Theory of Composite Beams and of Composite Columns.

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A beam composed of longitudinal prismatic elements of different materials and of sufficient adhesive resistance at their surfaces of contact for withstanding the longitudinal shearing action may be called a composite beam. In such a beam contiguous fibres of adjacent prismatic elements undergo the same longitudinal strains and consequently the corresponding stresses are in the proportion of their respective moduli of elasticity. Bending action on the beam being limited within the elastic limit of any of the materials composing the beam, a plane cross section of it may be assumed to remain plane after bending.

Referring to Fig. 1 let a composite beam be so placed that the bending takes place in a vertical plane XOY with its longitudinal neutral axis coinciding with the axis of X . In the cross sectional view the beam is shown, as an example, to consist of six prismatic elements symmetrically disposed with respect to the plane of bending XOY . Of these six elements No. 2 and No. 5 each consist of a pair of equal round bars of the same material, also symmetrically placed. Now the horizontal ordinate bc of any point such as y_1 on the cross section, measured to the oblique straight line COC' represents the longitudinal strain at that point due to bending. The horizontal ordinate measured to the stepped straight line, $EF, GH, KL, MO, OM', L'K', H'G', F'E'$, all passing through O , represent the bending stress at any point situated in the line $QPROY$ on the cross section.

Hence we have

$$\frac{1}{\rho} = \frac{f}{Ey} = \frac{f_1}{E_1y_1} = \frac{f_2}{E_2y_2} \dots = \frac{f_r}{E_r y_r} \dots, \dots \dots (1)$$

in which f_r is the variable normal stress due to bending, on the section of the element

Fig. 2.

indicated by the suffix r , corresponding to the variable ordinate y_r , and E_r the stretch modulus of the material of the element r .

All the external forces on the beam being supposed to be parallel to the axis of Y and symmetrical with reference to the plane of bending, one of statical conditions of equilibrium requires that the bending moment M on the beam at the cross section YOZ must be equal to the moment of resistance of the section,

$$M = f_1 \frac{I_1}{y_1} + f_2 \frac{I_2}{y_2} + \dots + f_r \frac{I_r}{y_r} + \dots$$

$$M = \frac{1}{\rho} (E_1 I_1 + E_2 I_2 + \dots + E_r I_r + \dots)$$

$$= \frac{1}{\rho} \Sigma (EI),$$

Hence

$$\frac{1}{\rho} = \frac{M}{\Sigma(EI)}, \quad \dots \dots \dots (2)$$

Combining this with (1) we obtain

$$M = \frac{f}{y} \cdot \frac{\Sigma(EI)}{E}, \quad \dots \dots \dots (3)$$

in which f is the normal bending stress at variable distance y , on the section of an element chosen for reference, and E the denominator of the fraction is the stretch modulus of that element. The equation (2) may be written in the form

$$\frac{d^2 v}{dx^2} = \frac{M}{\Sigma(EI)}, \quad \dots \dots \dots (4)$$

in which v is the ordinate of the elastic curve which the longitudinal neutral axis of the beam assumes after bending. Hence the amount of deflection of a composite beam can be calculated in the same manner as for a homogeneous beam, except that $\Sigma(EI)$ must be used instead of EI .

To find the intensity of shear stress at a point in a composite beam we consider a portion bounded by two cross sections dx apart and by horizontal cylindrical (in the general sense) surface such as $\alpha\beta\beta'\alpha'$, Fig. 2. For the equilibrium in horizontal direction of all the stresses on the slice thus formed, the total shear stress on the horizontal surface must be equal and opposite to the total difference of normal stresses on the two cross sections, that is

$$qzdx = \int df_1 dA_1 + \int df_2 dA_2 + \int df dA,$$

in which q is the mean intensity of shear stress on the surface $\alpha\beta$, z the perimeter $\alpha\beta$, df the difference of variable normal stresses on the two sections at a distance y from the neutral axis of the section, and dA an element of area of the cross section at y . Since the variation df corresponds to the variation of x only, we have by (1)

$$df = \frac{df_r}{E_r y_r} E y,$$

Hence the above equation may be put thus

$$q z dx = \frac{df_r}{E_r y_r} \left(E_4 \int y_4 dA_4 + E_5 \int y_5 dA_5 + E_6 \int y_6 dA_6 \right),$$

$$q = \frac{1}{z} \cdot \frac{1}{E_r y_r} \cdot \frac{df_r}{dx} \Sigma (EG),$$

in which

$$G = \int y dA,$$

is the moment of area, the integration extending over the cross section of each of the elements lying on one side of the perimeter z . Differentiating (3), we have

$$\frac{dM}{dx} = \text{shearing force } F = \frac{1}{E_r y_r} \cdot \frac{df_r}{dx} \Sigma (EI),$$

Hence the equation above for the mean intensity of shear stress becomes

$$q = \frac{F \Sigma (EG)}{z \Sigma (EI)}, \quad \dots \dots \dots (5)$$

The summation of EG 's includes in it those elements which lie on one side of the perimeter z . This perimeter may be the complete outline of the cross section of a single prismatic element.

