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On the Moment of the Force acting on a Flat Plate placed in a Stream between Two Parallel Walls.

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I. Introduction.

§ 1. In a previous paper[1] I have calculated, in conjunction with Mr. M. INANUMA, the moment[2] of the fluid pressure acting on a flat plate which is placed obliquely in a two-dimensional steady continuous irrotational flow of an incompressible perfect fluid bounded by two parallel plane walls. From the view-point of the application of the result to practical problems, we have confined ourselves to a special case in which the mid-point of the plate is on the centre line of the channel and the ratio of the breadth of the plate to that of the channel is fairely smaller than unity.

The result obtained was that the magnitude of the moment of the fluid pressure for the plate in the channel is always greater than that for a plate in an unlimited stream; in other words, the boundary walls

(1) S. TOMOTIKA and M. INANUMA, On the Moment of the Force acting on a Flat Plate placed in a Stream between Two Parallel Walls. Proc. Phys.-Math. Soc. Japan, [3] 14 (1932), 543—569.

(2) "The moment about the mid-point of the plate" will often be abbreviated as "the moment" in this paper, as in the previous paper just cited.

of the channel have the effect of increasing the moment of the fluid pressure exerting on the plate.

Although the problem has thus been solved quite rigorously, it seems to me not meaningless to treat the problem again in a somewhat different way. Thus, in the present paper I wish to investigate the problem, applying the same method of analysis as that employed in one of my previous papers.(1)

II. The Conformal Transformations.

§ 2. Although the conformal transformations necessary for the present problem have been shown in detail in my previous paper cited above, it may not be superfluous, for the sake of reference, to sketch here the procedure of the said conformal transformations as briefly as possible.

We take the plane of fluid motion, which is assumed to be two-dimensional, as the z-plane and we consider a steady irrotational continuous flow of a non-viscous fluid past a plate AA$'$ between two parallel walls, the circulation round the plate being assumed to be zero. A part of a particular stream line coincides with the surface of the plate. Let this stream line be defined by $\psi=0$. The flow pattern in the z-plane may become as shown in Fig. 1, where the fluid at infinity is considered to flow with the velocity U from left to right, i.e. in the positive direction of the x-axis, which is drawn parallel to the walls.

Let the value of ψ on the upper and lower boundaries be denoted by ψ_1 and $-\psi_2$ respectively. Then we evidently have

$$UD=\psi_1+\psi_2\,, \tag{1}$$

D being the distance between the two parallel walls.

(1) S. Tomotika, The Forces on a Flat Plate placed in a Stream of Fluid between Two Parallel Walls, Proc. Phys.-Math. Soc., Japan, [3] 14 (1932), 139–167.

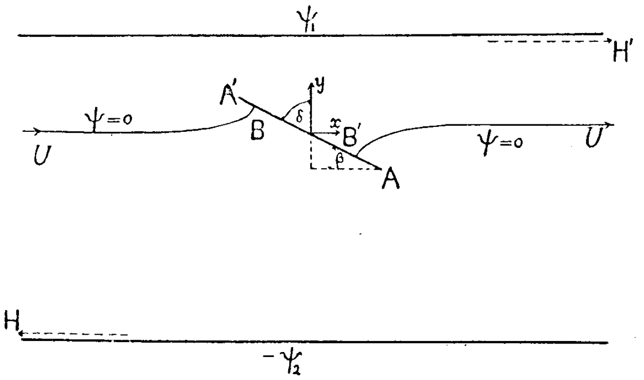

Fig. 1. z-plane.

Denoting the complex velocity potential for the irrotational continuous flow under consideration by $f=\phi+i\psi$, the f-plane is shown in Fig. 2.

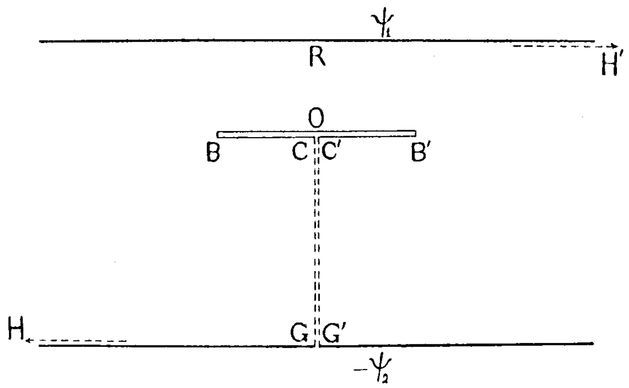

Fig. 2. f-plane.

By making a cut along C G G′ C′ as shown in the figure, we first transform the f-plane on to the upper half of a t-plane by the well-known SCHWARZ-CHRISTOFFEL's method, as in my previous paper.(1)

Next, by introducing $\wp$ function with periods $2\omega_1$, $2\omega_3$, where $\omega_1>0$ and $\omega_3/i>0$, we transform conformally the upper half of the t-plane into a rectangle of sides $2\omega_1$ and ω_3/i in a s-plane by the relation:

(1) S. TOMOTIKA, The Forces on a Flat Plate placed in a Stream of Fluid between Two Parallel Walls. Proc. Phys.-Math. Soc., Japan, [3] **14** (1932), 148.

$$t^2 = \wp(s) - e_3 . \qquad (2)$$

Then, the transformation equation for the direct transformation from the f-plane to the s-plane assumes the form :[1]

$$\frac{df}{ds} = \frac{\psi_1+\psi_2}{\pi} \frac{\wp'(\nu)}{\wp(\nu)-\wp(\mu)} \frac{\wp(s)-\wp(\mu)}{\wp(s)-\wp(\nu)}$$

$$= \frac{\psi_1+\psi_2}{\pi}\Big[\zeta(\mu+\nu)-\zeta(\mu-\nu)-\zeta(s+\nu)+\zeta(s-\nu)\Big] , \qquad (3)$$

where the points $s=\mu,\ -\mu,\ \nu,\ -\nu$ correspond to the points B, B′, H, H′ respectively.

The s-plane is illustrated in Fig. 3, where the correspondence of other points is also shown.

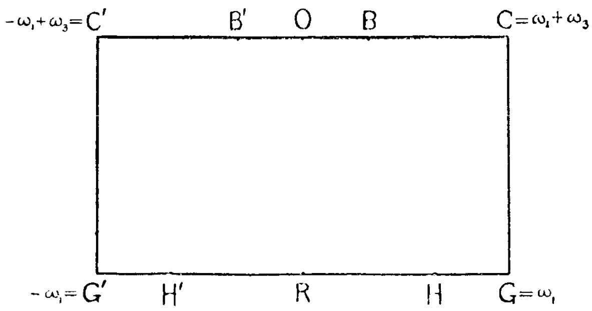

Fig. 3. s-plane.

The differential equation (3) can be integrated immediately, and if we choose the constant of integration such that $f=0$ at $s=\mu$, we get

$$f = \frac{\psi_1+\psi_2}{\pi}\left\{\Big[\zeta(\mu+\nu)-\zeta(\mu-\nu)\Big](s-\mu) - \log\frac{\sigma(s+\nu)\sigma(\mu-\nu)}{\sigma(s-\nu)\sigma(\mu+\nu)}\right\}. \qquad (4)$$

(1) S. Tomotika, The Forces on a Flat Plate placed in a Stream of Fluid between Two Parallel Walls. Proc. Phys.-Math. Soc., Japan, [3] 14 (1932), 149.

Since, however, we assume that the circulation round the plate is zero, the function f has a period $2\omega_1$, and this condition gives a relation between the constants as follows:

$$\zeta(\mu+\nu)-\zeta(\mu-\nu) = \frac{2\eta_1\nu}{\omega_1}. \qquad (5)$$

Another relation is obtained from the condition that f differs by $i\psi_2$ at $s=\omega_1$ and $s=\omega_1+\omega_3$, so that we get

$$f_C - f_G = i\psi_2 = \frac{\psi_1+\psi_2}{\pi}\left\{\left[\zeta(\mu+\nu)-\zeta(\mu-\nu)\right]\omega_3 - 2\eta_3\nu\right\}, \qquad (6)$$

which, in conjunction with (5), gives

$$\nu = \frac{\psi_2}{\psi_1+\psi_2}\omega_1. \qquad (7)$$

Finally, the inside of the rectangle in the s-plane is transformed conformally into a ring region in a Z-plane bounded by two concentric circles of radii 1 and $q\left[=\exp\left(\frac{\omega_3}{\omega_1}\pi i\right)<1\right]$ by the relation:

$$s = \omega_1+\omega_3-\frac{\omega_1}{i\pi}\log Z. \qquad (8)$$

The face of the flat plate corresponds to the outer circle and those of the two parallel walls to the inner circle, and the various points are transformed as follows:

$$\left.\begin{aligned} &A = e^{i\theta_4};\quad B = e^{i\theta_1};\quad A' = e^{i\theta_3};\quad B' = e^{i(2\pi-\theta_1)};\\ &H = qe^{i\theta_2};\quad H' = qe^{i(2\pi-\theta_2)}. \end{aligned}\right\}$$

If, further, we express μ and ν in terms of central angles in the Z-plane, we have

$$\left.\begin{aligned}\mu &= \omega_3 + \omega_1\left(1-\frac{\theta_1}{\pi}\right),\\ \nu &= \omega_1\left(1-\frac{\theta_2}{\pi}\right).\end{aligned}\right\} \qquad (9)$$

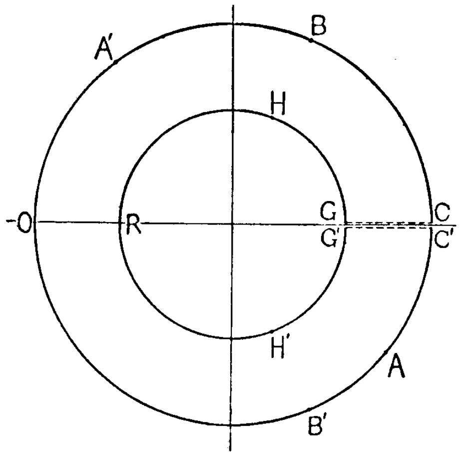

Fig. 4. Z-plane.

