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抄　録

單葉翼の空氣力學的特性を簡單に計算する方法

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この論文は，單葉翼の空氣力學的特性を簡單に計算する方法に就いて述べたものである．先づ單純な主翼を對象とし，プラントルの積分方程式を逐次近似的に解いて，翼幅に沿ふ循環の分布を決定する．この際誘導速度の値を，數個の代表點の循環の値の一次的結合として與へる事により（近似積分に於ける平均値法），計算の勞力を著しく縮少する事が出來た．循環分布が決定されるならば，更に同樣の方法を用ひて，揚力係數，誘導抵抗係數等を簡單に求める事が出來る．次に下げ翼或は補助翼操作の場合を取扱ふために，操作の影響を楕圓翼に就いて求め，その結果を比例の假定によつて問題の翼に移す方法を採用した．この樣にして，下げ翼操作に基く循環分布，並びに揚力及び誘導抵抗の增加，補助翼操作に基く循環分布，並びに橫搖及び偏搖モーメント等を簡單に導く事が出來る．同じ方法を用ひて，橫搖及び偏搖に基くモーメントを計算する事も出來る．尙典型的な先細翼を例題的に取扱ふことによつて，計算方法の說明を補ひ，併せてその精度の吟味を試みた．併し一方に於て，種々の空氣力學的特性の數値を知る事も，實用上必要と思はれるので，直線的先細翼（梯形翼）の系統に就いて計算を實行し，その結果を多くの線圖によつて示した．この論文に述べた方法は勿論近似的なものであるけれども，その結果は實用上十分な精度を持ち，しかも所要の計算は，從來の方法に比べて極めて簡單であるから，飛行機設計その他の實際問題に有效に應用されるものと思はれる．解法の骨子は，旣に二三の機會に發表したものが多いが，この論文は個々の提案を一つの形式に整へ，足らぬ分を補ひ，實際問題に應用する目的に適ふ樣に纏めたものである．

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A Simple Method of Calculating the Aerodynamic Characteristics of a Monoplane Wing.

By

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Abstract.

The present paper deals with a simple method of determining the aerodynamic characteristics of a monoplane wing. First, a method is developed for calculating the spanwise lift distribution and other aerodynamic characteristics of a simple wing. The solution is by means of successive approximations, the required induced angle being readily computed by the method of mean value as used for approximate quadrature. The effects of landing flap displacement and aileron displacement are next considered, the calculation being simplified by considering the effects on a reference wing of elliptic plan form, and by modifying the results by assuming a simple relation of proportionality. Worked numerical examples are added in order to explain the application of the method as well as to check the accuracy of the method. The lateral stability derivatives are calculated in much the same way. Finally, with the object of providing numerical data for a number of aerodynamic characteristics, a series of straight tapered wings is calculated by the method here discussed.

CONTENTS.

INTRODUCTION.

§ 1. A knowledge of the spanwise lift distribution over a wing is important not only from structural considerations, but also from the fact that certain conclusions regarding the behaviour of the wing near the stall may be drawn from it. Indirectly, lift distribution affects also matters relating to performance and stability, such as the magnitude of the induced drag, the angle of no lift, the pitching moment about the aerodynamic centre, the effects of flaps and ailerons, the derivatives of lateral stability, etc. Although an accurate knowledge of lift distribution could be obtained from wind tunnel experiments, it in almost every case be drawn also from theoretical calculation, which is based on Prandtl's well-known wing theory. While this calculation is not so simple as to lend itself readily to designing room work, in view of the great importance this problem has assumed of late, it cannot be dispensed with. And, since to make sufficiently extensive investigations, both theoretical and experimental, with the object of obtaining numerical data for the general case, is impossible, owing to the large number of variables involved, in order to determine the lift distribution over a wing, a special calculation becomes necessary for every design.

To repeat the calculation for every design requires that the process and the labour of calculation shall be reduced to the simplest possible. Numerous methods for computing lift distribution have been proposed, chief of which are those due to Betz (Ref. 1), Trefftz (Ref. 2), Glauert (Ref. 3), Gates (Ref. 4), Lotz (Ref. 5), Lippisch (Ref. 6), and Wada (Ref. 7). The methods of Betz and Lotz give accurate results, but the calculation, besides demanding experience, is time-consuming, while those due to Trefftz, Glauert, and Gates are less simple in procedure, and less accurate in practice. Those of Lippisch and Wada seem best suited for practical work. One fault, however, runs through all these methods, and that is, that considerable difficulty is encountered in applying them to that case in which stalling spreads over a part of the wing, or in which the flaps or ailerons are displaced.

The writer now proposes a simple method of calculating the lift distribution over a monoplane wing. The method, true, is approximate, but it is sufficiently accurate for all practical purposes; its simplicity commends itself to calculations of the designing room. Although parts of the method have already been published (Refs. 8, 9, 10), since they amount to nothing less than a gist of the process, its practical application leaves something to be desired. In this paper, they are recapitulated, with the addition of new proposals and data in a form suitable for practical purposes; numerical examples in order to illustrate the application of the method are also added.

§ 2. The present paper is divided into five chapters. In Chapter I the fundamental concepts of wing theory are treated in the same convenient form as adopted in the chapters that follow it. Explanations of the symbols used are also given. In Chapter II will be found a method for calculating the lift distribution and other aerodynamic characteristics of a simple wing, which is an improvement on the writer's proposal in his first paper (Ref. 8). At the end of the chapter, the calculation is extended to the case of a large angle of incidence in which stalling spreads locally over the wing. The method of attack is based on the proposition given in the writer's third paper (Ref. 10). In Chapters III and IV, non-simple wings are considered, i.e., a simple method for calculating the effects due to wash out and flap displacement is given in Chapter III, while the effect due to aileron displacement will be found in Chapter IV. The attempt that was made in the writer's second paper (Ref. 9) to simplify the calculation of aileron effect has since been improved and extended to such a degree as to constitute the contents of Chapters III and IV. The latter half of Chapter IV deals with the calculation of lateral stability derivatives along similar lines. Numerical examples are worked out in order to illustrate the arguments in Chapters from II to IV as well as to facilitate the application of the method. Finally, in Chapter V, a series of straight tapered wings is calculated according to the method discussed in preceeding chapters in order to provide general numerical data for a number of aerodynamic characteristics.

CHAPTER I.

FUNDAMENTALS OF WING THEORY.

§ 3. *Fundamental equations of wing theory.* In Prandtl's wing theory, the wing is represented by a vortex filament, which is perpendicular to the direction of flight and whose strength is equal to the circulation Γ at a station at distance y along the filament, i.e. along the span of the wing. Assuming that the free vortex, which forms as a result of the variation in Γ along the span, extends downstream in the form of a plane of discontinuity, it will be seen that the effect is to reduce the angle of incidence of the section by the small angle

$$\varphi(y) = \frac{1}{4\pi V}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{d\Gamma}{dy'}\frac{dy'}{y-y'}, \qquad (1.1)$$

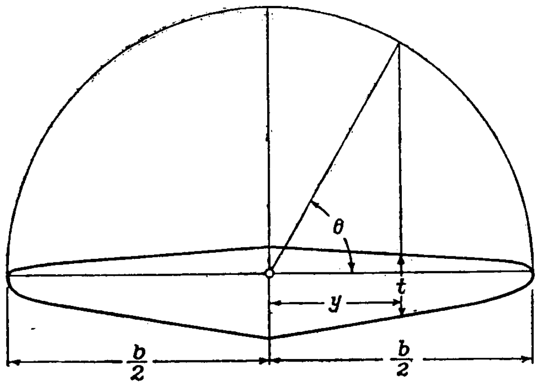

Fig. 1.

and that if α is the geometrical angle of incidence of the section, the effective incidence will be

$$\alpha_{\infty}(y) = \alpha(y) - \varphi(y) . \qquad (1.2)$$

Assuming that the aerofoil section behaves exactly the same as if it formed a part of a wing of infinite span at an angle of incidence α_{∞}, the lift coefficient of the section is given by the relation

$$c_z = c_z(\alpha_{\infty}) , \qquad (1.3)$$

which depends on the shape of the section. If the angles of incidence are measured from the attitude of no lift and are assumed to be small, the relation will be

$$c_z = a\alpha_{\infty} , \qquad (1.4)$$

where a is the slope of the curve of lift coefficient against the angle of incidence for the section in two dimensional flow. Moreover the circulation is given by

$$\Gamma = \frac{1}{2} Vtc_z , \qquad (1.5)$$

where V is the speed of flight and t is the chord of the section. Combining (1.2), (1.4), and (1.5); we have the equation

$$\Gamma(y) = \frac{1}{2} Va(y)t(y)\{\alpha(y)-\varphi(y)\} , \tag{1.6}$$

from which Γ is to be determined in terms of a, t, and α, which may of course vary across the span of the wing.

When the circulation Γ has been determined, the coefficients of lift and induced drag for the section are obtained by

$$c_z(y) = \frac{2\Gamma(y)}{Vt(y)} , \tag{1.7}$$

and

$$c_{xi}(y) = c_z(y)\varphi(y) = c_z(y)\left\{\alpha(y) - \frac{c_z(y)}{a(y)}\right\} \tag{1.8}$$

respectively. The coefficient of profile drag c_{xo} may be known in terms of c_z as section characteristics.

Although, strictly speaking, a should be regarded as a variable depending on the shape of the section, its variability may be neglected, and an appropriate mean value may be used without any appreciable loss of accuracy.

§ **4.** *Fourier representation of the circulation distribution.* Before dealing with the determination of Γ according to (1.6), it will be convenient here to write out the formulae giving the aerodynamic characteristics of the wing, when the coordinate y, measured to starboard along the span from its centre, is replaced by the angle θ by

$$y = -\frac{b}{2}\cos\theta , \tag{1.9}$$

and the circulation is expressed as the Fourier series

$$\Gamma = \frac{1}{2} Va\, t_0(K_1 \sin\theta + K_2 \sin 2\theta + K_3 \sin 3\theta + \cdots) , \tag{1.10}$$

where t_0 is the chord at the centre ($y = 0$, $\theta = \pi / 2$), a is the mean value of the slope of the curve of the lift coefficient across the span, and K_1, K_2, K_3, $\cdots$ are non-dimensional coefficients to be determined in accordance with (1.6). Substituting (1.9) and (1.10) in (1.1), we have

$$\varphi = \mu\left\{K_1 + 2K_2\frac{\sin 2\theta}{\sin\theta} + 3K_3\frac{\sin 3\theta}{\sin\theta} + \cdots\right\}, \qquad (1.11)$$

where

$$\mu = \frac{at_0}{4b}. \qquad (1.12)$$

The aerodynamic forces and moments of the wing are determined very readily in terms of the above Fourier coefficients. The lift of the wing is

$$Z = \int_{-\frac{b}{2}}^{\frac{b}{2}} \rho V\Gamma dy = \frac{\pi}{2}\rho V^2 b^2 \mu K_1, \qquad (1.13)$$

and the lift coefficient is

$$C_z = \frac{Z}{(\rho V^2/2)S} = \pi\mu A K_1, \qquad (1.14)$$

where A is the aspect ratio,

$$A = \frac{b^2}{S}, \qquad (1.15)$$

and S is the area of the wing. The induced drag and its coefficient are

$$X_i = \int_{-\frac{b}{2}}^{\frac{b}{2}} \rho V\Gamma\varphi dy = \frac{\pi}{2}\rho V^2 b^2 \mu^2 (K_1^2 + 2K_2^2 + 3K_3^2 + \cdots), \qquad (1.16)$$

$$C_{xi} = \frac{X_i}{(\rho V^2/2)S} = \pi\mu^2 A(K_1^2 + 2K_2^2 + 3K_3^2 + \cdots); \qquad (1.17)$$

the rolling moment and its coefficient are

$$L = -\int_{-\frac{b}{2}}^{\frac{b}{2}} \rho V\Gamma y\,dy = -\frac{\pi}{8}\rho V^2 b^3 \mu K_2\,, \tag{1.18}$$

$$C_l = \frac{L}{(\rho V^2/2)Sb} = -\frac{\pi}{4}\mu A K_2\,; \tag{1.19}$$

and the yawing moment and its coefficient are

$$N_i = \int_{-\frac{b}{2}}^{\frac{b}{2}} \rho V\Gamma\varphi y\,dy = \frac{\pi}{8}\rho V^2 b^3 \mu^2(3K_1K_2 + 5K_2K_3 + 7K_3K_4 + \cdots)\,, \tag{1.20}$$

$$C_{ni} = \frac{N_i}{(\rho V^2/2)Sb} = \frac{\pi}{4}\mu^2 A(3K_1K_2 + 5K_2K_3 + 7K_3K_4 + \cdots) \tag{1.21}$$

respectively. The moments are referred to the so-called wind axes, the positive moments being such that the starboard wing rolls down for L and yaws back for N_i.

The pitching moment M is referred to the transverse axis that passes through the leading edge of the centre of the wing, and which is reckoned as positive when the angle of incidence increases. In order to compute M, we assume the seat of circulation to be the locus of the aerodynamic centre of each section. Using the notations as shown in Fig. 2, we have then

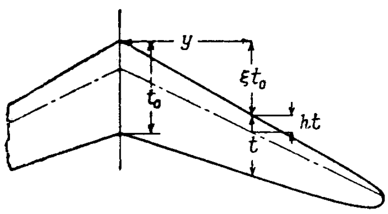

Fig. 2.

$$M = \int_{-\frac{b}{2}}^{\frac{b}{2}} \left\{\rho V\Gamma(\xi t_0 + ht) + c_{m0}\frac{\rho V^2}{2}t^2\right\} dy\,. \tag{1.22}$$

The coefficient of pitching moment is

$$C_m = \frac{M}{(\rho V^2/2)St_0}$$
$$= 4\mu A\int_0^1 (K_1 \sin\theta + K_3 \sin 3\theta + \cdots)\left(\xi + h\frac{t}{t_0}\right)d\left(\frac{y}{b/2}\right)$$
$$+\frac{4\mu A}{a}\int_0^1 c_{m0}\frac{t^2}{t_0^2}\, d\left(\frac{y}{b/2}\right), \tag{1.23}$$

in which h, the position of the aerodynamic centre, c_{mo}, the pitching moment cofficient about the aerodynamic centre, and ξ, the sweepback of the leading edge, are given as functions of y or θ.

In §5 we shall write the above expressions in a more convenient form.

§5. *Normal and zero distributions of circulation.* Since, generally speaking, the geometrical angle of incidence α varies across the span of the wing, it may be either expressed as

$$\alpha(y) = \alpha_m + f(\theta), \tag{1.24}$$

where α_m is a certain mean value, or as

$$\alpha(y)\sin\theta = D_1 \sin\theta + D_2 \sin 2\theta + D_3 \sin 3\theta + \cdots, \tag{1.25}$$

where D_1, D_2, $\cdots$ are constants.

In accordance with the expression (1.24), the distribution of circulation $\Gamma(y)$ may be regarded as consisting of two parts,

$$\Gamma(y) = \Gamma_a(y) + \Gamma_b(y), \tag{1.26}$$

where $\Gamma_a(y)$, the normal distribution, is the circulation when $\alpha = \alpha_m$, and $\Gamma_b(y)$, the zero distribution, is the circulation when $\alpha = f(\theta)$. The lift of the wing is determined by the normal distribution, the zero distribution modifying the shape of the circulation distribution withou altering the total lift. This definition determines the mean angle $\alpha_m^{(1)}$. It is convenient to write

$$K_j = A_j + B_j \quad (j = 1, 2, \ldots), \tag{1.27}$$

such that A_j and B_j are the Fourier coefficients of circulation distribution for Γ_a and Γ_b respectively. Since, according to (1.14), the lift is given by the coefficient K_1 and since the lift is given by the normal distribution alone, it follows that

$$B_1 = 0\,, \tag{1.28}$$

and since K_j for even j corresponds to the asymmetrical distribution of circulation,

$$A_2 = A_4 = A_6 = \cdots = 0\,. \tag{1.29}$$

Therefore the coefficients for normal distribution are $A_1, A_3, A_5, A_7, A_9, \cdots$, and the coefficients for zero distribution are $B_2, B_3, B_4, B_5, B_6, \cdots$, all the other coefficients being zero.

Since $A_1, A_3, A_5, \cdots$ are identical with $K_1, K_3, K_5, \cdots$ when α is taken to be equal to α_m in (1.8), (1.10), and (1.11), $A_1, A_3, A_5, \cdots$ are all proportional to the lift coefficient C_z. In other words, the normal distribution of circulation as a whole varies in proportion to C_z, the form of distribution remaining unchanged. Writing Γ_{a1} for the normal distribution for $C_z = 1$, (1.26) becomes

$$\Gamma(y) = \Gamma_{a1}(y)C_z + \Gamma_b(y)\,, \tag{1.30}$$

of which convenient use will be made in the next chapter.

We recapitulate here the various formulae of §4 in convenient forms. First, in order to treat the non-dimensional quantities, we replace y and Γ by η and G such that

$$\eta = \frac{y}{b/2}\,, \tag{1.31}$$

(1) See the end of this paragraph and "Relation between C_z and α_0" in § 10.

$$G(\eta) = \frac{\Gamma(y)}{bV} = 2\mu(K_1 \sin\theta + K_2 \sin 2\theta + K_3 \sin 3\theta + \cdots), \tag{1.32}$$

$$G(\eta) = G_{a1}(\eta)C_z + G_b(\eta), \tag{1.33}$$

$$G_{a1}(\eta) = \frac{G_a(\eta)}{C_z} = 2\mu\left(\frac{A_1}{C_z}\sin\theta + \frac{A_3}{C_z}\sin 3\theta + \frac{A_5}{C_z}\sin 5\theta + \cdots\right), \tag{1.34}$$

$$G_b(\eta) = 2\mu(B_2 \sin 2\theta + B_3 \sin 3\theta + B_4 \sin 4\theta + B_5 \sin 5\theta + \cdots). \tag{1.35}$$

Taking then (1.27), (1.28), and (1.29) into account, (1.14), (1.17), (1.19), and (1.21) may be written in the form

$$C_z = \pi\mu A A_1, \tag{1.36}$$

$$C_{xi} = \frac{C_z^2}{\pi A}\left\{1 + \frac{3K_3^2 + 5K_5^2 + \cdots}{A_1^2} + \frac{2B_2^2 + 4B_4^2 + \cdots}{A_1^2}\right\}, \tag{1.37}$$

$$C_l = -\frac{\pi}{4}\mu A B_2, \tag{1.38}$$

$$C_{ni} = -\frac{3C_zC_l}{\pi A}\left\{1 + \frac{5}{3}\frac{K_3}{A_1} + \frac{7}{3}\frac{K_3}{A_1}\frac{B_4}{B_2} + \frac{9}{3}\frac{K_5}{A_1}\frac{B_4}{B_2} + \cdots\right\} \tag{1.39}$$

respectively, and (1.23) transforms into

$$C_m = \Xi C_z + C_{m0}, \tag{1.40}$$

where

$$\Xi = 2A\int_0^1 G_{a1}\left(\xi + h\frac{t}{t_0}\right)d\eta, \tag{1.41}$$

$$C_{m0} = 2A\int_0^1 G_b\left(\xi + h\frac{t}{t_0}\right)d\eta + \frac{At_0}{b}\int_0^1 c_{m_0}\frac{t^2}{t_0^2}d\eta. \tag{1.42}$$

The last result shows that the position of the aerodynamic centre of the wing Ξ is determined by the normal distribution of circulation, and that the moment coefficient about the aerodynamic centre C_{m0} is determined by the zero distribution of circulation as well as by the c_{m0} of the section.

The angle α_m in (1·24) is the mean angle of incidence in the sense that when α is constant across the span (untwisted wing) α_m is equal to the constant value of α, and that when α varies across the span (twisted wing) α_m is equal to the angle of incidence of the untwisted wing which gives the same value for the lift coefficient C_z, whence it follows that the slope of lift coefficient against angle C_z/α_m is independent of the distribution of α, and that it depends solely on the plan form of the wing. Taking (1.12) and (1.36) into account, and writing

$$\frac{C_z}{\alpha_m} = \pi\mu A\frac{A_1}{\alpha_m} = ma\ , \qquad (1.43)$$

it will readily be seen that the coefficient m depends solely on the plan form of the wing.

Finally, the total drag coefficient of the wing C_x is the sum of the induced drag coefficient C_{xi}, (1.37), and the profile drag coefficient C_{x0}, which is obtained by

$$C_{x0} = \int_0^1 c_{x0}\frac{bt}{S}d\eta\ , \qquad (1.44)$$

where c_{x0} is the local profile drag coefficient of the section η, and is obtained by the local lift coefficient c_z, (1.7).

§ 6. *Explanations of the symbols used.*

ρ : density of air,
V : speed of flight,
b : span of wing,
S : area of wing,
$A = b^2/S$: aspect ratio of wing,

y : distance along the span from centre of wing, Fig. 1,

$\eta = 2y/b = \cos\theta$, Fig. 1,

t : chord of the section y, Fig. 1,

t_0 : chord of the centre section, Fig. 2,

α : geometrical angle of incidence of the section y, measured from the attitude of no lift, in radians,

α_m : mean angle of incidence in radians, see § 5,

α_∞ : effective angle of incidence of the section y, see Eq. (1.2),

φ : induced angle at the section y in radians, see Eq. (1.1),

a : slope of the curve of lift coefficient against angle of incidence in two dimensional flow, see Eq. (1.4),

$\lambda = \pi A/a$,

$\mu = at_0/4b$,

$\Gamma = bVG$: circulation of the section y,

$\Gamma = \Gamma_a + \Gamma_b = \Gamma_{a1}C_z + \Gamma_b$,

$\Gamma_a = bVG_a = bVG_{a1}C_z$: normal distribution of circulation, see § 5,

$\Gamma_b = bVG_b$: zero distribution of circulation, see § 5,

$K_j = A_j + B_j$: Fourier coefficient of circulation distribution,

A_j : Fourier coefficient of the normal distribution of circulation,

B_j : Fourier coefficient of the zero distribution of circulation,

D_j : Fourier coefficient of the distribution of the geometrical angle of incidence, see Eq. (1.25),

c_z : local lift coefficient of the section y,

c_x : local drag coefficient of the section y,

C_z : lift coefficient of the wing,

$C_x = C_{xi} + C_{x0}$: drag coefficient of the wing,

C_{xi} : induced drag coefficient of the wing,

C_{x0} : profile drag coefficient of the wing,

C_l : rolling moment coefficient of the wing, referred to wind axes, positive when the starboard wing is lowered,

C_m : pitching moment coefficient of the wing, referred to the transverse axis through the leading edge of the centre section, positive when the angle of incidence increases,

C_n : yawing moment coefficient of the wing, referred to wind axes, positive when the starboard wing is retarded,

C_{ni} : yawing moment coefficient of the wing caused by the aerodynamic induction, the moment due to asymmetry of profile drag being excluded,

$m = C_z / a\alpha'_m$, see end of § 5,

β : change in angle of no lift of the section due to flap displacement, see Fig. 17,

β_F : β for landing flap displacement, see § 14,

β_A : β for aileron displacement, see § 20,

p : rolling angular velocity, referred to wind axes,

r : yawing angular velocity, referred to wind axes,

k : taper ratio of the trapezoidal wing (straight tapered wing), see Fig. 22,

subscript (e): for the elliptic wing,

,, (F): for the landing flap,

,, (A): for the aileron,

,, (AS): for the starboard aileron,

,, (AP): for the port aileron,

symbol ($'$): for the landing flap displacement,

,, ($''$): for the wash out,

,, ($'''$): for the differential displacement of ailerons,

,, (*): for the basic wing, especially when landing flap is displaced.

CHAPTER 2.

CALCULATION OF A SIMPLE WING.

This chapter deals with a method of calculating the aerodynamic characteristics of a simple wing, that is, a wing without body or nacelle and without aileron or flap displacement.

