

# Optimization of Low Thrust Trajectories for Collinear Lagrange Point Mission

By

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**Summary:** Two kinds of optimization problems on low thrust transfer trajectories are discussed. One is a transfer from earth parking orbit to Sun-Earth interior libration point  $L_1$ . So-called J-minimum trajectories are obtained for this case. It is shown that, in spite of some injection error, spacecraft can reach  $L_1$  by optimizing low thrust programs. The other problem treated in this paper pertains to a transfer between collinear Lagrange points,  $L_1$  and  $L_2$ , of Sun-Earth system. In this case, two types of low thrust, that is, power-constant and thrust-constant trajectories are studied. In the former case, J-minimum trajectories are obtained, and, in the latter, fuel optimum trajectories with a single coasting arc were sought including optimal steering programs as a function of total mission time. Computational results show that spacecraft can travel between two points within a mission time that stands comparison with high thrust impulsive transfer.

*Keywords:* optimum transfer—low thrust transfer—Lagrange point

## 1. Introduction

Stationing of a scientific satellite in the vicinity of the Sun-Earth interior Lagrange point,  $L_1$  (see Fig. 1), was originally discussed by Farquhar in 1968 [1, 2]. And about one year later, Japan presented an independent review of this concept [3, 4]. In 1978, the ISEE-3 (International Sun-Earth Explorer) spacecraft was launched as a NASA/ESA joint mission, and was placed on so-called halo orbit around the Sun-Earth  $L_1$  point. It was the first satellite around a Lagrange point. The spacecraft attracted public attention by having setting out on a long journey last year so as to encounter Comet Giacobini-Zinner in September 1985.

Returning to the subject, with the advent of a variety of practical proposals with respect to the utilization of Lagrange points, navigation and guidance aspects associated with traveling in the Earth-Moon or the Sun-Earth space, together with the problem of stationkeeping at the Lagrange points, have already received much interest by many researchers [5-12]. The problem in this area will acquire greater importance from a practical point of view.

This paper discusses two kinds of low thrust optimum transfer problems. One is concerning a trajectory connecting the Earth parking orbit (EPO) with the Sun-Earth  $L_1$  point. The other is a transfer from  $L_1$  to  $L_2$ .

Section 2 of this paper reviews a dynamic model used here. As a dynamic

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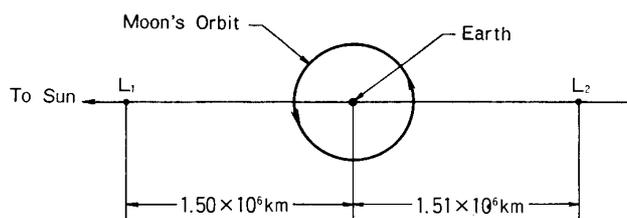


Fig. 1. Sun-Earth Collinear Libration Points

model, we have taken, for the present study, the well-known circular-restricted three-body problem in two dimensions. The Sun and the Earth (the Earth-Moon system, strictly speaking) are the only two acting bodies; they are assumed to have point-masses in circular orbits around each other.

Section 3 describes an above dynamic model as a two-point boundary-value problem.

Shown in Section 4 are the numerical results. Subsection 4.1 discusses a transfer from EPO to  $L_1$ . In this case, so-called J-minimum trajectories are obtained. And for the case where some error of injection velocity or position exists, it is shown to be able to escort a spacecraft to  $L_1$  so long as appropriate thrust program is carried out. In Subsection 4.2, after a preliminary study on the impulsive transfer between  $L_1$  and  $L_2$ , computational results for optimum low thrust transfer between them are shown. In this problem, two types of low thrust (power constant and thrust constant) are examined. Under the assumption of constant power, J-minimum trajectories are obtained. And in case of constant thrust, fuel optimum trajectories with a single coasting arc are sought together with optimal steering programs as a function of total mission time.

From the study on impulsive transfers, two families of trajectories are identified: one with higher  $\Delta V$  and the other with lower  $\Delta V$  requirements. And a similar grouping is made clear to exist for low thrust cases, too.

For both constant power and constant thrust cases, due to weakness of gravitational attractions, optimization will enable spacecraft to travel between  $L_1$  and  $L_2$  within a mission time that can bear comparison with high thrust impulsive case.

## 2. Dynamic Model

The well-known planar circular-restricted three-body problem is employed for the present study, that is, one of the three objects, say  $m_3$ , is supposed to have negligible mass compared with the two others,  $m_1$  and  $m_2$ . In the present context,  $m_1$  will be the Sun,  $m_2$  the Earth, and  $m_3$  the spacecraft. The motion of  $m_1$  and  $m_2$  around their center of mass is supposed to be the circular Keplerian motion. The motion of  $m_3$  is then obtained as a numerical solution to the equations of motion. As was mentioned above,  $m_3$  is assumed to remain in the plane of motion of two primaries,  $m_1$  and  $m_2$ .