§ 3. The conjugate complex velocity at any point in the z-plane is obtained simply by differentiating f with respect to z. If we denote it by v_1, we have

$$v_1 = v_x - iv_y = \frac{df}{dz}.$$

Let $|v_1|$ be the absolute magnitude of the velocity of the fluid element at any point and Θ be the angle which the direction of velocity at that point makes with the positive direction of the x-axis.

Then, since $v_1 = |v_1|\,e^{-i\Theta}$, putting

$$\Omega = \Theta + i\log|v_1|, \qquad (10)$$

we have

$$\frac{df}{dz} = e^{-i\Omega}. \qquad (11)$$

The direction of the fluid velocity at every point on the plate as well as on the two parallel walls being known, the real part of the function Ω is given from the beginning. Hence, in the Z-plane, the real part of Ω, which is now considered to be expressed as a function of Z, is defined for every point on the outer and inner circles of the ring region.

Thus, our next problem is to find an analytic function $\Omega(Z)$, which is everywhere regular in the ring region and whose real part on the bounding circles assumes the prescribed values.

According to H. VILLAT,(1) an analytic function $f(Z)$, which is everywhere regular in the annular region in the Z-plane bounded by two concentric circles of radii 1 and $q\left[=\exp\left(\frac{\omega_3}{\omega_1}\pi i\right)<1\right]^{(2)}$ and whose real part on the outer circle, expressed as a function of the central angle θ in the Z-plane, is given by $\Phi(\theta)$ and that on the inner circle by $\Psi(\theta)$, is expressed, in general, in the form:

$$f(Z)=\frac{i\omega_1}{\pi^2}\int_0^{2\pi}\Phi(\theta)\zeta\left(\frac{\omega_1}{i\pi}\log Z-\frac{\omega_1}{\pi}\theta\right)d\theta$$

$$-\frac{i\omega_1}{\pi^2}\int_0^{2\pi}\Psi(\theta)\zeta_3\left(\frac{\omega_1}{i\pi}\log Z-\frac{\omega_1}{\pi}\theta\right)d\theta+iC\,, \qquad (12)$$

where $\Phi(\theta+2\pi)=\Phi(\theta)$ and $\Psi(\theta+2\pi)=\Psi(\theta)$ and C is an arbitrary real constant to be determined by some boundary condition in each problem, and in addition we have as the condition for uniformity:

$$\int_0^{2\pi}\Phi(\theta)d\theta=\int_0^{2\pi}\Psi(\theta)d\theta\,. \qquad (13)$$

Thus, if we designate by $\Phi(\theta)$ the angle which the direction of the flow along the face of the plate makes with the positive direction of

(1) See, for example: H. VILLAT, Leçons sur l'hydrodynamique. (1929), 12—20.
(2) We assume here also that $\omega_1>0$ and $\omega_3/i>0$.

the x-axis, expressed as a function of the central angle θ in the Z-plane, and by $\Psi(\theta)$ the corresponding function for the two parallel walls, we have, applying VILLAT's formula just referred to,

$$\Omega(Z) = \frac{i\omega_1}{\pi^2}\int_0^{2\pi}\Phi(\theta)\zeta\left(\frac{\omega_1}{i\pi}\log Z - \frac{\omega_1}{\pi}\theta\right)d\theta$$

$$-\frac{i\omega_1}{\pi^2}\int_0^{2\pi}\Psi(\theta)\zeta_3\left(\frac{\omega_1}{i\pi}\log Z - \frac{\omega_1}{\pi}\theta\right)d\theta + iC\,, \qquad (14)$$

with the condition (13).

However, since the flow along the walls makes a zero angle with the positive direction of the x-axis, $\Psi(\theta)$ is zero for all values of θ and consequently we have from (13),

$$\int_0^{2\pi}\Phi(\theta)d\theta = 0\,,$$

or, remembering that if we denote by δ the acute angle between the plate and the y-axis, we have $\Phi(\theta) = -\frac{1}{2}\pi + \delta$ for $\theta_4 < \theta < \theta_1$, $\Phi(\theta) = \frac{1}{2}\pi + \delta$ for $\theta_1 < \theta < \theta_3$, $\Phi(\theta) = -\frac{1}{2}\pi + \delta$ for $\theta_3 < \theta < 2\pi - \theta_1$ and $\Phi(\theta) = \frac{1}{2}\pi + \delta$ for $2\pi - \theta_1 < \theta < 2\pi + \theta_4$, as can readily be seen from Figs. 1 and 4, we get

$$\int_{\theta_4}^{\theta_1}\left(-\frac{\pi}{2}+\delta\right)d\theta + \int_{\theta_1}^{\theta_3}\left(\frac{\pi}{2}+\delta\right)d\theta + \int_{\theta_3}^{2\pi-\theta_1}\left(-\frac{\pi}{2}+\delta\right)d\theta + \int_{2\pi-\theta_1}^{2\pi+\theta_4}\left(\frac{\pi}{2}+\delta\right)d\theta = 0\,,$$

from which we obtain

$$\theta_3 + \theta_4 = \pi - 2\delta = 2\beta\,, \qquad (15)$$

where β is the angle of attack of the plate.

In the next place, we get from (14),

$$\Omega(Z) = \frac{i\omega_1}{\pi^2}\int_0^{2\pi}\Phi(\theta)\zeta\left(\frac{\omega_1}{i\pi}\log Z - \frac{\omega_1}{\pi}\theta\right)d\theta + iC\,,$$

and taking account of the value of $\Phi(\theta)$ given above and carrying out the integration, we arrive at the result that

$$\Omega(Z) = iC - \frac{i}{\pi}\left[\delta\log\frac{\sigma\left(\frac{\omega_1}{i\pi}\log Z - \frac{\omega_1}{\pi}\theta_4 - 2\omega_1\right)}{\sigma\left(\frac{\omega_1}{i\pi}\log Z - \frac{\omega_1}{\pi}\theta_4\right)}\right.$$

$$\left.+\frac{\pi}{2}\log\frac{\left\{\sigma\left(\frac{\omega_1}{i\pi}\log Z - \frac{\omega_1}{\pi}\theta_3\right)\right\}^2\sigma\left(\frac{\omega_1}{i\pi}\log Z - \frac{\omega_1}{\pi}\theta_4\right)\sigma\left(\frac{\omega_1}{i\pi}\log Z - \frac{\omega_1}{\pi}\theta_4 - 2\omega_1\right)}{\left\{\sigma\left(\frac{\omega_1}{i\pi}\log Z - \frac{\omega_1}{\pi}\theta_1\right)\right\}^2\left\{\sigma\left(\frac{\omega_1}{i\pi}\log Z + \frac{\omega_1}{\pi}\theta_1 - 2\omega_1\right)\right\}^2}\right].$$

Putting

$$\left.\begin{aligned} s_3 &= \omega_1 + \omega_3 - \frac{\omega_1}{\pi}\theta_3\,, \\ s_4 &= \omega_1 + \omega_3 - \frac{\omega_1}{\pi}\theta_4\,, \end{aligned}\right\} \qquad (16)$$

we express Ω as a function of s, with the aid of (8), and then we simplify it by means of the formulae :

$$\sigma(u \pm 2\omega_\alpha) = -e^{\pm 2\eta_\alpha(u \pm \omega_\alpha)}\sigma(u)\,, \quad (\alpha = 1,\, 2,\, 3)\,.$$

We thus have

$$e^{i\Omega} = C_1\exp\left[\frac{2}{\pi}(\eta_1 s - \eta_3\omega_1)\left(\frac{\pi}{2} + \delta\right) + 2\eta_3 s\right]\frac{\sigma(s-s_3)\,\sigma(s-s_4)}{\sigma(s-\mu)\,\sigma(s+\mu)}\,, \quad (17)$$

C_1 being a constant. C_1 is evidently connected with C, so that for the present it is also arbitrary.

Also, the expression for df/ds can be put in the form:

$$\frac{df}{ds}=\frac{\psi_1+\psi_2}{\pi}\frac{\wp'(\nu)}{\wp(\nu)-\wp(\mu)}\left\{\frac{\sigma(\nu)}{\sigma(\mu)}\right\}^2\frac{\sigma(s+\mu)\,\sigma(s-\mu)}{\sigma(s+\nu)\,\sigma(s-\nu)}. \qquad (18)$$

Thus, we get finally, by combining (11), (17) and (18), the expression for dz/ds, namely:

$$\frac{dz}{ds}=\frac{dz}{df}\frac{df}{ds}=e^{i\Omega}\frac{df}{ds}$$

$$=-C'\exp\left[\frac{2}{\pi}(\eta_1 s-\eta_3\omega_1)\left(\frac{\pi}{2}+\delta\right)+2\eta_3 s\right]\frac{\sigma(s-s_3)\,\sigma(s-s_4)}{\sigma(s+\nu)\,\sigma(s-\nu)}, \quad (19)$$

where C' is connected with C_1 as follows:

$$C'=-C_1\frac{\psi_1+\psi_2}{\pi}\frac{\sigma(2\nu)}{\sigma(\nu+\mu)\,\sigma(\nu-\mu)}. \qquad (20)$$

§4. Next, we must integrate the expression for dz/ds. We put

$$\frac{dz}{ds}=F(s), \qquad (21)$$

where

$$F(s)=-C'\exp\left[\frac{2}{\pi}(\eta_1 s-\eta_3\omega_1)\left(\frac{\pi}{2}+\delta\right)+2\eta_3 s\right]\frac{\sigma(s-s_3)\,\sigma(s-s_4)}{\sigma(s+\nu)\,\sigma(s-\nu)}. \quad (22)$$