§ 7. *Solution by the method of successive approximations.* For calculating the circulation distribution Γ of a simple wing, the most practical way is to solve Eq. (1.6) by the method of successive approximations. It consists in making a suitable assumption for Γ for finding φ from (1.1), and in calculating a new value of Γ according to (1.6). Since, usually, the calculated Γ differs from that originally assumed, another assumption for Γ is then made, after which all the calculations are made over again. The process must be repeated until the assumed and calculated values of the circulation agree, although in practice three or four repetitions usually suffice for a simple wing.

The advantages of this method of solution are that the calculation is simple, that any errors are easily detected owing to repetitions of similar calculations, that since it is not necessary to assume a to be constant in (1.6), if it is required (1.3) may be used instead of (1.4), which is a convenient generalization in dealing with a wing of large angle of incidence when stalling spreads locally over the wing.

§ 8. *A simple method of calculating* $\varphi(y)$. Notwithstanding the advantages of the method of successive approximation as just enumerated, it is not practicable unless a simple method of calculating $\varphi(y)$ is used. Graphical and mechanical methods of calculating φ have already been proposed (Ref. 11, 12, 13, 14,), but the labour these methods entail is so great that it is no wonder that the method of successive approximation has not been used oftener.

To evaluate as simply as possible the definite integral (1.1), we adopt, as an approximation to the value of φ, the form

$$\varphi(\eta) = \kappa_1 G(\eta_1) + \kappa_2 G(\eta_2) + \cdots + \kappa_s G(\eta_s) , \qquad (2.1)$$

where $G(\eta_1)$, $G(\eta_2)$, $\cdots$, $G(\eta_s)$ are the values of the non-dimensional circulation G at the suitably chosen representative sections η_1, η_2, $\cdots$, η_s, and κ_1, κ_2, $\cdots$, κ_s are constants to be determined, which evidently depend on the value of η.(1)

In the case of a simple wing, since the distribution of circulation

$$G(\eta) = 2\mu(K_1 \sin\theta + K_3 \sin 3\theta + \cdots) \qquad (2.2)$$

is symmetrical about the centre ($\eta = 0$, $\theta = \pi/2$), it is sufficient to consider only the starboard half ($0 \leqq \eta \leqq 1$, $\pi/2 \geqq \theta \geqq 0$) of the wing. Taking $s = 4$, that is, the four points $\eta_1 = \cos^{-1}\theta_1$, $\eta_2 = \cos^{-1}\theta_2$, $\eta_3 = \cos^{-1}\theta_3$, $\eta_4 = \cos^{-1}\theta_4$ on the starboard wing, and putting (1.11) and (2.2) into (2.1), we get

$$\begin{aligned} K_1 + 3K_3 \frac{\sin 3\theta}{\sin\theta} + 5K_5 \frac{\sin 5\theta}{\sin\theta} + 7K_7 \frac{\sin 7\theta}{\sin\theta} &\\ = 2\kappa_1(K_1 \sin\theta_1 + K_3 \sin 3\theta_1 + K_5 \sin 5\theta_1 + K_7 \sin 7\theta_1) &\\ + 2\kappa_2(K_1 \sin\theta_2 + K_3 \sin 3\theta_2 + K_5 \sin 5\theta_2 + K_7 \sin 7\theta_2) &\\ + 2\kappa_3(K_1 \sin\theta_3 + K_3 \sin 3\theta_3 + K_5 \sin 5\theta_3 + K_7 \sin 7\theta_3) &\\ + 2\kappa_4(K_1 \sin\theta_4 + K_3 \sin 3\theta_4 + K_5 \sin 5\theta_4 + K_7 \sin 7\theta_4) . & \end{aligned}$$

Equating the coefficients of K_1, K_3, K_5, K_7 on both sides of the equation we have a set of linear equations,

(1) Recently, Moriya succeeded in calculating, in a similar way, φ due to helical vortices of a propeller. T. Moriya: Formulae for propeller characteristics calculation and a method to obtain the best pitch distribution. Jour. Soc. Aero. Sci., Nippon, Vol. 5 (1938), p. 995; Proc. Fifth Int. Congr. App. Mech. (1938), p. 504.

$$\left.\begin{aligned}
\frac{1}{2} &= \kappa_1 \sin\theta_1 + \kappa_2 \sin\theta_2 + \kappa_3 \sin\theta_3 + \kappa_4 \sin\theta_4 \,, \\
\frac{3}{2}\frac{\sin 3\theta}{\sin\theta} &= \kappa_1 \sin 3\theta_1 + \kappa_2 \sin 3\theta_2 + \kappa_3 \sin 3\theta_3 + \kappa_4 \sin 3\theta_4 \,, \\
\frac{5}{2}\frac{\sin 5\theta}{\sin\theta} &= \kappa_1 \sin 5\theta_1 + \kappa_2 \sin 5\theta_2 + \kappa_3 \sin 5\theta_3 + \kappa_4 \sin 5\theta_4 \,, \\
\frac{7}{2}\frac{\sin 7\theta}{\sin\theta} &= \kappa_1 \sin 7\theta_1 + \kappa_2 \sin 7\theta_2 + \kappa_3 \sin 7\theta_3 + \kappa_4 \sin 7\theta_4 \,,
\end{aligned}\right\} \qquad (2.3)$$

from which $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ may be determined. Considering the nature of the distribution of G, we take $\eta_1 = 0{\cdot}1$, $\eta_2 = 0{\cdot}4$, $\eta_3 = 0{\cdot}7$, $\eta_4 = 0{\cdot}9$, and get

$$\begin{aligned}
\eta = 0{\cdot}1: \quad & \kappa_1 = +1{\cdot}8376, \; \kappa_2 = -1{\cdot}3627, \; \kappa_3 = -0{\cdot}0477, \; \kappa_4 = -0{\cdot}1042, \\
\eta = 0{\cdot}4: \quad & \kappa_1 = -0{\cdot}9822, \; \kappa_2 = +2{\cdot}3818, \; \kappa_3 = -0{\cdot}9717, \; \kappa_4 = -0{\cdot}0268, \\
\eta = 0{\cdot}7: \quad & \kappa_1 = -0{\cdot}0276, \; \kappa_2 = -1{\cdot}2781, \; \kappa_3 = +3{\cdot}0778, \; \kappa_4 = -1{\cdot}1451, \\
\eta = 0{\cdot}9: \quad & \kappa_1 = +0{\cdot}1360, \; \kappa_2 = -0{\cdot}7007, \; \kappa_3 = -1{\cdot}0974, \; \kappa_4 = +4{\cdot}1080.
\end{aligned}$$

With these values of κ_s, φ is given by (2.1) in radians. Multiplying κ_s by $180/\pi$, we have the induced angle expressed in degrees, as follows:

$$\left.\begin{aligned}
\varphi^\circ(0{\cdot}1) &= +105{\cdot}3\,G(0{\cdot}1) - 78{\cdot}1\,G(0{\cdot}4) - 2{\cdot}7\,G(0{\cdot}7) - 6{\cdot}0\,G(0{\cdot}9)\,, \\
\varphi^\circ(0{\cdot}4) &= -56{\cdot}3\,G(0{\cdot}1) + 136{\cdot}5\,G(0{\cdot}4) - 55{\cdot}7\,G(0{\cdot}7) - 1{\cdot}5\,G(0.9)\,, \\
\varphi^\circ(0{\cdot}7) &= -1{\cdot}6\,G(0{\cdot}1) - 73{\cdot}2\,G(0{\cdot}4) + 176{\cdot}3\,G(0{\cdot}7) - 65{\cdot}6\,G(0{\cdot}9)\,, \\
\varphi^\circ(0{\cdot}9) &= +7{\cdot}8\,G(0{\cdot}1) - 40{\cdot}2\,G(0{\cdot}4) - 62{\cdot}9\,G(0{\cdot}7) + 235{\cdot}4\,G(0{\cdot}9)\,.
\end{aligned}\right\} \qquad (2.4)$$

A direct connection clearly holds between the number of terms taken in (2.1) and the degree of accuracy attained by that approximation. To take $s = 4$ is equivalent to neglecting coefficients higher than K_7 which, in practice, is an approximation sufficiently accurate for a simple wing. When, however, stalling spreads locally over the wing (§§ 12, 13), the distribution of Γ deforms into a complex shape, so that it is better to take $s = 5$. Thus

$$\left.\begin{aligned}
\varphi^{\circ}(0) &= +182{\cdot}73\,G(0{\cdot}1) - 166{\cdot}81\,G(0{\cdot}3) + 20{\cdot}83\,G(0{\cdot}5) \\
&\quad - 15{\cdot}09\,G(0{\cdot}7) - 2{\cdot}97\,G(0{\cdot}9), \\
\varphi^{\circ}(0{\cdot}1) &= +132{\cdot}71\,G(0{\cdot}1) - 88{\cdot}63\,G(0{\cdot}3) - 12{\cdot}69\,G(0{\cdot}5) \\
&\quad - 8{\cdot}48\,G(0{\cdot}7) - 4{\cdot}14\,G(0.9), \\
\varphi^{\circ}(0{\cdot}2) &= +15{\cdot}69\,G(0{\cdot}1) + 84{\cdot}73\,G(0{\cdot}3) - 75.81\,G(0{\cdot}5) \\
&\quad + 0{\cdot}73\,G(0{\cdot}7) - 6{\cdot}11\,G(0{\cdot}9), \\
\varphi^{\circ}(0{\cdot}3) &= -88{\cdot}46\,G(0{\cdot}1) + 208{\cdot}28\,G(0{\cdot}3) - 83{\cdot}98\,G(0{\cdot}5) \\
&\quad - 9{\cdot}26\,G(0{\cdot}7) - 6{\cdot}14\,G(0{\cdot}9), \\
\varphi^{\circ}(0{\cdot}4) &= -103{\cdot}88\,G(0{\cdot}1) + 154{\cdot}22\,G(0{\cdot}3) + 22{\cdot}72\,G(0{\cdot}5) \\
&\quad - 45{\cdot}86\,G(0{\cdot}7) - 4{\cdot}67\,G(0{\cdot}9), \\
\varphi^{\circ}(0{\cdot}5) &= -19{\cdot}67\,G(0{\cdot}1) - 68{\cdot}94\,G(0{\cdot}3) + 198{\cdot}85\,G(0{\cdot}5) \\
&\quad - 76{\cdot}07\,G(0{\cdot}7) - 8{\cdot}94\,G(0{\cdot}9), \\
\varphi^{\circ}(0{\cdot}6) &= +74{\cdot}23\,G(0.1) - 253{\cdot}21\,G(0{\cdot}3) + 261{\cdot}92\,G(0.5) \\
&\quad - 24{\cdot}55\,G(0{\cdot}7) - 29{\cdot}74\,G(0{\cdot}9), \\
\varphi^{\circ}(0{\cdot}7) &= +26{\cdot}07\,G(0{\cdot}1) - 87{\cdot}17\,G(0{\cdot}3) - 8{\cdot}72\,G(0{\cdot}5) \\
&\quad + 169{\cdot}61\,G(0{\cdot}7) - 63{\cdot}58\,G(0{\cdot}9), \\
\varphi^{\circ}(0{\cdot}8) &= -200{\cdot}22\,G(0{\cdot}1) + 455{\cdot}64\,G(0{\cdot}3) - 537{\cdot}90\,G(0{\cdot}5) \\
&\quad + 393{\cdot}00\,G(0{\cdot}7) - 49{\cdot}60\,G(0{\cdot}9), \\
\varphi^{\circ}(0{\cdot}9) &= -130{\cdot}99\,G(0{\cdot}1) + 273{\cdot}35\,G(0{\cdot}3) - 257{\cdot}46\,G(0{\cdot}5) \\
&\quad + 40{\cdot}67\,G(0{\cdot}7) + 211{\cdot}39\,G(0{\cdot}9).
\end{aligned}\right\} \tag{2.5}$$

The more terms we take, the more accurate the results. This holds, however, in cases wherein the values of $G(\eta_1)$, $G(\eta_2)$, $\cdots$, $G(\eta_s)$ are accurately known, i.e. to as many figures as may be desired. Since, in practice, G is not known in more than three figures, any increase in s is likely to vitiate the results, owing to the fact that large quantities of nearly equal magnitude but of opposite signs have to be added, for which reason it would seem a disadvantage to take s more than 5.

It is supposed that the accuracy of approximation depends to a certain extent on the choice of the representative values $\eta_1, \eta_2, \cdots, \eta_s$. The problem presented here is analogous to that of Gauss's method of approximate quadrature. However, we shall not pursue this inquiry further, contenting ourselves by calculating for several sets of values $\eta_1, \eta_2, \cdots, \eta_s$, chosen by taking into account the nature of distribution of G, and selecting the most desirable sets, as given in (2.4) and (2.5).

§ 9. *Harmonic analysis.* With the foregoing method, the circulation distribution for a simple wing may be calculated very easily. The result is expressed, however, as the curve of G, the Fourier coefficients $K_1, K_3, \cdots$ in (2.2) not being given explicitly. In view of the importance of these coefficients for calculating the aerodynamic characteristics of the wing, a convenient method of finding them from the curve of G may be given. After the method of attack in § 8, assuming, for instance, the form

$$K_1 = \kappa_1' G(\eta_1) + \kappa_2' G(\eta_2) + \kappa_3' G(\eta_3) + \kappa_4' G(\eta_4)\,,$$

we have the equations

$$\left.\begin{aligned} \frac{1}{2\mu} &= \kappa_1' \sin \theta_1 + \kappa_2' \sin \theta_2 + \kappa_3' \sin \theta_3 + \kappa_4' \sin \theta_4\,, \\ 0 &= \kappa_1' \sin 3\theta_1 + \kappa_2' \sin 3\theta_2 + \kappa_3' \sin 3\theta_3 + \kappa_4' \sin 3\theta_4\,, \\ 0 &= \kappa_1' \sin 5\theta_1 + \kappa_2' \sin 5\theta_2 + \kappa_3' \sin 5\theta_3 + \kappa_4' \sin 5\theta_4\,, \\ 0 &= \kappa_1' \sin 7\theta_1 + \kappa_2' \sin 7\theta_2 + \kappa_3' \sin 7\theta_3 + \kappa_4' \sin 7\theta_4\,, \end{aligned}\right\}$$

by which κ_1', κ_2', κ_3', κ_4', may be determined. Taking $\eta_1 = 0{\cdot}1$, $\eta_2 = 0{\cdot}4$, $\eta_3 = 0{\cdot}7$, $\eta_4 = 0{\cdot}9$, we have

$$K_1 = A_1$$
$$= \frac{1}{\mu}\{0{\cdot}1453\,G(0{\cdot}1)+0{\cdot}2184\,G(0{\cdot}4)+0{\cdot}1498\,G(0{\cdot}7)+0{\cdot}1108\,G(0{\cdot}9)\}, \tag{2.6}$$

whence by (1.14)

$$C_z = A\{0{\cdot}4565\,G(0{\cdot}1)+0{\cdot}6861\,G(0{\cdot}4)+0{\cdot}4707\,G(0.7)+0{\cdot}3479\,G(0{\cdot}9)\}\,. \tag{2.7}$$

The corresponding expression for $s = 5$ becomes

$$C_z = A\{0{\cdot}45368\,G(0{\cdot}1)+0{\cdot}28197\,G(0{\cdot}3)+0{\cdot}50246\,G(0{\cdot}5)$$
$$+0{\cdot}35521\,G(0{\cdot}7)+0{\cdot}37073\,G(0{\cdot}9)\}\,. \tag{2.8}$$

K_3 and K_5 may be obtained quite similarly, thus

$$\left.\begin{aligned} K_3 &= \frac{1}{\mu}\{-4{\cdot}7233\,G(0{\cdot}1)+6{\cdot}8828\,G(0{\cdot}4)-1{\cdot}9062\,G(0{\cdot}7) \\ &\qquad +0{\cdot}5798\,G(0{\cdot}9)\}\,, \\ K_5 &= \frac{1}{\mu}\{+7{\cdot}4602\,G(0{\cdot}1)-15{\cdot}2617\,G(0{\cdot}4)+11{\cdot}6124\,G(0{\cdot}7) \\ &\qquad -3{\cdot}9644\,G(0{\cdot}9)\}\,. \end{aligned}\right\} \tag{2.9}$$

Owing to the limited number of figures composing G, however, the calculation of these coefficients is satisfactorily performed with $s = 3$ rather than with $s = 4$. We have

$$\left.\begin{aligned} K_3 &= \frac{1}{\mu}\{-0{\cdot}2028\,G(0.1)+0.0752\,G(0{\cdot}5)+0{\cdot}3137\,G(0.9)\}, \\ K_5 &= \frac{1}{\mu}\{+0{\cdot}1636\,G(0{\cdot}1)-0{\cdot}2685\,G(0{\cdot}5)+0{\cdot}1600\,G(0{\cdot}9)\}. \end{aligned}\right\} \tag{2.10}$$

Seeing that it is not necessary, in practice, to treat coefficients higher than K_5, the harmonic analysis is very easily performed according to (2.6) and (2.10). It should be added that the lift coefficient of the wing C_z is calculated by (2.7), and the induced drag coefficient C_{xi} by (1.37), (2.6), and (2.10).

§ **10.** *Numerical example.* In order to illustrate the foregoing method of solution, it will be supposed that the aerodynamic characteristics are required for the tapered wing shown in Fig. 3, and Tables 1, 2. This wing has an aspect ratio $A = 8{\cdot}7036$, a taper ratio 0·3237, and a linear wash out of 2°. Although the slope of lift curve a varies

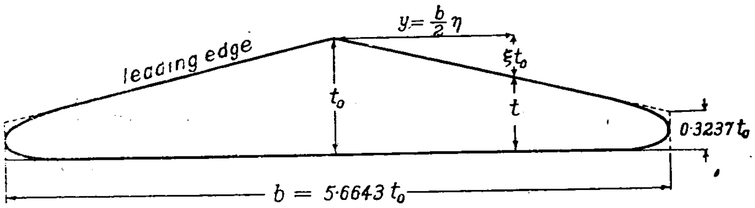

Fig. 3.

across the span, this minor variation may be neglected and a mean value of 5·67 applied throughout every section. Since $b/t_0 = 5{\cdot}6643$, $\mu = at_0/4b \doteqdot 0{\cdot}25025$, the quantity $(\pi/180)\,(at/2b)$ appearing in the equation

TABLE 1.

η	$\frac{t}{t_0}$	ξ	ψ°	α°
0	1·0000	0	2·33°	15·00°
0·1	0·9324	0·0676	2·32°	14·79°
0·2	0·8647	0·1353	2·30°	14·57°
0·3	0·7971	0·2029	2·28°	14·35°
0·4	0·7295	0·2705	2·25°	14·12°
0·5	0·6618	0·3382	2·22°	13·89°
0·6	0·5942	0·4058	2·17°	13·64°
0·7	0·5266	0·4734	2·11°	13·38°
0·8	0·4587	0·5412	2·00°	13·07°
0·9	0·3660	0·6214	1·73°	12·60°
0·95	0·2740	0·6842	1·45°	12·22°

— ψ° is the angle of no lift. α° is the angle of incidence of each section when the angle at $\eta = 0$ is 15°; $\alpha^\circ(\eta) = 15^\circ - 2^\circ\eta - \{\psi^\circ(0) - \psi^\circ(\eta)\}$.

TABLE 2.

η	a	$c_{z\max}$	cm_0	h	$c_{x0} \times 10^2$									
					$c_z = -0{\cdot}2$	0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6
0·1	5·59	1·67	-0 015	0·235	0 95	0·86	0 84	0·89	1·01	1·19	1·44	1·79	2·36	3·75
0·3	5·63	1·70	-0·016	0·236	0·92	0·83	0·81	0·86	0·97	1·14	1·38	1·71	2 23	3·30
0·5	5·67	1·75	-0 016	0·237	0·90	0·81	0·78	0·82	0·92	1·07	1·30	1·61	2·06	2·90
0·7	5·71	1·80	-0·015	0·238	0·82	0·75	0·72	0·76	0 86	1·00	1·22	1·50	1·90	2·60
0·9	5·75	1·80	-0 014	0·240	0·72	0·67	0·67	0·72	0·83	0·98	1·20	1·48	1·88	2·59

$$G(\eta) = \frac{\pi}{180}\frac{at}{2b}(\alpha^\circ - \varphi^\circ), \qquad (2.11)$$

which is the non-dimensional alternative of (1.6), is given in Table 3.

TABLE 3.

η	α°	$\frac{\pi}{180}\frac{at}{2b}$
0·1	14·79°	0·00815
0·4	14·12°	0·00638
0·7	13·38°	0·00460
0·9	12·60°	0·00320

First approximation. First, in order to make a rough estimate, the distribution of G is assumed to be elliptic, or

$$G(\eta) = \frac{2C_z}{\pi A}\sqrt{1-\eta^2},$$

although it might be possible to make a closer approximation as, for example, by using the known circulation distribution for a straight tapered wing, as given in Chapter V, the elliptic distribution is here adopted as an example of a somewhat crude assumption, but with the advantage that φ is constant across the span and is calculated very easily. Since the mean of α is roughly 14°, and since the value m, corresponding to $k = 0{\cdot}3237$ and $A/a = 8{\cdot}7036 / 5{\cdot}67 = 1{\cdot}534$ in Fig. 25, is 0·82, the lift coefficient of the wing will be nearly

$$C_z = m\alpha\alpha = 0{\cdot}82 \times 5{\cdot}67 \times \frac{14}{57{\cdot}3} = 1.14\,.$$

Therefore the first approximate values of circulation are

$$G(\eta) = 0{\cdot}083, \quad 0{\cdot}076, \quad 0{\cdot}060, \quad 0{\cdot}036$$

for $\eta = 0{\cdot}1$, $0{\cdot}4$, $0{\cdot}7$, $0{\cdot}9$ respectively. Substituting, at the same time,

$$\varphi^\circ = 57{\cdot}3 \frac{C_z}{\pi A} = 2{\cdot}4^\circ$$

in (2.11), we have a new distribution of circulation,

$$G(\eta) = 0{\cdot}101\,, \quad 0{\cdot}075\,, \quad 0{\cdot}051\,, \quad 0{\cdot}033\,,$$

for the four sections. Since these calculated values differ from those originally assumed, the mean of the assumed and calculated values is used for the second approximation.

Second approximation. The assumed values of G for the second approximation are,

$$G(\eta) = 0{\cdot}0920\,, \quad 0{\cdot}0760\,, \quad 0{\cdot}0550\,, \quad 0{\cdot}0340$$

respectively. With these values of G, we obtain φ from (2·4):

$$\varphi^\circ(\eta) = 3{\cdot}40^\circ, \quad 2{\cdot}08^\circ, \quad 1{\cdot}75^\circ, \quad 2{\cdot}21^\circ,$$

and substituting φ in (2.11), we have

$$G(\eta) = 0.0928\,, \quad 0{\cdot}0768\,, \quad 0{\cdot}0535, \quad 0{\cdot}0332\,,$$

which differ slightly from those originally assumed. Taking the mean of the two, we have

$$G(\eta) = 0{\cdot}0924\,, \quad 0{\cdot}0764\,, \quad 0{\cdot}0542\,, \quad 0{\cdot}0336\,,$$

TABLE 4.

