Big\}\,. \tag{107}$$

In the right-hand side of this equation we put $s_1=x_1+\omega_3$, $s_2=x_2+\omega_3$ and substitute the values of the constants C_0, C_1, C_2, C_3, B_1, B_2, B_3 given by (98), (99) and (100) and we perform various calculations. Then we get

$$z_{A'}-z_A=\frac{2\psi_0\omega_1 e^{-i\beta}}{\pi U}\Big[\sqrt{e_1-e_3}\Big\{\xi_{23}(x_1)-\xi_{23}(x_2)\Big\}\sin\beta$$
$$\times\Big\{(x_1-x_2)\Big[\zeta_3(x_1)-\zeta(x_1-x_2)\Big]-\log\frac{\sigma_3(x_1)}{\sigma_3(x_2)}+1$$
$$+\log\frac{1}{2}\Big[\sigma_2(x_1-x_2)+\sigma_3(x_1-x_2)\Big]\Big\}$$
$$+\sqrt{e_1-e_3}\Big\{\xi_{23}(x_1)-\xi_{23}(x_2)\Big\}\cos\beta$$
$$\times\Big\{\tan^{-1}\Big(\sqrt{e_1-e_3}\,\xi_{01}(x_1)\Big)-\tan^{-1}\Big(\sqrt{e_1-e_3}\,\xi_{01}(x_2)\Big)\Big\}$$
$$+\cos\beta\Big\{-(x_1-x_2)\zeta_3'(x_1)-\zeta_3(x_2)+\zeta_3(x_1)\Big\}\Big]\,. \tag{108}$$

Thus, combining this with (106), we get the expression for $2a$ in the form:

$$
\begin{aligned}
2a = -\frac{2\psi_0\omega_1}{\pi U}\Bigg[&\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\sin\beta \\
&\times\left\{(x_1-x_2)\Big[\zeta_3(x_1)-\zeta(x_1-x_2)\Big]-\log\frac{\sigma_3(x_1)}{\sigma_3(x_2)}+1\right. \\
&\left.+\log\frac{1}{2}\Big[\sigma_2(x_1-x_2)+\sigma_3(x_1-x_2)\Big]\right\} \\
&+\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\cos\beta \\
&\times\left\{\tan^{-1}\left(\sqrt{e_1-e_3}\,\xi_{01}(x_1)\right)-\tan^{-1}\left(\sqrt{e_1-e_3}\,\xi_{01}(x_2)\right)\right\} \\
&+\cos\beta\left\{-(x_1-x_2)\zeta_3'(x_1)-\zeta_3(x_2)+\zeta_3(x_1)\right\}\Bigg]. \qquad (109)
\end{aligned}
$$

This expression can however be simplified to some extent, by eliminating $\zeta_3'(x_1)$ by the aid of the relation:

$$
\begin{aligned}
-(x_1-x_2)\zeta_3'(x_1)\cos\beta = &\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\sin\beta \\
&\times\left\{-(x_1-x_2)\Big[\zeta_3(x_1)-\zeta(x_1-x_2)\Big]+\frac{\eta_1}{\omega_1}x_2(x_1-x_2)\right\} \\
&-(x_1-x_2)\cos\beta\left[\frac{\eta_1}{\omega_1}+\frac{\pi}{2\omega_1}\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\right],
\end{aligned}
$$

which follows immediately from (101). We have

$$
\begin{aligned}
2a = -\frac{2\psi_0\omega_1}{\pi U}\Bigg[&\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\sin\beta \\
&\times\left\{1+\frac{\eta_1}{\omega_1}x_2(x_1-x_2)-\log\frac{\sigma_3(x_1)}{\sigma_3(x_2)}+\log\frac{1}{2}\Big[\sigma_2(x_1-x_2)+\sigma_3(x_1-x_2)\Big]\right\} \\
&+\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\cos\beta \\
&\times\left\{\tan^{-1}\left(\sqrt{e_1-e_3}\,\xi_{01}(x_1)\right)-\tan^{-1}\left(\sqrt{e_1-e_3}\,\xi_{01}(x_2)\right)-\frac{\pi}{2\omega_1}(x_1-x_2)\right\} \\
&+\cos\beta\left\{\left[\zeta_3(x_1)-\frac{\eta_1}{\omega_1}x_1\right]-\left[\zeta_3(x_2)-\frac{\eta_1}{\omega_1}x_2\right]\right\}\Bigg]. \qquad (110)
\end{aligned}
$$

Now, let D be the distance at infinity upstream between the stream line $\psi = 0$ and the free surface. It will easily be seen that this quantity may be taken as a measure of the distance of the mid-point of the plate from the free surface in its undisturbed condition. Then, we have

$$\psi_0 = UD\,. \tag{111}$$

Substituting this in (110), we get

$$\begin{aligned}\frac{2a}{D} = -\frac{2\omega_1}{\pi}\Bigg[& \sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\sin\beta \\ & \quad\times\left\{1+\frac{\eta_1}{\omega_1}x_2(x_1-x_2)-\log\frac{\sigma_3(x_1)}{\sigma_3(x_2)}+\log\frac{1}{2}\Big[\sigma_2(x_1-x_2)+\sigma_3(x_1-x_2)\Big]\right\} \\ & +\sqrt{e_1-e_3}\left\{\xi_{23}(x_1)-\xi_{23}(x_2)\right\}\cos\beta \\ & \quad\times\left\{\tan^{-1}\left(\sqrt{e_1-e_3}\,\xi_{01}(x_1)\right)-\tan^{-1}\left(\sqrt{e_1-e_3}\,\xi_{01}(x_2)\right)-\frac{\pi}{2\omega_1}(x_1-x_2)\right\} \\ & +\cos\beta\left\{\left[\zeta_3(x_1)-\frac{\eta_1}{\omega_1}x_1\right]-\left[\zeta_3(x_2)-\frac{\eta_1}{\omega_1}x_2\right]\right\}\Bigg]\,. \end{aligned} \tag{112}$$

Further, by making use of (102), (103) and similar formulae, this can be expressed in terms of ϑ functions. We have

$$\begin{aligned}\frac{2a}{D} = \vartheta_3(0)\vartheta_4(0) & \left\{\frac{\vartheta_3\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}-\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}\right\} \\ & \times\left[\left\{1-\log\frac{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}+\log\frac{1}{2}\left[\frac{\vartheta_4\left(\frac{x_1-x_2}{2\omega_1}\right)}{\vartheta_4(0)}+\frac{\vartheta_3\left(\frac{x_1-x_2}{2\omega_1}\right)}{\vartheta_3(0)}\right]\right\}\sin\beta\right. \\ & \quad+\left\{\tan^{-1}\left[\frac{\vartheta_3(0)}{\vartheta_4(0)}\frac{\vartheta_1\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_2\left(\frac{x_1}{2\omega_1}\right)}\right]-\tan^{-1}\left[\frac{\vartheta_3(0)}{\vartheta_4(0)}\frac{\vartheta_1\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_2\left(\frac{x_2}{2\omega_1}\right)}\right]\right. \\ & \quad\left.\left.-\frac{\pi}{2\omega_1}(x_1-x_2)\right\}\cos\beta\right]+\frac{1}{\pi}\left\{\frac{\vartheta_4'\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}-\frac{\vartheta_4'\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}\right\}\cos\beta\,. \end{aligned} \tag{113}$$

IV. Calculation of the Lift.

§ 11. We shall proceed to the calculation of the resultant fluid pressure acting on the plate.