The addition formulae for the periodic functions give

$$\left.\begin{aligned}F(s+2\omega_1)&=F(s),\\ F(s+2\omega_3)&=\exp\left[2i\left(\frac{1}{2}\pi+\delta\right)\right]F(s),\end{aligned}\right\} \qquad (23)$$

and hence $F(s)$ is an elliptic function of the second kind with simple poles at $s=\pm\nu$. It can, therefore, be split up into simple elements by introducing a function $A(s)$[1] where

(1) It can easily be shown that this function has a simple pole at $s=0$ and also has exactly the same periodicity property as the function $F(s)$.

$$A(s) = -\frac{\sigma\left[s-\frac{2\,\omega_1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]}{\sigma[s]\sigma\left[\frac{2\,\omega_1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]}\,e^{\frac{2\eta_1}{\pi}\left(\frac{\pi}{2}+\delta\right)s}$$

$$= \frac{1}{2\,\omega_1}\frac{\vartheta_1'(0)\vartheta_2\left[\frac{s}{2\,\omega_1}-\frac{\delta}{\pi}\right]}{\vartheta_1\left[\frac{s}{2\,\omega_1}\right]\vartheta_2\left[\frac{\delta}{\pi}\right]}, \tag{24}$$

and as a result of this the function $F(s)$ can be integrated.

The law for the decomposition of $F(s)$ is

$$F(s) = C_\nu A(s-\nu)+C_{-\nu}A(s+\nu)\,, \tag{25}$$

where C_ν and $C_{-\nu}$ are the residues of $F(s)$ at the poles $+\nu$ and $-\nu$ respectively. We find, from (22), that

$$\left.\begin{aligned} C_\nu &= -C'\frac{[\sigma(\omega_3)]^2}{[\vartheta_4(0)]^2\sigma(2\nu)}\exp\left[\frac{\eta_1\omega_1}{2\pi^2}\left(\theta_3^2+\theta_4^2+4\,\pi\delta+\frac{2\pi^2\nu^2}{\omega_1^2}\right)\right] \\ &\qquad\times\vartheta_4\left(\frac{\theta_3-\theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4-\theta_2}{2\pi}\right), \\ C_{-\nu} &= +C'\frac{[\sigma(\omega_3)]^2}{[\vartheta_4(0)]^2\sigma(2\nu)}\exp\left[\frac{\eta_1\omega_1}{2\pi^2}\left(\theta_3^2+\theta_4^2+4\pi\delta+\frac{2\pi^2\nu^2}{\omega_1^2}\right)\right] \\ &\qquad\times\vartheta_4\left(\frac{\theta_3+\theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4+\theta_2}{2\pi}\right). \end{aligned}\right\} \tag{26}$$

Now, we have to determine the hitherto arbitrary constant C' in such a way that the function df/dz given by (11), together with (17) and (20), may satisfy the conditions at infinity. Since we have assumed that at infinity upstream H as well as at infinity downstream H′ the fluid flows with the constant velocity U parallel to the channel walls, i.e. parallel to the x-axis in the positive direction, the conditions at infinity can be written in the forms:

$$\left(\frac{df}{dz}\right)_{\mathrm{H}} = U\,, \qquad \left(\frac{df}{dz}\right)_{\mathrm{H}'} = U\,. \tag{27}$$

Remembering that H corresponds to the point $s = \nu$ in the s-plane and also H′ to the point $s = -\nu$, and carrying out some calculations, we get, from the first condition in (27),

$$\frac{1}{C'} = -\frac{\pi}{D}\frac{[\sigma(\omega_3)]^2}{[\vartheta_4(0)]^2\sigma(2\nu)}\exp\left[\frac{\eta_1\omega_1}{2\pi^2}\left(\theta_3^2+\theta_4^2+4\pi\delta+\frac{2\pi^2\nu^2}{\omega_1^2}\right)\right]$$
$$\times\,\vartheta_4\left(\frac{\theta_3-\theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4-\theta_2}{2\pi}\right), \tag{28}$$

while, from the second condition in (27) we have

$$\frac{1}{C'} = -\frac{\pi}{D}\frac{[\sigma(\omega_3)]^2}{[\vartheta_4(0)]^2\sigma(2\nu)}\exp\left[\frac{\eta_1\omega_1}{2\pi^2}\left(\theta_3^2+\theta_4^2+4\pi\delta+\frac{2\pi^2\nu^2}{\omega_1^2}\right)\right]$$
$$\times\,\vartheta_4\left(\frac{\theta_3+\theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4+\theta_2}{2\pi}\right). \tag{29}$$

Thus, we have obtained the two expressions for the unique constant C' and consequently there must necessarily exist a relation:

$$\vartheta_4\left(\frac{\theta_3-\theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4-\theta_2}{2\pi}\right) = \vartheta_4\left(\frac{\theta_3+\theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4+\theta_2}{2\pi}\right) \tag{30}$$

between the quantities θ_2, θ_3 and θ_4.

It is evident from the foregoing analysis that if we use the constant C' thus uniquely determined, all the boundary conditions of the problem are fulfilled.

Combining (26) with (28) and (29), we have

$$C_\nu = \frac{D}{\pi}\,, \qquad C_{-\nu} = -\frac{D}{\pi}\,, \tag{31}$$

and by inserting these in (25), the function $F(s)$ can be put in the quite simple form, namely:

$$F(s)=\frac{D}{\pi}\Big[A(s-\nu)-A(s+\nu)\Big]. \tag{32}$$

Now, when a complex quantity v satisfies the relation:

$$-\Re\left(\frac{\tau}{i}\right)<2\,\Re\left(\frac{v}{i}\right)<\Re\left(\frac{\tau}{i}\right),$$

in which $\tau=\omega_3/\omega_1$ and $\Re(z)$ means "the real part of z", we have the expansion formula:[1]

$$\begin{aligned}&\frac{\vartheta_1'(0)\vartheta_3(v+w)}{4\pi\vartheta_3(v)\,\vartheta_1(w)}\\ &=\frac{1}{4\sin\pi w}+\sum_{n=1}^{\infty}(-1)^n q^n\frac{\sin\pi(2nv+w)-q^{2n}\sin\pi(2nv-w)}{1-2q^{2n}\cos 2\pi w+q^{4n}}.\end{aligned} \tag{33}$$

Thus, if a complex quantity p satisfies the relation:

$$-\Re\left(\frac{\tau}{i}\right)<2\,\Re\left(\frac{1}{i}\left[\frac{p}{2\omega_1}-\frac{\tau}{2}\right]\right)<\Re\left(\frac{\tau}{i}\right), \tag{34}$$

the function $A(p)$ defined as:

$$\begin{aligned}A(p)&=-\frac{\sigma\left[p-\frac{2\omega_1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]}{\sigma[p]\,\sigma\left[\frac{2\omega_1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]}e^{\frac{2\eta_1}{\pi}\left(\frac{\pi}{2}+\delta\right)p}\\ &=\frac{1}{2\omega_1}\frac{\vartheta_1'(0)\,\vartheta_2\left[\frac{p}{2\omega_1}-\frac{\delta}{\pi}\right]}{\vartheta_1\left[\frac{p}{2\omega_1}\right]\vartheta_2\left[\frac{\delta}{\pi}\right]}\end{aligned}$$

(1) J. TANNERY et J. MOLK, Éléments de la théorie des fonctions elliptiques. 4 (1902) 104.

can be expanded in the following form:

$$A(p)=\frac{1}{2\omega_1}\frac{\vartheta_1'(0)\,\vartheta_3\left[\dfrac{p}{2\omega_1}-\dfrac{1+\tau}{2}-\dfrac{1}{\pi}\left(\dfrac{\pi}{2}+\delta\right)\right]}{\vartheta_3\left[\dfrac{p}{2\omega_1}-\dfrac{1+\tau}{2}\right]\vartheta_1\left[-\dfrac{1}{\pi}\left(\dfrac{\pi}{2}+\delta\right)\right]}\,e^{i\left(\frac{\pi}{2}+\delta\right)}$$

$$=\frac{2\pi}{\omega_1}e^{i\left(\frac{\pi}{2}+\delta\right)}\left[-\frac{1}{4\sin\left(\dfrac{\pi}{2}+\delta\right)}\right.$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{1-2q^{2n}\cos 2\left(\dfrac{\pi}{2}+\delta\right)+q^{4n}}$$

$$\times\left\{\sin\pi\left[2n\left(\frac{p}{2\omega_1}-\frac{1+\tau}{2}\right)-\frac{1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]\right.$$

$$\left.\left.-q^{2n}\sin\pi\left[2n\left(\frac{p}{2\omega_1}-\frac{1+\tau}{2}\right)+\frac{1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]\right\}\right].\tag{35}$$

When s varies from 0 to ω_3, the quantity $\left(\dfrac{s}{2\omega_1}-\dfrac{\tau}{2}\right)$ varies from $-\tau/2$ to 0, so that s satisfies the relation:

$$-\Re\left(\frac{\tau}{i}\right)<2\Re\left(\frac{1}{i}\left[\frac{s}{2\omega_1}-\frac{\tau}{2}\right]\right)<\Re\left(\frac{\tau}{i}\right),$$

which is the same relation as that for p in (34). Therefore, putting $p=(s-\nu)$ and $p=(s+\nu)$ in the expression for $A(p)$ consecutively and subtracting, we get

$$F(s)=\frac{D}{\pi}\frac{2\pi}{\omega_1}e^{i\left(\frac{\pi}{2}+\delta\right)}\sum_{n=1}^{\infty}\frac{i(-1)^{n+1}q^n\sin\dfrac{n\nu\pi}{\omega_1}}{1+2q^{2n}\cos 2\delta+q^{4n}}$$

$$\times\left[Z^n(e^{i\delta}+q^{2n}e^{-i\delta})-Z^{-n}(e^{-i\delta}+q^{2n}e^{i\delta})\right],\tag{36}$$

where s has been transformed to Z by means of equation (8).