	$\eta=0{\cdot}1$	$\eta=0{\cdot}4$	$\eta=0{\cdot}7$	$\eta=0{\cdot}9$
$G(0{\cdot}1)=0{\cdot}0920$	$(\times\ 105{\cdot}3)+9{\cdot}69$	$(\times -56{\cdot}3)-5{\cdot}18$	$(\times -1{\cdot}6)-0{\cdot}15$	$(\times\ 7{\cdot}8)+0{\cdot}72$
$G(0{\cdot}4)=0{\cdot}0760$	$(\times -78{\cdot}1)-5{\cdot}94$	$(\times\ 136{\cdot}5)+10{\cdot}37$	$(\times -73{\cdot}2)-5{\cdot}56$	$(\times -40{\cdot}2)-3{\cdot}06$
$G(0{\cdot}7)=0{\cdot}0550$	$(\times -2{\cdot}7)-0{\cdot}15$	$(\times -55{\cdot}7)-3{\cdot}06$	$(\times\ 176{\cdot}3)+9{\cdot}69$	$(\times -62{\cdot}9)-3{\cdot}46$
$G(0{\cdot}9)=0{\cdot}0340$	$(\times -6{\cdot}0)-0{\cdot}20$	$(\times -1{\cdot}5)-0{\cdot}05$	$(\times -65{\cdot}6)-2{\cdot}23$	$(\times\ 235{\cdot}4)+8{\cdot}01$
φ°	$3{\cdot}40^\circ$	$2{\cdot}08^\circ$	$1{\cdot}75^\circ$	$2{\cdot}21^\circ$
$\alpha^\circ-\varphi^\circ$	$11{\cdot}39^\circ$	$12{\cdot}04^\circ$	$11{\cdot}63^\circ$	$10{\cdot}39^\circ$
$G=\frac{\pi}{180}\frac{at}{2b}(\alpha^\circ-\varphi^\circ)$	$0{\cdot}0928$	$0{\cdot}0768$	$0{\cdot}0535$	$0{\cdot}0332$
G, revised	$0{\cdot}0924$	$0{\cdot}0764$	$0{\cdot}0542$	$0{\cdot}0336$

which are to be assumed in the subsequent approximation. The numerical calculation may be performed with a 50 cm slide rule, the steps being tabulated with advantage, as shown in Table 4.

Third approximation. The third approximation may be performed quite similarly to the preceding. The result is

$$G(\eta),\ \text{assumed} = 0{\cdot}0924,\quad 0{\cdot}0764,\quad 0{\cdot}0542,\quad 0{\cdot}0336,$$

$$G(\eta),\ \text{calculated} = 0{\cdot}0927,\quad 0{\cdot}0763,\quad 0{\cdot}0541,\quad 0{\cdot}0334,$$

$$G(\eta),\ \text{revised} = 0{\cdot}0925,\quad 0{\cdot}0764,\quad 0{\cdot}0542,\quad 0{\cdot}0335.$$

Since the calculated values almost agree with the assumed, we may accept the mean of the two as the final distribution without further checking. According to (2.7) the lift coefficient becomes

$$C_z = 8{\cdot}7036\times(0{\cdot}4565\times 0{\cdot}0925+0{\cdot}6861\times 0{\cdot}0764$$
$$+0{\cdot}4707\times 0{\cdot}0542+0{\cdot}3479\times 0{\cdot}0335) = 1{\cdot}146.$$

Normal and zero distributions. On the other hand, calculating similarly a fictitious wing with the same distribution of chord (i.e. plan form), but with a uniform distribution of angle of incidence across the span, $\alpha^\circ = 15^\circ$, we get for the distribution of circulation

$$G(\eta) = 0{\cdot}0951, \quad 0{\cdot}0810, \quad 0{\cdot}0601, \quad 0{\cdot}0389,$$

with $C_z = 1{\cdot}225$. Seeing that for such a wing circulation varies in proportion to C_z, the value of $G(\eta)$ for $C_z = 1{\cdot}146$ may be obtained by multiplying the above by $1{\cdot}146/1{\cdot}225$, giving

$$G(\eta) = 0{\cdot}0890, \quad 0{\cdot}0758, \quad 0{\cdot}0563, \quad 0{\cdot}0364,$$

which is nothing else but the normal distribution of the wing under investigation. Subtracting the normal distribution from the resultant distribution, previosuly calculated, we have the zero distribution

$$G_b(\eta) = 0{\cdot}0035, \quad 0{\cdot}0006, \quad -0{\cdot}0021, \quad -0{\cdot}0029.$$

The mean angle of incidence becomes

$$\alpha_m^\circ = 15^\circ \times \frac{1{\cdot}146}{1{\cdot}225} = 14{\cdot}03^\circ.$$

Calculation for other angles of incidence. Although the foregoing calculations apply to the case when the angle of incidence at the centre section $\alpha_0{}^\circ = 15^\circ$ and the lift coefficient $C_z = 1{\cdot}146$, the circulation distribution for other angles (hence for other values of C_z) may be very easily deduced from them. Multiplying the normal distribution previously given by $1/1{\cdot}146$, we have the normal distribution for $C_z = 1$, namely, G_{a1}:

$$G_{a1}(\eta) = 0{\cdot}0776, \quad 0{\cdot}0661, \quad 0{\cdot}0491, \quad 0{\cdot}0318.$$

The circulation distribution for any value of C_z is then given by

$$G(\eta) = G_{a1}(\eta)C_z + G_b(\eta) ,$$

whence we get the results shown in Fig. 4.

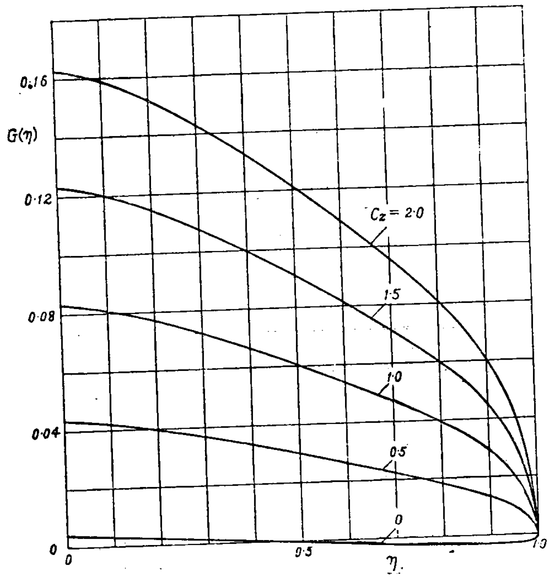

Fig. 4.

Relation between C_z and α_0. It has been shown that when C_z is 1·146, the mean angle of incidence α_m is 14·03°. Since the difference between α_m and the angle of incidence at the centre section α_0, which is equal to $15° - 14{\cdot}03° = 0{\cdot}97°$ in the present case, is independent of C_z, we have the following relation between C_z and α_0:

$$C_z = 1{\cdot}146 \times \frac{\alpha_m^\circ}{14{\cdot}03^\circ} = 0{\cdot}0817(\alpha_0^\circ - 0{\cdot}97^\circ) ,$$

the angle of no lift being 0·97°. It will be seen that the angle α_m means the angle of incidence at the centre, measured from the no lift attitude of the wing, or the angle of incidence of the untwisted wing, which has the same value of C_z as the wing in consideration. The slope of the curve of lift coefficient against angle of incidence is

$$ma = \frac{dC_z}{d\alpha_0} = 0{\cdot}0817 \times 57{\cdot}3 = 4{\cdot}68 ,$$

and the value of the coefficient m is

$$m = \frac{4\cdot 68}{5\cdot 67} = 0\cdot 826 \ .$$

Section lift coefficient. The relation between the section lift coefficient $c_z(\eta)$ and $G(\eta)$ is given by (1·7), namely,

$$c_z(\eta) = \frac{2b}{t(\eta)} G(\eta) \ ,$$

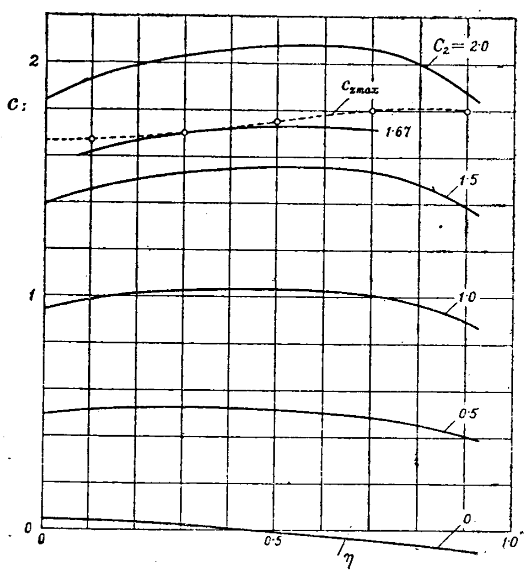

Fig. 5.

with the aid of which the distribution of c_z, Fig. 5, may be deduced from the distribution of G, Fig. 4. In Fig. 5, there is also a curve drawn representing the maximum lift coefficient $c_{z\max}$ of each section. The curve of c_z distribution becomes tangent to the curve of $c_{z\max}$ at $C_z = 1{\cdot}67$ and $\eta = 0{\cdot}3$, indicating the beginning of stall. Although the method of calculation described in §§ 12, 13 is preferably used for estimating the maximum lift of the wing, the method shown in Fig. 5 is sufficiently accurate for roughly estimating the condition near the stall.

Harmonic analysis. First, the leading coefficient A_1 is calculated by (1.36) as

$$\frac{A_1}{C_z} = \frac{1}{\pi\mu A} = 0{\cdot}14614\,.$$

In order to obtain other coefficients, the value of G at $\eta = 0{\cdot}5$ mus be found by interpolation. Substituting

$$G_{a1}(0{\cdot}1) = 0{\cdot}0776,\quad G_{a1}(0{\cdot}5) = 0{\cdot}0609,\quad G_{a1}(0{\cdot}9) = 0{\cdot}0318,$$
$$G_b(0{\cdot}1) = 0{\cdot}0035,\quad G_b(0{\cdot}5) = -0{\cdot}0005,\quad G_b(0{\cdot}9) = -0{\cdot}0029$$

in (2.10), we have

$$\frac{A_3}{C_z} = -0{\cdot}0047,\qquad \frac{A_5}{C_z} = 0{\cdot}0057\,,$$
$$B_3 = -0{\cdot}0066\,,\qquad B_5 = 0{\cdot}0010\,.$$

Induced drag. Substituting, then, the values calculated above in (1.37), where

$$K_3 = \frac{A_3}{C_z}C_z + B_3\,,\quad K_5 = \frac{A_5}{C_z}C_z + B_5\,,\quad B_2 = B_4 = \cdots = 0\,,$$

TABLE 5.

C_z	C_{xi}	C_{x0}	C_x
0	0·0002	0·0080	0·0082
0.2	0·0018	0·0078	0·0096
0·5	0·0096	0·0087	0·0183
1·0	0·0376	0·0133	0·0509
1·5	0·0839	0·0253	0·1092

we obtain the coefficient of induced drag,

$$C_{xi} = \frac{1}{3\cdot142 \times 8\cdot7036}(1\cdot0107\,C_z^2 + 0\cdot0114\,C_z + 0\cdot0063)$$
$$= 0\cdot0370\,C_z^2 + 0\cdot0004\,C_z + 0\cdot0002\;.$$

The value of C_{xi} calculated by this equation is given in Table 5.

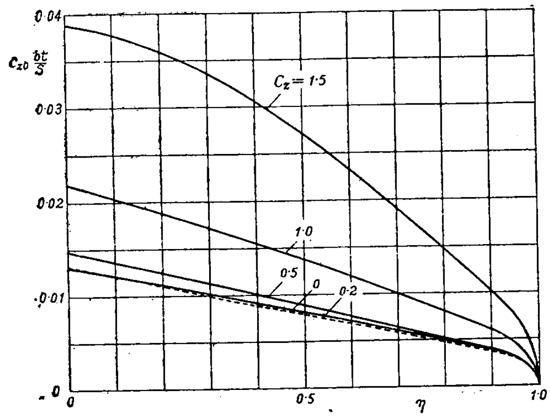

Fig. 6.

Profile drag. The coefficient of profile drag c_{x0} of each section being given in Table 2 as a function of the section lift coefficient c_z, the distribution of c_{x0} may be deduced from Fig. 5 and Table 2. The

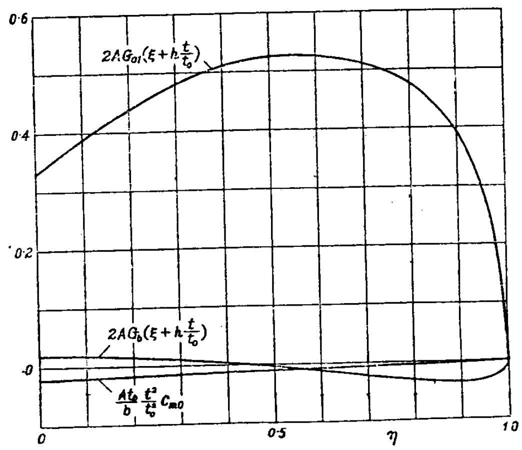

Fig. 7.

distribution of $c_{x0}(bt/S)$ is shown in Fig. 6, and the coefficient of profile drag of the wing calculated by (1.44) is given in Table 5.

Total drag. The coefficient of total drag is

$$C_x = C_{xi} + C_{x0} ,$$

which is given also in Table 5.

Pitching moment. The integrands in (1.41) and (1.42) are calculated by the numerical data in Tables 1, 2 and by the distributions G_{a1}, G^q above obtained. These are shown in Fig. 7. Integrating the curves, we have $\Xi = 0{\cdot}447$, $C_{m0} = -0{\cdot}017$, and the coefficient of pitching moment

$$C_m = 0{\cdot}447\, C_z - 0{\cdot}017 .$$

§ **11.** *Accuracy of the method.* As a test of the accuracy of the proposed method of calculation, numerical examples will be worked out also by the other methods to enable comparison of their results.

Comparison with Lotz's method. First, we shall treat the problem described in § 10 by Lotz's method (Ref. 5) which, although gives very accurate results, has the drawback that it is laborious. It is required to express the distribution of angle of incidence by the series (1.25), and the distribution of chord by the series

$$\frac{t_0}{t(\eta)} \sin\theta = E_0 + E_2 \cos 2\theta + E_4 \cos 4\theta + \cdots , \qquad (2.12)$$

the Fourier coefficients D_{2n+1} and E_{2j} being given in Table 6. Although for the present purpose, coefficients of higher order are unnecessary, they are nevertheless tabulated so that in the following chapters they may be applied to non-simple wings. The Fourier coefficients A_{2n+1} and B_{2n+1} of the circulation distribution are also given in the same table, while the values of G_{a1} and G_b are given in Table 7. The results of calculation are compared in Table 8 with those obtained in § 10.

TABLE 6.

n	D_{2n+1}	E_{2n}	E_{2n+2}	A_{2n+1}/C_z	B_{2n+1}
0	—0·01720	1·2091	—0·0142	0·14614	0·00000
1	—0 01186	—0·1387	0 0355	—0 00471	—0 00614
2	0·00116	0·0005	0 0115	0·00519	0·00046
3	—0·00174	—0 0100	0·0022	—0 00099	—0·00075
4	0·00056	—0·0043	0·0036	0·00037	0 00021
5	—0·00056	—0·0027	0·0016	—0 00045	—0 00016
6	0·00042	—0·0028	0·0013	0·00022	0·00011
7	—0·00032	—0·0018	0·0016	—0·00012	—0·00008
8	0·00030	—0·0015	0·0011	0·00012	0·00006
9	—0·00032	—0 0017	0·0011	—0·00009	—0·00006
10		—0·0014			

D_{2n+1} corresponds to the case when the angle of incidence at the centre α_0 is zero.

TABLE 7.

θ	η	G_{a1}	G_b
90°	0·0000	0·07928	0·00402
81°	0·1564	0·07621	0·00296
72°	0·3090	0·07029	0·00142
63°	0·4540	0·06333	—0 00001
54°	0·5878	0·05594	—0·00119
45°	0 7071	0·04851	—0·00206
36°	0·8090	0·04105	—0·00263
27°	0·8910	0·03286	—0·00286
18°	0·9511	0·02295	—0·00253
9°	0·9877	0·01181	—0·00152

TABLE 8.

	Lotz solution	Approx. solution
A_3+B_3	$-0{\cdot}00471\ C_z-0{\cdot}00614$	$-0{\cdot}0047\ C_z-0{\cdot}0066$
A_5+B_5	$0{\cdot}00519\ C_z+0{\cdot}00046$	$0{\cdot}0057\ C_z+0{\cdot}0010$
G_a+G_b $(\eta=0{\cdot}1)$	$0{\cdot}0777\ C_z+0{\cdot}0034$	$0{\cdot}0776\ C_z+0{\cdot}0035$
„ $(\eta=0{\cdot}4)$	$0{\cdot}0661\ C_z+0{\cdot}0005$	$0{\cdot}0661\ C_z+0{\cdot}0006$
„ $(\eta=0{\cdot}7)$	$0{\cdot}0490\ C_z-0{\cdot}0020$	$0{\cdot}0491\ C_z-0{\cdot}0021$
„ $(\eta=0{\cdot}9)$	$0{\cdot}0317\ C_z-0{\cdot}0029$	$0{\cdot}0318\ C_z-0{\cdot}0029$
$\alpha_0^\circ-\alpha_m^\circ$	$0{\cdot}961^\circ$	$0{\cdot}97^\circ$
C_{xi}	$0{\cdot}03693\ C_z^2+0{\cdot}00037C_z+0{\cdot}00020$	$0{\cdot}0370\ C_z^2+0{\cdot}0004C_z+0{\cdot}0002$

Comparison with Wada's method. The second example is that of a wing, the particulars of which are the same as that of § 10, except that a is 5·00 instead of 5·67, and that the distribution of no lift angle

TABLE 9.

η	Zero lift angle
0	$-3{\cdot}20^\circ$
0·2	$-3{\cdot}07^\circ$
0·4	$-2{\cdot}92^\circ$
0·6	$-2{\cdot}75^\circ$
0·7	$-2{\cdot}62^\circ$
0·8	$-2{\cdot}44^\circ$
0·9	$-2{\cdot}07^\circ$
0·95	$-1{\cdot}75^\circ$

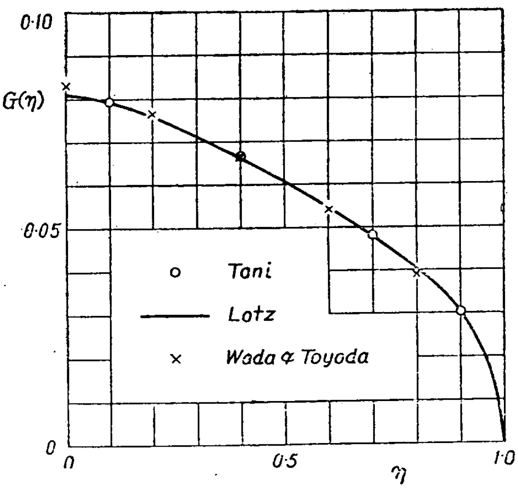

Fig. 8.

is as shown in Table 9, instead of as in Table 1. The distribution of circulation obtained by the writer's method is compared in Fig. 8 with those by Lotz and Wada.

Comparison with Lippisch's method. The third example is that of a straight tapered wing with a central parallel part and with a wash out of 9° (Fig. 9). This is the illustration given in the writer's previous paper (Ref. 8). The distribution of circulation obtained by the writer's method is compared with that by Lippisch, as worked out by Mr. Lippisch himself in his private communication to the writer, dated Oct. 5, 1935.

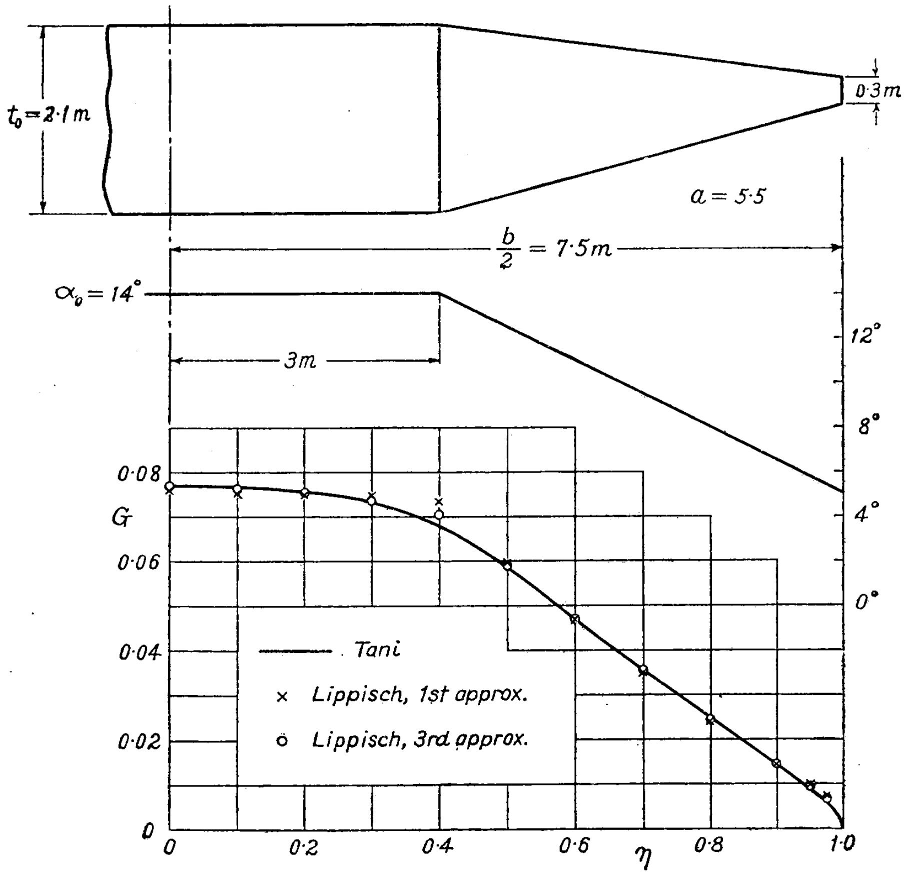

Fig. 9.

From the comparisons just mentioned, it will be seen that the proposed method of solution gives results sufficiently accurate for practical purposes.

Accuracy of the harmonic analysis. In order to check the accuracy of the harmonic analysis as described in §9, the Fourier coefficients A_1, A_3, A_5 are calculated by applying formulae (2.7) and (2.10) to the distribution of circulation for a series of straight tapered wings (trapezoidal wings) with constant angle of incidence, as given in the works of Betz (Ref. 1) and Hueber (Ref. 15). The results of calculation are compared with the values as tabulated in Glauert's text book

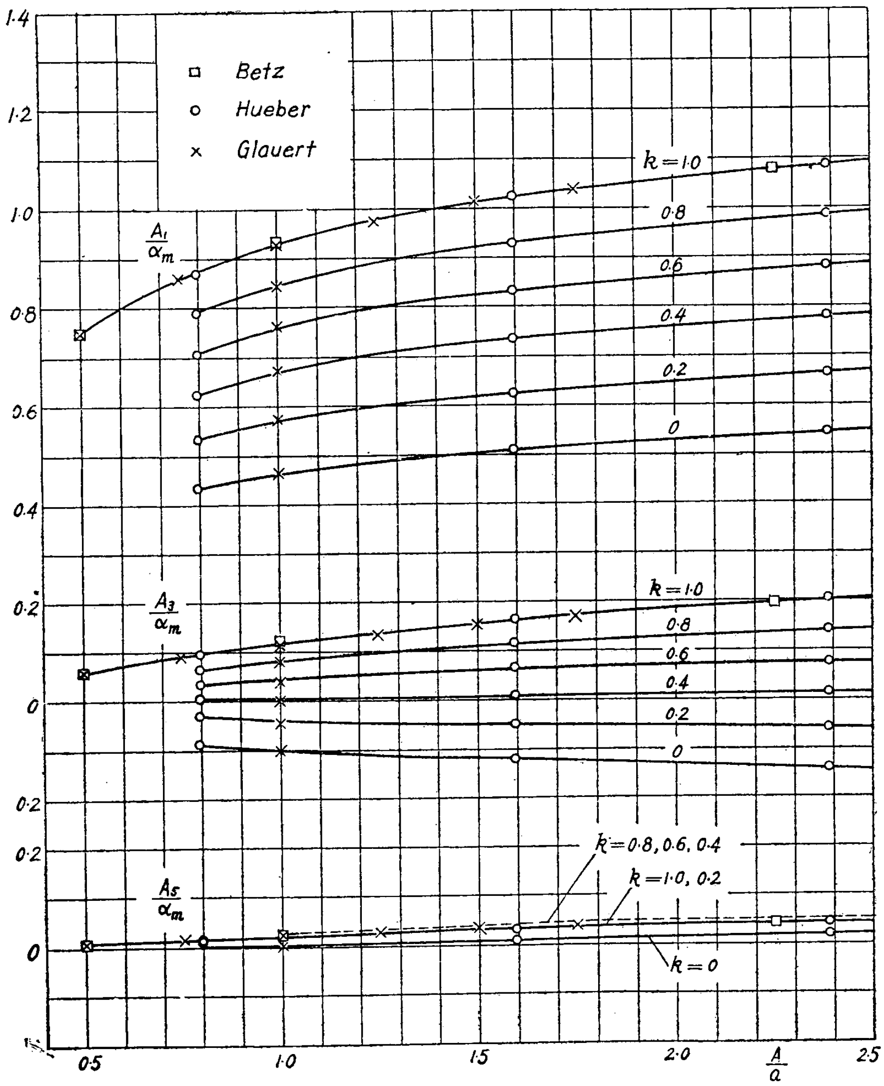

Fig. 10.