Our model corresponds to the Sun-Earth mass ratio

$$m_1/m_2 = 1.9891 \times 10^{30} \text{ kg} / (5.9742 \times 10^{24} + 7.3483 \times 10^{22}) \text{ kg} = 3.2889 \times 10^5 \quad (1)$$

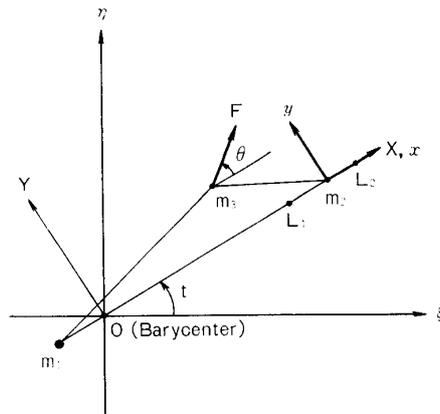


Fig. 2. Coordinate System

If we represent the masses of the Sun and the Earth by  $1-\mu$  and  $\mu$ , respectively, we have

$$\mu = m_2 / (m_1 + m_2) = 1 / (3.2889 \times 10^5) = 3.0404 \times 10^{-6}. \quad (2)$$

Length is nondimensionalized by average Sun-Earth distance ( $1.49598 \times 10^8$  km) and the unit of time is taken in such a way that the period of the Sun and the Earth in their circular motion is exactly  $2\pi$ .

For our problem, it is much convenient to use "synodical" coordinate system  $(xy)$ , which rotates around the center of mass with unit angular velocity, taking the Earth-Moon barycenter as its origin. In this rotating system, depicted in Fig. 2 with relevant parameters, the Sun lies at  $x_1 = -1$  and the Earth at  $x_2 = 0$ , both permanently on the  $x$  axis.

### 2.1. Transfer from EPO to $L_1$

We start from near-Earth circular parking orbit ( $Z_0 = 200$  km). A velocity increment  $\Delta V_0$  is added to spacecraft tangentially to the orbit at some point on the orbit. Then it leaves the parking orbit and, during approximately one day, the motion of the spacecraft can be said to be governed by only the Earth gravitation because of its dominant strength over the gravitational pull of the Sun. During this period, as a convenience, spacecraft is assumed to continue a coasting flight to be solved by two-body problem model. At the position where the coasting is terminated, powered flight using low thrust is commenced.

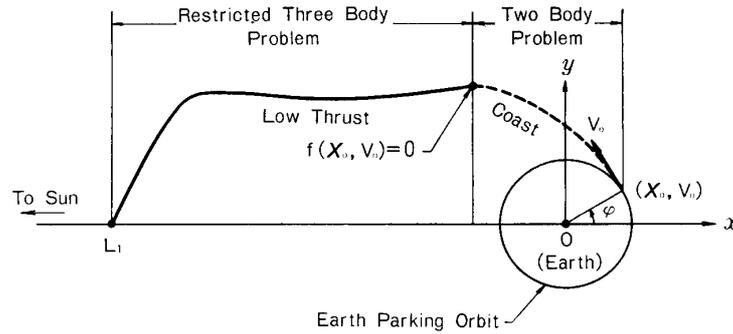
From this time point, dynamic model is replaced by above-mentioned planar circular-restricted three-body problem. In this phase, optimum thrust programs are sought so as to reach  $L_1$  point at rest.

When solving this two-point boundary value problem, initial condition at the time point just one day after starting EPO can be expressed as a function of  $(\mathbf{x}_0, V_0)$ , that is

$$f = f(\mathbf{x}_0, V_0) \quad \text{at } t = 0 \quad (3)$$

where  $\mathbf{x}_0$  denotes a starting position from EPO and spacecraft is supposed to be thrown off with a velocity  $V_0$ .

The transfer problem treated here is illustrated schematically in Fig. 3.

Fig. 3. Transfer Scheme from EPO to  $L_1$ .

Referring to Fig. 2, the equations of motion are described as follows:

$$\ddot{x} - 2\dot{y} = x + 1 - \mu - V_{px} + a \cos \theta, \quad \ddot{y} + 2\dot{x} = y - V_{py} + a \sin \theta \quad (4)$$

where

$$V_p = -\frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \quad V_{px} = \frac{\partial V_p}{\partial x}, \quad V_{py} = \frac{\partial V_p}{\partial y}, \quad a = F/m_3.$$

From the nature of the transfer trajectory, mission time for powered flight is confined to approximately 100~120 days. We specified 110 days as the powered arc duration. Accordingly, total mission time becomes 111 days including a coasting time (one day).

### 2.2. Transfer between $L_1$ and $L_2$

The second study of our optimization problems is related to a coplanar rest-to-rest transfer between  $L_1$  and  $L_2$  points. Leaving  $L_1$  from a stationary state, spacecraft is transferred to  $L_2$  to stand still. A dynamic model of a planar circular-restricted three-body problem is used.

The coordinate system shown in Fig. 2 are adopted in this transfer again, and the equations of motion are expressed as the form identical with Eq. (4).

## 3. Trajectory Optimization

Two point boundary value problems hitherto described are solved by generalized Newton-Raphson method [13]. The initial assumptions of Lagrangian multipliers time history required to start the iteration are obtained through solving an auxiliary minimization problem [14].

### 3.1. Transfer from EPO to $L_1$

The purpose of the this problem is to study the minimization of the functional

$$J = \int_0^r [a(t)]^2 dt \quad (5)$$

with respect to the state  $(x, y, \dot{x}, \dot{y})$  and the control  $(\theta, a)$  which satisfy the differential constraint (4) and the following boundary conditions

$$f(\mathbf{x}_0, V_0) = 0 \quad \text{at } t = 0 \quad (6)$$

$$\left. \begin{aligned} x_f &= -0.010011 \\ y_f &= 0 \\ \dot{x}_f &= \dot{y}_f = 0. \end{aligned} \right\} \quad \text{at } t = \tau \quad (7)$$

In this problem, transfer time  $\tau$  is fixed to be 110 days and initial acceleration is parametrically varied.