Although the singular point at the trailing edge of the plate can be removed by adopting the value given by (20) for the circulation κ, yet the leading edge remains still as the singular point of the flow and the fluid velocity there is infinite. If, however, we do not take this circumstance into consideration, as usually done in the force calculation in several aerodynamical problems relating to flat plates, where at least one edge of each plate is always a singular point, we can calculate the components (P_x, P_y) of the resultant fluid pressure exerting on the plate under consideration with the aid of the well-known BLASIUS's formula:

$$P_x - iP_y = \frac{1}{2} i\rho \oint_C \left(\frac{d\chi}{dz}\right)^2 dz, \tag{114}$$

where ρ is the density of the fluid concerned and C is any closed contour surrounding the plate. The integral is taken round C in the counter-clockwise sense, as indicated in the formula. Adopting this procedure we shall calculate the components of the force experienced by the plate, in the following lines.

For convenience in the evaluation of the integral in (114) we transform the integrand in such a way that the integration takes place in the Z-plane. We have in general

$$\int \left(\frac{d\chi}{dz}\right)^2 dz = \int \left(\frac{d\chi}{ds}\right)^2 \frac{ds}{dz} \frac{ds}{dZ} dZ = -\frac{\omega_1}{i\pi} \int \frac{d\chi}{dz} \frac{d\chi}{ds} \frac{dZ}{Z}. \tag{115}$$

Thus, on taking account of the fact that the process of going round the contour C in the z-plane in the counter-clockwise sense is equivalent to that of going round the corresponding contour Σ surrounding the inner circle in the Z-plane in the clockwise sense, we have

$$P_x - iP_y = -\frac{1}{2} \rho \frac{\omega_1}{\pi} \oint_\Sigma e^{-i\Omega} \frac{d\chi}{ds} \frac{dZ}{Z}, \tag{116}$$

since $d\chi/dz = e^{-i\Omega}$.

Now, we have from (41),

$$e^{-i\Omega} = \frac{Ue^{i\beta}}{e_2 - e_3}\Big[\xi_{20}(s+s_1) + \xi_{30}(s+s_1)\Big]\Big[\xi_{20}(s-s_2) + \xi_{30}(s-s_2)\Big],$$

and combining this with (22),

$$e^{-i\Omega}\frac{d\chi}{ds} = -\frac{2\psi_0\omega_1 Ue^{i\beta}}{\pi(e_2-e_3)} \times \frac{\sigma(s-s_1)}{\sigma(s-s_2)}\frac{[\sigma_2(s+s_1)+\sigma_3(s+s_1)][\sigma_2(s-s_2)+\sigma_3(s-s_2)]}{[\sigma(s_1)\sigma(s)]^2}. \quad (117)$$

We put

$$e^{-i\Omega}\frac{d\chi}{ds} = -\frac{2\psi_0\omega_1 Ue^{i\beta}}{\pi(e_2-e_3)}G(s), \quad (118)$$

where

$$G(s) = \frac{\sigma(s-s_1)}{\sigma(s-s_2)}\frac{[\sigma_2(s+s_1)+\sigma_3(s+s_1)][\sigma_2(s-s_2)+\sigma_3(s-s_2)]}{[\sigma(s_1)\sigma(s)]^2}. \quad (119)$$

Then, it will be seen that

$$G(s) = F_1(s) + F_2(s), \quad (120)$$

where $F_1(s)$ and $F_2(s)$ are the functions given by (55).

Also, we have

$$P_x - iP_y = \frac{\rho\psi_0\omega_1^2 Ue^{i\beta}}{\pi^2(e_2-e_3)}\oint_{\Sigma} G(s)\frac{dZ}{Z}. \quad (121)$$

To this integral, however, only the constant term in the Laurent expansion of the function $G(s)$ makes a contribution.

Since the expansions in powers of Z have been obtained for the functions $F_1(s)$ and $F_2(s)$, we can easily obtain the expansion for $G(s)$. The result is

$$G(s) = C_0 + \frac{i\pi}{\omega_1} C_3 + \frac{\eta_1}{\omega_1}(C_2 - C_3 s_2) + \frac{\pi}{2\omega_1 i}(B_1 - B_3)$$

$$+(\text{positive and negative integral powers of } Z)\,, \qquad (122)$$

or, since

$$C_0 + \frac{i\pi}{\omega_1} C_3 + \frac{\eta_1}{\omega_1}(C_2 - C_3 s_2) = \frac{\pi}{2\omega_1 i}(B_1 - B_3)\,,$$

which follows from (96),

$$G(s) = \frac{\pi}{\omega_1 i}(B_1 - B_3)$$

$$+(\text{positive and negative integral powers of } Z)\,. \qquad (123)$$

Therefore,

$$\oint_{\Sigma} G(s) \frac{dZ}{Z} = -\frac{2\pi^2}{\omega_1}(B_1 - B_3)\,, \qquad (124)$$

and

$$P_x - iP_y = -\frac{2\rho\psi_0 U\omega_1 e^{i\beta}}{e_2 - e_3}(B_1 - B_3)\,. \qquad (125)$$

From (99) we have

$$B_1 - B_3 = -i(e_2 - e_3)\sqrt{e_1 - e_3}\left\{\xi_{23}(x_1) - \xi_{23}(x_2)\right\}(\cos\beta - i\sin\beta)$$

$$= -i(e_2 - e_3)\sqrt{e_1 - e_3}\left\{\xi_{23}(x_1) - \xi_{23}(x_2)\right\}e^{-i\beta}\,.$$

Putting this in the right-hand side of (125) we get

$$P_x - iP_y = i2\rho\psi_0 U\omega_1\sqrt{e_1 - e_3}\left\{\xi_{23}(x_1) - \xi_{23}(x_2)\right\}\,, \qquad (126)$$

or, when expressed in terms of ϑ functions with the aid of (102),

$$P_x - iP_y = i\pi\rho\psi_0 U\vartheta_3(0)\vartheta_4(0)\left\{\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)} - \frac{\vartheta_3\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}\right\}. \qquad (127)$$