(Ref. 3), Fig. 10 showing the accuracy of the proposed method of harmonic analysis. The results shown in Fig. 10 will serve as basic data for the systematic calculations in Chapter V.

§ **12.** *Calculation for the condition near the stall.* The calculation described above (and those in Chapters from III to V) requires that we assume a linear relation

$$c_z = a\alpha_\infty \qquad (2.13)$$

between the section lift coefficient and effective angle of incidence. Although in most practical cases, the slope a, for simplicity, is taken to be constant across the span, the variation in a may, if desired, be taken into account by replacing the chord $t(y)$ by a fictitious chord $t(y)\,a\,(y)\,/\,a_{\text{mean}}$ and then performing similar calculations.

At large angles of incidence near the stall, however, the simple relation (2·13) no longer holds; the more general relation

$$c_z = c_z(\alpha_\infty) \qquad (2.14)$$

must be used, where the functional form is to be determined by experiment and may vary across the span. Such a general case can be solved only by the method of successive approximation, which, however, has never been used in practice owing to the great amount of labour involved for calculating the induced angle φ, except for the solitary case in which Lachmann (Ref. 16) investigated the problem of lateral control near the stall. Besides, the section characteristics of a wing are deduced from the results of wind tunnel experiments on an untwisted rectangular wing of aspect ratio A by applying the correction of angle (5·4), regardless of the fact that the maximum lift coefficient $C_{z\,\max}$ of an untwisted rectangular wing increases as A increases. In recent N.A.C.A. reports, an empirical correction is adopted, according to which the maximum lift coefficient for a rectangular wing of aspect ratio 6 is to be increased 7 percent to allow for the value of the infinite aspect ratio (the section characteristics).

The problems encountered may be treated with advantage by the writer's solution by successive approximation. The method of attack has already been suggested in the writer's third paper (Ref. 10), while a similar calculation was later performed by Wieselsberger (Ref. 17) by

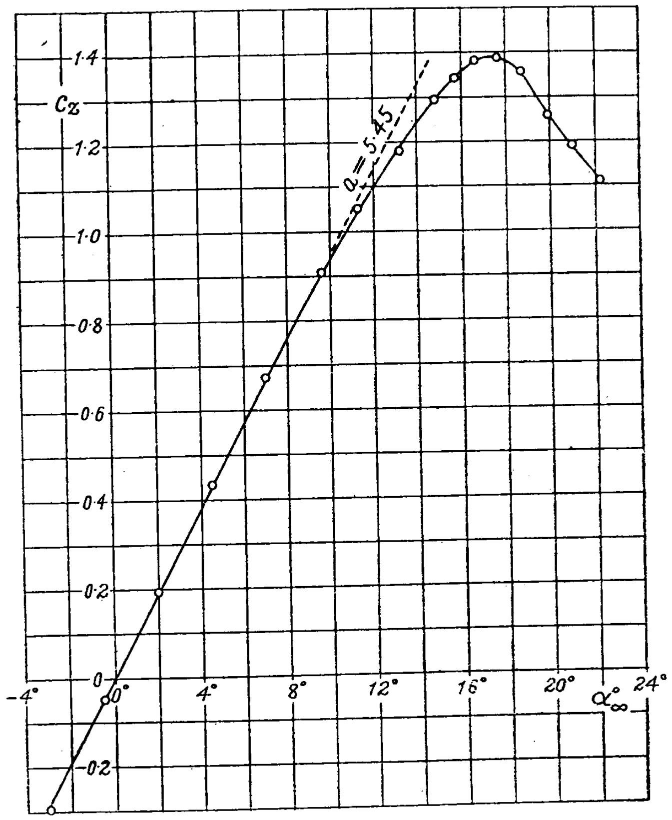

Fig. 11.

utilizing the results of the writer's first paper (Ref. 8). Wind tunnel experiments have since been performed by the writer, the results of which will now be analized.

§ 13. *Continuation. Numerical example.* The calculation for large angles may be made similarly to that as when a is assumed to be constant. Calculating $\varphi(\eta)$ by (2.4) or (2.5) from the assumed distribution of $G(\eta)$ and then $c_z(\eta)$ by (2.14) from the value of $\alpha_\infty(\eta) = \alpha(\eta) - \varphi(\eta)$, we get $G(\eta) = c_z(\eta)t(\eta)/2b$. Comparing the calculated values of $G(\eta)$ with those originally assumed, another assumption for $G(\eta)$ is then made, after which all the calculations are made over again.

In order to prepare the characteristics for the infinite aspect ratio (2.14), the pressure distribution was measured along the centre section of

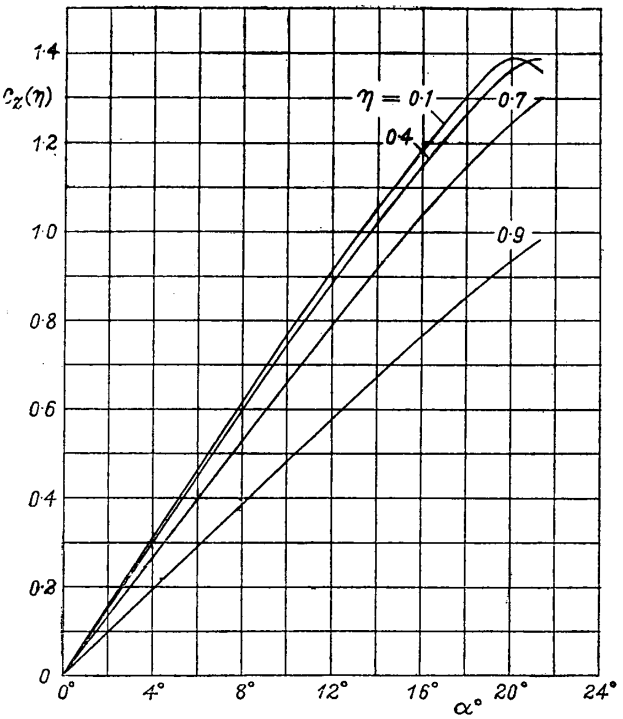

Fig. 12.

a rectangular wing of 28 cm span, 28 cm chord, of section Clark Y, and placed between large vertical walls in the 2 m wind tunnel of the Institute. The main purpose of the measurements was to study the effect of ground on the aerodynamic characteristics of a wing; descriptions of the apparatus together with the results of measurements for wind speed 40 m/s have already been published in the Rep. Aero. Res. Inst., No. 156 (1937). In Fig. 11, the measured characteristics

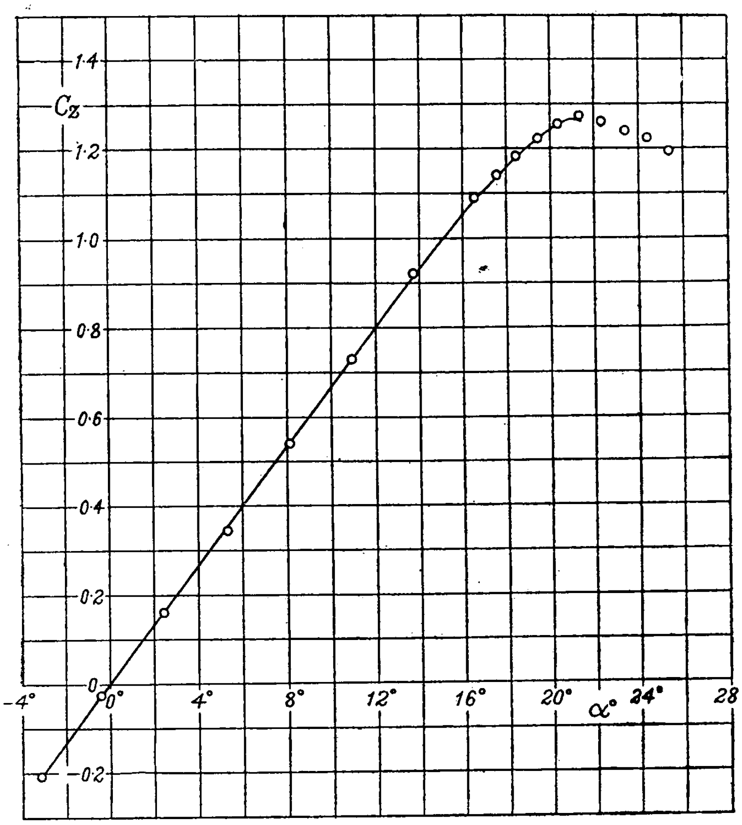

Fig. 13.

for a wind speed of 30 m/s is shown, where α_∞ means the angle of incidence measured from no lift attitude.

Using the curve drawn through the measured points as the relation (2.14), we obtain the value of c_z for sections $\eta = 0{\cdot}1$, $0{\cdot}4$, $0{\cdot}7$, and $0{\cdot}9$, of a rectangular wing with a span of 140 cm, chord 28 cm, and section Clark Y, as shown in Fig. 12. Since the relation (2.14) is non-linear, the curves of c_z against α are also non-linear, and, before everything else, c_z for $\eta = 0{\cdot}1$ attains a maximum value of 1.39, indicating the well known fact that a rectangular wing stalls first from the centre. Integrating, then, the distribution $G = c_z t/2b$ by (2.7), we have the lift coefficient of the wing c_z, which is shown as a curve in Fig. 13. The maximum value of c_z is attained at $\alpha \approx 21°$, where the section inboard from $\eta = 0{\cdot}1$ is already stalled. With further increase of α, the stall spreads as far as $\eta = 0{\cdot}4$ and the section $\eta = 0{\cdot}1$ will be completely burbled. Although, in view of the doubtful accuracy of the section characteristics for such a large angle of incidence, the calculation will not be continued further, it nevertheless seems certain that the maximum lift occurs somewhere near $\alpha = 21^{\circ}$.

The measured values of the lift coefficient of the rectangular wing are also shown by small circles in Fig. 13, which are in close agreement with the calculated curve. The typical example, therefore, shows the usefulness of the proposed method. Comparison of Fig. 11 with Fig. 13 moreover shows that the maximum lift coefficient of a rectangular wing of aspect ratio 5 is reduced by 9 percent as compared with that of the infinite aspect ratio.

CHAPTER III.

CALCULATION OF A WING WITH SYMMETRICAL DISTRIBUTION OF ANGLE OF INCIDENCE.

This chapter deals with the method of calculating the aerodynamic characteristics of a wing, whose angle of incidence varies symmetrically across the span, with or without discontinuities. It is therefore concerned with the effects of the landing flap and wash out on the characteristics of a simple wing.

§14. *Basis of calculation of the flap effect.* When the landing flap that is located at the centre of a wing is working, as shown in Fig. 14, the angle of incidence measured from the attitude of no lift is increased by β over that part occupied by the flap, where β depends on the flap displacement and the ratio of flap chord to wing chord, so that it is to be regarded as a function of θ. Consider now, for simplicity, the case when the basic wing has no twist and the angle of incidence is zero. Although this is a special case of flap effect, it

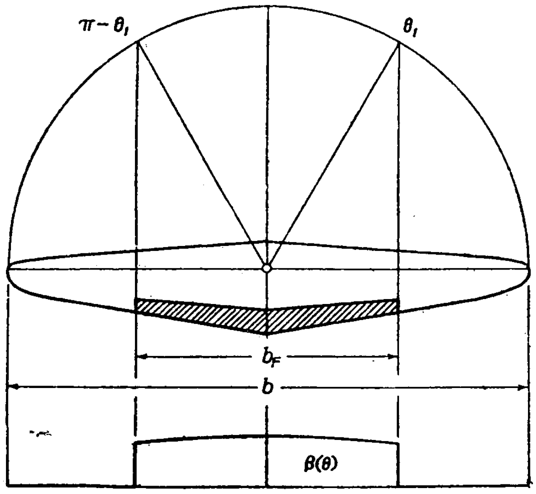

Fig. 14.

will be seen from § 17 that the general case when the wing has any distribution of angle of incidence may easily be deduced from it.

Since the distribution of angle of incidence α' is such that $\alpha'=0$ for $0\leqq\theta\leqq\theta_1$ and $\pi-\theta_1\leqq\theta\leqq\pi$, and $\alpha'=\beta$ for $\theta_1\leqq\theta\leqq\pi-\theta_1$, the coefficient of the series

$$\alpha'\sin\theta = D_1'\sin\theta + D_3'\sin 3\theta + D_5'\sin 5\theta + \cdots \qquad (3.1)$$

may be given by

$$D_{2n+1}' = \frac{4}{\pi}\int_{\theta_1}^{\frac{\pi}{2}}\beta(\theta)\sin(2n+1)\theta d\theta \quad (n=0,\ 1,\ 2,\ \ldots)\,. \qquad (3.2)$$

(3.1) is the special case of (1.25), the prime denoting the special case mentioned above. Should the ratio of flap chord to wing chord be maintained constant across the flap span, β may be taken as being constant ($=\beta_F$), and

$$\left.\begin{aligned} D_1' &= \frac{2\beta_F}{\pi}\left\{\frac{\pi}{2}-\theta_1+\frac{\sin 2\theta_1}{2}\right\}, \\ D_{2n+1}' &= \frac{2\beta_F}{\pi}\left\{\frac{\sin(2n+2)\theta_1}{2n+2}-\frac{\sin 2n\theta_1}{2n}\right\}\ (n>0)\,. \end{aligned}\right\} \qquad (3.3)$$

The calculation of the flap effect is therefore reduced to the problem of a wing whose distribution of angle of incidence is given by (3.1), (3.2), or (3.3). There being nothing particularly difficult about the calculation, it may be successfully performed by the method due to Lotz (Ref. 5), although a considerable number of Fourier coefficients in the series expression for the circulation will have to be taken into account, because a sudden change in angle of incidence can be reproduced only by retaining the Fourier coefficients D'_{2n+1} of the higher order. Considering the limited range of application of the theoretical calculation in practice, it is doubtful whether its use warrants the extra labour involved. An attempt will therefore be made in § 16 to develop an approximate method that would give results sufficiently accurate for most practical purposes.

§ **15.** *Flap effect for an elliptic wing.* Before describing the approximate method of solution, we might consider the case of an elliptic wing, the exact solution of the flap effect of which is very easily obtained, and of which use will be made in subsequent paragraphs.

Since the chord is distributed according to the relation

$$t(y) = t_0\sqrt{1-\eta^2} = t_0 \sin\theta\ , \tag{3.4}$$

the coefficients in (2.12) are: $E_0 = 1$, $E_2 = E_4 = \cdots = 0$. Substituting (3.1) and (3.4) in (1.6) and bearing in mind (1.11) and (1.32), we have at once the Fourier coefficients of circulation

$$K'_{2n+1,\,e} = \frac{D'_{2n+1}}{1+(2n+1)\mu_e}\ (n = 0,\ 1,\ 2,\ \cdots)\ , \tag{3.5}$$

where the prime denotes that the circulation distribution

$$G'_e(\eta) = 2\mu_e(K'_{1,\,e}\sin\theta + K'_{3,\,e}\sin 3\theta + \cdots) \tag{3.6}$$

corresponds to the special distribution of angle of incidence (3.1), and subscript e denotes the elliptic wing, for example,

$$\mu_e = \left(\frac{at_0}{4b}\right)_e = \left(\frac{a}{\pi A}\right)_e. \tag{3.7}$$

The normal distribution of an elliptic wing is given by

$$G_{a,e}(\eta) = \frac{2\mu_e \alpha_{m,e}}{1+\mu_e}\sin\theta. \tag{3.8}$$

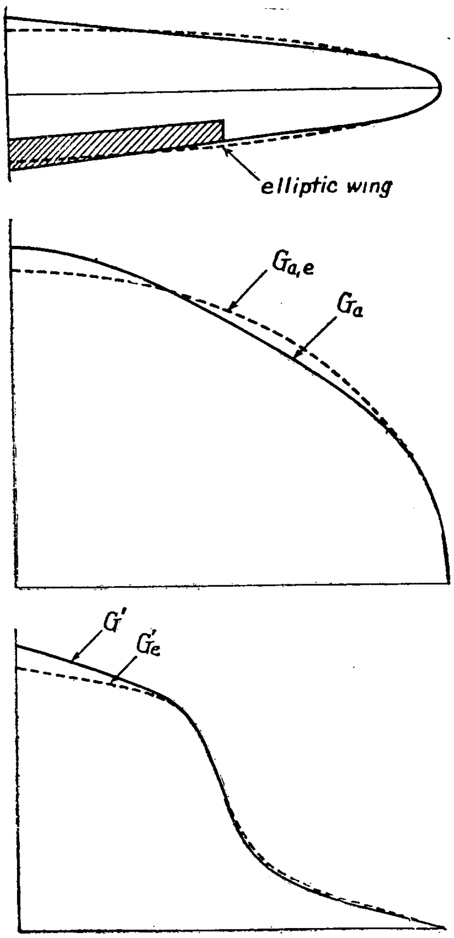

Fig. 15.

§ **16.** *Approximate solution of the flap effect.* In order to calculate the flap effect generally, but approximately, it is now assumed that the ratio of the circulation distribution $G'(\eta)$, due to flap displacement, to the normal distribution $G_a(\eta)$ for the mean angle α_m, is the same for wings of the same span b, the same aspect ratio A, and the same distribution of angle of incidence (3.1). Taking now one wing to be the elliptic wing (with subscript e) and the other to be the wing in question (without subscript), and dispensing with the subscript for that quantity which is the same for both wings, the proposed assumption may be expressed by

$$\frac{G'_e(\eta)}{G_{a,e}(\eta)} = \frac{G'(\eta)}{G_a(\eta)}, \tag{3.9}$$

where

$$G_{a,e}(\eta) = \frac{2}{\lambda}\frac{\alpha_m}{1+1/\lambda}\sin\theta, \tag{3.10}$$

$$G'_e(\eta) = \frac{2}{\lambda}\left\{\frac{D'_1}{1+1/\lambda}\sin\theta + \frac{D'_3}{1+3/\lambda}\sin 3\theta + \cdots\right\}, \tag{3.11}$$

$$\lambda = \frac{\pi A}{a}, \tag{3.12}$$

$$G_a(\eta) = 2\mu(A_1\sin\theta + A_3\sin 3\theta + \cdots). \tag{3.13}$$

The relation (3.9), true, is approximate, but that it is adequate may be shown to a certain extent theoretically, and, as will be seen later, its application gives results with sufficient accuracy. The practical importance of the relation (3.9) is then that $G'(\eta)$, which is laborious to calculate, may be deduced from $G_{o,e}(\eta)$, $G_a(\eta)$, $G'_e(\eta)$, all of which may be calculated without difficulty. As a matter of fact, by substituting (3.10), (3.11), and (3.13) in (3.9), we have

$$G'(\eta) = 2\mu(K'_1\sin\theta + K'_3\sin 3\theta + \cdots), \tag{3.14}$$

$$\left.\begin{aligned}
K'_1 &= \frac{D'_1 A_1}{\alpha_m}\left\{1 + \frac{\lambda+1}{\lambda+3}\frac{D'_3}{D'_1}\frac{A_3}{A_1} + \frac{\lambda+1}{\lambda+5}\frac{D'_5}{D'_1}\frac{A_5}{A_1} + \cdots\right\}, \\
K'_3 &= \frac{D'_1 A_1}{\alpha_m}\left\{\frac{A_3}{A_1} + \frac{\lambda+1}{\lambda+3}\frac{D'_3}{D'_1}\left(1 + \frac{A_3}{A_1} + \frac{A_5}{A_1}\right)\right. \\
&\qquad \left. + \frac{\lambda+1}{\lambda+5}\frac{D'_5}{D'_1}\left(\frac{A_3}{A_1} + \frac{A_5}{A_1} + \cdots\right) + \cdots\right\}, \\
K'_5 &= \frac{D'_1 A_1}{\alpha_m}\left\{\frac{A_5}{A_1} + \frac{\lambda+1}{\lambda+3}\frac{D'_3}{D'_1}\left(\frac{A_3}{A_1} + \frac{A_5}{A_1} + \cdots\right)\right. \\
&\qquad \left. + \frac{\lambda+1}{\lambda+5}\frac{D'_5}{D'_1}\left(1 + \frac{A_3}{A_1} + \frac{A_5}{A_1} + \cdots\right) + \cdots\right\}, \\
K'_7 &= \frac{D'_1 A_1}{\alpha_m}\left\{\frac{A_7}{A_1} + \frac{\lambda+1}{\lambda+3}\frac{D'_3}{D'_1}\left(\frac{A_5}{A_1} + \cdots\right)\right. \\
&\qquad + \frac{\lambda+1}{\lambda+5}\frac{D'_5}{D'_1}\left(\frac{A_3}{A_1} + \frac{A_5}{A_1} + \cdots\right) \\
&\qquad \left. + \frac{\lambda+1}{\lambda+7}\frac{D'_7}{D'_1}\left(1 + \frac{A_3}{A_1} + \frac{A_5}{A_1} + \cdots\right) + \cdots\right\}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{3.15}$$

In most practical cases, coefficient A_7 and those of higher order may be ignored without any loss of accuracy. Finally, using (1.36) and (1.43), the lift coefficient of the circulation distribution $G'(\eta)$, namely, the increase in lift coefficient due to flap displacement, becomes

$$C'_z = \pi\mu A K'_1 = maD'_1(1+\varpi)\,, \qquad (3.16)$$

where

$$\varpi = \frac{\lambda+1}{\lambda+3}\frac{D'_3}{D'_1}\frac{A_3}{A_1}+\frac{\lambda+1}{\lambda+5}\frac{D'_5}{D'_1}\frac{A_5}{A_1}+\cdots\,. \qquad (3.17)$$

It will readily be seen that $\varpi = 0$ for the elliptic wing.

§ **17.** *Flap effect for the general case.* The calculation in §§ 14–16 refers to the special case in which the angle of incidence of the basic wing is zero. If, more generally, the distribution of angle of incidence of the basic wing is not constant across the span, and it has the mean value α^*_m such that the distribution of circulation is given by

$$G^*(\eta) = 2\mu\{A^*_1\sin\theta+(A^*_3+B^*_3)\sin 3\theta+(A^*_5+B^*_5)\sin 5\theta+\cdots\}\,, \qquad (3.18)$$

the resultant distribution when the flap is displaced is obtained by

$$G(\eta) = G^*(\eta)+G'(\eta)\,, \qquad (3.19)$$

and the resultant lift coefficient by

$$C_z = C^*_z+C'_z = \pi\mu A(A^*_1+K'_1)\,. \qquad (3.20)$$

The asterisk is used to denote the basic wing, for which, however, letters without asterisk were used in Chapter II.

