A Kinetic Theory Analysis of Transient Evaporation Problem

By

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Summary: A transient evaporation from a condensed phase surface was studied based upon a kinetic model equation that yields correct Prandtl number Pr. The kinetic equation was reduced to a set of linear differential equations with the aid of half-range Hermite polynomials and the resultant equations were solved by use of the Laplace transformation. A quasisteady solution for $t \gg 1$ was obtained and the results showed that the evaporation rate and the uniform flow behind the evaporation wave or adjacent to the Knudsen layer were little dependent upon Pr but the behavior of the dissipative wave (contact surface) was rightly dependent upon Pr. The evaporation rate obtained showed a good agreement with the one in the previous paper.

1. INTRODUCTION

When a condensed phase surface at temperature T_s and corresponding saturated vapor pressure p_s is contact with its vapor at pressure p_u and temperature T_u an evaporation from or a condensation onto the condensed phase surface takes place. A kinetic theory analysis of transient evaporation (condensation) from (onto) a liquid surface was first done by Shankar and Marble [*1*] applying a moment method to the Boltzmann equation. They found that an evaporation wave (or expansion wave) propagated in the vapor and after the wave proceeded far away from the surface a quasisteady vapor motion** took place in the vicinity of the condensed phase surface.

Murakami and Oshima [2] carried out a Monte Carlo simulation to the Boltzmann equation for arbitrary values of p_s/p_u. Obtained transient behavior of vapor had similar features to those of shock tube problem. Present author [*3*] treated the problem as a quesisteady one with the aid of shock tube relation and jump conditions derived from an entropy balance relation and the obtained results showed a good agreement with those of Monte Carlo simulation.

If the evaporation is weak, i.e., Mach number M_∞ of quasisteady flow is small ($M_\infty < 1$), the results of Monte Carlo simulation showed innegligible scatterings atributed to the insufficient simulation time. These shortcomings of the previous papers are the main reason why the author treats the problem in this paper. A linearized hierarchy kinetic model equation [*4*] which resembles the Boltzmann equation correctly within 13 moments is reduced to a steady state equation with the aid of Laplace transformation and the resultant equation is solved using the half-range Hermite polynomials [*5*]. Analytical forms of the solution can be obtained for two limits, $s \gg 1$ and $1/s \gg 1$, where s is the

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** Shankar *et al*. did not find any quasisteady flow adjacent to the Kunudsen layer for the evaporation problem.

variable in the image function: An initial stage of transient evaporation may be corresponding to the solution for $1/s \gg 1$ and a solution for $s \ll 1$ must be corresponding to the solution for $t \ll 1$ (t is the time after the evaporation started up).

Since the solution for $t \gg 1$ must be physically interesting, the solution for $s \ll 1$ and the original function of it are obtained. The results will reveal the features of the transient evaporation and will show the differences between the hierarchy kinetic model equation and the BGK [6] model equation.

2. Formulation of the Problem

2.1 Basic Equation

We consider a transient evaporation from a plane surface of liquid which occupies $x<0$ and is kept at temperature T_s (corresponding pressure is p_s) is contact with ambient vapor which occupies $x>0$ and is kept at T_u (corresponding pressure is given by p_u); $T_u = T_s(1+\Delta t_u)$ and $p_u = p_s(1+\Delta p_u)$.

Due to the pressure difference $p_s \Delta p_u$ maintained between the liquid surface and ambient vapor, mass* and energy fluxes take place from the liquid surface (Fig. 1). If

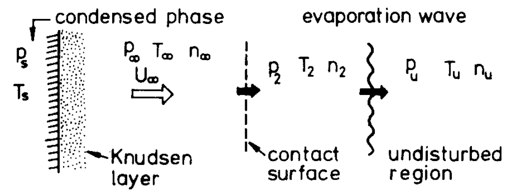

Fig. 1. Schematic drawing of evaporation from a plane surface.

Δp_u is small, i.e., the mass and heat fluxes are small, the phenomenon can be described by a linearized version of the Boltzmann equation. Instead of the linearized Boltzmann equation, for simplicity, a linearized kinetic model equation [4] which correctly resembles the Boltzmann equation within 13 moments so that it may yield the correct Prandtl number Pr is applied to the present analysis.

In the one-dimensional problem the kinetic model equation is given by

$$\frac{\partial \phi}{\partial t} + c_x \frac{\partial \phi}{\partial x} = N + 2c_x U + \tau\left(\boldsymbol{c}^2 - \frac{3}{2}\right) + \frac{3}{2}\left(1 - \frac{1}{\beta}\right) p_{xx}\left(c_x^2 - \frac{1}{3}\boldsymbol{c}^2\right)$$
$$+ \frac{4}{5}\left(1 - \frac{Pr}{\beta}\right) q_x c_x \left(\boldsymbol{c}^2 - \frac{5}{2}\right) - \phi, \tag{1}$$

where $f_s(1+\phi)$ is the distribution function with

$$f_s = \rho_s (2\pi R T_s)^{-3/2} \exp(-\boldsymbol{c}^2),$$

* If $\dot{m}<0$, it implies condensation onto the condensed phase surface.