The position of the neutral axis of the section may be determined in the following manner. Considering the bending action only, the sum of normal stresses on a cross section is equal to zero and we have the following equation of condition,

$$\Sigma \int f_x d(\text{area}) = 0,$$

in which the integration extends over all the points on the section of each prismatic element and the summation includes in it all the prismatic elements composing the beam.

By (1) we have

$$f_r = \frac{E_r y_r}{\rho},$$

But E_r is constant for all points on the cross section of any one prismatic element and ρ is constant for all points on the whole cross section.

The equation of condition therefore becomes

$$\frac{1}{\rho} \sum \left\{ E_r \int y_r d(\text{area}) \right\} = 0,$$

that is,

$$\sum (EG) = 0, \dots\dots\dots (6)$$

When the position of the neutral axis is not known, any convenient horizontal line lying on the section may be taken as a reference axis such as the horizontal tangent at the lowest edge of the section. Let $\bar{y}$ be the distance of the neutral axis measured from such a reference axis.

Then G in (6) becomes

$$\begin{aligned} G &= \int (y_r - \bar{y}) d(\text{area}) \\ &= \int y_r d(\text{area}) - \bar{y} \int d(\text{area}) \\ &= G' - \bar{y} A, \text{ say,} \end{aligned}$$

Hence the equation of condition (6) becomes

$$\bar{y} = \frac{\sum (EG')}{\sum (EA)}, \dots\dots\dots (7)$$

in which G' is the moment of area referred to the assumed axis.

The foregoing principal equations for a composite beam are remarkably analogous to those for a homogeneous beam. In fact all well known beam equation are at once obtained from those given above by striking out the sign of summation and cancelling the letters for elasticity. An important fact worthy of note is that the deflection of a composite beam can be found from (4) in exactly the same manner as for a homogeneous beam. We now proceed to consider a composite column and it will be found that the same analogy exists also in this case.

If a composite bar is capable of acting as a beam, it is also capable of acting as a column of great length. In a column of short length, however, the adhesive resistance on the contiguous surfaces of prismatic elements need not be as strong as in a beam. In a short

column under a heavy axial load, tensile and compressive stresses of appreciable intensity may be developed at right angles to the axis of the column. This will happen if the materials composing the column have the values of Poisson's ratio very much different from one another.

In a short column the axial strains of contiguous fibres of different prismatic elements are the same and the compressive stresses at any point on a cross section due to direct axial load are in the proportion of the respective moduli of elasticity. It may be assumed that a plane cross section of the column remains a plane after the application of the axial load. Denoting by λ the compressive strain per unit length, we have

$$\lambda = \frac{f}{E} = \frac{f_1}{E_1} = \frac{f_r}{E_r} \dots = \frac{f_r}{E_r} \dots, \dots (8)$$

The total axial load P carried by the composite column is

$$\begin{aligned} P &= f_1 A_1 + f_2 A_2 + \dots + f_r A_r + \dots \\ &= \lambda (E_1 A_1 + E_2 A_2 + \dots + E_r A_r + \dots) \\ &= \lambda \Sigma (EA), \dots (9) \end{aligned}$$

and by (1),

$$P = f \frac{\Sigma (EA)}{E} \dots (10)$$

in which the E the denominator is the modulus of elasticity of the material of any one particular element which is taken as the standard of reference and f is the corresponding compressive stress produced in it by the load P . The quotient $\Sigma(EA)/E$ may be called an equivalent area of the column referred to the standard prismatic element.

For the theory of long composite columns of uniform section, equation (4) may be used and we arrive at Euler's equations very slightly modified, namely,

$$P = (\pi^2 \text{ or } 4\pi^2 \text{ or } 2\pi^2) \frac{\Sigma(EI)}{l^2}, \dots (11)$$

for columns in which both ends are jointed, or both ends are fixed, or one end is jointed and the other fixed. Dividing both sides of each of three equations (11) by the equivalent area and writing f for the safe intensity of the compressive resistance of the standard material, we obtain

$$f = (\pi^2 \text{ or } 4\pi^2 \text{ or } 2\pi^2) \cdot \frac{E}{n} \cdot \frac{k^2}{l^2}, \dots\dots\dots (12)$$

in which

$$k^2 = \frac{\Sigma (EI)}{\Sigma (EA)}, \dots\dots\dots (13)$$

and n is a factor of safety corresponding to the safe resistance f . The quantity k may be called an equivalent radius of gyration of the section of a composite column. Equations (12) enable us to determine the limiting length l below which the modified Euler equations cease to be applicable. For columns of which the length is smaller than the value determined by (12), we may make use of the well known semi-rational formula given by Rankine

$$P = \frac{fA}{1 + 4a \frac{l^2}{k^2}}, \dots\dots\dots (14)$$

substituting for A the equivalent area and for k the equivalent radius of gyration of the section. To obtain reliable values of the constants f and a for a given kind of composite column, a series of experiments on column of that kind would be required. Equation (14) refers to the case in which both ends are jointed; for the two other end conditions, a and $2a$ have to be put in place of $4a$.

Very many formulae relating to reinforced concrete beams and columns can be easily derived from the foregoing general equations.

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