Resolving the resultant distribution in the normal and zero distributions, such that

$$G(\eta) = G_a(\eta)+G_b(\eta)\,, \qquad (3.21)$$

we have

$$G_a(\eta) = 2\mu(A_1 \sin\theta + A_3 \sin 3\theta + A_5 \sin 5\theta + \cdots), \quad (3.22)$$

$$G_b(\eta) = 2\mu(\qquad B_3 \sin 3\theta + B_5 \sin 5\theta + \cdots), \quad (3.23)$$

$$A_1 = A_1^* + A_1', \qquad A_1' = K_1', \quad (3.24)$$

$$\left.\begin{aligned} A_3 &= A_3^* + A_3', \quad A_3' = K_1'\frac{A_3^*}{A_1^*}\left[= K_1'\frac{A_3}{A_1}\right], \\ B_3 &= B_3^* + B_3', \quad B_3' = K_3' - A_3', \\ A_5 &= A_5^* + A_5', \quad A_5' = K_1'\frac{A_5^*}{A_1^*}\left[= K_1'\frac{A_5}{A_1}\right], \\ B_5 &= B_5^* + B_5', \quad B_5' = K_5' - A_5', \text{ etc.} \end{aligned}\right\} \quad (3.25)$$

It will be seen therefore that the flap displacement affects both normal and zero distributions of circulation. By (1.37) the coefficient of induced drag is given by

$$C_{xi} = \frac{1+\delta}{\pi A}C_z^2 + \left(\mu w_1 + \frac{f_1}{\pi A}C_z'\right)C_z' + \mu^2 A w_2 + \mu f_w C_z' + \frac{f_2}{\pi A}C_z'^2, \quad (3.26)$$

where

$$\delta = 3\left(\frac{A_3^*}{A_1^*}\right)^2 + 5\left(\frac{A_5^*}{A_1^*}\right)^2 + \cdots, \quad (3.27)$$

$$w_1 = 6\left(B_3^*\frac{A_3^*}{A_1^*} + \frac{5}{3}B_5^*\frac{A_5^*}{A_1^*} + \frac{7}{3}B_7^*\frac{A_7^*}{A_1^*} + \cdots\right), \quad (3.28)$$

$$w_2 = 3\pi\left(B_3^{*2} + \frac{5}{3}B_5^{*2} + \frac{7}{3}B_7^{*2} + \cdots\right), \quad (3.29)$$

$$f_1 = 6\left(\frac{B_3'}{A_1'}\frac{A_3^*}{A_1^*} + \frac{5}{3}\frac{B_5'}{A_1'}\frac{A_5^*}{A_1^*} + \frac{7}{3}\frac{B_7'}{A_1'}\frac{A_7^*}{A_1^*} + \cdots\right), \quad (3.30)$$

$$f_2 = 3\left(\frac{B_3'^2}{A_1'^2} + \frac{5}{3}\frac{B_5'^2}{A_1'^2} + \frac{7}{3}\frac{B_7'^2}{A_1'^2} + \cdots\right), \quad (3.31)$$

$$f_w = 6\left(B_3^* \frac{B_3'}{A_1'} + \frac{5}{3} B_5^* \frac{B_5'}{A_1'} + \frac{7}{3} B_7^* \frac{B_7'}{A_1'} + \cdots \right), \tag{3.32}$$

and where δ is the well known factor for the wing without flap and without wash out, while w_1 and w_2 represent the effect of wash out, f_1 and f_2 the effect of flap displacement, and f_w the combined effect of wash out and flap displacement.

Since we have

$$\frac{A_1}{\alpha_m} = \frac{A_1^*}{\alpha_m^*}, \quad \frac{A_3}{A_1} = \frac{A_3^*}{A_1^*}, \quad \frac{A_5}{A_1} = \frac{A_5^*}{A_1^*}, \quad \cdots \tag{3.33}$$

from the definition of the normal distribution, the asterisks for A_{2n+1} in equations (3.27), (3.28), (3.30) may be omitted.

As will be seen from (3.16) and (3.26), although the calculation of C_z' requires K_1' only, that of $C_{x i}$ requires, in addition, B_3', B_5', $\cdots$. The coefficients B_3', B_5', $\cdots$ that represent the flap effect usually converge more slowly than the coefficients B_3^*, B_5^*, $\cdots$ that represent the effect of wash out, so that, for example, B_7^* may be neglected in calculating w_1, w_2, although as far as B_7' or B_9' must be retained in calculating f_1, f_2. The coefficients B_3^*, B_5^*, $\cdots$, as well as B_3', B_5', $\cdots$ are however easily found.

To summarise, in order to calculate the flap effect according to the assumption (3.9), all that is necessary is to provide the Fourier coefficients of the circulation distribution when the flap is closed, A_1^*, A_3^*, A_5^*, $\cdots$, B_3^*, B_5^*, $\cdots$, as well as the Fourier coefficients of the distribution of angle of incidence (3.1), D_1', D_3', D_5', $\cdots$, which depend only on the flap displacement. The calculation, which is performed by means of formulae (3.15)—(3.32), is very simple compared with those hitherto proposed, the latter requiring the solution of linear simultaneous equations or other processes equally laborious.

Finally, it is worth noting that the flap with central cutaway (Fig. 16) may be treated similarly. If the ends of the flap are defined by parameters θ_1 and θ_2, the Fourier coefficients of the series (3.1) is given by

$$D'_{2n+1} = \frac{4}{\pi}\int_{\theta_1}^{\theta_2} \beta(\theta) \sin\theta \sin(2n+1)\theta d\theta \quad (n = 0, 1, 2, \ldots). \quad (3.34)$$

It appears therefore that the value of D'_{2n+1}, and consequently the increase in lift coefficient due to such a flap, is expressed as that difference

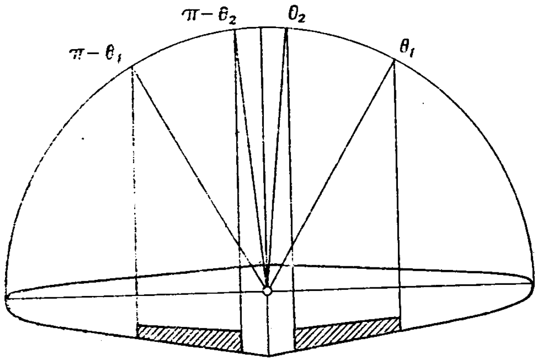

Fig. 16.

between the value due to a flap without cutaway and that due to a flap occupying the part of the cutaway.

§ 18. *Numerical example.* As an illustration of the method described above, consider the tapered wing of § 10 as being fitted with a central flap of 'split' type. The flap span b_F is 0·4928 b; the ratio of flap chord to wing chord is 20%, and flap displacement is 45°, both of which are assumed to be constant across the span. The change in angle of no lift β may therefore be substituted by a constant value β_F. The data for β are best obtained from wind tunnel experiments, but, in their absence, a rough estimate may be made from Fig. 17, in which available experimental results are plotted to show the general relation between β and flap displacement and chord ratio. Referring to Fig. 17a, we have $\beta = 11^\circ$, and then $\beta_F = 11 / 57{\cdot}3 = 0{\cdot}192$.

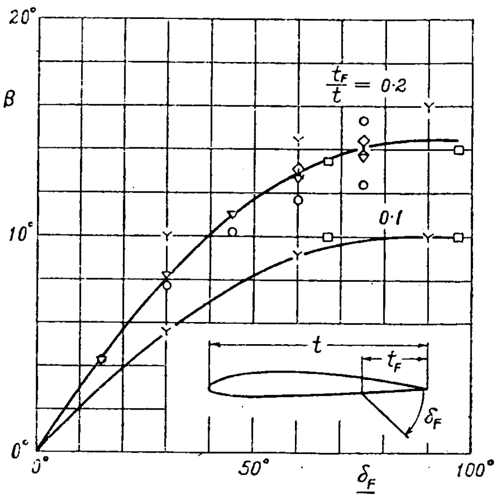

Fig. 17 a.

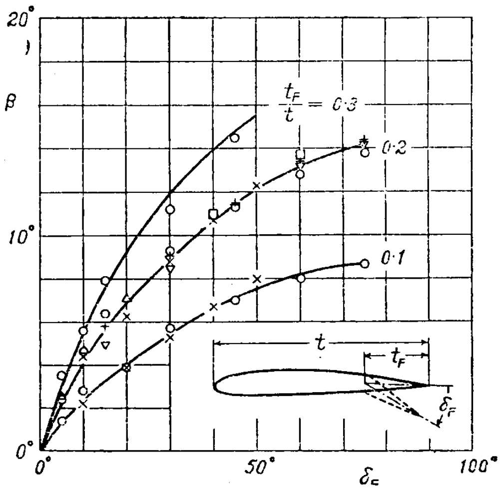

Fig. 17 b.

Since $\cos\theta_1 = 0{\cdot}4928$, $\theta_1 = 60^\circ 28'$, and D'_{2n+1}, as given by (3.3), is calculated as shown in Table 10. Referring to (3.12), $\lambda = 3{\cdot}1416 \times 8{\cdot}7036 \div 5{\cdot}67 = 4{\cdot}822$, and according to (3·11), in which the coefficients as far as D'_{19} are retained, $G'_e(\eta)$ is calculated as shown in Table 11. Next, $G_{a,e}(\eta)$ and $G_a(\eta)$ are deduced from results given in 'Normal and zero distributions' of § 10, i.e. $G_{a,e}(\eta) = (2/4{\cdot}822)\{14{\cdot}03/57{\cdot}3/(1+1/4{\cdot}822)\}\sin\theta$ and $G_a(\eta) = 0{\cdot}0890, 0{\cdot}0758, 0{\cdot}0563, 0{\cdot}0364$ for $\eta = 0{\cdot}1, 0{\cdot}4, 0{\cdot}7, 0{\cdot}9$ respectively. With a view, then, to obtaining $G'(\eta)$ by multiplying $G'_e(\eta)$ by $G_a(\eta)/G_{a,e}(\eta)$ according to the assumption (3.9), the values of $G_a(\eta)/G_{a,e}(\eta)$ could have been obtained by interpolating the values obtained above. Although this is the method proposed above, the calculation illustrated in Table 11 does not follow the step recommended, the values of $G_a(\eta) / G_{a,e}(\eta)$ being calculated by Lotz's method in § 11, not by the

TABLE 10.

n	D'_{2n+1}	K'_{2n+1} (approx.)	K'_{2n+1} (Lotz)
0	0·11542	0·0808	0·08131
1	—0.07936	—0·0431	—0·04325
2	0·02794	0·0141	0·01395
3	0 01171	0·0025	0·00243
4	—0·02395	—	—0 00612
5	0·01204	—	0·00272
6	0·00602	—	0·00146
7	—0·01407	—	—0·00289
8	0·00805	—	0·00157
9	0·00373	—	0·00053

writer's approximate method in § 10. This is because the purpose of the illustration is not only to demonstrate the application of the method, but also to check the accuracy of the assumption (3.9), and because, for the latter purpose, it is desirable that all causes for error other than the assumption (3.9) shall be excluded as far as possible. Since it has been shown in § 11 that the writer's approximate method gives results almost agreeing with those obtained by Lotz's method, the end results of the present calculation would scarcely have been altered had the values of $G_a(\eta) / G_{a,e}(\eta)$ been taken from the approximate solution.

We have thus obtained distribution of circulation due to the flap displacement, which is given in column 'G' (approx)' of Table 11. The resultant distribution of circulation when the basic wing has the lift coefficient C_z^* is obtained by $G(\eta) = G_{a\,1}^*(\eta) C_z^* + G_b^*(\eta) + G'(\eta)$, where $G_{a\,1}^*(\eta)$ and $G_b^*(\eta)$ have already given in § 10.

Although coefficients D'_{2n+1} of the higher order have to be retained for plotting the curve of $G'(\eta)$, the few leading coefficients suffice for other purposes. As to A_{2n+1}, only the first three coefficients A_1, A_3, A_5 are required. From the results of § 11, we have

TABLE 11.

θ	η	G'_e	$G_a/G_{a,e}$	G'(approx.)	G' (Lotz)
90°	0	0·0612	1·081	0·0662	0·0664
81°	0·1564	0·0585	1 052	0·0615	0·0618
72°	0·3090	0·0553	1·008	0·0557	0·0561
63°	0·4540	0·0392	0·969	0·0380	0·0385
54°	0·5878	0·0196	0 942	0·0137	0·0139
45°	0·7071	0·0082	0·935	0·0077	0·0077
36°	0·8090	0·0048	0 952	0·0046	0·0046
27°	0·8910	0·0031	0·986	0·0031	0·0031
18°	0·9511	0·0018	1·012	0·0018	0·0018
9°	0·9877	0·0009	1·030	0 0009	0·0009

$$\frac{A_1}{\alpha_m} = 0{\cdot}684\,, \quad \frac{A_3}{A_1} = -0{\cdot}032\,, \quad \frac{A_5}{A_1} = 0{\cdot}036\,,$$
$$B_3^* = -0{\cdot}0061\,, \quad B_5^* = 0{\cdot}0005\,.$$

These values could have been obtained from the results of §10, but with a view to checking assumption (3.9), they were purposely taken from Lotz's solution. From the values D_1', D_3', D_5', D_7' in Table 10, K_1', K_3', K_5', K_7' are calculated by (3.15), as given in the colum 'K'_{2n+1} (approx.)' of Table 10. We then have

$$B_3' = -0.0405\,, \quad B_5' = 0{\cdot}0112\,, \quad B_7' = 0{\cdot}0025$$

from (3.25), and

$$\varpi = 0{\cdot}022\,, \qquad C_z' = 0{\cdot}554$$

from (3.16), (3.17), and finally,

$$\frac{1+\delta}{\pi A} = 0{\cdot}0370\,, \quad \mu w_1 = 0{\cdot}0004\,, \quad \mu^2 A w_2 = 0{\cdot}0002\,,$$
$$\mu f_w = 0{\cdot}0047\,, \quad \frac{f_1}{\pi A} = 0{\cdot}0052\,, \quad \frac{f_2}{\pi A} = 0{\cdot}0313\,,$$
$$C_{xi} = 0{\cdot}0370\,C_z^2 + 0{\cdot}0033\,C_z + 0{\cdot}0124$$

from (3.26)—(3.32), where $C_z = C_z^* + 0{\cdot}554$.

Comparison with Lotz's solution. As a check on the accuracy of the approximate solution, particularly the validity of the assumption (3.9), Lotz's method is applied to the same problem. The distribution of circulation $G'(\eta)$ and the corresponding Fourier coefficients K'_{2n-1} are given in Tables 11 and 10 respectively. Besides

$$B'_3 = -0{\cdot}04063\,,\quad B'_5 = 0{\cdot}01107\,,\quad B'_7 = 0{\cdot}00298\,,$$

$$C'_z = 0{\cdot}5564\,,\quad \frac{1+\delta}{\pi A} = 0{\cdot}03693\,,\quad \mu w_1 = 0{\cdot}00037\,,\quad \mu^2 A w_2 = 0{\cdot}00020\,,$$

$$\mu f_w = 0{\cdot}00460\,,\quad \frac{f_1}{\pi A} = 0{\cdot}00502\,,\quad \frac{f_2}{\pi A} = 0{\cdot}0347\,.$$

Comparing these results with those obtained above, it appears that the approximate solution gives results sufficiently accurate for practical purposes.

§ **19.** *Effect of wash out.* In §§ 14–18, the wing with displaced flap is treated as a typical example of a wing with symmetrical distribution of angle of incidence. The wing with wash out may be regarded as another typical example, of which a short account will now be given. Now, while it may appear unnecessary to resort to calculation based on assumption (3.9), seeing that the simple wing with wash out could be dealt with satisfactorily with the aid of the method developed in Chapter II, it nevertheless seems quite appropriate to consider an alternate way of calculating the change in lift and induced drag due to wash out, of calculating the coefficients B_3^*, B_5^*, $\cdots$, and so on.

Consider now the distribution of circulation corresponding to the distribution of angle of incidence

$$\alpha'' \sin\theta = D''_1 \sin\theta + D''_3 \sin 3\theta + D''_5 \sin 5\theta + \cdots\,, \qquad (3{\cdot}35)$$

where

$$D''_{2n+1} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \alpha''(\theta)\sin\theta\sin(2n+1)\theta d\theta\ (n = 0,\ 1,\ 2,\ \ldots)\,, \qquad (3{\cdot}36)$$

and α'' denotes the angle of incidence when the value at the centre of wing is zero. After the assumption (3.9), it is assumed that

$$\frac{G_e''(\eta)}{G_{a,e}(\eta)} = \frac{G''(\eta)}{G_a(\eta)}, \tag{3.37}$$

from which $G''(\eta)$ may be obtained from $G_{a,e}(\eta)$, $G_a(\eta)$ and $G_e''(\eta)$. $G_{a,e}(\eta)$ and $G_a(\eta)$ are given by (3.10) and (3.13) respectively, while $G''(\eta)$ may be deduced from (3.14) when K'_{2n+1} is replaced by K''_{2n+1} where K''_{2n+1} is obtained from (3.15) when D'_{2n+1} is replaced by D''_{2n+1}. Also the lift coefficient C_z'' due to the distribution $G''(\eta)$ may be obtained from (3.16) and (3.17) when D'_{2n+1} is replaced by D''_{2n+1}. Since C_z'' represents the change in lift coefficient due to wash out, and since the slope of lift coefficient ma is independent of the amount of wash out, the change in angle of no lift becomes

$$\Delta^*\alpha = D_1''\left\{1 + \frac{\lambda+1}{\lambda+3}\frac{D_3''}{D_1''}\frac{A_3}{A_1} + \frac{\lambda+1}{\lambda+5}\frac{D_5''}{D_1''}\frac{A_5}{A_1} + \cdots\right\}. \tag{3.38}$$

The Fourier coefficients that appear in (3.18) are given by

$$B_3^* = K_3'' - K_1''\frac{A_3}{A_1}, \quad B_5^* = K_5'' - K_1''\frac{A_5}{A_1}, \quad \text{etc.} \tag{3.39}$$

To summarize, in order to calculate the effect of wash out according to the assumption (3.37), all that is necessary is to provide the Fourier coefficients of the normal distribution of circulation (i.e. the circulation distribution of the untwisted wing) A_1, A_3, A_5, $\cdots$, as well as the Fourier coefficients of the distribution of angle of incidence D_1'', D_3'', D_5'', $\cdots$. As to the wing calculated in § 10, we have from Table 6,

$$\frac{A_1}{\alpha_m} = 0{\cdot}6844, \qquad ma = 4{\cdot}6829,$$

$$\frac{A_3}{A_1} = -0{\cdot}0322, \quad \frac{A_5}{A_1} = 0{\cdot}0355,$$

$$D_1''(=D_1 \text{ in Table 6}) = -0{\cdot}01720\,,$$
$$D_3''(=D_3 \text{ in Table 6}) = -0{\cdot}01186\,,$$
$$D_5''(=D_5 \text{ in Table 6}) = +0{\cdot}00116\,.$$

Substituting these values in (3.38) and (3.39), we have

$$\Delta^*\alpha^\circ = -0{\cdot}968^\circ, \quad B_3^* = -0{\cdot}00608\,, \quad B_5^* = 0{\cdot}00044\,,$$

which agree very well with the exact values given in Table 8, which agreement seems to indicate the adequacy of the assumption (3.37) and consequently of (3.9).

CHAPTER IV.

CALCULATION OF A WING WITH ASYMMETRICAL DISTRIBUTION OF ANGLE OF INCIDENCE.

This chapter is concerned with the calculation of the effect of aileron displacement, and with certain stability derivatives in the lateral motion of an aeroplane.

§ 20. *Basis of calculation of the aileron effect.* The wing with displaced ailerons is represented as the result of a sudden change in angle of incidence over that part occupied by the ailerons. Since it is possible to consider the circulation distribution of a wing without ailerons separately from the effect of aileron displacement, its treatment may be simplified by assuming the angle of incidence of the basic wing equal to zero, and by considering the effect of the ailerons by themselves.

Consider now the starboard aileron displaced upward while the port aileron displaced downward by an equal amount. Then, since the distribution of angle of incidence as measured from the attitude of no lift α_{asym} is such that $\alpha_{\text{asym}} = -\beta$ for $\theta_3 \leqq \theta \leqq \theta_4$, $\alpha_{\text{asym}} = \beta$ for $\pi - \theta_4 \leqq \theta \leqq \pi - \theta_3$, and $\alpha_{\text{asym}} = 0$ for the remaining range of θ, the coefficients of the series

$$\alpha_{\text{asym}} \sin \theta = D_2 \sin 2\theta + D_4 \sin 4\theta + D_6 \sin 6\theta + \cdots \qquad (4.1)$$

may be given by

$$D_{2n} = -\frac{4}{\pi}\int_{\theta_3}^{\theta_4} \beta(\theta) \sin \theta \sin 2n\theta d\theta \ (n = 1, 2, \ldots) . \qquad (4.2)$$

Although β generally varies across the aileron span, being dependent on the aileron displacement and the ratio of aileron chord to wing chord, a constant value β_A may be given for β when the aileron displacement and the chord ratio are maintained constant across the span. We then have

$$D_{2n} = \frac{2\beta_A}{\pi}\left\{\frac{\sin(2n-1)\theta_3}{2n-1} - \frac{\sin(2n+1)\theta_3}{2n+1} - \frac{\sin(2n-1)\theta_4}{2n-1} + \frac{\sin(2n+1)\theta_4}{2n+1}\right\} . \qquad (4.3)$$

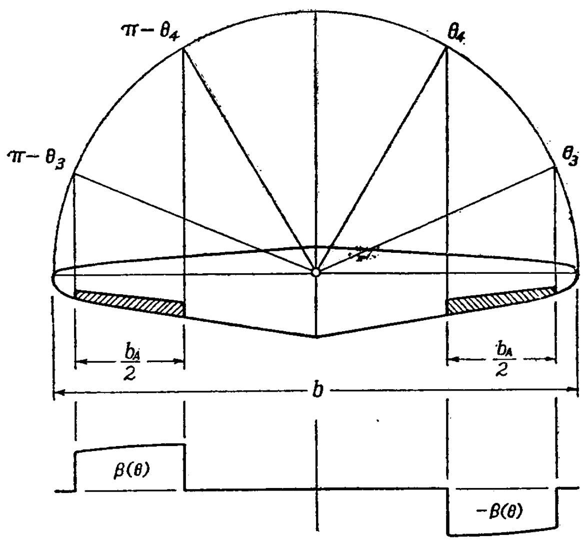

Fig. 18.

The calculation of the aileron effect is therefore reduced to the problem of a wing whose distribution of angle of incidence is given by (4.1), (4.2), or (4.3). There being nothing particularly difficult about the calculation, it may be successfully performed by the method due to Lotz (Ref. 5), although a considerable number of Fourier coefficients in the series expression for the circulation will have to be taken into account, because a sudden change in angle of incidence can be reproduced only by retaining the Fourier coefficients D_{2n} of the higher order. Considering the limited range within which the theoretical calculation can be applied in practice, it is doubtful whether its use justifies the extra labour entailed. An attempt will therefore be made in § 22 to develop an approximate method that would give results sufficiently accurate for most practical purposes.

