TECHNICAL REPORT OF NATIONAL AEROSPACE LABORATORY

TR-611T

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May 1980

NATIONAL AEROSPACE LABORATORY

CHŌFU, TOKYO, JAPAN

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ABSTRACT

Miner's rule regarding the fatigue behavior of materials is discussed from a statistical standpoint. The conditions required for both the parameters of the log-normal and Weibull distribution functions of fatigue life and the cumulative damage rule to satisfy statistical Miner's rule are derived. For the case when the conditions obtained are not satisfied, the reliability of the population and the mean cumulative cycle ratio to failure are considered for two-step fatigue tests, and calculation examples are given. It is found that fatigue life scatter has a fairly large effect on the mean cumulative cycle ratio to failure.

概 要

本研究は材料の疲労に関するMiner則を統計的観点から検討したものである。疲労寿命分布関数として対数正規分布およびWeibull分布を採り上げ、これらの分布形における母数ならびに累積損傷則に関する統計的平均としてのMiner則成立条件を導いた。また、得られた条件が成立しない場合、2段2重重複荷重に対する母集団の信頼度と破断までの平均累積寿命比について考察し、計算例を示した。この結果、疲労寿命のばらつきは破断までの平均累積寿命比にかなり大きな影響を与えることが明らかとなった。

1. INTRODUCTION

Miner's rule [1] is very simple and useful in estimating fatigue life of a specimen subjected to variable amplitude loading from an $S-N$ relation. Experimental results obtained by either constant or variable amplitude fatigue tests show wide scattering of fatigue life. Therefore, Miner's rule should be considered statistically.

A few papers have discussed Miner's rule from statistical standpoints. Yokobori [2] derived statis-

tical Miner's rule based on a theory of stochastic processes assuming constant transition probability. A probabilistic interpretation of Miner's rule was attempted by Birnbaum and Saunders [3], and Saunders [4]. Ichikawa [5] obtained statistical Miner's rule based on a theory of successive stochastic process. Tanaka et al. [6], and Tanaka and Akita [7] discussed Miner's rule statistically by using their experimental results.

In the papers referred to above, however, the effects of fatigue life distributions and the cumulative damage rule on statistical Miner's rule are not sufficiently clarified. On the bases of the conditions required to satisfy statistical Miner's rule, which were found by Tanaka et al. [6], this paper derives the conditions with respect to both the parameters of the log-normal and Weibull distribution functions of fatigue life and the cumulative damage rule. Numerical calculation examples of reliability curves of the population and the mean cumulative cycle ratio to failure are given for two-step fatigue tests when the introduced conditions are not satisfied.

* Received March 14, 1980

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A rewrite of the original paper published in the Journal of the Japanese Society for Strength and Fracture of Materials, Vol. 9, No. 1 (1974), pp. 29-43 and as NAL TR-370 (1974) in Japanese.

2. CONDITIONS REQUIRED TO SATISFY STATISTICAL MINER'S RULE

2.1 General Conditions

When a specimen is fatigue-tested under programmed loading of m stress levels and fractured, original Miner's rule is

$$\sum_{i=1}^m \frac{n_i}{N_i} = 1 \quad (1)$$

regardless of the order of applied stress levels. In Eq. (1) i indicates the order number, S_i is the stress level of the i -th order, n_i is the number of cycles repeated at S_i , and N_i is the number of cycles to failure at S_i only. However, fatigue life inevitably has a wide scatter, and N_i of a particular specimen is unknown before the test.

Statistical Miner's rule is defined as

$$E \left(\sum_{i=1}^m \frac{n_i}{\bar{N}_i} \right) = 1, \quad (2)$$

that is, N_i in Eq. (1) is replaced by $\bar{N}_i$, which is the mean life at S_i only, and the expected value of the left-hand side of Eq. (1) is set to be unity. Let $n_i/\bar{N}_i$ be the cycle ratio and the cumulative cycle ratio x be

$$x = \sum_{i=1}^j \frac{n_i}{\bar{N}_i} \quad (3)$$

where j is an arbitrary integer larger than zero. Consider continuous probability density functions $p_i(x)$ and $p_v(x)$, where $p_v(x)$ is probability density function of the cumulative cycle ratio to failure for programmed loading and $p_i(x)$ is that at S_i in programmed loading. The mean cumulative cycle ratio to failure $\bar{x}$ is

$$\bar{x} \equiv E \left[\sum_{i=1}^m \frac{n_i}{\bar{N}_i} \right], \quad (4)$$

or

$$\begin{aligned} \bar{x} &= \int_0^\infty x p_i(x) dx \\ &= \int_0^{x_1} x p_1(x) dx + \int_{x_1}^{x_2} x p_2(x) dx + \cdots \\ &\quad + \int_{x_{m-1}}^\infty x p_m(x) dx, \end{aligned} \quad (5)$$