In order to start the generalized Newton-Raphson method, following nominal trajectory is chosen:

$$\begin{aligned} x &= k_x \sin(2\pi/\tau) + x_0(1-t/\tau) + x_f t/\tau, \\ y &= k_y \sin(\pi/\tau) + y_0(1-t/\tau). \end{aligned} \quad (8)$$

The nominal trajectory can be altered by changing the values of  $k_x$  and  $k_y$ .

### 3.2. Transfer between $L_1$ and $L_2$

#### *Impulsive Transfer*

The value of  $a$  in Eq. (4) is essentially zero, and starting and stopping impulses are applied at both ends. Given a mission time, transfer trajectory is determined almost uniquely (as will be mentioned later, two kinds of trajectories exist).

#### *Low Thrust Transfer*

Two types of thrusting mechanisms are considered, one constant thrust and the other constant power system.

In the former, steering history  $\theta(t)$  and coasting duration should be optimized so as to give minimum transfer time with respect to the following nondimensional boundary conditions:

$$x_0 = -0.010011, \quad x_f = 0.010078, \quad y_0 = y_f = 0. \quad (9)$$

In our analysis, it is assumed that only one coasting arc is included.

In the transfer with constant power system, on the other hand, so-called J-minimum problem results and steering,  $\theta(t)$ , and acceleration,  $a(t)$ , should be optimized in terms of Eq. (9).

To begin iteration steps to solve above two-point boundary value problems using generalized Newton-Raphson method, nominal trajectory is described as

$$x = x_0 + (x_f - x_0)[3(t/\tau)^2 - 2(t/\tau)^3], \quad y = 16y_{\max}(t/\tau)^2(1/\tau - 1)^2 \quad (10)$$

The trajectory given above is so constructed as to satisfy boundary conditions in Eq. (9) and to make a maximum excursion of  $y_{\max}$  in  $y$  direction at  $t = \tau/2$ .

## 4. Numerical Results

### 4.1. Transfer from EPO to $L_1$

Injection scheme from the initial point of the two-point boundary value problem is shown in Fig. 4. The starting point  $\mathbf{x}_0 = (x_0, y_0)$  from EPO is expressed by

$$x_0 = r_0 \cos \varphi_0, \quad y_0 = r_0 \sin \varphi_0, \quad (11)$$

where  $r_0$  is the distance of starting point from the center of the Earth and its altitude  $Z_0$  from the Earth surface is specified as  $Z_0=200$  km here, that is,

$$r_0 = 6378 + 200 = 6578 \text{ (km)}. \quad (12)$$

And injection is assumed to be tangential to EPO with the velocity of

$$V_0 = 10.95 \text{ (km/s)} \quad (13)$$

One day after the spacecraft is thrown off from EPO, the situation of the spacecraft is described to be

$$f(x_0, V_0) = 0. \quad (14)$$

Then the J-minimum problem to  $L_1$  mentioned in the previous section was solved. The results are shown in Fig. 5, on which two-impulse trajectory is superimposed. As described in Section 3, transfer time is fixed to be  $\tau=110$  days and, adding one day coasting from EPO, total mission time is 111 days. Arrows in Fig. 5 represent thrust direction at each time point.

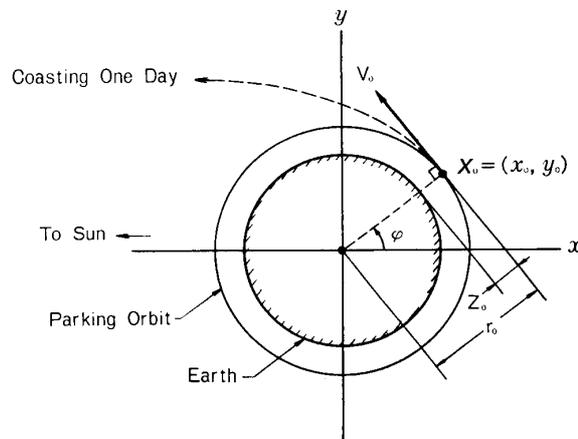


Fig. 4. Injection Scheme from EPO.

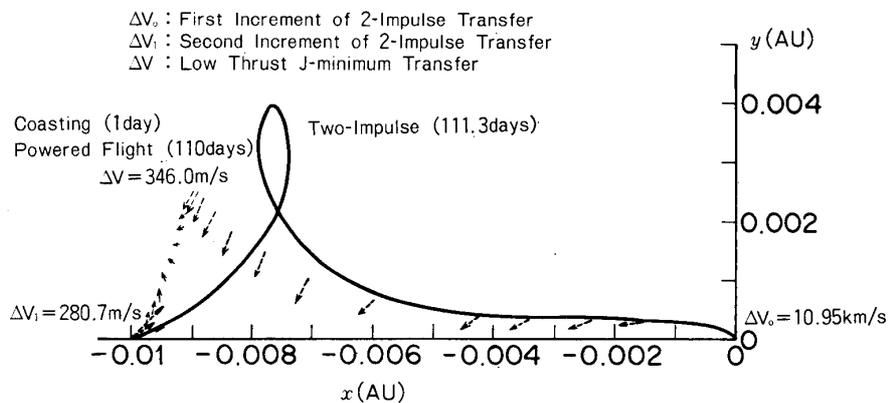


Fig. 5. J-minimum Transfer Trajectory and Two-Impulse Transfer

Figure 6 shows the trajectories when the injection error,  $\delta V$ , exists with respect to the velocity magnitude. The trajectories in Fig. 6 are drawn from one day after injection from EPO, that is, from the time when the correction of low thrust began. Potential near the Earth is so sensitive that the trajectory for  $\delta V=30\text{m/s}$  effects so different from that of  $\delta V=-20\text{m/s}$ .