Separating real and imaginary parts on both sides of this equation, we have

$$\left.\begin{aligned} &P_x = 0\,, \\ &P_y = \pi\rho\psi_0 U\vartheta_3(0)\vartheta_4(0)\left\{\frac{\vartheta_3\left(\dfrac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\dfrac{x_2}{2\omega_1}\right)}-\frac{\vartheta_3\left(\dfrac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\dfrac{x_1}{2\omega_1}\right)}\right\}.\end{aligned}\right\} \tag{128}$$

Thus, if we denote by L the lift on the plate under consideration, L is equal to P_y. Hence we have

$$L = \pi\rho\psi_0 U\vartheta_3(0)\vartheta_4(0)\left\{\frac{\vartheta_3\left(\dfrac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\dfrac{x_2}{2\omega_1}\right)}-\frac{\vartheta_3\left(\dfrac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\dfrac{x_1}{2\omega_1}\right)}\right\}, \tag{129}$$

or, since $\psi_0 = UD$,

$$L = \pi\rho U^2 D\vartheta_3(0)\vartheta_4(0)\left\{\frac{\vartheta_3\left(\dfrac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\dfrac{x_2}{2\omega_1}\right)}-\frac{\vartheta_3\left(\dfrac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\dfrac{x_1}{2\omega_1}\right)}\right\}. \tag{130}$$

Now, it will be expected that when the distance D becomes infinitely large, i.e., when the bounding free surface removes to infinity, the expression (130) degenerates into the well-known expression for the lift acting on a flat plate placed in an unbounded stream.

When D becomes very large, $2a/D$ becomes very small, and it will easily be seen that q becomes also very small. From (113) we have approximately

$$\frac{2a}{D} = 8q\,. \tag{131}$$

Also, it follows from the two equations (50) and (105) which determine x_1 and x_2, that

$$\left.\begin{aligned} \frac{\pi x_1}{\omega_1} &\rightarrow -\left(\frac{\pi}{2}+\beta\right), \\ \frac{\pi x_2}{\omega_1} &\rightarrow \left(\frac{\pi}{2}-\beta\right), \end{aligned}\right\} \tag{132}$$

as $q \rightarrow 0$.

Further, using these limiting values of $\pi x_1/\omega_1$ and $\pi x_2/\omega_1$ we have approximately

$$\vartheta_3(0)\vartheta_4(0)\left\{\frac{\vartheta_3\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}-\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}\right\} = 8q\sin\beta\,, \tag{133}$$

when q is sufficiently small(1).

Thus, if we write

$$\lim_{D\to\infty} L = L_0\,, \tag{134}$$

we get, by (130), (131) and (133),

$$L_0 = 2\pi a U^2 \rho \sin\beta\,. \tag{135}$$

This is the well-known expression for the lift experienced by a flat plate with the breadth $2a$ when it is immersed in an unbounded stream of an incompressible perfect fluid of density ρ, its angle of attack and the fluid velocity at infinity being equal to β and U respectively.

Finally, dividing L by L_0 we have

$$\frac{L}{L_0} = \frac{D}{2a}\frac{1}{\sin\beta}\vartheta_3(0)\vartheta_4(0)\left\{\frac{\vartheta_3\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}-\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}\right\}\,. \tag{136}$$

(1) For these results, reference should be made to the approximate calculations developed on later pages.

When the values of x_1 and x_2 can be determined by solving the two equations (50) and (105), the value of the ratio $2a/D$ will be obtained by the formula (113), and then the value of L/L_0 will be calculated numerically by the above formula (136).

V. Approximate Formulae.

§ 12. Before proceeding to the numerical discussions of the values of $2a/D$ and L/L_0 by using the exact formulae (113) and (136), it will be of interest to obtain the approximate formulae for these quantities.

Now, it is convenient to collect here all those relations between various quantities that are going to be of use in the following lines. First come two equations which must be satisfied by the two quantities x_1 and x_2 simultaneously. They are given by (50) and (105), namely:

$$\tan^{-1}\left[\frac{\vartheta_3(0)}{\vartheta_4(0)}\frac{\vartheta_1\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_2\left(\frac{x_1}{2\omega_1}\right)}\right]+\tan^{-1}\left[\frac{\vartheta_3(0)}{\vartheta_4(0)}\frac{\vartheta_1\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_2\left(\frac{x_2}{2\omega_1}\right)}\right]+\beta=0, \quad (137)$$

and

$$\vartheta_3(0)\vartheta_4(0)\left\{\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}-\frac{\vartheta_3\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}\right\}\frac{1}{\pi}\left\{\frac{\vartheta_4'\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}-\frac{\vartheta_1'\left(\frac{x_1-x_2}{2\omega_1}\right)}{\vartheta_1\left(\frac{x_1-x_2}{2\omega_1}\right)}\right\}\sin\beta$$

$$=\left\{\left[\vartheta_3(0)\vartheta_4(0)\right]^2\left[\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}\right]^2+\frac{1}{\pi^2}\frac{\vartheta_2''(0)}{\vartheta_2(0)}\right.$$

$$\left.-\vartheta_3(0)\vartheta_4(0)\left[\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}-\frac{\vartheta_3\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}\right]\right\}\cos\beta. \quad (138)$$

Then comes the expression for $2a/D$ which is given by (113). We have

$$\frac{2a}{D}=\vartheta_3(0)\vartheta_4(0)\left\{\frac{\vartheta_3\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}-\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}\right\}$$

$$\times\left[\left\{1-\log\frac{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}+\log\frac{1}{2}\left[\frac{\vartheta_4\left(\frac{x_1-x_2}{2\omega_1}\right)}{\vartheta_4(0)}+\frac{\vartheta_3\left(\frac{x_1-x_2}{2\omega_1}\right)}{\vartheta_3(0)}\right]\right\}\sin\beta\right.$$

$$+\left\{\tan^{-1}\left[\frac{\vartheta_3(0)}{\vartheta_4(0)}\frac{\vartheta_1\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_2\left(\frac{x_1}{2\omega_1}\right)}\right]-\tan^{-1}\left[\frac{\vartheta_3(0)}{\vartheta_4(0)}\frac{\vartheta_1\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_2\left(\frac{x_2}{2\omega_1}\right)}\right]\right.$$

$$\left.\left.-\frac{\pi}{2\omega_1}(x_1-x_2)\right\}\cos\beta\right]+\frac{1}{\pi}\left\{\frac{\vartheta_4'\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}-\frac{\vartheta_4'\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}\right\}\cos\beta\,. \tag{139}$$

Lastly, the expression for L/L_0 is

$$\frac{L}{L_0}=\frac{D}{2a}\frac{1}{\sin\beta}\vartheta_3(0)\,\vartheta_4(0)\left\{\frac{\vartheta_3\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}-\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}\right\}. \tag{140}$$

To solve the two equations (137) and (138) approximately, we assume

$$\left.\begin{aligned}\frac{\pi x_1}{\omega_1}&=a_0+a_1q+a_2q^2+a_3q^3+\cdots\cdots,\\ \frac{\pi x_2}{\omega_1}&=b_0+b_1q+b_2q^2+b_3q^3+\cdots\cdots,\end{aligned}\right\} \tag{141}$$