where $x_1 = n_1/\bar{N}_1$, $x_2 = n_1/\bar{N}_1 + n_2/\bar{N}_2$, $\cdots$, $x_{m-1} = n_1/\bar{N}_1 + n_2/\bar{N}_2 + \cdots + n_{m-1}/\bar{N}_{m-1}$. Reliability $R_i(x)$ and $R_v(x)$ are defined as $dR_v(x)/dx = -p_v(x)$ and $dR_i(x)/dx = -p_i(x)$ respectively. Then,

$$\left. \begin{aligned} \bar{x} &= \int_0^{x_1} R_1(x) dx + \int_{x_1}^{x_2} R_2(x) dx + \cdots \\ &\quad + \int_{x_{m-1}}^\infty R_m(x) dx, \\ \text{or} \\ &= \int_0^\infty R_v(x) dx, \end{aligned} \right\} \quad (6)$$

because $R_1(x_1) = R_2(x_1)$, $R_2(x_2) = R_3(x_2)$, $\cdots$, and $R_{m-1}(x_{m-1}) = R_m(x_{m-1})$. Eq. (6) shows that the mean cumulative cycle ratio to failure is calculated by integrating the reliability $R_v(x)$. For constant amplitude fatigue tests Eq. (6) becomes unity. If Eq. (4) instead of Eq. (6) is used in calculating $\bar{x}$, n_i for a specimen fractured during S_i cycles must be substituted by the actual number of cycles applied at S_i to this specimen.

Tanaka et al. [6] found the following two conditions required to satisfy statistical Miner's rule.

I. A reliability R curve as a function of x under constant amplitude loading is denoted by $R_c(x)$ and shown in Fig. 1, where x is the ratio of fatigue life to its mean, i.e., $N/\bar{N}$. The first condition is that the reliability curve is independent of stress levels.

Fig. 1 Reliability curve as a function of cycle ratio under constant amplitude loading

II. The second condition is that "the transfer law of reliability curve" is satisfied. This law is denoted by the acronym TLRC. TLRC for two-step fatigue tests is illustrated in Fig. 2. Reliability curves as a function of fatigue life N at stress levels S_1 and S_2 of constant amplitude loading are depicted as ABC and ADE respectively. Suppose that a group of specimens are loaded n_1 cycles at the first stress level S_1 , then the applied stress level is changed from S_1 to the second stress level S_2 , and consider the reliability curve $R_v(N)$ obtained for this stress sequence. $R_v(N)$

Fig. 2 Transfer law of reliability curve (TLRC)

naturally follows ABC as far as S_1 is applied, and reaches R_1 at n_1 of S_1 . After stress level is changed to S_2 , $R_v(N)$ is obtained as BF . If BF coincides with DE by parallel shifting, TLRC is said to be satisfied.

The definition of TLRC by Tanaka et al. [6] is not enough for a distribution with a non-zero location parameter. TLRC should involve such an additional condition that $n_1/\bar{N}_1 = n_2/\bar{N}_2$ when the stress level change is made during $R_v(N) = 1$, or n_1 is smaller than the location parameter at the first stress level. TLRC can be extended to any variable amplitude loading in the similar manner.

Under both the first and second conditions the reliability curve for variable amplitude loading, $R_v(x)$, agrees with that for any constant amplitude loading, $R_c(x)$, and Eq. (6) becomes unity. Thus, statistical Miner's rule is satisfied.

2.2 Parametric Conditions in Fatigue Life Distributions

Parametric conditions required to satisfy the first condition are determined for the log-normal and Weibull distributions, which are frequently used for analyzing fatigue life data. In this section x means the ratio of fatigue life to its mean, i.e., $N/\bar{N}$, since the first condition is concerned with fatigue life distributions under constant amplitude loading.

2.2.1 The log-normal distribution

The probability density function of x is given as

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma_L x} \exp \left\{ -\frac{(\ln x + \sigma_L^2/2)^2}{2\sigma_L^2} \right\}, \quad x > 0, \quad (7)$$

where σ_L is the standard deviation of logarithm of fatigue life, $\ln N$. If σ_L is independent of stress levels, the first condition is satisfied.

2.2.2 The Weibull distribution

The reliability function of x is given as

$$R(x) = \exp \left[-\left\{ \Gamma(1 + 1/\alpha)(x - k)/(1 - k) \right\}^\alpha \right], \quad x \geq 0, \quad (8)$$

where α is the shape parameter and k is the ratio of the location parameter to the mean life. If α and k are constant regardless of stress levels, the first condition is satisfied.