§ **21.** *Aileron effect for an elliptic wing.* For the case of an elliptic wing, the exact solution of the aileron effect is obtained very easily, very similarly to that for the flap effect (§ 15). The distribution of circulation corresponding to the distribution of angle of incidence (4.1) is given by

$$G_{\mathrm{asym},e}(\eta) = 2\mu_e(B_{2,e}\sin 2\theta + B_{4,e}\sin 4\theta + \cdots)\,, \qquad (4.4)$$

where

$$B_{2n,e} = \frac{D_{2n}}{1+2n\mu_e} \quad (n = 1,\ 2,\ \cdots)\,, \qquad (4.5)$$

and

$$\mu_e = \left(\frac{at_0}{4b}\right)_e = \left(\frac{a}{\pi A}\right)_e\,, \qquad (4.6)$$

of which convenient use will be made in calculating the aileron effect for the general wing.

§ **22.** *Approximate solution of the aileron effect.* In order to calculate the aileron effect generally, but approximately, it is now assumed, exactly as in the case of flap effect, that the ratio of the circulation distribution $G_{\mathrm{asym}}(\eta)$, due to aileron displacement, to the normal distribution for the mean angle α_m is the same for the wings of the same span b, the same aspect ratio A, and the same distribution of angle of incidence (4.1).

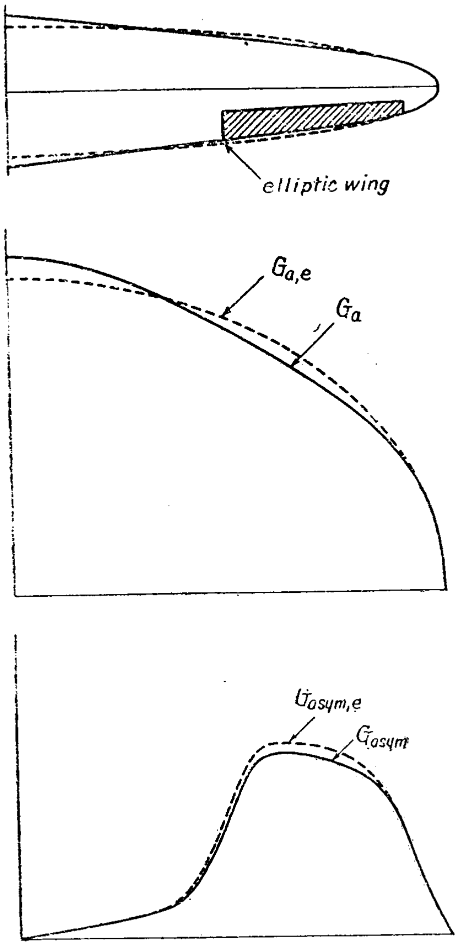

Fig. 19.

Considering now one wing as the elliptic wing (with subscript e) and the other to be the wing in question (without subscript), and dispensing with the subscript for the quantity, which is the same for both wings, the proposed assumption may be expressed by

$$\frac{G_{\mathrm{asym},e}(\eta)}{G_{a,e}(\eta)} = \frac{G_{\mathrm{asym}}(\eta)}{G_a(\eta)} , \qquad (4.7)$$

where

$$G_{a,e}(\eta) = \frac{2}{\lambda}\frac{\alpha_m}{1+1/\lambda}\sin\theta , \qquad (4.8)$$

$$G_{\mathrm{asym},e}(\eta) = \frac{2}{\lambda}\left\{\frac{D_2}{1+2/\lambda}\sin 2\theta + \frac{D_4}{1+4/\lambda}\sin 4\theta + \cdots\right\}, \qquad (4.9)$$

$$\lambda = \frac{\pi A}{a} , \qquad (4.10)$$

$$G_a(\eta) = 2\mu(A_1\sin\theta + A_3\sin 3\theta + \cdots) . \qquad (4.11)$$

The relation (4.7), true, is approximate, but its adequacy may be proved, as will be shown later, by the fact that its application gives results agreeing very well with those by exact solution. The practical importance of the relation (4.7) is then that $G_{\mathrm{asym}}(\eta)$, which is laborious to calculate, may be deduced from $G_{a,e}(\eta)$, $G_a(\eta)$, $G_{\mathrm{asym},e}(\eta)$, all of which may be calculated without difficulty. As a matter of fact, by substituting (4.8), (4.9), and (4.11) in (4.7), we have

$$G_{\mathrm{asym}}(\eta) = 2\mu(B_2\sin 2\theta + B_4\sin 4\theta + \cdots) , \qquad (4.12)$$

where

$$\left.\begin{aligned}
B_2 &= \frac{D_2 A_1}{\alpha_m}\frac{\lambda+1}{\lambda+2}\left\{\left(1+\frac{A_3}{A_1}\right)+\frac{\lambda+2}{\lambda+4}\frac{D_4}{D_2}\left(\frac{A_3}{A_1}+\frac{A_5}{A_1}\right)\right.\\
&\qquad\left.+\frac{\lambda+2}{\lambda+6}\frac{D_6}{D_2}\left(\frac{A_5}{A_1}+\cdots\right)+\cdots\right\},\\
B_4 &= \frac{D_2 A_1}{\alpha_m}\frac{\lambda+1}{\lambda+2}\left\{\left(\frac{A_3}{A_1}+\frac{A_5}{A_1}\right)+\frac{\lambda+2}{\lambda+4}\frac{D_4}{D_2}\left(1+\frac{A_3}{A_1}+\frac{A_5}{A_1}+\cdots\right)\right.\\
&\qquad\left.+\frac{\lambda+2}{\lambda+6}\frac{D_6}{D_2}\left(\frac{A_3}{A_1}+\frac{A_5}{A_1}+\cdots\right)+\cdots\right\},\\
B_6 &= \frac{D_2 A_1}{\alpha_m}\frac{\lambda+1}{\lambda+2}\left\{\left(\frac{A_5}{A_1}+\cdots\right)+\frac{\lambda+2}{\lambda+4}\frac{D_4}{D_2}\left(\frac{A_3}{A_1}+\frac{A_5}{A_1}+\cdots\right)\right.\\
&\qquad\left.+\frac{\lambda+2}{\lambda+6}\frac{D_6}{D_2}\left(1+\frac{A_3}{A_1}+\frac{A_5}{A_1}+\cdots\right)+\cdots\right\},\\
B_8 &= \frac{D_2 A_1}{\alpha_m}\frac{\lambda+1}{\lambda+2}\left\{\left(\frac{A_7}{A_1}+\cdots\right)+\frac{\lambda+2}{\lambda+4}\frac{D_4}{D_2}\left(\frac{A_5}{A_1}+\cdots\right)\right.\\
&\qquad+\frac{\lambda+2}{\lambda+6}\frac{D_6}{D_2}\left(\frac{A_3}{A_1}+\frac{A_5}{A_1}+\cdots\right)\\
&\qquad\left.+\frac{\lambda+2}{\lambda+8}\frac{D_8}{D_2}\left(1+\frac{A_3}{A_1}+\frac{A_5}{A_1}+\cdots\right)+\cdots\right\},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \quad (4.13)$$

In most practical cases, coefficients A_7 and those of higher order may be ignored without any loss of accuracy. Finally, using (1.36), (1.38), and (1.43), the rolling moment coefficient of the circulation distribution $G_{\mathrm{asym}}(\eta)$, namely, that due to the aileron displacement, becomes

$$C_l = -\frac{\pi}{4}\mu A B_2 = -\frac{ma}{4}\frac{\lambda+1}{\lambda+2}D_2(1+\sigma), \qquad (4.14)$$

where

$$\sigma = \frac{A_3}{A_1}+\frac{\lambda+2}{\lambda+4}\frac{D_4}{D_2}\left(\frac{A_3}{A_1}+\frac{A_5}{A_1}\right)+\frac{\lambda+2}{\lambda+6}\frac{D_6}{D_2}\left(\frac{A_5}{A_1}+\frac{A_7}{A_1}\right)+\cdots. \qquad (4.15)$$

It will readily be seen that $\sigma = 0$ for the elliptic wing.

§ 23. *Continuation. Induced yawing moment.* In the preceding paragraphs, only the asymmetrical part of the circulation distribution was considered, the reason being that the rolling moment is determined by the asymmetrical distribution alone. In calculating the yawing moment, the symmetrical distribution

$$G_{\text{sym}}(\eta) = 2\mu\{A_1 \sin\theta + (A_3 + B_3)\sin 3\theta + (A_5 + B_5)\sin 5\theta + \cdots\} \tag{4.16}$$

must be considered together with the asymmetrical part $G_{\text{asym}}(\eta)$, given in (4.12). Referring to (1.39), we have the coefficient of induced yawing moment

$$C_{ni} = -\frac{3C_z C_l}{\pi A}\left\{1 + \frac{A_3 + B_3}{A_1}\left(\frac{5}{3} + \frac{7}{3}\frac{B_4}{B_2}\right) + \frac{A_5 + B_5}{A_1}\left(\frac{9}{3}\frac{B_4}{B_2} + \frac{11}{3}\frac{B_6}{B_2}\right) + \cdots\right\}. \tag{4.17}$$

The total yawing moment is obtained by adding together the induced yawing moment and the profile yawing moment, the latter being caused by asymmetry in the profile drags of both ailerons.

It would appear from (4.17) that C_{ni} consists of two parts, the first due to normal distribution with coefficients A_1, A_3, $\cdots$, and the second to symmetrical zero distribution with coefficients B_3, B_5, $\cdots$, in view of which, it is convenient to write (4.17) in the form

$$\frac{C_{ni}}{C_l} = -\frac{3C_z}{\pi A}(1 + \nu) - \mu T\,, \tag{4.18}$$

where

$$\nu = \frac{5}{3}\frac{A_3}{A_1}\left(1 + \frac{7}{5}\frac{B_4}{B_2}\right) + \frac{9}{3}\frac{A_5}{A_1}\left(\frac{B_4}{B_2} + \frac{11}{9}\frac{B_6}{B_2}\right) + \cdots\,, \tag{4.19}$$

and

$$T = 5B_3\left(1 + \frac{7}{5}\frac{B_4}{B_2}\right) + 9B_5\left(\frac{B_4}{B_2} + \frac{11}{9}\frac{B_6}{B_2}\right) + \cdots. \tag{4.20}$$

Since (4.16) represents the circulation distribution for *any* symmetrical distribution of angle of incidence, it may be presumed for example, that the wing has a wash out and a landing flap displaced over that part between the ailerons ($\theta_1 \leqq \theta \leqq \pi - \theta_1$, § 16). The coefficients B_3, B_5, $\cdots$ will then consist of those due to wash out (with asterisk) and those due to landing flap (with prime), and reference to (3.25) and (3.39) gives

$$\left.\begin{aligned} B_3 &= B_3^* + B_3' = \left(K_3'' - K_1''\frac{A_3}{A_1}\right) + \left(K_3' - K_1'\frac{A_3}{A_1}\right), \\ B_5 &= B_5^* + B_5' = \left(K_5'' - K_1''\frac{A_5}{A_1}\right) + \left(K_5' - K_1'\frac{A_5}{A_1}\right), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad (4.21)$$

Since B_3^* and B_3' are negative, $-\mu T$ in (4.18) becomes positive, indicating that the adverse yawing moment due to aileron displacement is somewhat nullified by the effects of the wash out as well as by the central flap displacement.

As will be seen from (4.14) and (4.18), although the calculation of C_l requires B_2 only, that of C_{ni} requires, in addition, B_4, B_6, $\cdots$. Since the coefficients B_2, B_4, $\cdots$ converge somewhat slowly, it is usually necessary to take them as far as B_8 or B_{10}, which can very easily be found.

To summarise, in order to calculate the aileron effect according to the assumption (4.7), all that is necessary is to provide the Fourier coefficients of the circulation distribution when the aileron is not displaced, A_1, A_3, A_5, $\cdots$, B_3, B_5, $\cdots$, as well as the Fourier coefficients of the distribution of angle of incidence (4.1) D_2, D_4, D_6, $\cdots$, which depend only on the aileron displacement. The calculation, which is performed by means of the formulae (4.13)—(4.21), is very simple compared with those hitherto proposed, the latter requiring the solution of linear simultaneous equations or other equally processes laborious.

§ **24.** *Differentially displaced ailerons.* In the preceeding paragraphs it was assumed that the starboard aileron is displaced upward by an amount equal to that by which the port aileron is displaced downward. The effect is thus a pure rolling moment which does not alter the symmetrical distribution of circulation. If the ailerons were displaced differentially, say the starboard aileron was displaced upward by an amount exceeding that by which the port aileron was displaced downward, the distribution of angle of incidence would become such that $\alpha = -\beta_{AS}(\theta)$ for $\theta_3 \leqq \theta \leqq \theta_4$, $\alpha = \beta_{AP}(\theta)$ for $\pi - \theta_4 \leqq \theta \leqq \pi - \theta_3$, and $\alpha = 0$ for the remaining range of θ, where $\beta_{AS}(\theta) > \beta_{AP}(\pi - \theta)$. The effect of such an aileron displacement may be regarded as being composed of the symmetric circulation distribution due to the two ailerons having been displaced upward by the same amount and the asymmetric distribution due to starboard and port ailerons having been displaced upward and downward respectively also by the same amount. The asymmetric distribution may be calculated, as explaned in §§ 22–23, by using

$$D_{2n} = -\frac{4}{\pi}\int_{\theta_3}^{\theta_4} \frac{\beta_{AS}(\theta) + \beta_{AP}(\pi - \theta)}{2} \sin\theta \sin 2n\theta d\theta \quad (4.22)$$

instead of (4.2), while the symmetric distribution may be calculated, as explained in §§ 16–17, by using

$$D'''_{2n+1} = -\int_{\theta_3}^{\theta_4} \frac{\beta_{AS}(\theta) - \beta_{AP}(\pi - \theta)}{2} \sin\theta \sin(2n+1)\theta d\theta \quad (4.23)$$

instead of (3.2). It will thus be seen that the diffenrential displacement does not affect the rolling moment so long as the value of $\frac{1}{2}\left\{\beta_{AS}(\theta) + \beta_{AP}(\pi - \theta)\right\}$ remains constant. To the symmetric distribution, however, will be added the coefficients

$$B'''_3 = K'''_3 - K'''_1 \frac{A_3}{A_1}, \quad B'''_5 = K'''_5 - K'''_1 \frac{A_5}{A_1}, \quad \ldots, \quad (4.24)$$

where K_1''', K_3''', $\cdots$ are obtained from (3.15) if D_1', D_3', $\cdots$ are replaced by D_1''', D_3''', $\cdots$. Since B_3''' is negative, the effect is to make B_3 more negative, showing that the differential displacement nullifies somewhat the adverse yawing moment due to aileron rolling moment.

§25. *Numerical example.* As an illustration of the method described above, consider the tapered wing that is calculated in §§10, 18 as being fitted with ailerons, whose inner ends coincide with the ends of the landing flap and whose outer ends are 0·0786 $(b/2)$ inboard from the wing tips, whence

$$\theta_4 = \theta_1 = 60^\circ 28', \quad \theta_3 = \cos^{-1}(1-0{\cdot}0786) = 22^\circ 52',$$

and the total aileron span becomes 0·4286b. Besides, the ratio of aileron chord to wing chord is assumed constant across the span and that it is equal to 25%.

We shall now consider first the simple case in which the starboard and port ailerons are respectively displaced 15° upward and downward. Referring to Fig. 17b, we have $\beta^\circ = 6{\cdot}6^\circ$, and then $\beta_A = 6{\cdot}6^\circ/57{\cdot}3^\circ = 0{\cdot}115$. Using the foregoing values of θ_3, θ_4, and β_A, values of D_2, D_4, $\cdots$, are calculated as given in Table 12. $G_{\text{asym},\,e}$ is then calculated according

TABLE 12.

n	D_{2n}	B_{2n} (approx.)	B_{2n} (Lotz)
1	−0·05857	−0·0327	−0·03252
2	−0·00235	−0·0012	−0·00141
3	0·03149	0·0104	0·01074
4	−0·00283	−0·0009	−0·00071
5	−0·00207	—	−0·00034
6	0·00928	—	0·00216
7	−0·00932	—	−0·00199
8	−0·00630	—	−0·00120
9	0·00535	—	0·00091
10	−0·00348	—	−0·00056

to (4.9), in doing which the coefficients as far as D_{20} are retained, and 'G_{asym} (approx)' is deduced by multiplying $G_{asym,e}$ by the factor $G_a/G_{a,e}$ that was previously obtained. The resultant circulation is obtained by adding G_{asym} to G_{sym}, which latter is already known from §§10, 18.

Although the coefficient D_{2n} of the higher order have to be retained for plotting the curve of G_{asym}, for other purposes only the first three or four coefficients are needed, D_2, D_4, D_6, D_8 are retained here. Substituting in (4.13) the coefficients for the normal distribution,

$$\frac{A_1}{\alpha_m} = 0{\cdot}684 , \quad \frac{A_3}{A_1} = -0{\cdot}032 , \quad \frac{A_5}{A_1} = 0{\cdot}036 ,$$

which were obtained in §10, we have the coefficients B_2, B_4, B_6, B_8, as shown in the column 'B_{2n} (approx)' of Table 12. Substituting these values in (4.14) and (4.15), we have the coefficient of rolling moment

$$\sigma = -0{\cdot}044 , \qquad C_l = 0{\cdot}0559 .$$

Moreover, we have

$$B_3^* = -0{\cdot}0061 , \qquad B_5^* = 0{\cdot}0005$$

for the zero distribution due to wash out, and

$$B_3' = -0{\cdot}0405 , \quad B_5' = 0{\cdot}0112 , \quad B_7' = 0{\cdot}0025$$

for the zero distribution due to landing flap displacement. The computation according to (4.19) and (4.20) then gives

$$\nu = -0{\cdot}094 , \quad \mu T = -0{\cdot}009 - 0{\cdot}067 ,$$

from which the coefficient of induced yawing moment becomes

$$C_{ni} = -0{\cdot}0056\, C_z + 0{\cdot}0005 + 0{\cdot}0037 ,$$

where the terms on the righthand side represent the effects due to normal distribution, wash out, and landing flap displacement respectively.

TABLE 13.

θ	η	$G_{asym,e}$	$G_a/G_{a,e}$	G_{asym} (approx.)	G_{asym} (Lotz)
90°	0	0	1·081	0	0
81°	0·1564	−0·0015	1·052	−0 0016	−0·0014
72°	0·3090	−0·0030	1·008	−0·0030	−0·0028
63°	0·4540	−0 0102	0·969	−0·0099	−0·0096
54°	0·5878	−0·0222	0·942	−0·0209	−0·0210
45°	0·7071	−0.0218	0·935	−0·0204	−0·0204
36°	0·8090	−0·0192	0·952	−0 0183	−0·0184
27°	0 8910	−0·0132	0·986	−0 0130	−0·0130
18°	0·9511	−0·0047	1·012	−0·0048	−0 0047
9°	0·9877	−0·0015	1·030	−0.0015	−0·0015

As an example of differential displacement, consider next that case in which the starboard aileron is displaced 20° upward while the port aileron is displaced 10° downward. From Fig. 17b,

$$\beta^0_{AS} = 8 \cdot 1^\circ, \qquad \beta^0_{AP} = 4 \cdot 9^\circ,$$

hence

$$\frac{1}{2}(\beta^0_{AS} + \beta^0_{AP}) = 6.5^\circ, \quad \frac{1}{2}(\beta^0_{AS} - \beta^0_{AP}) = 1 \cdot 6^\circ.$$

The value of $\frac{1}{2}(\beta^0_{AS} + \beta^0_{AP})$ is slightly smaller than the value $\beta^0 = 6 \cdot 6^\circ$, obtained for the equally displaced ailerons, because the change in angle of no lift does not vary in proportion to displacement (Fig. 17). For this reason, the rolling moment for the differential displacement is reduced by a factor $6 \cdot 5^\circ / 6 \cdot 6^\circ = 0 \cdot 985$, as compared with the previously calculated value, namely,

$$C_l = 0 \cdot 0559 \times 0 \cdot 985 = 0 \cdot 0551 \,.$$

Substituting the values

$$D_1''' = -0 \cdot 0104\,, \quad D_3''' = -0 \cdot 0096\,, \quad D_5''' = 0 \cdot 0065\,, \quad D_7''' = 0 \cdot 0038$$

from (4.23) for D_1', D_3', D_5', D_7' of (3.15), we have

$$K_1''' = -0{\cdot}0069\,,\quad K_3''' = -0{\cdot}0047\,,\quad K_5''' = 0{\cdot}0025\,,\quad K_7''' = 0{\cdot}0011\,.$$

Hence from (4.24) we have

$$B_3''' = -0{\cdot}0049\,,\quad B_5''' = 0{\cdot}0027\,,\quad B_7''' = 0{\cdot}0011\,,$$

which must be added to B_3^*, B_5^*, $\cdots$, B_3', B_5', $\cdots$ above obtained, whereupon the coefficient of induced yawing moment becomes

$$C_{ni} = -0{\cdot}0055\,C_z + 0{\cdot}0005 + 0{\cdot}0037 + 0{\cdot}0021\,,$$

the last term on the righthand side representing the favourable moment due to differential displacement.

In these calculations, the results from Lotz's solution were used for the characteristics of the basic wing, namely the values of $G_a/G_{a,\,e}$ A_1/α_m, A_3/A_1, A_5/A_1, B_3^*, B_5^*, etc. These values could have been obtained from the approximate solution of §10, but with a view to checking the assumption (4.7), it is desirable that all causes for error other than the assumption (4.7) shall be excluded as far as possible. Since, however, it has been shown in §11 that the approximate solution gives results almost agreeing with those obtained by Lotz's solution, the end results of the present calculation would scarecely have differed had the basic values been taken from the approximate solution.

Comparison with Lotz's solution. As a check of the accuracy of the approximate solution, Lotz's method is applied to the same problem for the case when both ailerons are displaced by the same amount. The distribution of circulation G_{asym} and the corresponding Fourier coefficients B_{2n} are respectively given in Table 13 and 12; the respective coefficients of rolling moment and induced yawing moment are

$$C_l = 0{\cdot}05563\,,\quad C_{ni} = -0{\cdot}00558\,C_z + 0{\cdot}00043 + 0{\cdot}00388\,.$$

Comparing these results with those obtained above, it will be found that the approximate solution gives results sufficiently accurate for practical purposes.

§26. *Effect of dihedral.* The function of dihedral is to import a rolling moment in order to recover the incidental roll. Suppose, for example, that the starboard wing rolls down, resulting in a side slip that adds to the angle of incidence of the starboard wing an approximate

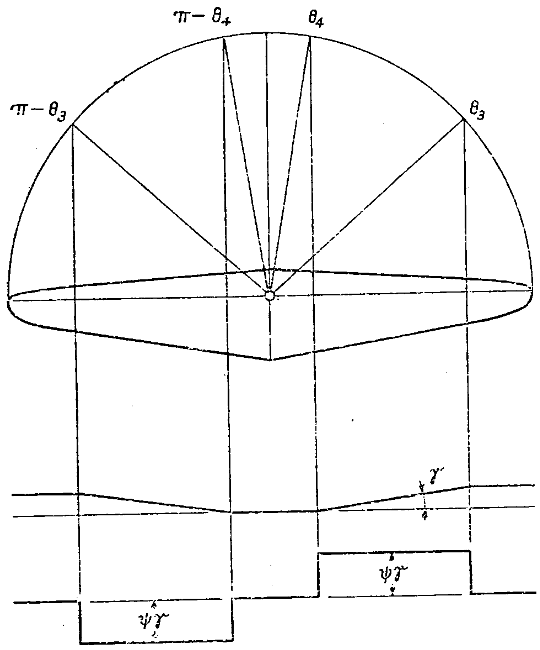

Fig. 20.

amount $\psi\gamma$, and then subtract the same amount from the angle of the port wing, where ψ is the angle of side slip and γ is the angle of dihedral, both being assumed small. The effect of dihedral may then be treated quite similarly to the effect of the ailerons. Let the dihedral be given over the parts $\theta_3 \leqq \theta \leqq \theta_4$ and $\pi-\theta_4 \leqq \theta \leqq \pi-\theta_3$. Then by referring to (4.14) and (4.18), the coefficients of rolling and yawing moments become

$$C_l^{(D)} = -\frac{\pi}{4}\mu A B_2^{(D)}, \tag{4.25}$$

$$C_{ni}^{(D)} = -C_l^{(D)}\left\{\frac{3C_z}{\pi A}(1+\nu^{(D)})+\mu T^{(D)}\right\} \tag{4.26}$$

respectively, where (D) refers to the dihedral and $B_2^{(D)}$, $\nu^{(D)}$, and $T^{(D)}$ are obtained by putting $\beta_A = -\psi\gamma$ in (4.3), (4.13), (4.19) and (4.20) respectively. Although these moments are expressed by reference to wind axes, they can be easily converted to chord axes. In normal flight condition, in which the angle of incidence α_0 is small and $C_{ni}^{(D)}$ is small compared with $C_l^{(D)}$, $C_l^{(D)}$ will have sensibly the same numerical value whichever system of axes is used, but $C_{ni}^{(D)}$ referred to chord axes can be obtained by adding $C_l^{(D)}\alpha_0$ to the value that is referred to wind axes.