We can now integrate the differential equation (21) and express z as a function of Z. The result is

$$z = \int \frac{dz}{ds} ds = \int F(s) \frac{ds}{dZ} dZ = -\frac{\omega_1}{i\pi} \int F(s) \frac{dZ}{Z}$$

$$= -\frac{2D}{\pi} e^{i\left(\frac{\pi}{2}+\delta\right)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^n \sin \frac{n\nu\pi}{\omega_1}}{n(1+2q^{2n}\cos 2\delta + q^{4n})}$$

$$\times \left[Z^n (e^{i\delta} + q^{2n} e^{-i\delta}) + Z^{-n}(e^{-i\delta} + q^{2n} e^{i\delta}) \right] + C_0, \quad (37)$$

where C_0 is an arbitrary constant of integration.

If the breadth of the plate be denoted by $2a$, we know that

$$z_{A'} - z_A = 2a e^{i\left(\frac{\pi}{2}+\delta\right)}, \quad (38)$$

and since the points A′ and A in the Z-plane correspond to $Z = e^{i\theta_3}$ and $Z = e^{i\theta_4}$ respectively, putting $Z = e^{i\theta_3}$ and $Z = e^{i\theta_4}$ in equation (37) and then subtracting, we get

$$2a = \frac{8D}{\pi} \sum_{n=1}^{\infty} \frac{q^n \sin n\theta_2 \sin \frac{1}{2} n(\theta_3 - \theta_4)}{n(1 - 2q^{2n}\cos 2\beta + q^{4n})} \left[\cos(n-1)\beta - q^{2n}\cos(n+1)\beta \right], \quad (39)$$

where the angle of attack β has been introduced by the relation $\beta = \frac{1}{2}\pi - \delta$.

Lastly, we see, by (37), that the co-ordinates of the mid-point of the plate AA′ are given by

$$z_m = \frac{1}{2}(z_A + z_{A'})$$

$$= \frac{4D}{\pi} e^{i\left(\frac{\pi}{2}+\delta\right)} \sum_{n=1}^{\infty} \frac{q^n \sin n\theta_2 \cos \frac{1}{2} n(\theta_3 - \theta_4)}{n(1 - 2q^{2n}\cos 2\beta + q^{4n})}$$

$$\times \left[\sin(n-1)\beta - q^{2n}\sin(n+1)\beta \right] + C_0. \quad (40)$$

§ 5. So far we have discussed the problem in the most general manner. From the practical standpoint, however, it is important to consider a rather special case in which the mid-point lies on the central line of the channel. Therefore, we shall hereafter confine ourselves to such a special case.

When the mid-point of the plate lies on the central line of the channel, ψ_1 is equal to ψ_2 and consequently we have, from (7),

$$\nu = \frac{1}{2}\omega_1 . \tag{41}$$

Putting this in (5), we get

$$\zeta\left(\mu+\frac{1}{2}\omega_1\right)-\zeta\left(\mu-\frac{1}{2}\omega_1\right) = \eta_1 ,$$

and comparing this equation with the well-known relation $\eta_1+\eta_2+\eta_3 = 0$ we easily obtain

$$\mu = \frac{1}{2}\omega_1+\omega_3 . \tag{42}$$

Thus, by comparing (41) and (42) with (9) we have

$$\left.\begin{aligned} \theta_1 &= \frac{\pi}{2} , \\ \theta_2 &= \frac{\pi}{2} \end{aligned}\right\} \tag{43}$$

Next, in the present special case the relation (30) takes the form:

$$\vartheta_4\left(\frac{\theta_3}{2\pi}-\frac{1}{4}\right)\vartheta_4\left(\frac{\theta_4}{2\pi}-\frac{1}{4}\right) = \vartheta_4\left(\frac{\theta_3}{2\pi}+\frac{1}{4}\right)\vartheta_4\left(\frac{\theta_4}{2\pi}+\frac{1}{4}\right) ,$$

and this relation is proved without difficulty to be satisfied by

$$\theta_3-\theta_4 = \pi . \tag{44}$$

Thus, combining this with (15) we get the values of θ_3 and θ_4 as:

$$\left.\begin{aligned}\theta_3 &= \frac{\pi}{2}+\beta\,,\\ \theta_4 &= -\frac{\pi}{2}+\beta\,.\end{aligned}\right\} \tag{45}$$

Now, since $\sin\left(\frac{1}{2}n\pi\right)=0$ when n is an even integer and also $\cos\left(\frac{1}{2}n\pi\right)=0$ if n is an odd integer, by putting (43) and (44) in (40) we have

$$z_m = C_0\,.$$

Hence, if we take the mid-point of the plate as the origin of the co-ordinate axes (x, y), we have

$$C_0 = 0\,,$$

and the expression for z becomes:

$$z = -\frac{2D}{\pi}e^{i\left(\frac{\pi}{2}+\delta\right)}\sum_{n=1}^{\infty}{}'\frac{(-1)^{n+1}q^n\sin\left(\frac{1}{2}n\pi\right)}{n(1+2q^{2n}\cos 2\delta+q^{4n})}\times\left[Z^n(e^{i\delta}+q^{2n}e^{-i\delta})+Z^{-n}(e^{-i\delta}+q^{2n}e^{i\delta})\right], \tag{46}$$

or

$$z = \frac{2D}{\pi}ie^{-i\beta}\sum_{n=1}^{\infty}{}'\frac{(-1)^{n+1}q^n\sin\left(\frac{1}{2}n\pi\right)}{n(1-2q^{2n}\cos 2\beta+q^{4n})}\times\left[Z^n(e^{-i\beta}-q^{2n}e^{i\beta})-Z^{-n}(e^{i\beta}-q^{2n}e^{-i\beta})\right], \tag{47}$$

where the angle of attack β of the plate has been introduced by the relation $\frac{\pi}{2}-\delta=\beta$.

Next, on putting $\psi_1 = \psi_2$, $\nu = \frac{1}{2}\omega_1$ and $\mu = \frac{1}{2}\omega_1 + \omega_3$ in (18) and (19) we get the expressions for df/ds and dz/ds in the following forms:

$$\frac{df}{ds} = \frac{2\psi_1}{\pi} \frac{\wp'\left(\frac{\omega_1}{2}\right)}{\wp\left(\frac{\omega_1}{2}\right) - \wp\left(\frac{\omega_1}{2} + \omega_3\right)} \left\{\frac{\sigma\left(\frac{\omega_1}{2}\right)}{\sigma\left(\frac{\omega_1}{2} + \omega_3\right)}\right\}^2 \frac{\sigma\left(s + \frac{\omega_1}{2} + \omega_3\right)\sigma\left(s - \frac{\omega_1}{2} - \omega_3\right)}{\sigma\left(s + \frac{\omega_1}{2}\right)\sigma\left(s - \frac{\omega_1}{2}\right)}$$

$$= \frac{UD}{\pi} \frac{\wp'\left(\frac{\omega_1}{2}\right)}{\wp\left(\frac{\omega_1}{2}\right) - \wp\left(\frac{\omega_1}{2} + \omega_3\right)} \left\{\frac{\sigma\left(\frac{\omega_1}{2}\right)}{\sigma\left(\frac{\omega_1}{2} + \omega_3\right)}\right\}^2 \frac{\sigma\left(s + \frac{\omega_1}{2} + \omega_3\right)\sigma\left(s - \frac{\omega_1}{2} - \omega_3\right)}{\sigma\left(s + \frac{\omega_1}{2}\right)\sigma\left(s - \frac{\omega_1}{2}\right)}, \tag{48}$$

$$\frac{dz}{ds} = -C' \exp\left[\frac{2}{\pi}(\eta_1 s - \eta_3 \omega_1)\left(\frac{\pi}{2} + \delta\right) + 2\eta_3 s\right] \frac{\sigma(s - s_3)\,\sigma(s - s_4)}{\sigma\left(s + \frac{\omega_1}{2}\right)\sigma\left(s - \frac{\omega_1}{2}\right)}, \tag{49}$$

where

$$s_3 = \left(\frac{1}{2} - \frac{\beta}{\pi}\right)\omega_1 + \omega_3, \quad s_4 = \left(\frac{3}{2} - \frac{\beta}{\pi}\right)\omega_1 + \omega_3. \tag{50}$$

In the former, we have used the relation $2\psi_1 = UD$ which is obtained from (1) by putting $\psi_1 = \psi_2$, and in the latter, the value of C' can be found from (29), which now takes the form:

$$\frac{1}{C'} = -\frac{\pi}{D} \frac{[\sigma(\omega_3)]^2}{[\vartheta_4(0)]^2 \sigma(\omega_1)} \exp\left[\frac{\eta_1\omega_1}{2\pi^2}(3\pi^2 - 4\pi\beta + 2\beta^2)\right] \vartheta_3\left(\frac{\beta}{2\pi}\right)\vartheta_4\left(\frac{\beta}{2\pi}\right). \tag{51}$$

Lastly, the expression (39) for $2a$ assumes the following form:

$$2a = \frac{8D}{\pi} \sum_{n=1}^{\infty}{}_{\circ} \frac{q^n}{n(1 - 2q^{2n}\cos 2\beta + q^{4n})} \left\{\cos(n-1)\beta - q^{2n}\cos(n+1)\beta\right\}, \tag{52}$$

where Σ_o indicates that only the positive odd integral values of n are to be included in the summation.