Putting these in (137) and (138) and performing lengthy and tedious calculations the values of the coefficients a_0, b_0, a_1, b_1, etc. have been found. The results are

$$\left.\begin{aligned} a_0 &= -\left(\frac{\pi}{2}+\beta\right), \\ b_0 &= \left(\frac{\pi}{2}-\beta\right); \end{aligned}\right\} \tag{142}$$

$$\left.\begin{aligned} a_1 &= 2\cos\beta, \\ b_1 &= -2\cos\beta; \end{aligned}\right\} \tag{143}$$

$$\left.\begin{aligned} a_2 &= 4\sin 2\beta, \\ b_2 &= 0; \end{aligned}\right\} \tag{144}$$

$$\left.\begin{aligned} a_3 &= 5\cos\beta-\frac{23}{3}\cos 3\beta, \\ b_3 &= 3\cos\beta-\frac{1}{3}\cos 3\beta. \end{aligned}\right\} \tag{145}$$

Thus,

$$\left.\begin{aligned} \frac{\pi x_1}{\omega_1} &= -\left(\frac{\pi}{2}+\beta\right)+2q\cos\beta+4q^2\sin 2\beta \\ &\qquad +q^3\left(5\cos\beta-\frac{23}{3}\cos 3\beta\right)+\cdots\cdots, \\ \frac{\pi x_2}{\omega_1} &= \left(\frac{\pi}{2}-\beta\right)-2q\cos\beta+q^3\left(3\cos\beta-\frac{1}{3}\cos 3\beta\right)+\cdots\cdots. \end{aligned}\right\} \tag{146}$$

Next, on substituting these q-expansion formulae for $\pi x_1/\omega_1$ and $\pi x_2/\omega_1$ in (139) we get the approximate expression for $2a/D$ in the form of a power series of q, namely:

$$\begin{aligned} \frac{2a}{D} &= 8q-32q^2\sin\beta+q^3(64-48\cos^2\beta) \\ &\qquad +\frac{64}{3}q^4\sin\beta(7\cos^2\beta-2)+\cdots\cdots. \end{aligned} \tag{147}$$

From this equation we can obtain q as a power series of $2a/D$. The result is

$$q = \frac{1}{8}\left(\frac{2a}{D}\right) + \frac{1}{16}\sin\beta\left(\frac{2a}{D}\right)^2 + \left(\frac{3}{64} - \frac{13}{256}\cos^2\beta\right)\left(\frac{2a}{D}\right)^3$$
$$+ \left(\frac{31}{768} - \frac{41}{768}\cos^2\beta\right)\sin\beta\left(\frac{2a}{D}\right)^4 + \cdots\cdot \qquad (148)$$

In a similar manner, we have, with the help of (146),

$$\vartheta_3(0)\vartheta_4(0)\left\{\frac{\vartheta_3\left(\dfrac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\dfrac{x_2}{2\omega_1}\right)} - \frac{\vartheta_3\left(\dfrac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\dfrac{x_1}{2\omega_1}\right)}\right\}$$
$$= \sin\beta\left\{8q - 16q^3\cos^2\beta - 64q^4\cos^2\beta\sin\beta + \cdots\cdot\right\},$$

and therefore, substituting this in (140),

$$\frac{L}{L_0} = \frac{D}{2a}\left\{8q - 16q^3\cos^2\beta - 64q^4\cos^2\beta\sin\beta + \cdots\cdot\right\}. \qquad (149)$$

Then, if, in the right-hand side of this equation, we substitute for q the corresponding expansion formula given by (148), we get ultimately the expression for L/L_0 as a power series of $2a/D$:

$$\frac{L}{L_0} = 1 + \frac{1}{2}\sin\beta\left(\frac{2a}{D}\right) + \frac{1}{16}(6 - 7\cos^2\beta)\left(\frac{2a}{D}\right)^2$$
$$+ \frac{1}{96}(31 - 47\cos^2\beta)\sin\beta\left(\frac{2a}{D}\right)^3 + \cdots\cdot, \qquad (150)$$

or,

$$\frac{L}{L_0} = 1 + \frac{1}{2}\sin\beta\left(\frac{2a}{D}\right) + \frac{1}{16}(7\sin^2\beta - 1)\left(\frac{2a}{D}\right)^2$$
$$+ \frac{1}{96}(47\sin^2\beta - 16)\sin\beta\left(\frac{2a}{D}\right)^3 + \cdots\cdot. \qquad (151)$$

From this formula we can calculate the approximate values for L/L_0, and by comparing these approximate values with the accurate ones which would be obtained from the exact formula (140) we can find out the range in which the above approximate formula can be applied.

VI. Numerical Discussions.

§ 13. In order to calculate the exact values of $2a/D$ and L/L_0 by the respective exact formulae (139) and (140), we have to find the exact values of x_1 and x_2 by solving the two equations (137) and (138) simultaneously. However, these equations are so complicated that it is not possible to solve them straightforward with respect to the two quantities x_1 and x_2.

But, the values of x_1 and x_2 can be calculated approximately by their respective approximate expressions given in (146), which have been obtained from (137) and (138), and the approximation is especially good for small values of q. Therefore, starting from those approximate values we can obtain the exact values in the following manner.

Taking those approximate values as the first approximation we denote them by x_1^0 and x_2^0 respectively. Then, the exact values x_1, x_2 will be given by

$$\left.\begin{aligned} x_1 &= x_1^0+\delta_1 , \\ x_2 &= x_2^0+\delta_2 . \end{aligned}\right\} \qquad (152)$$

We put these in the equations (137) and (138), and then assuming that both δ_1 and δ_2 are small, we expand various functions in power series of δ_1 and δ_2. Then, if we retain only the first powers of δ_1 and δ_2 and neglect their second and higher powers, we get two simultaneous linear equations having δ_1, δ_2 as variables. Calculating the values of δ_1 and δ_2 from those linear equations and adding them to x_1^0 and x_2^0 respectively, we can obtain, by (152), the more accurate values for x_1 and x_2, which will be taken as the second approximation.

Next, starting from those second approximate values, we can obtain, by a similar process, the third approximate values.

Repeating similar calculations two or three times we can obtain the exact values of x_1 and x_2 which satisfy the two equations (137) and (138) simultaneously. The formulae necessary for those calculations will be described in the Appendix.

The exact values of x_1 and x_2 determined in this way for cases in which the angle of attack β is equal to 2°, 5°, 10° and 15° respectively are tabulated in the following tables.

TABLE I.