N the perturbed density defined by $\rho=\rho_s(1+N)$, $c_{ms}U$ the flow velocity with $c_{ms}=(2RT_s)^{1/2}$, R the gas constant, τ the perturbed temperature defined by $T=T_s(1+\tau)$, $\rho_s RT_s p_{xx}$ the shear stress, $\rho_s c_{ms}{}^3 q_x/2$ the heat flux, and $c_{ms}c_x$, $c_{ms}c_y$, and $c_{ms}c_z$, are the x, y, and z components of the molecular velocity $c_{ms}\mathbf{c}$; $\mathbf{c}^2=c_x{}^2+c_y{}^2+c_z{}^2$. In Eq. (1) the distance x is nondimensionalized by $l_s\beta$ where l_s is the reduced mean free path defined by

$$l_s=\frac{\mu_s}{\rho_s}\left(\frac{\pi}{RT_s}\right)^{1/2},$$

μ the viscosity, and β the scaling factor of the collision frequency and the time t by $l_s/\beta c_{ms}$.

The perturbed values are given by

$$\begin{bmatrix} N \\ U \\ \tau \\ p_{xx} \\ q_x \end{bmatrix}=\pi^{-3/2}\iiint_{-\infty}^{\infty}\begin{bmatrix} 1 \\ c_x \\ \frac{2}{3}\mathbf{c}^2-1 \\ 2c_x^2-\frac{2}{3}\mathbf{c}^2 \\ c_x\mathbf{c}^2-\frac{5}{2}c_x \end{bmatrix}\phi\exp(-\mathbf{c}^2)d\mathbf{c},\quad p=N+\tau. \tag{2}$$

If the evaporation coefficient $\sigma_e=1$, the distribution function of the emitted molecules is given by [7]

$$\phi(t,\, x,\, c_x>0)=0. \tag{3a}$$

Since the ambient vapor at $x\to\infty$ is at rest and in thermal equilibrium, the perturbed distribution function at $x\to\infty$ can be given by

$$\phi(t,\, \infty,\, \mathbf{c})=\Delta N_u+\Delta t_u(\mathbf{c}^2-\tfrac{3}{2});\ \Delta N_u=\Delta p_u-\Delta t_u. \tag{3b}$$

Conveniently we introduce half-range distribution functions $\phi_0{}^{\pm}(t,\ x,\ c_x\lesseqgtr 0)$ and $\phi_2{}^{\pm}(t,\ x,\ c_x\lesseqgtr 0)$;

$$\begin{pmatrix}\phi_0^{\pm}\\ \phi_2^{\pm}\end{pmatrix}=\pi^{-1}\iint_{-\infty}^{\infty}\begin{pmatrix}1\\ c_y^2+c_z^2-1\end{pmatrix}\phi\exp(-c_y^2-c_z^2)dc_y dc_z, \tag{4}$$

and expand them with the aid of half-range Hermite polynomials $H_h(\eta)$ [5] as

$$\begin{pmatrix}\phi_0^{\pm}\\ \phi_2^{\pm}\end{pmatrix}=\sum_{k=1}^{n}H_k(\eta)\begin{pmatrix}a_k^{\pm}\\ b_k^{\pm}\end{pmatrix};\quad \begin{pmatrix}a_k^{\pm}\\ b_k^{\pm}\end{pmatrix}=\int_0^{\infty}H_k(\eta)\begin{pmatrix}\phi_0^{\pm}\\ \phi_2^{\pm}\end{pmatrix}\exp(-\eta^2)d\eta, \tag{5}$$

where $\eta=|c_x|$.

Substituting the expansion form (5) into Eq. (1) and using the orthonormal relation of H_k [5], we obtain

$$\frac{\partial X}{\partial t}+\mathbf{M}\frac{\partial X}{\partial x}=\Lambda X; \tag{6}$$

$$X=(a_1^+, a_2^+, \cdots, a_n^+, b_1^+, b_2^+, \cdots, b_n^+, a_1^-, a_2^-, \cdots, a_n^-, b_1^-, b_2^-, \cdots, b_n^-)^t,$$

where $\mathbf{M}$ is a constant matrix [5], the matrix Λ is obtained from the righthand side of Eq. (1) and the superscript t denotes the transpose of a vector or a matrix.

In terms of vector X the boundry conditions (3) yield

$$X^+(t, 0)=\mathbf{0}, \tag{7a}$$

$$X(\infty)=X_u=X_1\varDelta N_u+X_3\varDelta t_u, \tag{7b}$$

where X_1 and X_2 are the vectorial forms of 1 and $c^2-3/2$, respectively.