2.3 Explanation of TLRC by the Cumulative Damage Rule and Fatigue Life Distributions

TLRC for variable amplitude loading can be explained by the cumulative damage rule and fatigue life distributions, because a reliability curve is drawn by fatigue lives of specimens. TLRC is considered for two-step fatigue tests.

Suppose that a population consisting of infinite number of specimens is simultaneously fatigue-tested under the same constant amplitude loading. Since a fatigue life distribution is obtained after completing tests, the distribution of damage having been accumulated in each specimen by any time during tests must be present. Consider the ascending order of damage in an arbitrary specimen from the population. Now introduce the *first assumption* that the order of damage in an arbitrary specimen is kept unchanged under any stress history. This assumption is illustrated in Fig. 3 for two-step fatigue tests. The order of

Fig. 3 Cumulative damage in a particular specimen corresponding to the population reliability R for two-step fatigue tests

damage in an arbitrary specimen can be expressed by the reliability of the population at the time of this specimen's failure. The assumption identical with this was first used by Levy [8].

The damage D_R in a particular specimen corresponding to the population reliability R at stress level S_i is defined as a function of the inherent cycle ratio n_i/N_{Ri} , that is,

$$D_R = f(n_i/N_{Ri}), (0 \leq D_R \leq 1), \quad (9)$$

where n_i and N_{Ri} are the number of cycles repeated and the inherent fatigue life of this specimen at S_i respectively. Let's introduce *the second assumption* that the transfer law of damage curve is satisfied for variable amplitude loading, which is illustrated in Fig. 4 for two-step fatigue tests. This law means no stress-interaction effect in damage accumulation. One more condition is indispensable to satisfy TLRC. *The third condition* is that a single damage curve is obtained regardless of stress levels and the order of damage accumulated in an arbitrary specimen, i.e., D_R of Eq. (9) is independent of R and S_i . The first and third conditions under the first and second assumptions give the physical bases for TLRC. When the first condition does not hold, TLRC is not satisfied even if the two assumptions and the third condition are valid, as shown in Example 1 later. Therefore, TLRC is only one condition required to satisfy statistical Miner's rule.

Fig. 4 Transfer law of damage curve

3. RELIABILITY CURVES AND THE MEAN CUMULATIVE CYCLE RATIO TO FAILURE: REQUIRED CONDITIONS NOT SATISFIED

Under the two assumptions the first and third conditions are required to satisfy statistical Miner's rule by considering the cumulative damage rule. In subsequent discussions reliability curves and the mean cumulative cycle ratio to failure for two-step fatigue tests are derived when the first and third conditions are not satisfied. When the third condition is not satisfied, introduce *the third assumption* that a single damage curve is obtained regardless of the order of damage accumulated in an arbitrary specimen from the population under constant amplitude loading, in other words, independent of the population reliability at the time of this specimen's failure, or independent of specimens. The log-normal and two-parameter Weibull distribution functions are used for representing fatigue life distributions.

The coefficient of variation of fatigue life, V_i , can be a common measure of fatigue life scatter for the two distribution functions. The cycle ratio to failure X_{Ri} with respect to a particular specimen corresponding to R at stress level S_i is given as

$$X_{Ri} = N_{Ri}/\bar{N}_i. \quad (10)$$

V_i and X_{Ri} have the following relations with shape parameters. For the log-normal distribution,

$$\left. \begin{aligned} V_i &= \sqrt{\exp(\sigma_{Li}^2) - 1} \\ \text{and} \\ X_{Ri} &= \exp(\sigma_{Li} t_R - \sigma_{Li}^2/2), \end{aligned} \right\} \quad (11)$$

where t_R is the standardized normal variable corresponding to R . For the two-parameter Weibull distribution,

$$\left. \begin{aligned} V_i &= \sqrt{\Gamma(1 + 2/\alpha_i) / \Gamma^2(1 + 1/\alpha_i) - 1} \\ \text{and} \\ X_{Ri} &= \{\ln(1/R)\}^{1/\alpha_i} / \Gamma(1 + 1/\alpha_i). \end{aligned} \right\} \quad (12)$$

Many models have been proposed to specify the cumulative damage rule under constant amplitude loading. This paper uses Macro-Starkey's model [9] which is simple and involves stress dependency. This model is

$$D = (n_i/N_{Ri})^{c_i} = (x_i/X_{Ri})^{c_i}, \quad (13)$$

where the exponent c_i is a function of the stress level and constant regardless of R at S_i from the third assumption. Suppose that each specimen is loaded n_1 cycles at S_1 first, then loaded at S_2 to failure if it does not fail before the stress level change.