III. Calculation of the Moment of the Force about the Mid-point of the Plate.

§ 6. We shall now proceed to the calculation of the moment of the fluid pressure acting on the flat plate about its mid-point, that is, about the origin of the co-ordinate axes,(1) under the assumption that the mid-point of the plate lies on the central line of the channel.

It is well-known that the moment about the origin of the co-ordinate axes of the resultant fluid pressure acting on a cylindrical body can be calculated, in general, by using BLASIUS' formula, namely:

$$M = -\frac{1}{2}\rho\,\mathfrak{R}\oint_S \left(\frac{df}{dz}\right)^2 z\,dz\ ,$$

where M is the moment per unit length of the body and $\mathfrak{R}$ means that the real part of the value of the integral is taken.

In this formula, the integral must, of course, be taken round the surface S of the body in the counter-clockwise sense as indicated above. However, when the surface does not contain any singular point at which the fluid velocity becomes infinitely large, the integral taken round the surface can be replaced simply by that taken round any closed contour C surrounding the body.

Further, it can be proved(2) without difficulty that in the calculation of the moment of the fluid pressure acting on the body with the aid of the above-mentioned BLASIUS' formula, we can replace the integral taken round the surface of the body by that taken round any closed contour surrounding it, even when the surface contains some singular points as in the case of a flat plate with which we are dealing in the present paper.

(1) As mentioned already, the mid-point of the plate is taken as the origin of the co-ordinate axes in the present paper.

(2) For the case of the flat plate, reference may be made to the following paper. S. TOMOTIKA and M. INANUMA, On the Moment of the Force acting on a Flat Plate placed in a Stream between Two Parallel Walls. Proc. Phys.-Math. Soc., Japan, [3], **14** (1932), 543—569.

Thus, if we designate by C an arbitrary closed contour surrounding the plate under consideration, the moment of the force exerting on it can be calculated by the formula:

$$M = -\frac{1}{2}\rho\Re\oint_{C}\left(\frac{df}{dz}\right)^2 z dz\,. \tag{53}$$

Now, we have in general

$$\int\left(\frac{df}{dz}\right)^2 z dz = \int\left(\frac{df}{ds}\right)^2\left(\frac{ds}{dz}\frac{ds}{dZ}\right) z dZ = -\frac{\omega_1}{i\pi}\int\left(\frac{df}{ds}\right)^2\left(\frac{ds}{dz}\right) z\frac{dZ}{Z}\,.$$

Consequently, since the procedure of encircling the plate in the z-plane in the counter-clockwise sense is equivalent to that of encircling the inner circle in the Z-plane in the clockwise sense, we have

$$M = \frac{1}{2}\rho\frac{\omega_1}{\pi}\Re\left[\frac{1}{i}\oint_{\Sigma}\left(\frac{df}{ds}\right)^2\left(\frac{ds}{dz}\right) z\frac{dZ}{Z}\right], \tag{54}$$

where Σ is a closed curve in the ring region in the Z-plane, which corresponds to the curve C in the z-plane.

In this way, we can calculate the required moment by the formula (54), if we evaluate the integral in it.

§ 7. Before proceeding to the evaluation of the integral in (54), we shall examine the property of the function $(df/ds)^2(ds/dz)$.

From (48) and (49) we have

$$\left(\frac{df}{ds}\right)^2\frac{ds}{dz} = C''\exp\left[-\frac{2}{\pi}(\eta_1 s - \eta_3\omega_1)\left(\frac{\pi}{2}+\delta\right) - 2\eta_3 s\right]$$
$$\times\frac{\left[\sigma\left(s+\frac{\omega_1}{2}+\omega_3\right)\sigma\left(s-\frac{\omega_1}{2}-\omega_3\right)\right]^2}{\sigma\left(s+\frac{\omega_1}{2}\right)\sigma\left(s-\frac{\omega_1}{2}\right)\sigma(s-s_3)\,\sigma(s-s_4)}\,, \tag{55}$$

where we put

$$C'' = -\frac{1}{C'}\left(\frac{UD}{\pi}\right)^2 \left\{ \frac{\wp'\left(\frac{\omega_1}{2}\right)}{\wp\left(\frac{\omega_1}{2}\right) - \wp\left(\frac{\omega_1}{2} + \omega_3\right)} \right\}^2 \left\{ \frac{\sigma\left(\frac{\omega_1}{2}\right)}{\sigma\left(\frac{\omega_1}{2} + \omega_3\right)} \right\}^4, \quad (56)$$

for the sake of simplicity.

This new constant C'' can however be written in another simple form, if we substitute the value for C' given by (50) and then simplify with the aid of the various well-known formulae for elliptic functions such as:

$$\wp(u) - \wp(v) = -\frac{\sigma(u+v)\,\sigma(u-v)}{[\,\sigma(u)\,\sigma(v)\,]^2},$$

$$\sigma(2u) = -\wp'(u)[\,\sigma(u)\,]^4 .$$

The result is

$$C'' = \frac{U^2D}{\pi}\frac{\sigma(\omega_1)}{[\,\vartheta_4(0)\,]^2[\,\sigma(\omega_1+\omega_3)\,]^2}\exp\left[\frac{\eta_1\omega_1}{2\,\pi^2}(3\pi^2 - 4\pi\beta + 2\beta^2)\right]\vartheta_3\left(\frac{\beta}{2\pi}\right)\vartheta_4\left(\frac{\beta}{2\pi}\right). \quad (57)$$

Next, if we put

$$G(s) = C''\exp\left[-\frac{2}{\pi}(\eta_1 s - \eta_3\omega_1)\left(\frac{\pi}{2} + \delta\right) - 2\eta_3 s\right]$$

$$\times \frac{\left[\sigma\left(s + \frac{\omega_1}{2} + \omega_3\right)\sigma\left(s - \frac{\omega_1}{2} - \omega_3\right)\right]^2}{\sigma\left(s + \frac{\omega_1}{2}\right)\sigma\left(s - \frac{\omega_1}{2}\right)\sigma(s-s_3)\,\sigma(s-s_4)}, \quad (58)$$

we have

$$\left(\frac{df}{ds}\right)^2\frac{ds}{dz} = G(s)\,. \quad (59)$$

The function $G(s)$ thus defined has simple poles at the four points $s = \omega_1/2$, $-\omega_1/2$, s_3 and s_4, and it can readily be shown that

$$\left.\begin{aligned} G(s+2\omega_1) &= G(s), \\ G(s+2\omega_3) &= \exp\left[-2i\left(\tfrac{1}{2}\pi+\delta\right)\right]G(s) . \end{aligned}\right\} \qquad (60)$$

Therefore, $G(s)$ is an elliptic function of the second kind, and if we introduce a function $B(s)$ defined as:

$$B(s) = \frac{\sigma\left[s+\frac{2\omega_1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]}{\sigma[s]\,\sigma\left[\frac{2\omega_1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]}e^{-\frac{2\eta_1}{\pi}\left(\frac{\pi}{2}+\delta\right)s} , \qquad (61)$$

which has a simple pole at $s=0$ and has exactly the same periodicity property as the function $G(s)$, we can decompose $G(s)$ in the form:

$$G(s) = R_1 B\left(s-\frac{\omega_1}{2}\right)+R_2 B\left(s+\frac{\omega_1}{2}\right)+R_3 B(s-s_3)+R_4 B(s-s_4) , \quad (62)$$

in which R_1, R_2, R_3 and R_4 denote the residues of $G(s)$ at its simple poles $\omega_1/2$, $-\omega_1/2$, s_3 and s_4 respectively.

After some tedious calculations, the values of these residues are found to be as follows:

$$\left.\begin{aligned} R_1 &= \frac{U^2 D}{\pi} , \\ R_2 &= -\frac{U^2 D}{\pi} , \end{aligned}\right\} \qquad (63)$$

$$\left.\begin{aligned} R_3 &= \frac{U^2 D}{\pi}\frac{\left[\vartheta_1\left(\frac{\beta}{2\pi}\right)\vartheta_2\left(\frac{\beta}{2\pi}\right)\right]^2}{[\vartheta_3(0)\,\vartheta_4(0)]^2}e^{i\beta} , \\ R_4 &= -\frac{U^2 D}{\pi}\frac{\left[\vartheta_1\left(\frac{\beta}{2\pi}\right)\vartheta_2\left(\frac{\beta}{2\pi}\right)\right]^2}{[\vartheta_3(0)\,\vartheta_4(0)]^2}e^{i\beta} . \end{aligned}\right\} \qquad (63')$$

Thus we have

$$R_1 = -R_2\,, \qquad R_3 = -R_4\,,$$

and equation (62) becomes:

$$G(s) = R_1\left\{B\left(s-\frac{\omega_1}{2}\right)-B\left(s+\frac{\omega_1}{2}\right)\right\}+R_3\left\{B(s-s_3)-B(s-s_4)\right\}. \quad (64)$$

§ 8. We are now able to evaluate the integral in the formula (54). The integrand:

$$\left(\frac{df}{ds}\right)^2\left(\frac{ds}{dz}\right)z\frac{1}{Z} \equiv G(s)\,z\frac{1}{Z}$$

has, in the contour Σ, only one simple pole at $Z=0$ and if we denote by R_0 the residue of this function at the said simple pole, we have, by CAUCHY's theorem,

$$\oint_{\Sigma}\left(\frac{df}{ds}\right)^2\left(\frac{ds}{dz}\right)z\frac{dZ}{Z} = -2\pi i R_0\,, \qquad (65)$$

and substituting this in (54) we get

$$M = -\rho\,\omega_1\,\Re[R_0]\,. \qquad (66)$$

It is easily seen, however, that R_0 is equal to the constant term in the expansion of the function $G(s)\,z$ in a power series of Z.

The expansion for z being already known, we have now to expand $G(s)$ in a power series of Z.