For each in Fig. 6, acceleration time history is illustrated in Fig. 7, followed by Fig. 8 where the values of J-minimum are given.

Thus the spacecraft has arrived at  $L_1$ . Now the transfer to  $L_2$  point is discussed in the subsequent subsection.

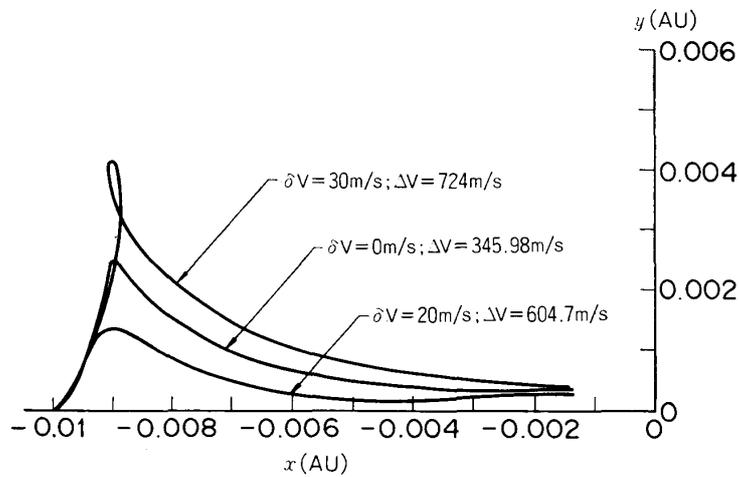


Fig. 6. Transfer Trajectory to  $L_1$ .

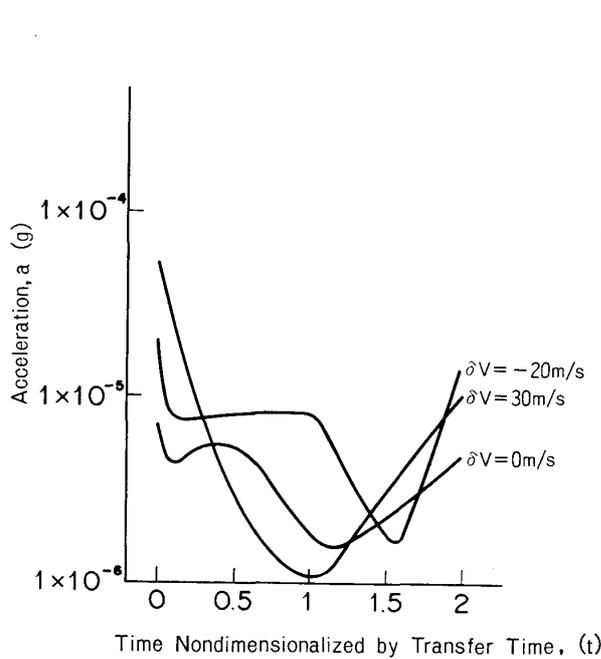


Fig. 7. Acceleration time history.

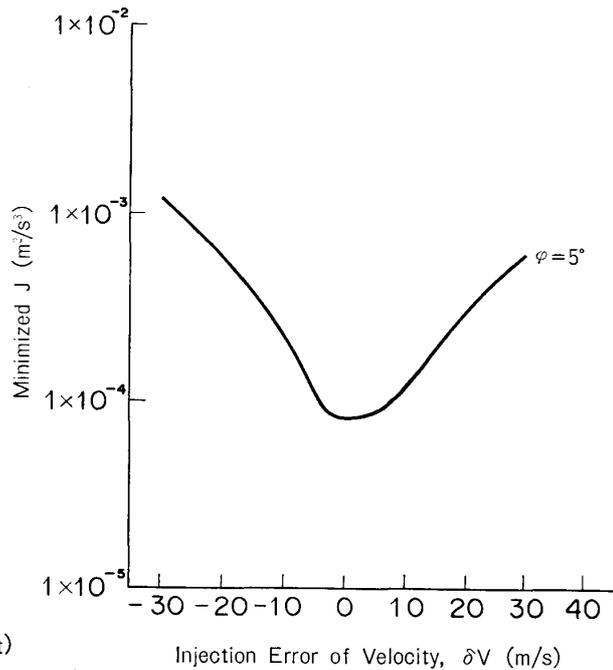


Fig. 8. Minimized  $J$  vs. injection error of velocity.

4.2. Transfer between  $L_1$  and  $L_2$

To get insight into general feature of this mission, two-impulse transfer was investigated in the first place. As shown in Fig. 9, two families of trajectories were identified as far as we could find. One flies retrograde with respect to the Sun-Earth rotation (A-type) and the other posigrade (B-type).

For reference, all-propulsion transfer trajectories are shown in Fig. 10 for some selected initial accelerations together with arrows representing thrust direction in case of  $a_0 = 1.0 \times 10^{-5} \text{ g}$ .

Constant Low Thrust Transfer

Optimization of constant low thrust rest-to-rest transfer between  $L_1$  and  $L_2$  was solved in terms of some selected initial acceleration values. Specific impulse of 5000 sec was assumed through this study.

A minimum time transfer problem without coasting was solved first.