$(\beta = 2^\circ)$

q	$\frac{\pi x_1}{\omega_1}$	$\frac{\pi x_2}{\omega_1}$
0·05	$-86^\circ\ 15'\ 12''$	$82^\circ\ 17'\ 35''$
0·10	$-80^\circ\ 32'\ 39''$	$76^\circ\ 41'\ 44''$
0·20	$-69^\circ\ 37'\ 28''$	$66^\circ\ 8'\ 18''$
0·30	$-59^\circ\ 38'\ 50''$	$56^\circ\ 34'\ 42''$
0·40	$-50^\circ\ 32'\ 44''$	$47^\circ\ 52'\ 14''$

TABLE II.

$(\beta = 5^\circ)$

q	$\frac{\pi x_1}{\omega_1}$	$\frac{\pi x_2}{\omega_1}$
0·05	$-89^\circ\ 12'\ 43''$	$79^\circ\ 18'\ 42''$
0·10	$-83^\circ\ 21'\ 8''$	$73^\circ\ 44'\ 12''$
0·20	$-71^\circ\ 56'\ 27''$	$63^\circ\ 15'\ 37''$
0·30	$-61^\circ\ 26'\ 44''$	$53^\circ\ 50'\ 50''$
0·40	$-51^\circ\ 54'\ 54''$	$45^\circ\ 20'\ 11''$

TABLE III.

($\beta = 10^\circ$)

q	$\frac{\pi x_1}{\omega_1}$	$\frac{\pi x_2}{\omega_1}$
0·05	$-94^\circ\ 10'\ 41''$	$74^\circ\ 22'\ 38''$
0·10	$-88^\circ\ 5'\ 30''$	$68^\circ\ 52'\ 25''$
0·20	$-75^\circ\ 51'\ 38''$	$58^\circ\ 36'\ 2''$
0·30	$-64^\circ\ 28'\ 21''$	$49^\circ\ 29'\ 47''$
0·40	$-54^\circ\ 12'\ 10''$	$41^\circ\ 22'\ 15''$

TABLE IV.

($\beta = 15^\circ$)

q	$\frac{\pi x_1}{\omega_1}$	$\frac{\pi x_2}{\omega_1}$
0·05	$-99^\circ\ 11'\ 21''$	$69^\circ\ 29'\ 9''$
0·10	$-92^\circ\ 54'\ 50''$	$64^\circ\ 5'\ 45''$
0·20	$-79^\circ\ 52'\ 23''$	$54^\circ\ 6'\ 11''$
0·30	$-67^\circ\ 33'\ 21''$	$45^\circ\ 22'\ 32''$
0·40	$-56^\circ\ 30'\ 56''$	$37^\circ\ 41'\ 17''$

Next, using these values for x_1 and x_2 we have calculated the values of $2a/D$ and L/L_0 by the aid of the respective exact formulae (139) and (140). The results are shown in the following tables.

TABLE V.

($\beta = 2^\circ$)

q	$\frac{2a}{D}$	$\frac{L}{L_0}$
0·05	0·3992	0·9970
0·10	0·8052	0·9756
0·20	1·6887	0·8987
0·30	2·7586	0·8134
0·40	4·1506	0·7428

TABLE VI.

($\beta = 5°$)

q	$\frac{2a}{D}$	$\frac{L}{L_0}$
0·05	0·3951	1·0073
0 10	0·7893	0·9950
0·20	1·6306	0·9293
0·30	2·6373	0·8483
0·40	3·9423	0·7791

TABLE VII.

($\beta = 10°$)

q	$\frac{2a}{D}$	$\frac{L}{L_0}$
0·05	0·3884	1·0249
0·10	0·7635	1·0285
0·20	1·5377	0·9834
0·30	2·4479	0·9101
0·40	3·6219	0·8433

TABLE VIII.

($\beta = 15°$)

q	$\frac{2a}{D}$	$\frac{L}{L_0}$
0·05	0·3818	1·0425
0 10	0·7387	1·0631
0·20	1·4507	1·0410
0·30	2·2736	0·9765
0·40	3·3317	0·9124

The manner of variation of L/L_0 with respect to $2a/D$ is shown graphically in Fig. 7 by four curves corresponding to $\beta = 2^\circ$, 5°, 10° and 15° respectively.

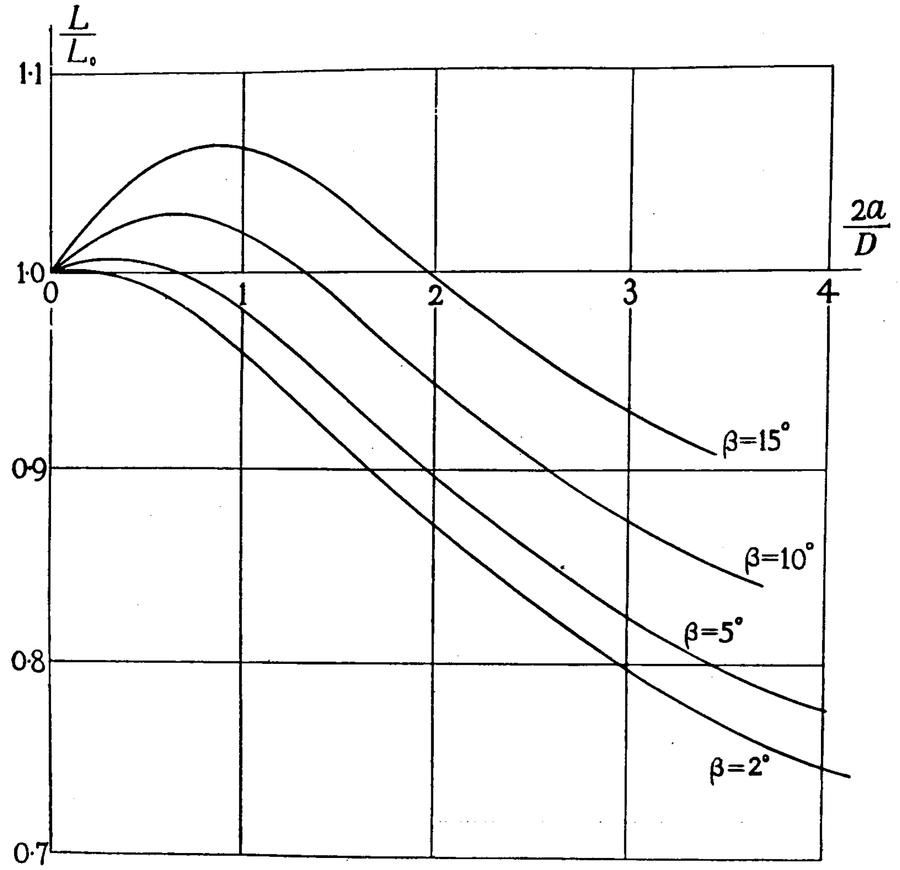


Fig. 7.