§ 27. *Moments due to rate of roll.* Consider a wing with circulation distribution (4.16) rotating about the longitudinal axis with a rate of roll p. Since the positive sign of p corresponds to the lowering of the starboard wing, the effect of rolling is expressed by the asymmetrical distribution of angle of incidence

$$\alpha_{\text{asym}}^{(p)} = \frac{py}{V} = \frac{pb}{2V}\cos\theta\,, \tag{4.27}$$

where (p) indicates the effect of rolling. (4.27) is a special case of (4.1), and the Fourier coefficients of (4.2) and (4.13) become

$$D_2^{(p)} = \frac{pb}{4V}\,,\quad D_4^{(p)} = D_6^{(p)} = \cdots = 0\,, \tag{4.28}$$

and

$$\left.\begin{aligned} B_2^{(p)} &= \frac{A_1}{\alpha_m}\frac{\lambda+1}{\lambda+2}\left(1+\frac{A_3}{A_1}\right)\frac{pb}{4V}\,,\\ B_4^{(p)} &= \frac{A_1}{\alpha_m}\frac{\lambda+1}{\lambda+2}\left(\frac{A_3}{A_1}+\frac{A_5}{A_1}\right)\frac{pb}{4V}\,,\quad\cdots \end{aligned}\right\} \tag{4.29}$$

respectively. Then (4.14) gives the coefficient of rolling moment

$$C_l^{(p)} = -\frac{ma}{8}\frac{\lambda+1}{\lambda+2}\left(1+\frac{A_3}{A_1}\right)\frac{pb}{2V}, \tag{4.30}$$

which is negative, representing the damping moment for raising the starboard wing.

Care, however, must be exercised in calculating the induced yawing moment due to rate of roll. The actual flow at a starboard section behaves as shown in Fig. 21a, the flight speed V being horizontal, the resultant of V and py being inclined upward. In calculation, however, the wing is expressed as a twisted wing such that the

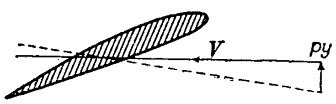

Fig. 21 a.

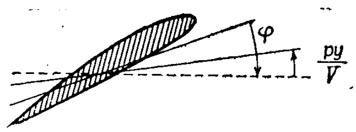

Fig. 21 b.

resultant of V and py lies horizontaly, as shown in Fig. 21b. Since the effective inflow is inclined downward by the induced angle φ from the horizontal, the yawing moment must be calculated by considering the horizontal component (horizontal in Fig. 21a, but not in Fig. 21b) of the lift $\rho V\Gamma$, which is perpendicular to the effective inflow. It becomes then

$$N_i^{(p)} = \int_{-\frac{b}{2}}^{\frac{b}{2}} \rho V\Gamma\left(\varphi - \frac{py}{V}\right) y\,dy\,, \tag{4.31}$$

giving the coefficient

$$\begin{aligned} C_{ni}^{(p)} = -\frac{pb}{2V}\Bigg[&\frac{C_z}{8}\left\{\left(1+\frac{A_3}{A_1}+\frac{B_3}{A_1}\right)-\frac{3ma}{\pi A}\frac{\lambda+1}{\lambda+2}\left(1+\frac{A_3}{A_1}\right)\left(1+\frac{5}{3}\frac{A_3}{A_1}\right)\right\} \\ &-\frac{\mu ma}{8}\frac{\lambda+1}{\lambda+2}\left\{B_3\left(5+12\frac{A_5}{A_1}+7\frac{A_5}{A_1}\right)+B_5\left(9\frac{A_3}{A_1}+20\frac{A_5}{A_1}+\cdots\right) \right. \\ &\left. +\cdots\right\}\Bigg] \end{aligned} \tag{4.32}$$

Since $C_{ni}^{(p)}$ is negative, it is evident that the lowering starboard wing is accelerated forward. A mistake that is frequently made is to assume that an increase in drag of the lowering wing results in a yawing moment retarding that wing, that is, in a positive value of $C_{ni}^{(p)}$ — erroneous because it omits to take into account the conditions explained above, namely, that it really results from taking the horizontal component of lift in the sense as shown in Fig. 21b.

The above calculations are obviously not precise, seeing that they take no account of changes in the trailing vortices due to rolling, and do not allow for the moment due to section profile drag. For most practical purposes, however, these effects are so small that they could be neglected altogether. The stability derivatives due to rate of roll are then given by

$$\frac{dC_l}{d\dfrac{pb}{2V}} = -\frac{ma}{8}\frac{\lambda+1}{\lambda+2}\left(1+\frac{A_3}{A_1}\right), \qquad (4.33)$$

$$\begin{aligned}\frac{dC_n}{d\dfrac{pb}{2V}} = &-\frac{C_z}{8}\left(1+\frac{A_3}{A_1}\right)\left\{1-\frac{3ma}{\pi A}\frac{\lambda+1}{\lambda+2}\left(1+\frac{5}{3}\frac{A_3}{A_1}\right)\right\} \\ &-\frac{\mu}{8}\left[\pi AB_3-ma\frac{\lambda+1}{\lambda+2}\left\{B_3\left(5+12\frac{A_3}{A_1}+7\frac{A_5}{A_1}\right)\right.\right. \\ &\left.\left.+B_5\left(9\frac{A_3}{A_1}+20\frac{A_5}{A_1}+\cdots\right)+\cdots\right\}\right]. \qquad (4.34)\end{aligned}$$

Apparently, the zero distribution of circulation does not affect $dC_l/d(pb/2V)$. Finally the derivatives for the typical wing considered in §§10, 18 are

$$\frac{dC_l}{d\dfrac{pb}{2V}} = -0.484\,,$$

$$\frac{dC_n}{d\dfrac{pb}{2V}} = -0.071\,C_z + 0.002\left(\begin{matrix}\text{due to}\\ \text{wash out}\end{matrix}\right) + 0.011\left(\begin{matrix}\text{due to flap}\\ \text{displacement}\end{matrix}\right).$$

§ 28. *Moments due to rate of yaw.* The wing in yawing motion about the vertical axis with a rate of yaw r experiences rolling and yawing moments, which may be calculated in much the same way. We shall assume that the wing has the distribution of angle of incidence α_{sym}, such that

$$\alpha_{sym} \sin \theta = D_1 \sin \theta + D_3 \sin 3\theta + D_5 \sin 5\theta + \cdots , \qquad (4.35)$$

and the distribution of circulation as given by (4.16).

Writing the circulation distribution in yawing motion

$$\Gamma = \Gamma_{sym} + \Gamma^{(r)}_{asym} = bV(G_{sym} + G^{(r)}_{asym}) , \qquad (4.36)$$

where G_{sym} is given by (4.16) and (r) indicates the effect of yawing, and taking the rate of yaw r to be positive when the starboard wing is retarded, we have approximately the equation for determining Γ, namely,

$$\Gamma = \frac{1}{2} at (V - ry) \left\{ \alpha_{sym} - \frac{1}{4\pi (V - ry)} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{d\Gamma}{dy'} \frac{dy'}{y - y'} \right\} , \qquad (4.37)$$

or

$$\Gamma = \frac{1}{2} at V \left\{ \alpha_{sym} \left(1 - \frac{ry}{V} \right) - \frac{1}{4\pi V} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{d\Gamma}{dy'} \frac{dy'}{y - y'} \right\} . \qquad (4.38)$$

It would seem from the last equation that the effect of yawing is equivalent to adding to the symmetric distribution of angle of incidence the asymmetric distribution

$$\alpha^{(r)}_{asym} = - \frac{ry}{V} \alpha_{sym} . \qquad (4.39)$$

Since (4.39) is a special case of (4.1), and since the Fourier coefficients of (4.2) become

$$D^{(r)}_2 = - \frac{rb}{4V} (D_1 + D_3) , \quad D^{(r)}_4 = - \frac{rb}{4V} (D_3 + D_5) , \; \ldots , \qquad (4.40)$$

it is obvious that the problem may be treated as a special case of aileron effect. Substituting $D_{2n}^{(r)}$ for D_{2n} in (4.13), we have

$$\left.\begin{aligned} B_2^{(r)} &= -\frac{rb}{4V}\frac{A_1}{\alpha_m}\left\{\frac{\lambda+1}{\lambda+2}\left(1+\frac{A_3}{A_1}\right)(D_1+D_3)\right. \\ &\qquad \left.+\frac{\lambda+1}{\lambda+4}\left(\frac{A_3}{A_1}+\frac{A_5}{A_1}\right)(D_3+D_5)+\cdots\right\}, \\ B_4^{(r)} &= -\frac{rb}{4V}\frac{A_1}{\alpha_m}\left\{\frac{\lambda+1}{\lambda+2}\left(\frac{A_3}{A_1}+\frac{A_5}{A_1}\right)(D_1+D_3)\right. \\ &\qquad \left.+\frac{\lambda+1}{\lambda+4}\left(1+\frac{A_3}{A_1}+\frac{A_5}{A_1}+\cdots\right)(D_3+D_5)+\cdots\right\}, \\ &\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad (4.41)$$

For calculating the rolling moment we must take into account the effective longitudinal velocity, thus

$$L^{(r)} = -\int_{-\frac{b}{2}}^{\frac{b}{2}}\rho(V-ry)\Gamma y dy = \int_{-\frac{b}{2}}^{\frac{b}{2}}\rho V\Gamma_{\text{sym}}\frac{ry}{V}ydy - \int_{-\frac{b}{2}}^{\frac{b}{2}}\rho V\Gamma_{\text{asym}}ydy\,, \quad (4.42)$$

where the first term on the right hand side can be easily integrated and the second term can be calculated with the aid of (4.14). The coefficient of rolling moment then becomes

$$C_l^{(r)} = \frac{rb}{2V}\frac{C_z}{8}\left[\left(1+\frac{A_3}{A_1}+\frac{B_3}{A_1}\right)+\left\{\frac{\lambda+1}{\lambda+2}\left(1+\frac{A_3}{A_1}\right)\left(\frac{D_1}{\alpha_m}+\frac{D_3}{\alpha_m}\right)\right.\right.$$
$$\left.\left.+\frac{\lambda+1}{\lambda+4}\left(\frac{A_3}{A_1}+\frac{A_5}{A_1}\right)\left(\frac{D_3}{\alpha_m}+\frac{D_5}{\alpha_m}\right)+\cdots\right\}\right], \quad (4.43)$$

which is positive, indicating that the retarding starboard wing is lowered.

The induced yawing moment may be calculated in much the same way as that for the aileron displacement. Referring to (4.14), (4.18), (4.19) and (4.20), we have

$$C_{ni}^{(r)} = \frac{\pi}{4}\mu A B_2^{(r)}\left[\frac{3C_z}{\pi A}\left\{1+\frac{5}{3}\frac{A_3}{A_1}\left(1+\frac{7}{5}\frac{B_4^{(r)}}{B_2^{(r)}}\right)\right.\right.$$
$$\left.+\frac{9}{3}\frac{A_5}{A_1}\left(\frac{B_4^{(r)}}{B_2^{(r)}}+\frac{11}{9}\frac{B_6^{(r)}}{B_2^{(r)}}\right)+\cdots\right\}$$
$$\left.+\mu\left\{5B_3\left(1+\frac{7}{5}\frac{B_4^{(r)}}{B_2^{(r)}}\right)+9B_5\left(\frac{B_4^{(r)}}{B_2^{(r)}}+\frac{11}{9}\frac{B_6^{(r)}}{B_2^{(r)}}\right)+\cdots\right\}\right], \quad (4.44)$$

which is negative, i.e., the damping yawing moment.

Besides, the wing experiences the rolling and yawing moments caused by the section profile drag. Although the former is negligible, the latter in coefficient form is given by

$$C_{n0}^{(r)} = \frac{1}{(\rho V^2/2)Sb}\int_{-\frac{b}{2}}^{\frac{b}{2}}\frac{\rho}{2}(V-ry)^2 t c_{x0}\, y\, dy = -\frac{rb}{2V}\int_0^1 c_{x0}\frac{bt}{S}\eta^2 d\eta\,. \quad (4.45)$$

Although the foregoing calculations take no account of changes in trailing vortices due to yawing, estimates obtained by means of them are sufficiently accurate for most practical cases. Thus the stability derivatives due to yawing become

$$\frac{dC_l}{d\frac{rb}{2V}} = \frac{C_z}{8}\left[\left(1+\frac{A_3}{A_1}+\frac{B_3}{A_1}\right)+\left\{\frac{\lambda+1}{\lambda+2}\left(1+\frac{A_3}{A_1}\right)\left(\frac{D_1}{\alpha_m}+\frac{D_3}{\alpha_m}\right)\right.\right.$$
$$\left.\left.+\frac{\lambda+1}{\lambda+4}\left(\frac{A_3}{A_1}+\frac{A_5}{A_1}\right)\left(\frac{D_3}{\alpha_m}+\frac{D_5}{\alpha_m}\right)+\cdots\right\}\right], \quad (4.46)$$

$$\frac{dC_n}{d\frac{rb}{2V}} = -\int_0^1 c_{x0}\frac{bt}{S}\eta^2 d\eta - \frac{C_z}{8}\left\{\frac{\lambda+1}{\lambda+2}\left(1+\frac{A_3}{A_1}\right)\left(\frac{D_1}{\alpha_m}+\frac{D_3}{\alpha_m}\right)\right.$$

$$\left. +\frac{\lambda+1}{\lambda+4}\left(\frac{A_3}{A_1}+\frac{A_5}{A_1}\right)\left(\frac{D_3}{\alpha_m}+\frac{D_5}{\alpha_m}\right)+\cdots\right\}\times$$

$$\times\left[\frac{3C_z}{\pi A}\left\{1+\frac{5}{3}\frac{A_3}{A_1}\left(1+\frac{7}{5}\frac{B_4^{(r)}}{B_2^{(r)}}\right)+\cdots\right\}\right.$$

$$\left.+\mu\left\{5B_3\left(1+\frac{7}{5}\frac{B_4^{(r)}}{B_2^{(r)}}\right)+\cdots\right\}\right]. \qquad (4.47)$$

If the angle of incidence is constant across the span, that is, if the wing has neither twist nor flap displacement, we have $D_1/\alpha_m = 1$, $D_3 = D_5 = \cdots = 0$, $B_3 = B_5 = \cdots = 0$. The values of the derivatives of the wing calculated in §§ 10, 18 are

$$\frac{dC_l}{d\frac{rb}{2V}} = 0{\cdot}224\,C_z - 0{\cdot}011\,,$$

$$\frac{dC_n}{d\frac{rb}{2V}} = -(0{\cdot}103\,C_z - 0{\cdot}006)(0{\cdot}110\,C_z - 0{\cdot}008) - 0{\cdot}213\,C_{x0}\,.$$

Note on change of axes. The four derivatives (4.33), (4.34), (4.46), (4.47) were derived by reference to wind axes. The corresponding derivatives with respect to chord axes, whose x-axis inclines by an angle of incidence α_0 to that of the wind axes, are calculated by means of the following relations:

$$\left.\begin{aligned}
\left[\frac{dC_l}{d\frac{pb}{2V}}\right]_{\substack{\text{chord}\\\text{axes}}} &= \frac{dC_l}{d\frac{pb}{2V}}\cos^2\alpha_0 - \left(\frac{dC_l}{d\frac{rb}{2V}}+\frac{dC_n}{d\frac{pb}{2V}}\right)\cos\alpha_0\sin\alpha_0 + \frac{dC_n}{d\frac{rb}{2V}}\sin^2\alpha_0\,,\\
\left[\frac{dC_n}{d\frac{pb}{2V}}\right]_{\substack{\text{chord}\\\text{axes}}} &= \frac{dC_n}{d\frac{pb}{2V}}\cos^2\alpha_0 + \left(\frac{dC_l}{d\frac{pb}{2V}}-\frac{dC_n}{d\frac{rb}{2V}}\right)\cos\alpha_0\sin\alpha_0 - \frac{dC_l}{d\frac{rb}{2V}}\sin^2\alpha_0\,,
\end{aligned}\right\}$$

$$\left[\frac{dC_l}{d\frac{rb}{2V}}\right]_{\substack{\text{chord}\\\text{axes}}}=\frac{dC_l}{d\frac{rb}{2V}}\cos^2\alpha_0+\left(\frac{dC_l}{d\frac{pb}{2V}}-\frac{dC_n}{d\frac{rb}{2V}}\right)\cos\alpha_0\sin\alpha_0-\frac{dC_n}{d\frac{rb}{2V}}\sin^2\alpha_0\,,$$

$$\left[\frac{dC_n}{d\frac{rb}{2V}}\right]_{\substack{\text{chord}\\\text{axes}}}=\frac{dC_n}{d\frac{rb}{2V}}\cos^2\alpha_0-\left(\frac{dC_l}{d\frac{rb}{2V}}+\frac{dC_n}{d\frac{bp}{2V}}\right)\cos\alpha_0\sin\alpha_0+\frac{dC_l}{d\frac{pb}{2V}}\sin^2\alpha_0\,. \qquad (4.48)$$

CHAPTER V.

NUMERICAL RESULTS FOR A SERIES OF STRAIGHT TAPERED WINGS.

In the preceeding chapters, numerical calculations were made for a typical wing in order to illustrate the arguments as well as to facilitate the application of the method. In view of the importance of providing the general numerical concept relating to a number of aerodynamic characteristics, this chapter deals with the numerical results for a series of straight tapered wings, calculated according to the method discussed

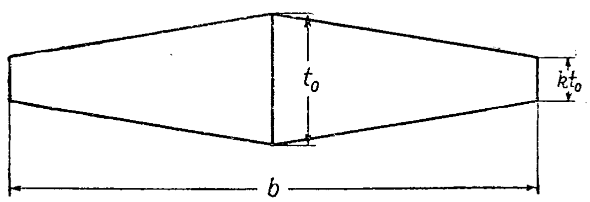

Fig. 22.

in the preceeding chapters. The straight tapered wing, also called trapezoidal wing, is one in which the chord diminishes uniformly from a maximum value t_0 at the centre of the wing, to kt_0 at the tips. The parameters involved are the taper ratio k, and the ratio of aspect ratio to the slope of the lift curve, $A/a[=\lambda/\pi$, see (3.12) (4·10)].

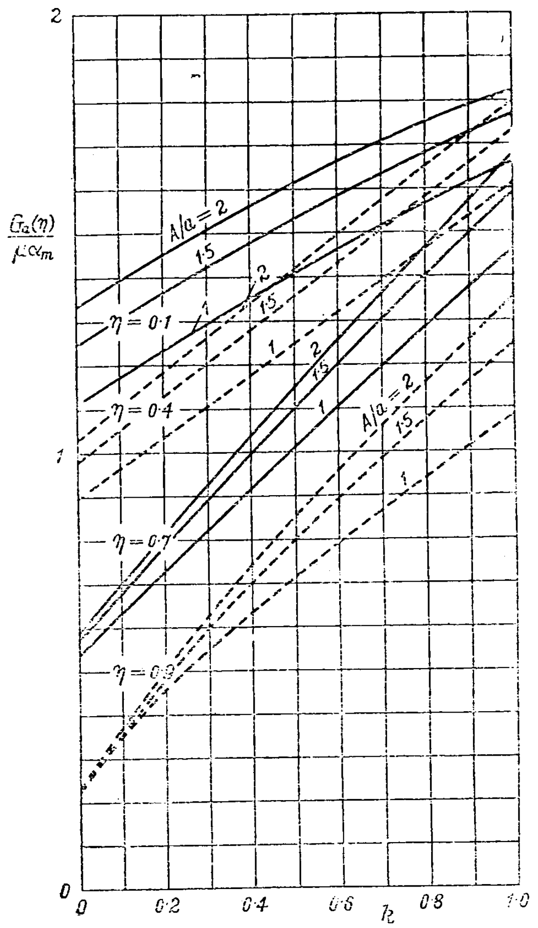

Fig. 23.

§ **29.** *Simple untwisted wing.* Numerical results for the circulation distribution of a straight tapered untwisted wing, or the normal distribution of circulation of a straight tapered wing, have been given by Betz (Ref. 1), Glauert (Ref. 3), and Hueber (Ref. 15). The values of $G_a/\mu\alpha_m$ for $\eta = 0{\cdot}1$, $0{\cdot}4$, $0{\cdot}7$, $0{\cdot}9$ have been interpolated from these results, as shown in Fig. 23. Here α_m is the angle of incidence, which is constant across the span, and G_a is the non-dimensional circulation, expressed by

$$G_a(\eta) = \frac{\Gamma_a(\eta)}{bV} = 2\mu(A_1 \sin\theta + A_3 \sin 3\theta + \cdots), \tag{5.1}$$

where

$$\mu = \frac{at_0}{4b} = \frac{a}{2A(1+k)}. \tag{5.2}$$

Using the above values of $G_a/\mu\alpha_m$, the Fourier coefficients A_1/α_m, A_3/α_m, A_5/α_m are derived by harmonic analysis as discussed in §9, see Fig. 10. From this result, the values of A_3/A_1, A_5/A_1, and the rate of reduction of the slope of lift curve as compared with that in two dimensional flow [see (1.43)],

$$m = \frac{C_z}{a\alpha_m} = \frac{\pi}{2(1+k)} \frac{A_1}{\alpha_m}, \tag{5.3}$$

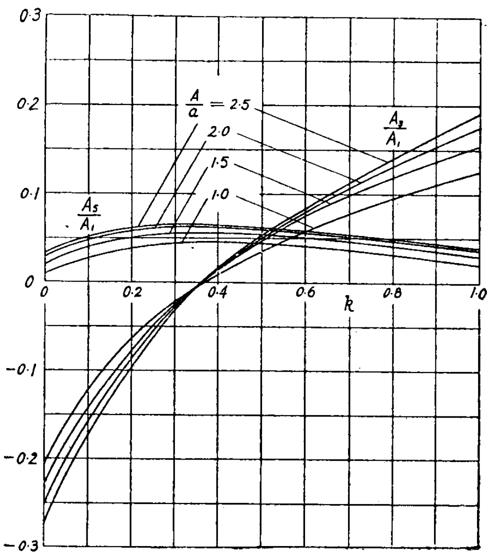

Fig. 24.

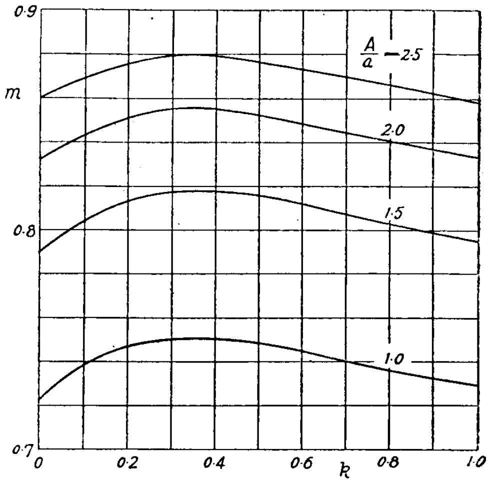

Fig. 25.

are calculated, and shown in Figs. 24, 25, which will serve as basic data for the calculation in §§30–32.