Consider a particular specimen corresponding to R . When $n_1 \geq N_{R1}$, the cycle ratio to failure X_f is

$$X_f = X_{R1}. \quad (14)$$

When $n_1 \leq N_{R1}$, the first stress cycles (n_1, S_1) give the cycle ratio x_1 as

$$x_1 = n_1 / \bar{N}_1. \quad (15)$$

On the basis of Eq. (13), x_1 provides equivalent cycle ratio $x_{2,0}$ at S_2 , that is,

$$x_{2,0} = X_{R2} \cdot (x_1 / X_{R1})^{c_1/c_2}. \quad (16)$$

The additional cycle ratio to failure after the stress level change is denoted by $x_{2,a}$ and given as

$$x_{2,a} = X_{R2} - x_{2,0}. \quad (17)$$

The cumulative cycle ratio to failure X_f is

$$\begin{aligned} X_f &= x_1 + x_{2,a} \\ &= x_1 + X_{R2} - X_{R2} \cdot (x_1 / X_{R1})^{c_1/c_2}. \end{aligned} \quad (18)$$

Eqs. (14) and (18) specify the relation between the population reliability and the cumulative cycle ratio to failure. For the log-normal and two-parameter

Weibull distributions R as a function of X_f can be calculated using Eqs. (11), (12), (14), and (18).

The mean cumulative cycle ratio to failure, $\bar{X}_f$, is determined from Eq. (6) using the R - X_f relation as

$$\bar{X}_f = \int_0^{X_1} R_1(X_f) dX_f + \int_{X_1}^{\infty} R_2(X_f) dX_f, \quad (19)$$

where the second term of the right-hand side of Eq. (19) can be calculated by numerical integration.

[Example 1] Consider a stress-independent damage curve which has constant c_i in Eq. (13) regardless of stress levels and the large difference in fatigue life scattering between two stress levels S_A and S_B . The values of the coefficient of variation of fatigue life are given as $V_A = 0.1$ at S_A and $V_B = 0.8$ at S_B . Reliability R curves as a function of X_f for the log-normal distribution are shown in Figs. 5 (a) and (b). Solid lines indicate reliability curves under constant stress conditions. A parameter in the figures is the cycle ratio x_1 given at the first stress level. The reliability curves after the stress level change are drawn by broken lines. Solid and broken lines are used in the same manner for drawing reliability curves in Examples 2 and 3 also. It is noted that the broken lines pass through between the two solid lines. Similar reliability curves are obtained for the two-parameter Weibull distribution. The mean cumulative cycle ratio to failure $\bar{X}_f$ is depicted in Fig. 6 as

Fig. 5-(a)

Fig. 5-(b)

Fig. 5 Relation between the population reliability and the cumulative cycle ratio to failure for two-step fatigue tests, when a stress-independent damage curve is given

Fig. 6 The mean cumulative cycle ratio to failure as a function of the cycle ratio given at the first stress level

a function of x_1 for the two distribution functions. The deviation of $\bar{X}_f$ from unity is small for the stress sequence from S_A to S_B , and relatively large for that from S_B to S_A . These results indicate the effect of the difference in fatigue life scattering between the two stress levels on the mean cumulative cycle ratio to failure.

[Example 2] Let high and low stress levels be S_H and S_L respectively. Suppose that $V_H = 0.2$ and $C_H = 2.0$ are given at S_H , and $V_L = 0.5$ and $C_L = 5.0$ at S_L . Damage curves obtained by substituting the values of C_H and C_L into Eq. (13) are illustrated in Fig. 7. For the log-normal distribution R - X_f curves are shown in Figs. 8 (a) and (b). The reliability curves for the two-parameter Weibull distribution are similar to those in Figs. 8 (a) and (b). $\bar{X}_f$ - x_1 relations are drawn in Fig. 9. When there is no scatter in fatigue life, simple cumulative cycle ratio to failure is calculated from Fig. 7 and also drawn in Fig. 9. The difference of the coefficient of variation of fatigue life between the two stress levels gives a fairly large effect on $\bar{X}_f$ when the stress sequence from S_L to S_H ;

however, a small effect for that from S_H to S_L . This trend corresponds to that of Fig. 6.