§ 14. Next, in order to find out the range in which the approximate formula (151) for L/L_0 can be used, the values of L/L_0 have been calculated by using the said approximate formula, which is given by, to the third power of $2a/D$,

$$\frac{L}{L_0} = 1 + \frac{1}{2}\sin\beta\left(\frac{2a}{D}\right) + \frac{1}{16}(7\sin^2\beta - 1)\left(\frac{2a}{D}\right)^2 + \frac{1}{96}(47\sin^2\beta - 16)\sin\beta\left(\frac{2a}{D}\right)^3. \quad (153)$$

Thus, when β is equal to 2°, 5°, 10° and 15° respectively, we have, in turn,

$$\frac{L}{L_0} = 1 + 0{\cdot}01745\left(\frac{2a}{D}\right) - 0{\cdot}06197\left(\frac{2a}{D}\right)^2 - 0{\cdot}00580\left(\frac{2a}{D}\right)^3, \quad (154)$$

$$\frac{L}{L_0} = 1 + 0{\cdot}04358\left(\frac{2a}{D}\right) - 0{\cdot}05918\left(\frac{2a}{D}\right)^2 - 0{\cdot}01420\left(\frac{2a}{D}\right)^3, \quad (155)$$

$$\frac{L}{L_0} = 1 + 0{\cdot}08682\left(\frac{2a}{D}\right) - 0{\cdot}04931\left(\frac{2a}{D}\right)^2 - 0{\cdot}02638\left(\frac{2a}{D}\right)^3, \quad (156)$$

and

$$\frac{L}{L_0} = 1 + 0{\cdot}12941\left(\frac{2a}{D}\right) - 0{\cdot}03319\left(\frac{2a}{D}\right)^2 - 0{\cdot}03465\left(\frac{2a}{D}\right)^3. \quad (157)$$

The approximate values of L/L_0 calculated by these formulae using the values of $2a/D$ given in Tables V—VIII are shown in the third column with the heading $(L/L_0)_{\text{appr.}}$ in each of the following four tables, where the exact values of L/L_0 have been reproduced for comparison in the fourth column with the heading L/L_0.

TABLE IX.

($\beta = 2^{\circ}$)

q	$\frac{2a}{D}$	$\left(\frac{L}{L_0}\right)_{\text{appr.}}$	$\frac{L}{L_0}$
0·05	0·3992	0·9967	0·9970
0·10	0·8052	0·9708	0·9756

TABLE X.

($\beta = 5^{\circ}$)

q	$\frac{2a}{D}$	$\left(\frac{L}{L_0}\right)_{\text{appr.}}$	$\frac{L}{L_0}$
0·05	0·3951	1·0071	1·0073
0·10	0·7893	0·9905	0·9950

TABLE XI.

($\beta = 10^\circ$)

q	$\frac{2a}{D}$	$\left(\frac{L}{L_0}\right)_{appr.}$	$\frac{L}{L_0}$
0·05	0·3884	1·0247	1·0249
0·10	0·7635	1 0258	1·0285

TABLE XII.

($\beta = 15^\circ$)

q	$\frac{2a}{D}$	$\left(\frac{L}{L_0}\right)_{appr.}$	$\frac{L}{L_0}$
0·05	0·3818	1·0426	1·0425
0·10	0·7387	1·0635	1·0631

From these tables it will be seen that when $2a/D < 0{\cdot}7$ the approximate formula (153) gives for L/L_0 good approximate values which are sufficient for practical purposes, and also it seems that the larger the angle of attack β, the better is the approximation.

Thus, using the above approximate formula, the values of L/L_0 have been calculated for various values of $2a/D$ less than 0·7, the results of which are shown graphically in Fig. 8, where some of the previous accurate results for larger values of q are also taken into account.

§ 15. From Figs. 7 and 8 it will be seen that the lift of a plate is considerably affected by the presence of a free surface which bounds the stream on the lower side of the plate. Thus, for small values of the angle of attack such as 2° or 5° the lift is somewhat increased when the distance D of the plate from the undisturbed free surface is large so that $2a/D$ is small, but the lift is rather decreased more and more due to the influence of the free surface as the plate approaches to the surface.

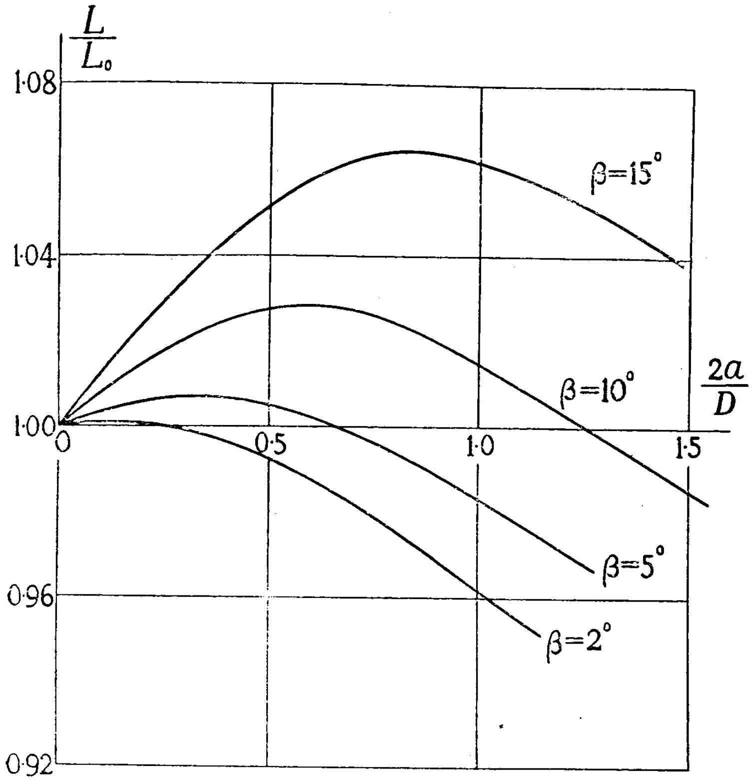


Fig. 8.

However, when the angle of attack takes a value such as 10° or 15° in the practically important range of values the lift is increased till the distance D becomes comparable with the breadth $2a$ of the plate. For instance, when $\beta = 10^\circ$ and $2a/D = 0{\cdot}6$, the increase in lift is about 3 per cent of the lift of the plate in an unbounded stream, and also when $\beta = 15^\circ$ and $2a/D = 0{\cdot}9$, the said increase is about 6·5 per cent. But, even when the angle of attack assumes practically important values such as 10° or 15°, the lift of the plate is rather decreased, as in the case of small values of the angle of attack, due to the effect of the free surface when the plate approaches sufficiently near to the surface.

Now, the result for a flat plate may be applied without serious error to the case of a thin aerofoil and therefore nearly the same conclusions may be given as to the effect of a free surface upon the lift of a thin aerofoil when it is placed in a semi-infinite stream bounded by the free surface on the lower side of the aerofoil.