When a complex quantity p satisfies the relation:

$$-\Re\left(\frac{\tau}{i}\right) < 2\,\Re\left(\frac{1}{i}\left[\frac{p}{2\omega_1}-\frac{\tau}{2}\right]\right) < \Re\left(\frac{\tau}{i}\right), \qquad (67)$$

in which $\tau = \omega_3/\omega_1$ as before, the function $B(p)$, namely:

$$B(p) = \frac{\sigma\left[p+\frac{2\omega_1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]}{\sigma[p]\,\sigma\left[\frac{2\omega_1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]}\, e^{-\frac{2\eta_1}{\pi}\left(\frac{\pi}{2}+\delta\right)p}$$

$$= \frac{1}{2\omega_1}\,\frac{\vartheta_1'(0)\vartheta_1\left[\frac{p}{2\omega_1}+\frac{1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]}{\vartheta_1\left[\frac{p}{2\omega_1}\right]\vartheta_1\left[\frac{1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]}$$

can be expanded, with the aid of the formula (33), in the form:

$$B(p) = \frac{1}{2\omega_1}\,\frac{\vartheta_1'(0)\vartheta_3\left[\frac{p}{2\omega_1}-\frac{1+\tau}{2}+\frac{1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]}{\vartheta_3\left[\frac{p}{2\omega_1}-\frac{1+\tau}{2}\right]\vartheta_1\left[\frac{1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]}\, e^{-\left(\frac{\pi}{2}+\delta\right)i}$$

$$= \frac{2\pi}{\omega_1} e^{-\left(\frac{\pi}{2}+\delta\right)i}\left[\frac{1}{4\sin\left(\frac{\pi}{2}+\delta\right)}\right.$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{1+2q^{2n}\cos 2\delta+q^{4n}}$$

$$\times\left\{\sin\pi\left[2n\left(\frac{p}{2\omega_1}-\frac{1+\tau}{2}\right)+\frac{1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]\right.$$

$$\left.\left.-q^{2n}\sin\pi\left[2n\left(\frac{p}{2\omega_1}-\frac{1+\tau}{2}\right)-\frac{1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]\right\}\right]. \tag{68}$$

We see easily that when s varies from 0 to ω_3, $s-\omega_1/2$ and $s+\omega_1/2$ satisfy the inequalities:

$$-\Re\left(\frac{\tau}{i}\right) < 2\Re\left(\frac{1}{i}\left[\frac{s-\omega_1/2}{2\omega_1}-\frac{\tau}{2}\right]\right) < \Re\left(\frac{\tau}{i}\right),$$

and

$$-\Re\left(\frac{\tau}{i}\right) < 2\Re\left(\frac{1}{i}\left[\frac{s+\omega_1/2}{2\omega_1}-\frac{\tau}{2}\right]\right) < \Re\left(\frac{\tau}{i}\right),$$

respectively and since these relations are exactly the same as that for p given in (67), both functions $B\left(s-\frac{\omega_1}{2}\right)$ and $B\left(s+\frac{\omega_1}{2}\right)$ can be expanded with the aid of (68) and if we transform s to Z by means of (8) and introduce the angle of attack β by the relation $\frac{\pi}{2}-\delta=\beta$, we have

$$B\left(s-\frac{\omega_1}{2}\right)-B\left(s+\frac{\omega_1}{2}\right)$$

$$=-\frac{2\pi}{\omega_1}e^{i\beta}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{1}{2}n\pi\right)q^n}{1-2q^{2n}\cos 2\beta+q^{4n}}\left\{Z^n(e^{i\beta}-q^{2n}e^{-i\beta})+Z^{-n}(e^{-i\beta}-q^{2n}e^{i\beta})\right\}. \tag{69}$$

In the next place, when p satisfies the relation:

$$-\Re\left(\frac{\tau}{i}\right) < 2\Re\left(\frac{1}{i}\left[\frac{p}{2\omega_1}+\frac{\tau}{2}\right]\right) < \Re\left(\frac{\tau}{i}\right), \tag{70}$$

the function $B(p)$ can be expanded in the form:

$$B(p)=\frac{1}{2\omega_1}\frac{\vartheta_1'(0)\vartheta_3\left[\frac{p}{2\omega_1}+\frac{1+\tau}{2}+\frac{1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]}{\vartheta_3\left[\frac{p}{2\omega_1}+\frac{1+\tau}{2}\right]\vartheta_1\left[\frac{1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]}$$

$$= \frac{2\pi}{\omega_1} e^{\left(\frac{\pi}{2}+\delta\right)i} \left[\frac{1}{4\sin\left(\frac{\pi}{2}+\delta\right)} \right.$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1+2q^{2n}\cos 2\delta + q^{4n}}$$

$$\times \left\{ \sin\pi\left[2n\left(\frac{p}{2\omega_1}+\frac{1+\tau}{2}\right)+\frac{1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right] \right.$$

$$\left.\left. -q^{2n}\sin\pi\left[2n\left(\frac{p}{2\omega_1}+\frac{1+\tau}{2}\right)-\frac{1}{\pi}\left(\frac{\pi}{2}+\delta\right)\right]\right\}\right], \tag{71}$$

and since $s-s_3$ and $s-s_4$ satisfy the relations:

$$-\Re\left(\frac{\tau}{i}\right) < 2\Re\left(\frac{1}{i}\left[\frac{s-s_3}{2\omega_1}+\frac{\tau}{2}\right]\right) < \Re\left(\frac{\tau}{i}\right),$$

and

$$-\Re\left(\frac{\tau}{i}\right) < 2\Re\left(\frac{1}{i}\left[\frac{s-s_4}{2\omega_1}+\frac{\tau}{2}\right]\right) < \Re\left(\frac{\tau}{i}\right),$$

respectively, which are the same relations as that for p in (70), putting $p = s-s_3$ and $p = s-s_4$ in the expression (71) for $B(p)$ consecutively and subtracting, we get

$$B(s-s_3)-B(s-s_4)$$

$$= \frac{i\pi}{\omega_1} e^{-i\beta} \sum_{n=1}^{\infty} \frac{\{1-(-1)^n\} q^n}{1-2q^{2n}\cos 2\beta + q^{4n}} \left\{ Z^n(q^{-n}e^{i\beta}-q^n e^{-i\beta}) e^{-in\pi\left(\frac{1}{2}+\frac{\beta}{\pi}\right)} \right.$$

$$\left. -Z^{-n}(q^n e^{-i\beta}-q^{3n}e^{i\beta}) e^{in\pi\left(\frac{1}{2}+\frac{\beta}{\pi}\right)} \right\}, \tag{72}$$

where s has been transformed to Z with the aid of (8) as before and also (50) has been taken into account.

§ 9. In the preceding paragraph we have obtained the series for

$$\left\{B\left(s-\frac{\omega_1}{2}\right)-B\left(s+\frac{\omega_1}{2}\right)\right\} \text{ and } \left\{B(s-s_3)-B(s-s_4)\right\}$$

and they are given by (69) and (72) respectively. Hence, by inserting them in the right-hand side of (64) we get a series expanded in integral powers of Z for the function $G(s)$. The result is:

$$G(s) = -R_1\frac{2\pi}{\omega_1}e^{i\beta}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{1}{2}n\pi\right)q^n}{1-2q^{2n}\cos 2\beta+q^{4n}}$$

$$\times\left\{Z^n(e^{i\beta}-q^{2n}e^{-i\beta})+Z^{-n}(e^{-i\beta}-q^{2n}e^{i\beta})\right\}$$

$$+R_3\frac{i\pi}{\omega_1}e^{-i\beta}\sum_{n=1}^{\infty}\frac{\{1-(-1)^n\}\,q^n}{1-2q^{2n}\cos 2\beta+q^{4n}}$$

$$\times\left\{Z^n(q^{-n}e^{i\beta}-q^ne^{-i\beta})\,e^{-in\pi\left(\frac{1}{2}+\frac{\beta}{\pi}\right)}\right.$$

$$\left.-Z^{-n}(q^ne^{-i\beta}-q^{3n}e^{i\beta})\,e^{in\pi\left(\frac{1}{2}+\frac{\beta}{\pi}\right)}\right\}, \qquad (73)$$

where the values of R_1 and R_3 are given by (63) and (63′) respectively.

Then, multiplying the power series for $G(s)$ just obtained by that for z given in (47) and inserting the values for R_1 and R_3 given by (63) and (63′) respectively, we get, after some reductions,

$$G(s)z = \frac{8U^2D^2}{\pi\omega_1}\sin 2\beta\sum_{n=1}^{\infty}\frac{\sin^2\left(\frac{1}{2}n\pi\right)q^{2n}(1-q^{4n})}{n(1-2q^{2n}\cos 2\beta+q^{4n})^2}$$

$$+\frac{2U^2D^2}{\pi\omega_1}\frac{\left[\vartheta_1\left(\frac{\beta}{2\pi}\right)\vartheta_2\left(\frac{\beta}{2\pi}\right)\right]^2}{[\vartheta_3(0)\vartheta_4(0)]^2}e^{-i\beta}$$

$$\times\sum_{n=1}^{\infty}\frac{\{1-(-1)^n\}\sin\left(\frac{1}{2}n\pi\right)q^{2n}}{n(1-2q^{2n}\cos 2\beta+q^{4n})^2}$$

$$\times\left\{(q^{-n}e^{2i\beta}-2q^n+q^{3n}e^{-2i\beta})e^{-in\pi\left(\frac{1}{2}+\frac{\beta}{\pi}\right)}\right.$$

$$\left.+(q^ne^{-2i\beta}-2q^{3n}+q^{5n}e^{2i\beta})e^{in\pi\left(\frac{1}{2}+\frac{\beta}{\pi}\right)}\right\}$$