Fig. 7 Damage curves at stress levels S_H and S_L

Fig. 8-(a)

Fig. 8-(b)

Fig. 8 Relation between the population reliability and the cumulative cycle ratio to failure for two-step fatigue tests

Fig. 9 The mean cumulative cycle ratio to failure as a function of the cycle ratio given at the first stress level

[Example 3] Consider two stress levels at which an equal coefficient of variation of fatigue life is given. Let them be S_E with $V_E = 0.5$ and $C_E = 2.0$ and S_G with $V_G = 0.5$ and $C_G = 5.0$. For the log-normal distribution R - X_f curves are shown in Fig. 10.

For the two-parameter Weibull distribution similar reliability curves are obtained. $\bar{X}_f$ - x_1 relations are shown in Fig. 11. The effect of fatigue life scatter on $\bar{X}_f$ appears in this example also. This effect becomes smaller as the scatter of fatigue life decreases.

Fig. 10 Relation between the population reliability and the cumulative cycle ratio to failure for two-step fatigue tests, when an equal coefficient of variation of fatigue life is given at two stress levels

Fig. 11 The mean cumulative cycle ratio to failure as a function of the cycle ratio given at the first stress level

4. DISCUSSION

The results of statistical fatigue tests provide experimental bases to some of the conditions and assumptions used in this paper.

1) As to the first condition, the fatigue life data by Shimokawa and Hamaguchi [10], [11], Tanaka et al. [6], Bastenaire [12], and others show that the coefficient of variation of fatigue life is constant within a certain range of stress levels.

2) As to the first assumption and third condition, Tanaka et al. [7] employed 25 low-carbon-steel plate specimens with a central notch in their constant amplitude fatigue tests at one stress level to study statistical properties of fatigue crack propagation. They concluded that the ascending order of crack length in each of 25 specimens is macroscopically unchanged in crack propagation process, and that crack propagation as a function of n/N_R are represented by a practically single curve independent of specimens, where n and N_R are the number of cycles repeated and the fatigue life of individual specimen respectively. In addition to the above results, Shimokawa and Hamaguchi [10] investigated damage curves of Eq. (9) represented by crack propagation and found that they are drawn by a single curve within a certain range of stress amplitude.

3) As to the second condition (TLRC), Tanaka et al. [6] discussed Miner's rule statistically by using their fatigue test data of large sample sizes and found "the transfer law of reliability curve" (TLRC). Statistical Miner's rule is satisfied in their test results. Another example of TLRC and statistical Miner's rule is found in the authors' analysis of the fatigue data on a structural steel obtained by Konishi and Shinozuka [13].

5. CONCLUDING REMARKS

The conditions required for both the parameters of the log-normal and Weibull distribution functions of fatigue life and the cumulative damage rule to satisfy statistical Miner's rule are derived on the bases of the two conditions found by Tanaka et al. [6].

1) The standard deviation of logarithm of fatigue life for the log-normal distribution, or the shape parameter and the ratio of the location parameter to the mean life for the Weibull distribution are required to

be constant regardless of stress levels.

2) Under two assumptions that the order of damage accumulated in an arbitrary specimen from the population is kept unchanged through any stress history and that the transfer law of damage curve is satisfied under variable stress amplitude loading, a single damage curve is required to be given regardless of stress levels and specimens.

3) The two conditions above lead the transfer law of reliability curve (TLRC). TLRC is clarified to be only one condition required to satisfy statistical Miner's rule.

When the conditions above are not satisfied, the reliability curves and the mean cumulative cycle ratio to failure are discussed for two-step fatigue tests, and calculation examples are given.

4) The difference of fatigue life scatters at two stress levels give a small effect on the mean cumulative cycle ratio to failure when the smaller scatter of fatigue life is associated with the first stress level, and a fairly large effect when the larger scatter of fatigue life is associated with the first stress level.

5) Equal fatigue life scatters at the two stress levels also have an effect on the mean cumulative cycle ratio to failure when damage curves are not independent of stress levels.

ACKNOWLEDGMENTS

The authors wish to express their hearty thanks to Mr. Tadao Kamiyama, a member of the Aircraft Accident Investigation Commission, the Ministry of Transportation, and Dr. Masahiro Ichikawa, Professor of the University of Electro-Communications, for their friendly advice and constructive criticisms.

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**TECHNICAL REPORT OF NATIONAL
AEROSPACE LABORATORY
TR-611T**

航空宇宙技術研究所報告611T号 (欧文)

昭和55年5月発行

発行所 航空宇宙技術研究所
東京都調布市深大寺町1880
電話武蔵野三鷹(0422)47-5911(大代表) ㊦182
印刷所 株式会社 共 進
東京都杉並区久我山5-6-17

**Published by
NATIONAL AEROSPACE LABORATORY
1,880 Jindaiji, Chōfu, Tokyo
JAPAN**