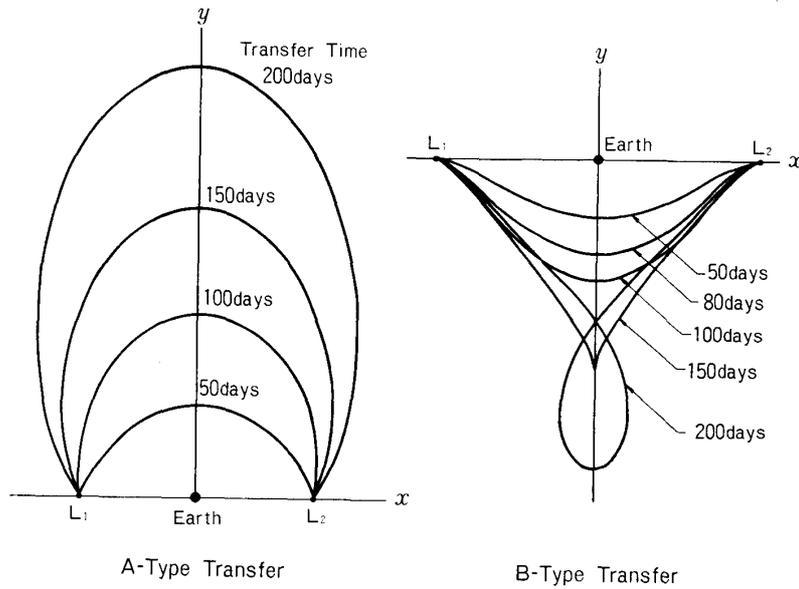


Fig. 9. Two-impulse transfer between  $L_1$  and  $L_2$ .

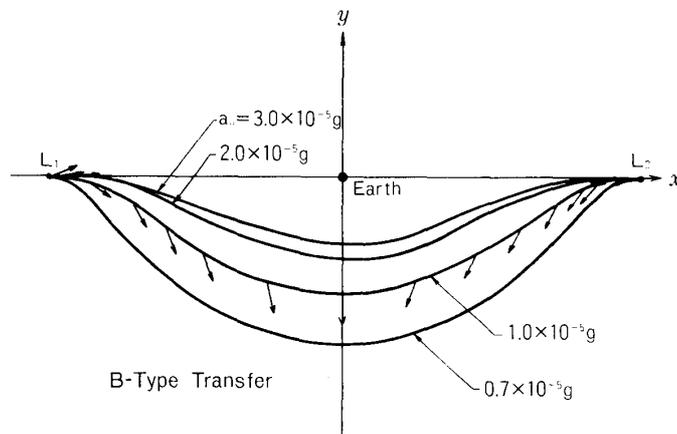
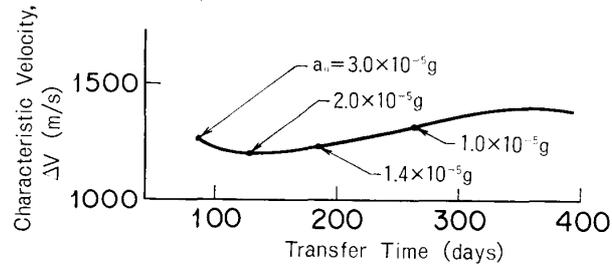
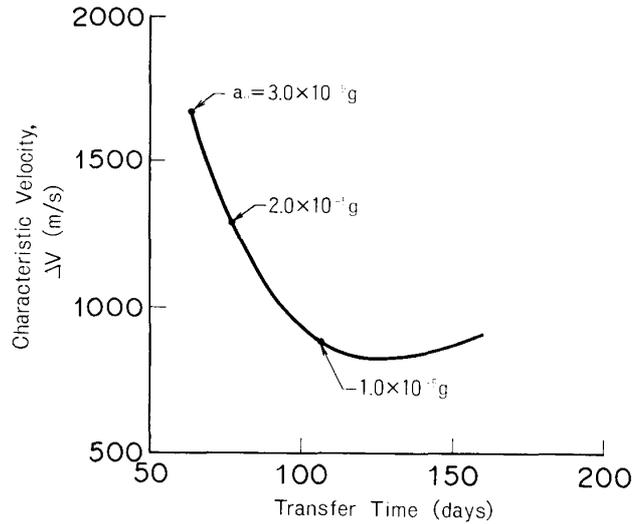


Fig. 10. Typical trajectories of all-propulsion transfer.

As was anticipated from the results of impulsive transfer, similar kind of grouping also exists for the case of low thrust transfer. Figure 11 gives the required characteristic velocity for A- and B-types transfers. Every point on the curves for 'all-propulsion' gives a minimum transfer time for a specified initial acceleration level.



(a) A-Type Transfer



(b) B-Type Transfer

Fig. 11. Characteristic velocity.

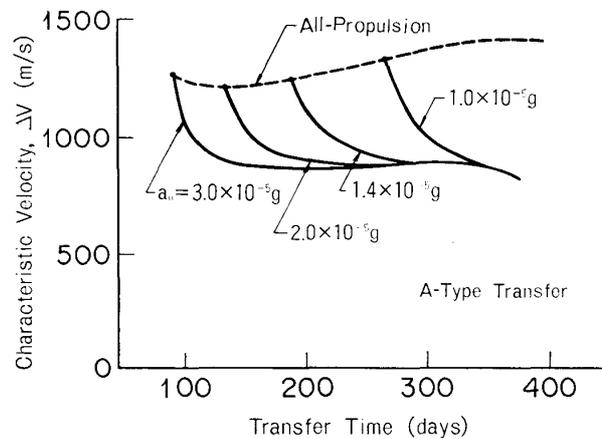


Fig. 12. Characteristic velocity of constant thrust transfer.

For this all-propulsion mode, B-type transfer is advantageous over A-type in terms of characteristic velocity,  $\Delta V$ .