$$+(\text{positive and negative integral powers of } Z). \quad (74)$$

The constant term in this series is the required residue R_0, as mentioned already, and if we simplify the various terms by using the obvious results that $\sin^2(n\pi/2)=1$ for all odd integral values of n and $\sin(n\pi/2)=0$, $1-(-1)^n=0$ for all even integral values of n, we have the result that

$$R_0 = \frac{8U^2D^2}{\pi\omega_1}\sin 2\beta\sum_{n=1}^{\infty}{}^{\circ}\frac{q^{2n}(1-q^{4n})}{n(1-2q^{2n}\cos 2\beta+q^{4n})^2}$$

$$+\frac{4U^2D^2}{\pi\omega_1}\frac{\left[\vartheta_1\left(\frac{\beta}{2\pi}\right)\vartheta_2\left(\frac{\beta}{2\pi}\right)\right]^2}{[\vartheta_3(0)\vartheta_4(0)]^2}$$

$$\times\sum_{n=1}^{\infty}{}^{\circ}\frac{q^{2n}}{n(1-2q^{2n}\cos 2\beta+q^{4n})^2}$$

$$\times\left\{\left[-q^{-n}\sin(n-1)\beta+q^n\{2\sin(n+1)\beta-\sin(n-3)\beta\}\right.\right.$$

$$\left.+q^{3n}\{2\sin(n-1)\beta-\sin(n+3)\beta\}-q^{5n}\sin(n+1)\beta\right]$$

$$\left.+(\text{a purely imaginary constant})\right\}, \quad (75)$$

where Σ_o indicates, as before, that only the positive odd integral values of n are to be included in the summation, and taking the real part of this quantity we obtain

$$\Re[R_0] = \frac{8U^2D^2}{\pi\omega_1}\sin 2\beta \sum_{n=1}^{\infty}{}_o \frac{q^{2n}(1-q^{4n})}{n(1-2q^{2n}\cos 2\beta+q^{4n})^2}$$

$$+\frac{4U^2D^2}{\pi\omega_1}\frac{\left[\vartheta_1\left(\frac{\beta}{2\pi}\right)\vartheta_2\left(\frac{\beta}{2\pi}\right)\right]^2}{[\vartheta_3(0)\,\vartheta_4(0)]^2}$$

$$\times\sum_{n=1}^{\infty}{}_o \frac{q^{2n}}{n(1-2q^{2n}\cos 2\beta+q^{4n})^2}$$

$$\times\Big[-q^{-n}\sin(n-1)\beta+q^n\{2\sin(n+1)\beta-\sin(n-3)\beta\}$$

$$+q^{3n}\{2\sin(n-1)\beta-\sin(n+3)\beta\}-q^{5n}\sin(n+1)\beta\Big]. \tag{76}$$

Finally, putting this in (66) we get the expression for the required moment M of the fluid pressure acting on the plate under consideration. The result is as follows:

$$M = -\frac{8\rho U^2D^2}{\pi}\sin 2\beta \sum_{n=1}^{\infty}{}_o \frac{q^{2n}(1-q^{4n})}{n(1-2q^{2n}\cos 2\beta+q^{4n})^2}$$

$$-\frac{4\rho U^2D^2}{\pi}\frac{\left[\vartheta_1\left(\frac{\beta}{2\pi}\right)\vartheta_2\left(\frac{\beta}{2\pi}\right)\right]^2}{[\vartheta_3(0)\,\vartheta_4(0)]^2}$$

$$\times\sum_{n=1}^{\infty}{}_o \frac{q^{2n}}{n(1-2q^{2n}\cos 2\beta+q^{4n})^2}$$

$$\times\Big[-q^{-n}\sin(n-1)\beta+q^n\{2\sin(n+1)\beta-\sin(n-3)\beta\}$$

$$+q^{3n}\{2\sin(n-1)\beta-\sin(n+3)\beta\}-q^{5n}\sin(n+1)\beta\Big], \tag{77}$$

which is exactly the same expression as that obtained for the moment in the previous paper, as we should have expected.

We see readily from this result that the moment is always negative and this shows, as we should naturally expect, that the fluid pressure acting on the plate has the effect of rotating it in the clockwise sense; in other words, the fluid pressure tends to set the plate broadside on to the stream.

§ 10. Now, it is expected that the limiting form of the expression for M given by (77) when the breadth D of the channel becomes infinitely large would give the well-known expression for the moment about the mid-point of the fluid pressure acting on a flat plate in an unbounded stream. In the present paragraph, we shall calculate the said limiting form.

We see from (52) that, when D tends to infinity,

$$\frac{2a}{D} \longrightarrow \frac{8q}{\pi}\,, \tag{78}$$

and by making use of the q-expansion formulae for various elliptic functions we find that as $D \to \infty$,

$$M \longrightarrow -\frac{8\rho U^2 D^2}{\pi} q^2 \sin 2\beta\,. \tag{79}$$

Thus, if we write

$$\lim_{D\to\infty} M = M_0\,, \tag{80}$$

we get finally

$$M_0 = -\frac{1}{2}\pi\rho a^2 U^2 \sin 2\beta\,. \tag{81}$$

This is the well-known expression for the moment about the mid-point of a plate of the fluid pressure exerting on that plate, which is placed in an unbounded stream of an incompressible perfect fluid, and in this way we can verify that our expression (77) for the moment of the force acting on the plate in the channel gives the correct limiting value when the walls of the channel remove to infinity.

In order to see clearly how the walls of the channel affect the moment of the force acting on the plate, it is convenient to discuss the magnitude of the ratio M/M_0.

We have, from (77) and (81),

$$\frac{M}{M_0}=\frac{64}{\pi^2}\left(\frac{D}{2a}\right)^2\sum_{n=1}^{\infty}{}^{\circ}\frac{q^{2n}(1-q^{4n})}{n(1-2q^{2n}\cos 2\beta+q^{4n})^2}$$

$$+\frac{32}{\pi^2}\left(\frac{D}{2a}\right)^2\frac{1}{\sin 2\beta}\frac{\left[\vartheta_1\left(\frac{\beta}{2\pi}\right)\vartheta_2\left(\frac{\beta}{2\pi}\right)\right]^2}{[\vartheta_3(0)\,\vartheta_4(0)]^2}$$

$$\times\sum_{n=1}^{\infty}{}^{\circ}\frac{q^{2n}}{n(1-2q^{2n}\cos 2\beta+q^{4n})^2}$$

$$\times\Big[-q^{-n}\sin(n-1)\beta+q^n\{2\sin(n+1)\beta-\sin(n-3)\beta\}$$

$$+q^{3n}\{2\sin(n-1)\beta-\sin(n+3)\beta\}-q^{5n}\sin(n+1)\beta\Big], \tag{82}$$

and using this together with (52) we can calculate the values of M/M_0 for various values of $2a/D$ and β to any required degree of accuracy.

IV. Approximate Expressions for the Moment. Numerical Discussions.

§ 11. Although the values of M/M_0 for various values of $2a/D$ and β can be calculated, as mentioned just in the above, by making use of (52) and (82) to any required degree of accuracy, it may not be useless to derive here approximate expressions for M/M_0 in terms of $2a/D$ and β. Without entering into the detailed tedious calculations, we shall now write down the results only.

When the ratio $2a/D$ is less than unity and especially when it is smaller than $1/2$, as in the practically important cases,[1] the ratio M/M_0 can be expressed as a power series of $2a/D$, and if we neglect powers of $2a/D$ greater than the sixth, the approximate expression for M/M_0 becomes:

$$\frac{M}{M_0} = 1 + \frac{\pi^2}{48}(1 + 2\sin^2\beta)\left(\frac{2a}{D}\right)^2 - \frac{\pi^4}{23040}(11 - 106\sin^2\beta - 66\sin^4\beta)\left(\frac{2a}{D}\right)^4. \tag{83}$$

Also, if we retain the sixth power of $2a/D$ but neglect powers of it greater than the eighth, we get

$$\begin{aligned}\frac{M}{M_0} = 1 &+ \frac{\pi^2}{48}(1 + 2\sin^2\beta)\left(\frac{2a}{D}\right)^2 \\ &- \frac{\pi^4}{23040}(11 - 106\sin^2\beta - 66\sin^4\beta)\left(\frac{2a}{D}\right)^4 \\ &+ \frac{\pi^6}{15482880}(327 - 3078\sin^2\beta + 10800\sin^4\beta + 3504\sin^6\beta)\left(\frac{2a}{D}\right)^6.\end{aligned} \tag{84}$$

Now, when q is equal to 0·2 we have $2a/D = 0{\cdot}5$ nearly and also when $q = 0{\cdot}3$ the ratio $2a/D$ becomes equal to 0·75 approximately in the case where $\beta = 45°$, as shown in the table given later. Thus, if

(1) In acoustical experiments with a RAYLEIGH disc, the ratio of the diameter ($2a$) of the disc to that (D) of a tube, in which the disc is suspended, is usually smaller than $1/2$. This remark is due to Dr. K. SATÔ.

we confine ourselves to the practically important cases in which $2a/D$ is fairly smaller than unity, the ratio M/M_0 can be expressed in a rapidly convergent power series of q in the form:

$$\frac{M}{M_0}=1+\frac{4}{3}(1+2\sin^2\beta)q^2+\frac{8}{45}(9+106\sin^2\beta-14\sin^4\beta)q^4$$
$$+\frac{1}{945}(640+42816\sin^2\beta+8832\sin^4\beta+4096\sin^6\beta)q^6+\cdots, \tag{85}$$

and it is not difficult to see from this result that in cases under consideration the ratio M/M_0 is not less than unity, that is,

$$\frac{M}{M_0}\geqq 1. \tag{86}$$

Thus, we can say that in the practically important cases mentioned above, there is always more or less increase in the moment of the force acting on the plate in the channel due to the presence of the channel walls, whatever the value of the angle of attack of the plate β may be.