It is known however that for ordinary aerofoils the maximum lift occurs when the angle of attack is nearly equal to 15°. Thus, applying our theoretical results to the case of a thin aerofoil, it may be concluded that the increase in maximum lift of the aerofoil due to the influence of a bounding free surface, which is on the lower side of the aerofoil, is about 6 per cent when the distance of the aerofoil from the free surface is nearly equal to its breadth.

As mentioned already in the Introduction, the boundary surface between the sea water and the air over it may be considered as a free surface, if the sea water is assumed to be at rest and the gravity is neglected, because the fluid pressure in the sea water is then everywhere constant so that it is constant also along the boundary surface. Thus, the results of our present theory may be applied to the important practical problem concerning the interference effect of the surface of the sea upon the lift of a seaplane flying over it, and therefore it may be expected that the maximum lift of a seaplane is increased by about 6 per cent due to the interference effect of the surface of the sea when the seaplane is taxi-ing over the surface so that the distance of the wing of the seaplane from the sea surface is comparable with the breadth of the wing.

This theoretical result may be compared with the result of full scale tests carried out at Felixstowe, England, which shows that the maximum lift of a seaplane is increased by about 10 per cent due to the interference effect of the sea surface(1). The agreement between the results of our theory and of full scale experiments is fairly satisfactory, in spite of the difference in the two cases that the flow is two-dimensional in our theoretical problem, while it is three-dimensional in full scale experiments.

(1) Report for the year 1933, National Physical Laboratory. 202.

VII. Summary.

§ 16. It is known that an appreciable interference effect occurs on the lift of a seaplane when it is flying near the surface of the sea. Full scale experiments carried out at Felixstowe, England, have shown that the maximum lift of a seaplane is increased by about 10 per cent due to the interference of the sea surface.

In view of the practical importance of the phenomenon it is of great interest to develop, if possible, a theory which could explain the phenomenon satisfactorily. It seems however that the ordinary vortex theory is incapable of explaining the said interference effect satisfactorily.

In the present paper, a rigorous mathematical analysis is developed for a hydrodynamical problem of calculating the lift on a flat plate placed in a two-dimensional continuous stream of fluid which is bounded by a free surface on the lower side of the plate. By carrying out long and tedious numerical calculations, the interference effect of the free surface upon the lift is discussed in detail, assuming various values for the angle of attack of the plate as well as for the distance of the plate from the free surface. It is shown that for practically important values of angle of attack such as 10° or 15° the lift is increased by a few per cent due to the presence of the free surface when the distance of the plate from the surface is of the same order of magnitude as the breadth of the plate.

Now, the boundary surface between the sea water and the air may be considered as a free surface along which the pressure is constant, if the sea water is assumed to be at rest and the gravity is neglected so that the fluid pressure is everywhere constant in the sea. Therefore, our problem has a close connection with the practical problem concerning the interference effect of the surface of the sea upon the lift of a seaplane while its taxi-ing over the sea surface, and the results of our theory may be applied to the practical case.

Thus, remembering that for ordinary aerofoil the maximum lift occurs when the angle of attack is nearly equal to 15°, it may be expected from our theory that the maximum lift of a seaplane is increased by about 6 per cent due to the interference effect of the surface of the sea when the aircraft is taxi-ing over the surface so that the distance of its wing from the surface is of the same order of magnitude as the breadth of the wing.

This theoretical result should be compared with the result of full scale tests mentioned above, and it will be seen that the agreement between the theory and experiments is satisfactory.

December, 1936.

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Appendix

Note on the Calculation of x_1 and x_2.

The two equations for determining the two real quantities x_1 and x_2 defined by (46) are given by (50) and (105). Expressing the first equation in a somewhat different form, we have

$$\frac{\vartheta_3(0)}{\vartheta_4(0)}\left\{\frac{\vartheta_1\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_2\left(\frac{x_1}{2\omega_1}\right)}+\frac{\vartheta_1\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_2\left(\frac{x_2}{2\omega_1}\right)}\right\}$$

$$+\tan\beta\left\{1-\left[\frac{\vartheta_3(0)}{\vartheta_4(0)}\right]^2\frac{\vartheta_1\left(\frac{x_1}{2\omega_1}\right)\vartheta_1\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_2\left(\frac{x_1}{2\omega_1}\right)\vartheta_2\left(\frac{x_2}{2\omega_1}\right)}\right\}=0,$$

and the second equation is

$$\vartheta_3(0)\vartheta_4(0)\left\{\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}-\frac{\vartheta_3\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}\right\}\frac{1}{\pi}\left\{\frac{\vartheta_4'\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}-\frac{\vartheta_1'\left(\frac{x_1-x_2}{2\omega_1}\right)}{\vartheta_1\left(\frac{x_1-x_2}{2\omega_1}\right)}\right\}\sin\beta$$

$$=\left\{\left[\vartheta_3(0)\vartheta_4(0)\right]^2\left[\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}\right]^2+\frac{1}{\pi^2}\frac{\vartheta_2''(0)}{\vartheta_2(0)}\right.$$

$$\left.-\vartheta_3(0)\vartheta_4(0)\left[\frac{\vartheta_3\left(\frac{x_1}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1}{2\omega_1}\right)}-\frac{\vartheta_3\left(\frac{x_2}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2}{2\omega_1}\right)}\right]\right\}\cos\beta\,.$$

To calculate the roots x_1 and x_2 of these simultaneous equations for given values of q and β, the angle of attack of the plate, the following procedure has been adopted in the present paper. First, we calculate the approximate values of x_1 and x_2 by using their respective approximate q-expansion formulae given in (146), namely:

$$\left.\begin{aligned}\frac{\pi x_1}{\omega_1} &= -\left(\frac{\pi}{2}+\beta\right)+2q\cos\beta+4q^2\sin 2\beta \\ &\qquad\qquad +q^3\left(5\cos\beta-\frac{23}{3}\cos 3\beta\right)+\cdots\cdots, \\ \frac{\pi x_2}{\omega_1} &= \left(\frac{\pi}{2}-\beta\right)-2q\cos\beta+q^3\left(3\cos\beta-\frac{1}{3}\cos 3\beta\right)+\cdots\cdots.\end{aligned}\right\}$$

We denote these first approximate values by x_1^0 and x_2^0 respectively. Then, the roots x_1 and x_2 will be expressed as:

$$\left.\begin{aligned}x_1 &= x_1^0+\delta_1\,, \\ x_2 &= x_2^0+\delta_2\,,\end{aligned}\right\}$$

where δ_1 and δ_2 are naturally small quantities in comparison with x_1^0 and x_2^0.