The reduction in the slope of lift may be expressed in an alternative form, that is, that the angle of incidence of the wing exceeds the angle of incidence in two dimensional flow, which would give the same lift coefficient, by an angle

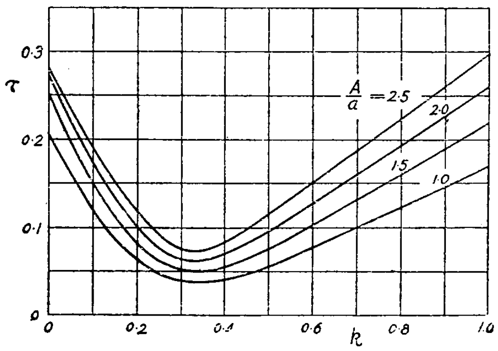

Fig. 26.

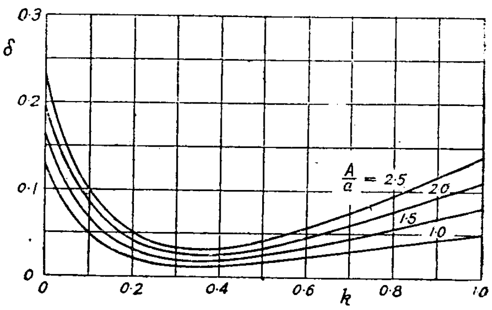

Fig. 27.

$$\Delta\alpha = \frac{1+\tau}{\pi A} C_z \,, \tag{5.4}$$

where

$$\tau = \frac{\pi A}{a} \frac{1-m}{m} - 1 \,, \tag{5.5}$$

as shown in Fig. 26. The induced drag coefficient is given in a form similar to (5.4), by

$$C_{xi} = \frac{1+\delta}{\pi A} C_z^2 \,, \tag{5.6}$$

where δ is shown in Fig. 27.

§ 30. *Effect of linear wash out.* As an illustration of a twisted wing, consider a straight tapered wing whose angle of incidence diminishes uniformly from the centre to the tips, the difference of the angles between the centre and the tips, or the amount of wash out, being ϵ (rad.). With the notations of § 19, we have

$$\alpha'' = -\epsilon \,|\cos\theta|\,, \tag{5.7}$$

$$D''_{2n+1} = \frac{4\epsilon}{\pi} \frac{(-1)^n}{(2n-1)(2n+3)} \quad (n = 0,\ 1,\ 2,\ \ldots)\,. \tag{5.8}$$

First, $G''(\eta)$ is calculated by using the assumption (3.37), after which the zero distribution due to wash out $G_b^*(\eta)$ is derived from the relation

$$G_b^*(\eta) = G''(\eta) - ma\,\Delta^*\alpha\, G_{a1}(\eta)\,, \tag{5.9}$$

where $\Delta^*\alpha$ is the change in angle of no lift due to wash out, and is obtained by (3.38). The values of $G_b^*/\mu\epsilon$ and $\Delta^*\alpha/\epsilon$ are given in Figs. 28, 29.

Finally, calculating B_3^*, B_5^*, $\cdots$ by the method of § 19, and substituting the values obtained in (3.28) and (3.29), we get the increase

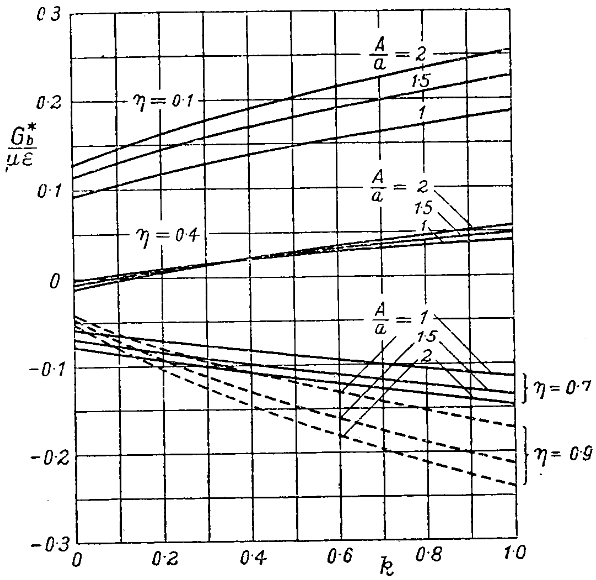

Fig. 28.

in induced drag coefficient due to wash out. The induced drag coefficient of the straight tapered twisted wing is written in the form

$$C_{xi} = \frac{1+\delta}{\pi A} C_z^2 + \mu \frac{w_1}{\epsilon} \epsilon C_z + \mu^2 A \frac{w_2}{\epsilon^2} \epsilon^2 , \qquad (5.10)$$

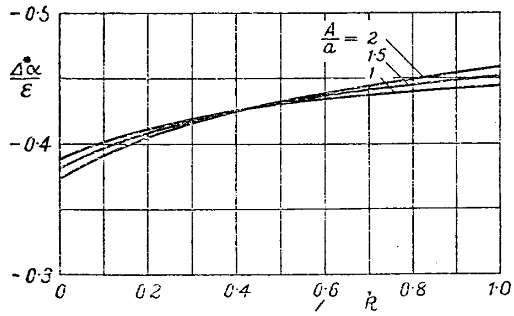

Fig. 29.

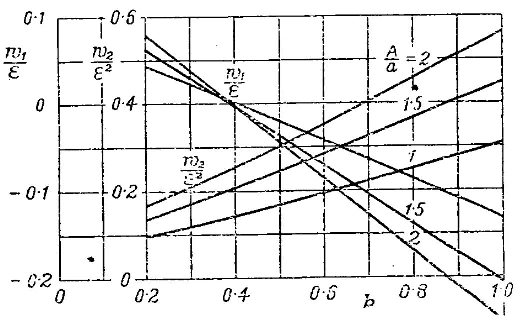

Fig. 30.

where the first and the third terms depend only on the normal and zero distributions respectively, while the second term depends on both. The values of w_1/ϵ and w_2/ϵ^2 are shown in Fig. 30.

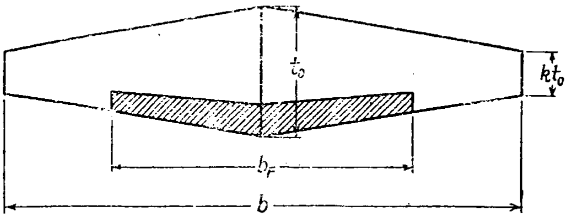

Fig. 31.

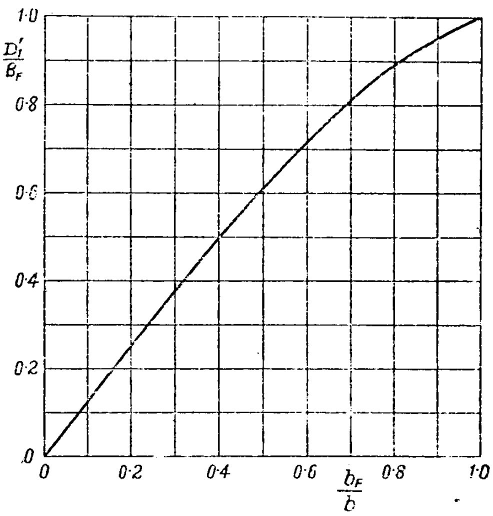

Fig. 32.

§31. *Effect of flap displacement.* Consider the straight tapered wing to be fitted with a central landing flap of span b_F, and of constant chord ratio such that the change in angle of no lift is constant and is equal to β_F (rad.). The increase in lift coefficient due to flap displacement (3.16) is

$$C_z' = ma\frac{D_1'}{\beta_F}\beta_F(1+\varpi), \tag{5.11}$$

where the values of D_1'/β_F and ϖ are shown in Figs. 32 and 33 respectively. The induced drag coefficient of an untwisted wing with displaced flap as expressed by (3.26), is

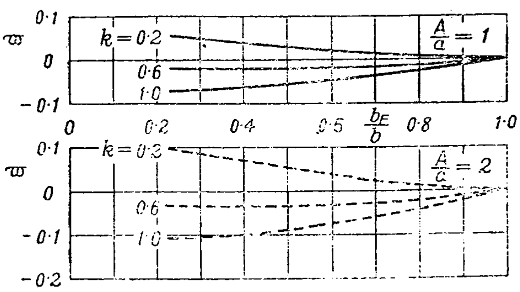

Fig. 33.

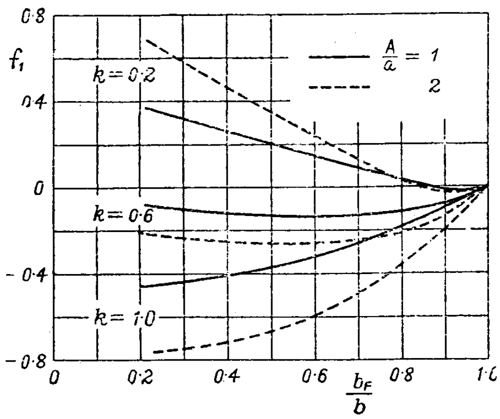

Fig. 34.

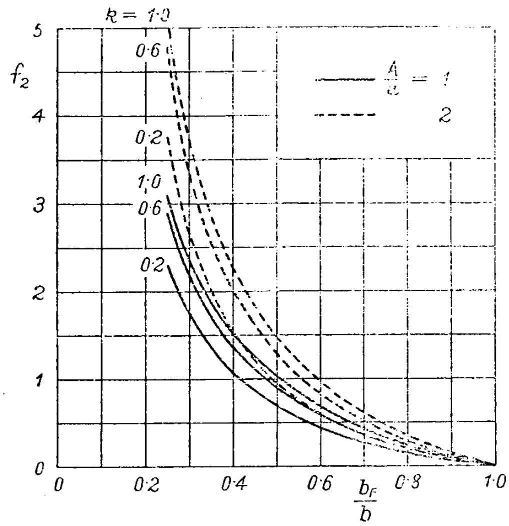

Fig. 35.

$$C_{xi} = \frac{1+\delta}{\pi A}C_z^2 + \frac{f_1}{\pi A}C_z C_z' + \frac{f_2}{\pi A}C_z'^2\,, \qquad (5.12)$$

the values of f_1 and f_2 being shown in Figs. 34 and 35 respectively.

§ 32. *Effect of aileron displacement.* Consider the straight tapered wing to be fitted with tip ailerons of span $b_A/2$ and of constant chord ratio. The starboard aileron is displaced upward and the port aileron downward, the changes in angle of no lift being β_{AS} and β_{AP} (rad.) respectively, and

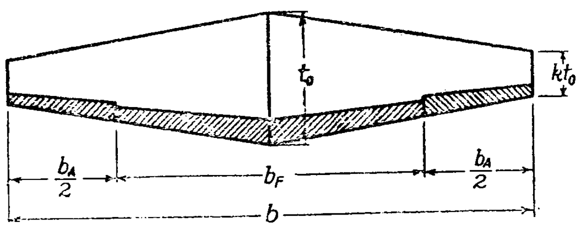

Fig. 36.

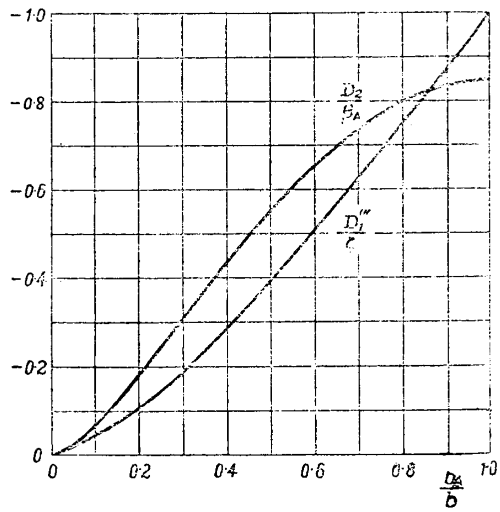

Fig. 37.

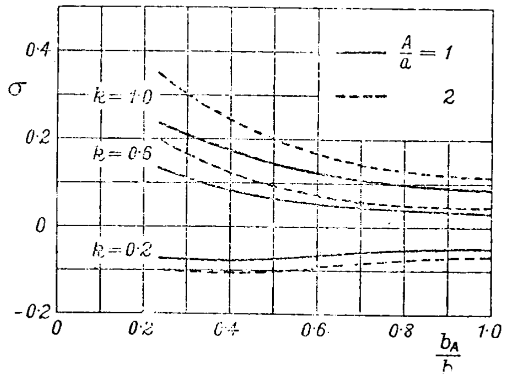

Fig. 38.

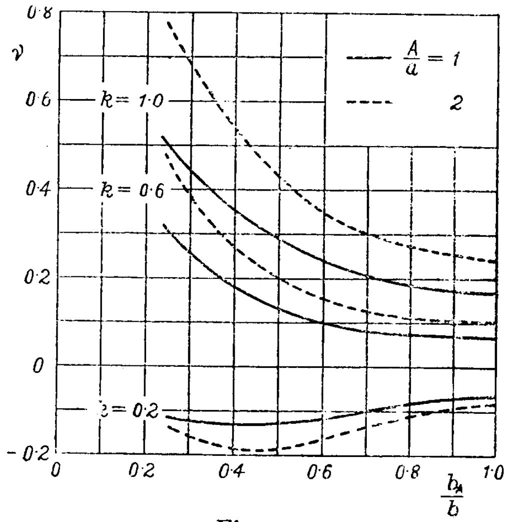

Fig. 39.

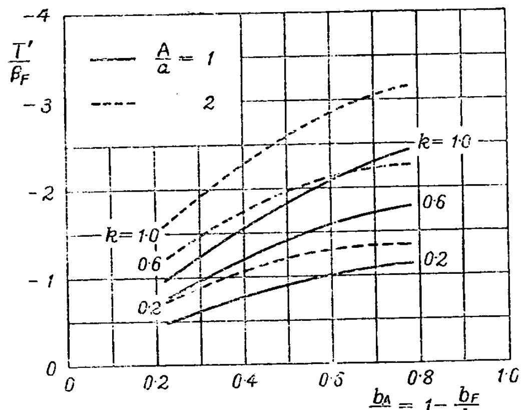

Fig. 40.

$$\beta_A = \frac{1}{2}(\beta_{AS}+\beta_{AP})\,, \quad \zeta = \frac{1}{2}(\beta_{AS}-\beta_{AP}) > 0\,. \qquad (5.13)$$

The wing has, moreover, a linear wash out ϵ (rad.), as in. §30, and a landing flap between the ailerons, as in §31 (span $= b_F = b - b_A$, the change in angle of no lift $= \beta_F$). Referring to (4.14) and (4.18), the coefficients of rolling and induced yawing moments are given by

$$C_l = -\frac{ma}{4}\frac{\pi A/a+1}{\pi A/a+2}\frac{D_2}{\beta_A}\beta_A(1+\sigma)\,, \qquad (5\cdot14)$$

$$C_{ni} = -C_l\left\{\frac{3C_z}{\pi A}(1+\nu)+\mu\left(\frac{T'}{\beta_F}\beta_F+\frac{T''}{\epsilon}\epsilon+\frac{T'''}{\zeta}\zeta\right)\right\} \qquad (5\cdot15)$$

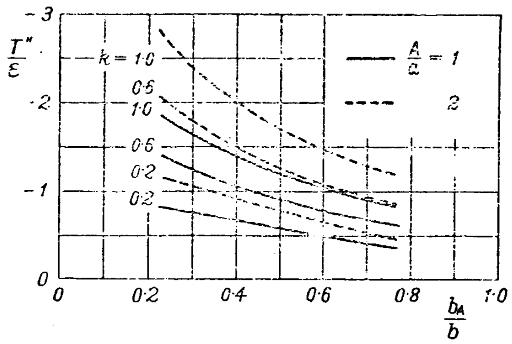

Fig. 41.

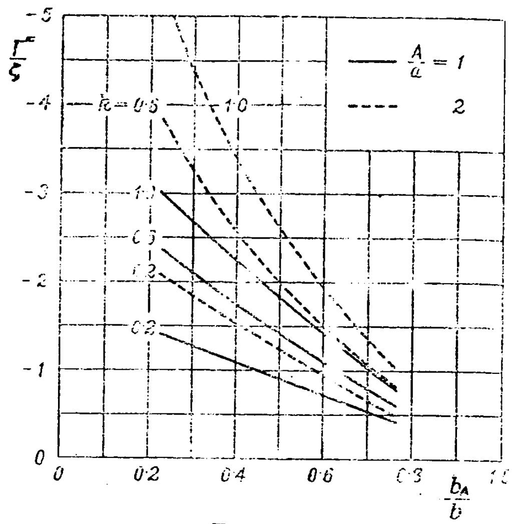

Fig. 42.

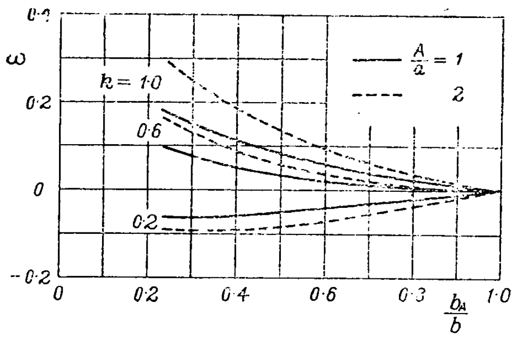

Fig. 43.

respectively. It will be seen that the lift coefficient C_z in (5.15) is the resultant of values for the basic untwisted wing and those due to the flap displacement

$$C'_z = ma\frac{D'_1}{\beta_F}\beta_F(1+\varpi)\,, \qquad (5.11)$$

due to wash out

$$C''_z = ma\,\Delta^*\alpha\,, \qquad (5.16)$$

and due to differential aileron displacement

$$C'''_z = ma\frac{D'''_1}{\zeta}\zeta(1+\omega) \qquad (5.17)$$

respectively. The values of D_2/β_A, D_1'''/ζ, σ, ν, T'/β_F, T''/ϵ, T'''/ζ, and ω are shown in Figs. 37–43 respectively, the other quantities appearing in the above formulae being shown elsewhere.

§ 33. *Derivatives of lateral stability.* Four derivatives of lateral stability are calculated for the simple untwisted wing, according to

$$\frac{dC_l}{d\dfrac{pb}{2V}} = -a\cdot\frac{m}{8}\frac{\pi A/a+1}{\pi A/a+2}\left(1+\frac{A_3}{A_1}\right), \tag{5.18}$$

$$\frac{dC_n}{d\dfrac{pb}{2V}} = -C_z\cdot\frac{1}{8}\left(1+\frac{A_3}{A_1}\right)\left\{1-\frac{3ma}{\pi A}\frac{\pi A/a+1}{\pi A/a+2}\left(1+\frac{5}{3}\frac{A_3}{A_1}\right)\right\}, \tag{5.19}$$

$$\frac{dC_l}{d\dfrac{rb}{2V}} = C_z\cdot\frac{1}{8}\frac{2\pi A/a+3}{\pi A/a+2}\left(1+\frac{A_3}{A_1}\right), \tag{5.20}$$

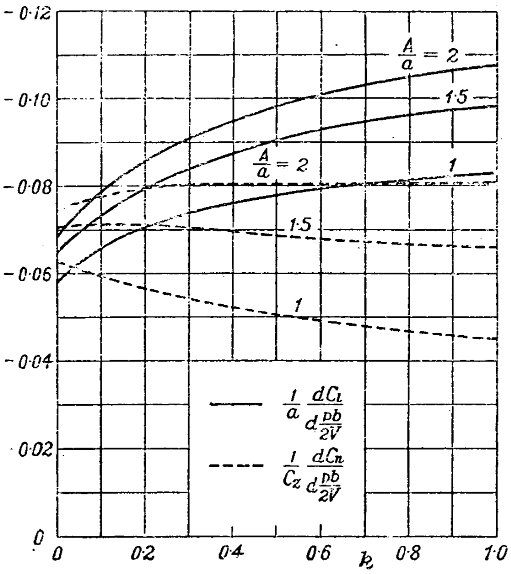

Fig. 44.

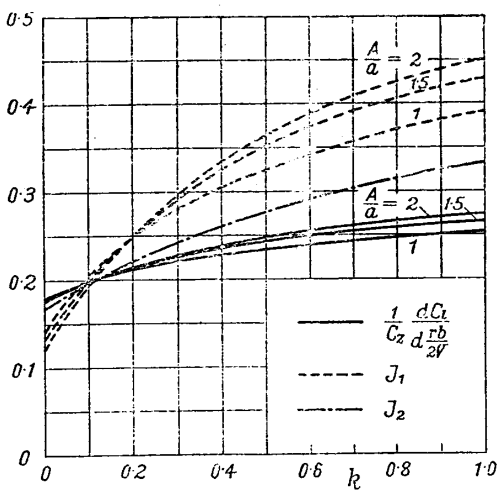

Fig. 45.

$$\left.\begin{aligned} \frac{dC_n}{d\frac{rb}{2V}} &= -J_1C_{xi}-J_2C_{x0}\,, \quad J_2 = \frac{1+3k}{6(1+k)}\,, \\ J_1 &= \frac{3}{8}\,\frac{\pi A/a+1}{\pi A/a+2}\,\frac{(1+A_3/A_1)(1+5A_3/3A_1)}{1+\delta}\,, \end{aligned}\right\} \quad (5.21)$$

where J_2C_{x0} is obtained by assuming that the section profile drag coefficient c_{x0} is constant and is equal to the over all coefficient c_{x0}. See Fig. 44 and 45 for the values of the derivatives.

§ 34. *Supplement.* Although, strictly speaking, the numerical results given in §§29–33 refer to the straight tapered wing, they can also be used for roughly estimating the characteristics of the tapered wing with rounded tips, for which reason the values of the coefficients are given separately; for example, the rolling moment coefficient due to aileron displacement instead of being given as the curve of $C_l/a\beta_A$, is given as curves of the constituent quantities m, D_2/β_A, σ, etc. The rolling moment of a tapered wing with rounded tips may be best deduced by applying the correction to the values of m and A/a, the value of σ, which is laborious to calculate, being estimated from Fig. 38.

In order to estimate the circulation distribution for a wing with rounded tips from the values of G_a in Fig. 23, it may be assumed that the shape of distribution of the local lift coefficient $c_z=(2b/t)G_a$ does not change, whether the tips are rounded or squared, the scale of the curve varying in proportion to the wing area S. Making use then of the bar to indicate the straight tapered wing, we get the circulation

$$G_a=\overline{G}_a\times\frac{St}{\overline{S}\overline{t}} \quad (5.22)$$

for the tapered wing with rounded tips. Although the conversion is only approximate, it gives results sufficiently accurate for most practical purposes, being particularly convenient in our estimation of the first approximation to G_a in the calculation shown in Chapter II. Anderson

(Ref. 18) has given the numerical results for wings having straight taper with rounded tips, the rounding off of which lies within the trapezoidal tip of length t_0.

The more general system of tapered wings is that of straight tapered wings with a central parallel part (for example, as in Fig. 9), which may cover the system of simple straight tapered wings as a special case. The circulation distribution for such wings within a limited range will be found in the paper of Koning and Boelen (Ref. 19) and in the text book of Fuchs, Hopf, and Seewald (Ref. 20), but a more extensive range of parameters is covered in the work of Hikita and Sinra (Ref. 21), who performed the calculation by means of the writer's method of solution discussed in Chapter II. With these results as basic data, it will be possible to obtain such numerical results as are given in this chapter for a number of aerodynamic characteristics.

May, 1939.

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