If transfer time is allowed to be longer than that of all-propulsion mode, some coasting arcs can be introduced and required characteristic velocity would be able to reduce as compared with that for all-propulsion transfer. Here the cases where only one coasting arc is included in between two thrusting arcs were investigated. The results are shown in Figs. 12 and 13 for A- and B-types, respectively. In the figures, solid lines, branched from all-propulsion curves, show the solutions of constant thrust transfer with an intermediate coasting period. Comparison of

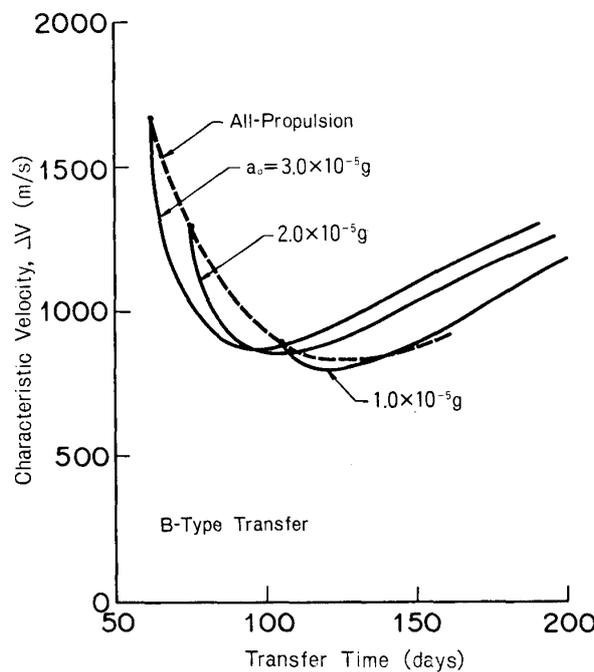


Fig. 13. Characteristic velocity of constant thrust transfer.

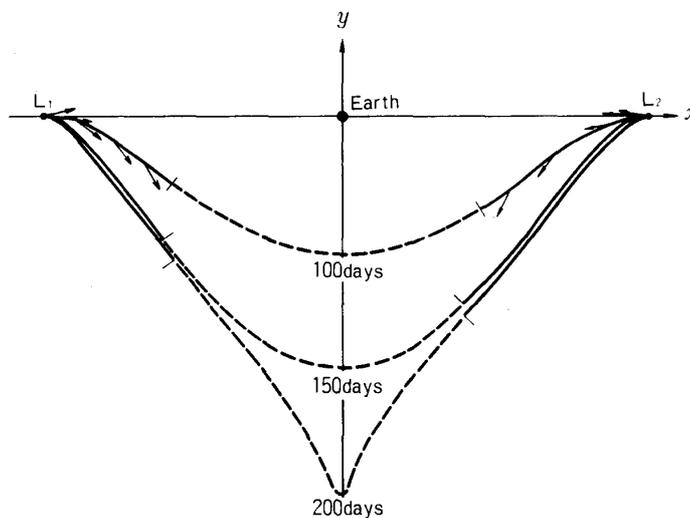


Fig. 14. Typical trajectories of constant low thrust transfer.

these two figures tells that, here again, B-type transfer is advantageous than A-type with respect to  $\Delta V$ . In average, the former requires 1000 m/s of  $\Delta V$ , while the latter requires 2000 m/s. Taking account of specific impulse of 5000 sec used here, even 2000 m/s of  $\Delta V$  means only 4% of fuel consumption. Accordingly, constant thrust solutions obtained here could be essentially considered as constant acceleration solutions.

Figure 14 shows typical trajectories of B-type, where broken lines correspond to the coasting period and arrows on 100 days line denote the thrust direction.

*Constant Power Transfer*

In this case, J-minimum transfer problem was solved in terms of transfer time  $\tau$ . Figure 15 gives minimized value of J as a function of transfer time,  $\tau$ . Typical time histories of acceleration magnitude is illustrated in Fig. 16 for only B-type transfer.

*Impulsive Transfer*

In Figs. 17 and 18 are summed the required characteristic velocities for low-

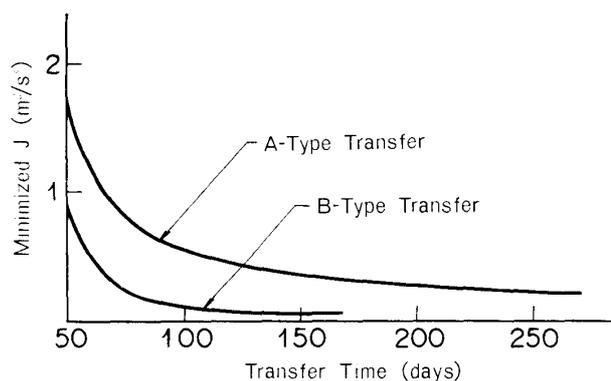


Fig. 15. J-minimum transfer.

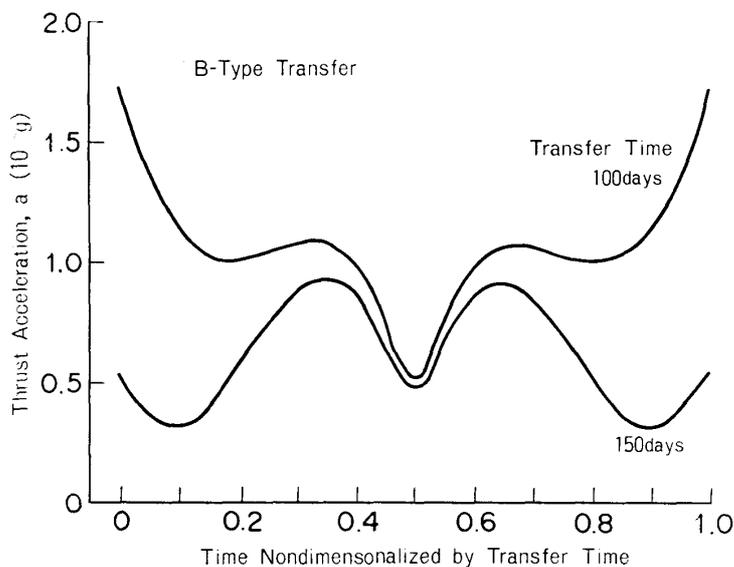


Fig. 16. Thrust acceleration time history for J-minimum transfer.