Putting x_1 and x_2 in the above two equations, we expand various functions in power series of δ_1 and δ_2. Thus, retaining only the first powers of δ_1 and δ_2, we get the simultaneous linear equations for determining the two quantities δ_1 and δ_2, namely:

$$\left.\begin{aligned}a_1\frac{\pi\delta_1}{2\omega_1}+a_2\frac{\pi\delta_2}{2\omega_1} &= a_3\,, \\ b_1\frac{\pi\delta_1}{2\omega_1}+b_2\frac{\pi\delta_2}{2\omega_1} &= b_3\,,\end{aligned}\right\}$$

from which we have

$$\left.\begin{aligned}\frac{\pi\delta_1}{2\omega_1} &= \frac{a_2b_3-a_3b_2}{a_2b_1-a_1b_2}\,, \\ \frac{\pi\delta_2}{2\omega_1} &= \frac{a_3b_1-a_1b_3}{a_2b_1-a_1b_2}\,.\end{aligned}\right\}$$

In these formulae, a_1, a_2, a_3, b_1, b_2, b_3 are the quantities defined as follows:

$$\left.\begin{aligned}
a_1 &= \left[\vartheta_2(0)\right]^2 \frac{\vartheta_3\left(\frac{x_1^0}{2\omega_1}\right)\vartheta_4\left(\frac{x_1^0}{2\omega_1}\right)}{\left[\vartheta_2\left(\frac{x_1^0}{2\omega_1}\right)\right]^2}\left\{1-\frac{\vartheta_3(0)}{\vartheta_4(0)}\,\frac{\vartheta_1\left(\frac{x_2^0}{2\omega_1}\right)}{\vartheta_2\left(\frac{x_2^0}{2\omega_1}\right)}\tan\beta\right\},\\
a_2 &= \left[\vartheta_2(0)\right]^2 \frac{\vartheta_3\left(\frac{x_2^0}{2\omega_1}\right)\vartheta_4\left(\frac{x_2^0}{2\omega_1}\right)}{\left[\vartheta_2\left(\frac{x_2^0}{2\omega_1}\right)\right]^2}\left\{1-\frac{\vartheta_3(0)}{\vartheta_4(0)}\,\frac{\vartheta_1\left(\frac{x_1^0}{2\omega_1}\right)}{\vartheta_2\left(\frac{x_1^0}{2\omega_1}\right)}\tan\beta\right\},\\
a_3 &= \left\{\frac{\vartheta_3(0)}{\vartheta_4(0)}\,\frac{\vartheta_1\left(\frac{x_1^0}{2\omega_1}\right)\vartheta_1\left(\frac{x_2^0}{2\omega_1}\right)}{\vartheta_2\left(\frac{x_1^0}{2\omega_1}\right)\vartheta_2\left(\frac{x_2^0}{2\omega_1}\right)}-\frac{\vartheta_4(0)}{\vartheta_3(0)}\right\}\tan\beta\\
&\qquad -\left\{\frac{\vartheta_1\left(\frac{x_1^0}{2\omega_1}\right)}{\vartheta_2\left(\frac{x_1^0}{2\omega_1}\right)}+\frac{\vartheta_1\left(\frac{x_2^0}{2\omega_1}\right)}{\vartheta_2\left(\frac{x_2^0}{2\omega_1}\right)}\right\};
\end{aligned}\right\}$$

$$\left.\begin{aligned}
b_1 &= AB_1+A_1B+C\cot\beta\,,\\
b_2 &= AB_2+A_2B\,,\\
b_3 &= AB-(B_1+B_2)\cot\beta\,,
\end{aligned}\right\}$$

where

$$A = \vartheta_3(0)\,\vartheta_4(0)\left\{\frac{\vartheta_3\left(\frac{x_2^0}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_2^0}{2\omega_1}\right)}-\frac{\vartheta_3\left(\frac{x_1^0}{2\omega_1}\right)}{\vartheta_4\left(\frac{x_1^0}{2\omega_1}\right)}\right\},$$

$$A_1 = -\vartheta_3(0)\,\vartheta_4(0)\left[\vartheta_2(0)\right]^2\frac{\vartheta_1\left(\frac{x_1^0}{2\omega_1}\right)\vartheta_2\left(\frac{x_1^0}{2\omega_1}\right)}{\left[\vartheta_4\left(\frac{x_1^0}{2\omega_1}\right)\right]^2},$$

$$A_2 = \vartheta_3(0)\,\vartheta_4(0)\left[\vartheta_2(0)\right]^2\frac{\vartheta_1\left(\frac{x_2^0}{2\omega_1}\right)\vartheta_2\left(\frac{x_2^0}{2\omega_1}\right)}{\left[\vartheta_4\left(\frac{x_2^0}{2\omega_1}\right)\right]^2};$$

$$B = \frac{1}{\pi}\left\{\frac{\vartheta_1'\left(\dfrac{x_1^0-x_2^0}{2\omega_1}\right)}{\vartheta_1\left(\dfrac{x_1^0-x_2^0}{2\omega_1}\right)} - \frac{\vartheta_4'\left(\dfrac{x_1^0}{2\omega_1}\right)}{\vartheta_4\left(\dfrac{x_1^0}{2\omega_1}\right)}\right\} - \cot\beta\ ,$$

$$B_1 = \left[\vartheta_3(0)\,\vartheta_4(0)\right]^2\left\{\left[\frac{\vartheta_2\left(\dfrac{x_1^0-x_2^0}{2\omega_1}\right)}{\vartheta_1\left(\dfrac{x_1^0-x_2^0}{2\omega_1}\right)}\right]^2 + \left[\frac{\vartheta_3\left(\dfrac{x_1^0}{2\omega_1}\right)}{\vartheta_4\left(\dfrac{x_1^0}{2\omega_1}\right)}\right]^2\right\},$$

$$B_2 = -\left[\vartheta_3(0)\vartheta_4(0)\right]^2\left[\frac{\vartheta_2\left(\dfrac{x_1^0-x_2^0}{2\omega_1}\right)}{\vartheta_1\left(\dfrac{x_1^0-x_2^0}{2\omega_1}\right)}\right]^2 + \frac{1}{\pi^2}\frac{\vartheta_2''(0)}{\vartheta_2(0)}\ ;$$

$$C = -2\left[\vartheta_2(0)\vartheta_3(0)\vartheta_4(0)\right]^2\frac{\vartheta_1\left(\dfrac{x_1^0}{2\omega_1}\right)\vartheta_2\left(\dfrac{x_1^0}{2\omega_1}\right)\vartheta_3\left(\dfrac{x_1^0}{2\omega_1}\right)}{\left[\vartheta_4\left(\dfrac{x_1^0}{2\omega_1}\right)\right]^3}\,.$$

If we add the values of δ_1 and δ_2 calculated in this way to x_1^0 and x_2^0 respectively we get the second approximation for the values of x_1 and x_2.

In a similar manner we can get the third and higher approximations, and finally the exact values of the roots x_1 and x_2 of the original equations can be obtained. However, it has been found from our calculations that for values of q less than 0·1, the second approximation gives the exact values for x_1 and x_2 and no further approximation is necessary, and that even for larger values of q the exact values of x_1 and x_2 can be obtained by repeating the above procedure twice or at most three times.