To show this more clearly, we shall here calculate numerically the values of M/M_0 for various values of $2a/D$ in the case in which $\beta=45°$, since the result for this case may possibly be applied, with some modifications, to the RAYLEIGH disc problem. In this case, we have, from (83),

$$\frac{M}{M_0}=1+\frac{\pi^2}{24}\left(\frac{2a}{D}\right)^2+\frac{13\pi^4}{5120}\left(\frac{2a}{D}\right)^4$$
$$=1+0{\cdot}41123\left(\frac{2a}{D}\right)^2+0{\cdot}24733\left(\frac{2a}{D}\right)^4, \tag{87}$$

when we neglect powers of $2a/D$ greater than the sixth, while if we retain the sixth power of $2a/D$, we have, from (84),

$$\frac{M}{M_0} = 1 + \frac{\pi^2}{24}\left(\frac{2a}{D}\right)^2 + \frac{13\pi^4}{5120}\left(\frac{2a}{D}\right)^4 + \frac{107\pi^6}{860160}\left(\frac{2a}{D}\right)^6$$

$$= 1 + 0\cdot41123\left(\frac{2a}{D}\right)^2 + 0\cdot24733\left(\frac{2a}{D}\right)^4 + 0\cdot11959\left(\frac{2a}{D}\right)^6. \quad (88)$$

The numerical values of M/M_0 for the case when $\beta = 45^\circ$ are tabulated in the annexed table. In this table, the values of $2a/D$ shown in the second column have been calculated by (52), and the third column gives the values of M/M_0 calculated by means of the exact expression (82), putting $\beta = 45^\circ$. Also, the fourth column with the heading $(M/M_0)_I$ gives the values of M/M_0 calculated by the approximate formula (87), while the last column with the heading $(M/M_0)_{II}$ gives those calculated by the approximate formula (88). We see from this table that even the first formula (87) gives a fairly good approximation, especially when $2a/D \leqq 0\cdot5$.

TABLE. ($\beta = 45^\circ$)

q	$\frac{2a}{D}$	$\frac{M}{M_0}$	$\left(\frac{M}{M_0}\right)_I$	$\left(\frac{M}{M_0}\right)_{II}$
0·005	0·012732	1·000067	1·000067	1·000067
0·050	0·12732	1·00673	1·00673	1·00673
0·075	0·19098	1·01533	1·01533	1·01533
0·100	0·25462	1·02773	1·02770	1·02773
0·150	0·38174	1·06558	1·06518	1·06555
0·200	0·50832	1·12516	1·12277	1·12483
0·300	0·75659	1·34841	1·31644	1·33888

We see also that in the present case the value of M/M_0 is greater than unity, as in the more general case. Fig. 5 shows the approximate curve of M/M_0 drawn against the ratio $2a/D$, when $\beta = 45^\circ$.

It is usual that in acoustical experiments with a RAYLEIGH disc, the disc is suspended in a tube with circular section in such a way that its centre lies on the axis of the tube and its angle of attack is 45°.

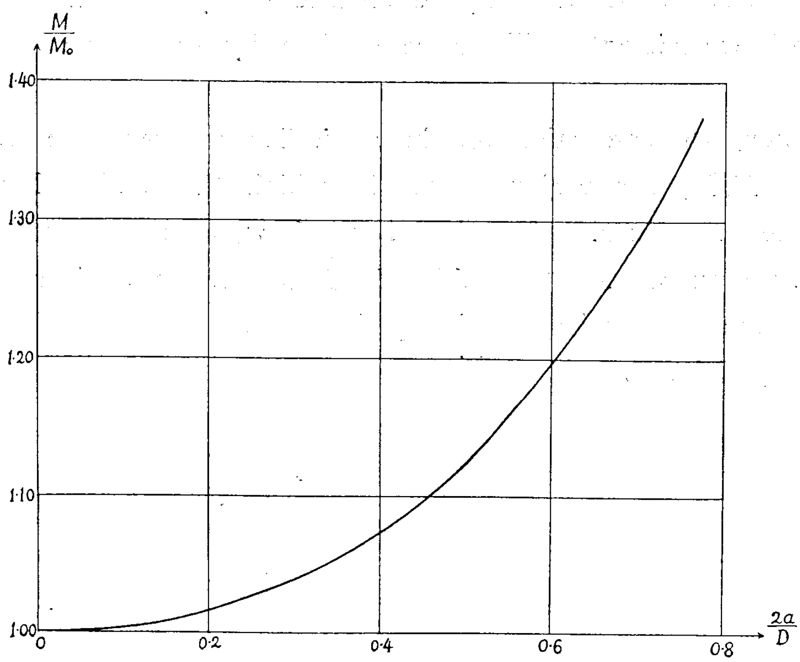

Fig. 5. ($\beta = 45°$)

The flow past the RAYLEIGH disc is, of course, not two-dimensional, and in reality the compressibility as well as the viscosity of the air, both of which have been neglected in this paper, may perhaps have appreciable effects upon the moment of the force acting on the RAYLEIGH disc, especially when the diameter of the disc is comparable with that of the tube, in which the disc is suspended.

Therefore, it may not be appropriate to apply the results obtained in the present paper directly to the RAYLEIGH disc problem. However, we may anticipate from our results that the tendency of variation of the moment of the force acting on the RAYLEIGH disc with respect to the ratio of its diameter to that of the tube will be similar to that for the plate in the two-dimensional flow, which has been discussed in this paper, and consequently there will be more or less increase of the moment of the force acting on the disc due to the presence of the wall of the tube.

This anticipation can only be ascertained by experiments, and we hope such experiments will be undertaken by some one in the near future.[1]

Further, we can calculate, in a similar manner, the values of the ratio M/M_0 for various cases in which the angle of attack β takes different values. We shall here show, as an addendum, only approximate curves for the said ratio M/M_0 drawn against the ratio $2a/D$ in five cases where β is equal to 10°, 20°, 30°, 45°, and 60° respectively (Fig. 6).

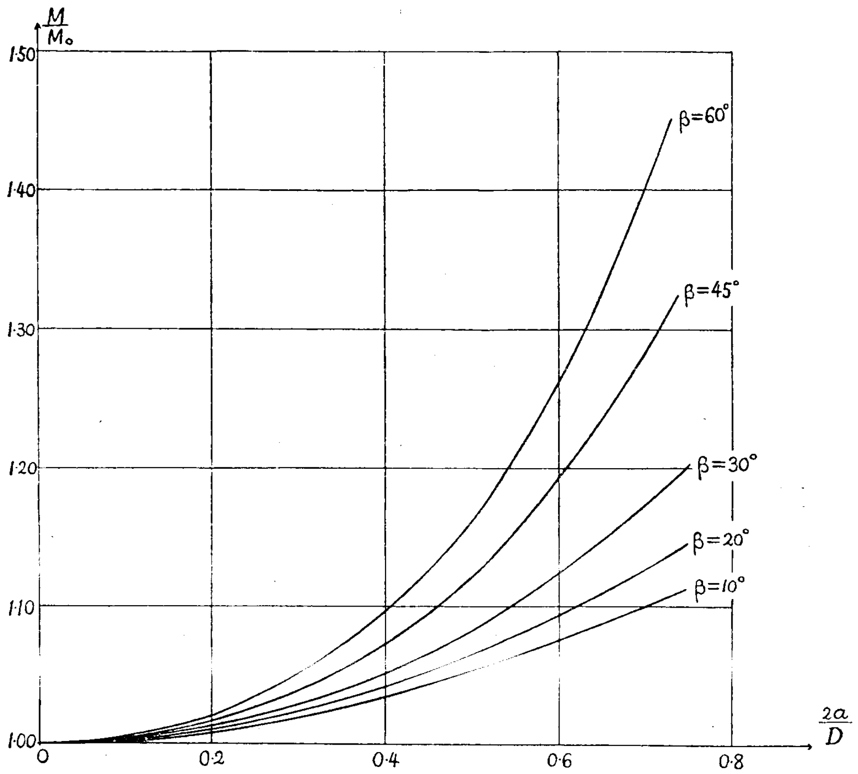

Fig. 6.

We see from this figure that the larger the value of β is, the greater is the rate of increase of the ratio M/M_0 with $2a/D$.

(1) It appears that Dr. K. SATÔ in this Institute has the intention of performing experiments on such lines in the near future.

V. Summary.

§ 12. In the present paper, we have calculated again, by using the well-known BLASIUS' formula, the moment M of the force acting on a flat plate about its mid-point, which is placed obliquely in a steady irrotational continuous flow of an incompressible perfect fluid bounded by two parallel plane walls, under the supposition that the mid-point of the plate lies on the central line of the channel. The method of analysis used in this paper was slightly different from that employed in my previous paper, but the result obtained was, of course, the same.

Considering only the practically important cases in which the ratio of the breadth $2a$ of the plate to the width D of the channel is fairly smaller than unity, we have arrived at the result that there is always an increase of the moment of the force due to the presence of the channel walls.

Approximate expressions for the moment have been given as power series of $2a/D$, retaining however only the first two or three important terms. Numerical calculations of the values of M/M_0 have been carried out for the case in which β is equal to 45°, where M_0 is the moment of the force acting on a flat plate with the same breadth $2a$ placed in an unlimited stream, and the approximate curve for M/M_0 was drawn against the ratio $2a/D$.

Then, basing upon our results, we have anticipated the tendency of variation of the moment of the force exerting on a RAYLEIGH disc with the ratio of its diameter to that of a tube, in which the disc is suspended obliquely in such a way that its angle of attack is equal to 45°.

Lastly, as an addendum, approximate curves for M/M_0 drawn against $2a/D$ were shown for the cases in which β is equal to 10°, 20°, 30°, 45° and 60° respectively.

January, 1933.