thrust transfer between  $L_1$  and  $L_2$  together with those for two-impulse transfer. As shown in these figures, for a given mission time,  $\Delta V$  requirement for two impulse transfer goes beyond that for low thrust transfer in the longer transfer time missions. This is due to the non-optimality of two-impulse transfer. To check the optimality, Lawden's primer histories were calculated for A- and B-type two-impulse transfers to be shown in Figs. 19 and 20, respectively. Should be the transfer be optimal at least locally, its magnitude should stay below unity throughout the mission time [15]. Figure 20, for example, tells that the two-impulse transfers, except for the 50-day mission, are not optimal even in a local

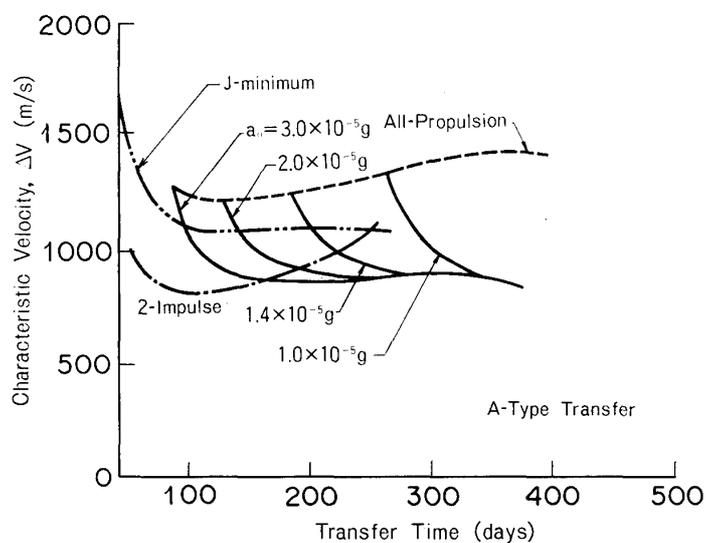


Fig. 17. Characteristic velocity of transfer between  $L_1$  and  $L_2$ .

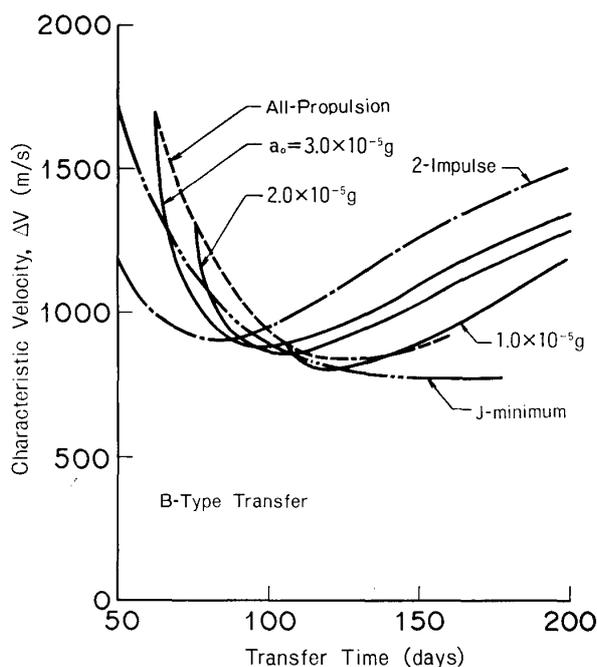


Fig. 18. Characteristic velocity of transfer between  $L_1$  and  $L_2$ .

sense and that application of two more impulses, possibly one near the departure point, the other near the arrival point, would reduce the total  $\Delta V$  requirement. The greatest improvement in  $\Delta V$ , to first order, can be realized by applying above midcourse impulses at the times the primer magnitude reaches its local maximum.

Thus four-impulse transfer between  $L_1$  and  $L_2$  was optimized for B-type and the resulting trajectory is shown in Fig. 21. In Fig. 22, computational results for impulsive transfer are shown. In case of four-impulse 100-day transfer of B-type, for example, it requires 760 m/s of  $\Delta V$  and a considerable saving in  $\Delta V$  resulted as compared with 946 m/s of  $\Delta V$  for two-impulse transfer (473 m/s at departure and 473 m/s at arrival). For A-type transfer, only one solution for three-impulse transfer is plotted in Fig. 21.

All these results strongly suggest that, also, in low thrust transfer, multiple coasting arc solution would be more economical than single coasting arc solution

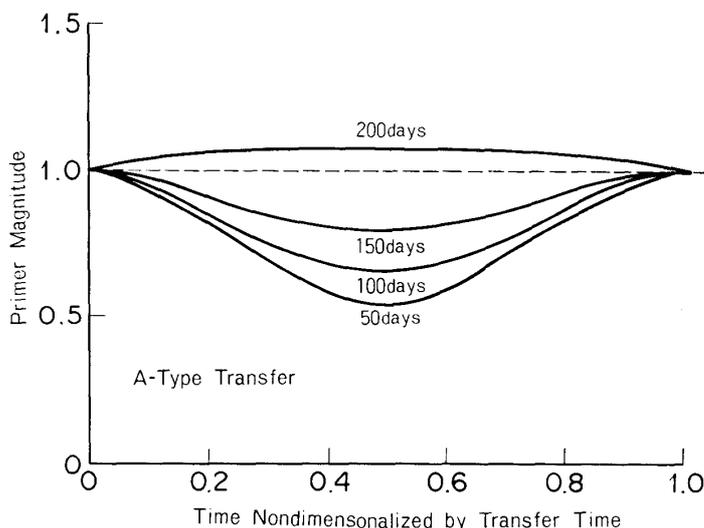


Fig. 19. Primer histories of two-impulse transfer.

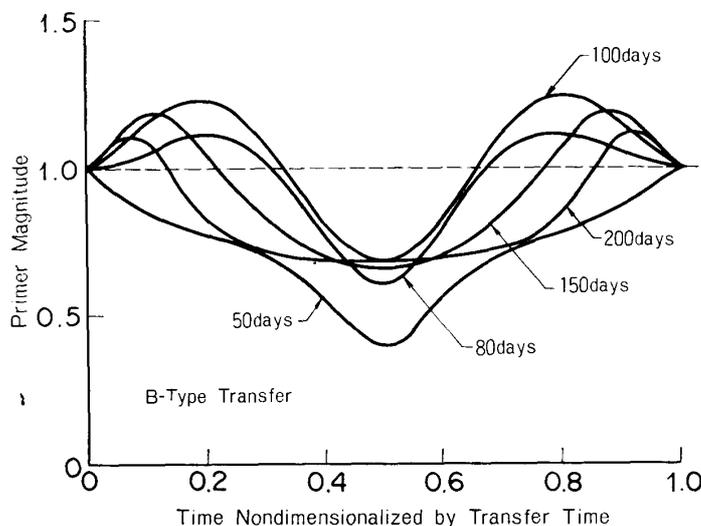


Fig. 20. Primer histories of two-impulse transfer.

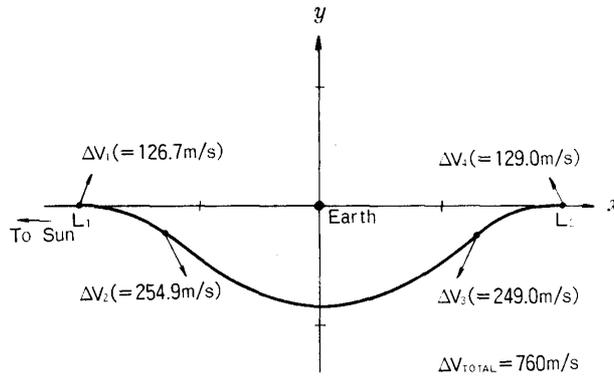


Fig. 21. Trajectory of 4-impulse transfer (100 days).

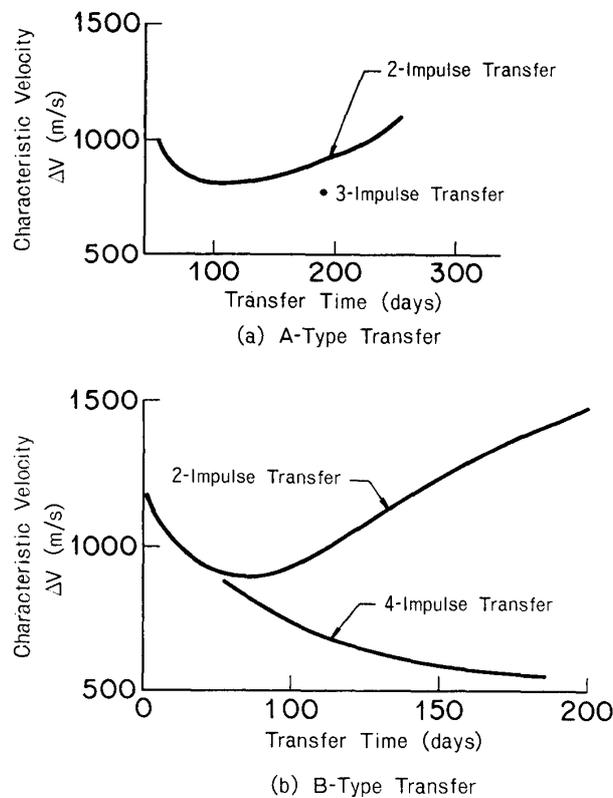


Fig. 22. Characteristic velocity of impulsive transfer.

especially in the longer transfer time missions. This problem is left for further studies and investigations.

## 5. Conclusions

Two kinds of optimum transfer problems were studied.

- (1) For transfer from EPO to  $L_1$ , J-minimum trajectory was obtained. Required velocity increment,  $\Delta V$ , was evaluated for the case where some error exists with respect to injection velocity from EPO.
- (2) The  $\Delta V$  required to transfer from  $L_1$  to  $L_2$  was evaluated.

Two transfer modes were found both in the impulsive and the low thrust transfers, one retrograde and the other posigrade. In general, the latter is more economical than the former.

In the two-impulse, the optimum transfer time is about 70 to 100 days.

For the constant thrust case, minimum transfer time was obtained as a function of the initial acceleration.

By permitting an intermediate coasting period,  $\Delta V$  can be considerably reduced.

For the constant power case, J-minimum trajectories were calculated and  $J$  was obtained as a function of the mission time. It is of a universal nature as far as this mission is concerned.

When longer transfer time is permitted, three or four impulse transfer becomes more economical than the two-impulse one. This suggests the superiority of the multi-coasting-arc transfer in the constant thrust case.

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