2.2 Laplace Transformation of the Basic Equation

Introducing the Laplace transformation

$$\tilde{X}=\int_0^\infty X\exp(-st)dt.$$

to Eqs. (6) and (7), we obtain

$$\frac{d\tilde{X}}{dx}=(\Gamma-s\Gamma_1)\tilde{X}+\mathbf{M}^{-1}X_u; \quad \Gamma=\mathbf{M}^{-1}\Lambda, \quad \Gamma_1=\mathbf{M}^{-1}, \tag{8}$$

$$\tilde{X}^+(0)=\mathbf{0}, \tag{9a}$$

$$\tilde{X}(\infty)=\frac{1}{s}X_u. \tag{9b}$$

A solution of Eq. (8) is formally obtained as*

$$\tilde{X}=\sum_{k=1}^{2n}U_k(s)\exp[-\gamma_i(s)x]p_k(s)+\frac{1}{s}X_u, \tag{10}$$

where $-\gamma_i(s)$ is the eigenvalue of the characteristic equation

$$|\Gamma-s\Gamma_1-\gamma\mathbf{I}|=\prod_{k=1}^{2n}(\gamma^2-\gamma_i^2)=0, \tag{11}$$

and U_k is the eigenvector corresponding to the eigenvalue $-\gamma_k(s)$.

If we choose the unknown parameters $p_k(s)$ so that X may satisfy the boundary conditions (9a-b), a general solution to Eq. (6) is given by

* In the solution (10), terms including positive eigenvalues should be omitted because these terms express waves propagating into the condensed phase.

$$\boldsymbol{X}=\frac{1}{2\pi i}\sum_{k=1}^{2n}\int_{\sigma-i\infty}^{\sigma+i\infty}U_k(s)\exp\left[-\gamma_i(s)\right]p_k(s)\exp(st)ds, \tag{12}$$

where σ is a constant ($\sigma>1$).

In order to know the behavior of the solution $\tilde{\boldsymbol{X}}$ we first obtain the eigenvalues as the function of s and the results are shown in Fig. 2 (see also Table 1). In the two limited cases Eq. (11) can be rewritten as follows:

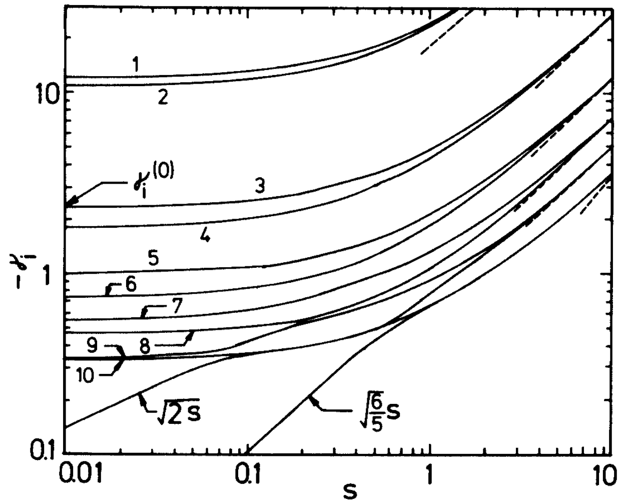

Fig. 2. Loci of $\gamma_i(s)$ (<0) for $\beta=Pr=1$ and $n=6$; --- asymptotic value of $\gamma_i(s)$ for $s\gg1$.

Table 1. Values of $\gamma_i^{(0)}$, $\gamma_i^{(1)}$, and $\gamma_i(\infty)$ for $\beta=Pr=1$ and $n=6$; $\gamma_{i+2n-2}^{(k)}=-\gamma_i^{(k)}$ ($1\leqq i\leqq 2n-2$, $k=0, 1$), $\gamma_i^{(k)}=0$ ($4n-3\leqq i\leqq 4n$, $k=0, 1$).

i	$-\gamma_i^{(0)}$	$-\gamma_i^{(1)}$	$-(\gamma_i/s)_\infty$
1	12.00246	12.73204	12.72368
2	10.75151	12.81236	12.72368
3	2.28586	2.58687	2.58593
4	1.74974	2.61480	2.58593
5	0.97746	1.16228	1.15427
	[1/sound speed $=(6/5)^{1/2}=1.09545$]		
6	0.72083	0.92699	1.15427
7	0.53769	0.77479	0.68234
8	0.42120	0.47133	0.68234
9	0.33504	0.39642	0.46030
10	0.32930	0.32939	0.46030
11	.....	.....	0.32932
12	.....	.....	0.32932

$$\text{i)}\quad 1/s\ll1;\quad |\Gamma-s\Gamma_1-\gamma\mathbf{I}|\doteqdot\prod_{k=1}^{n}(\gamma^4-s^4\gamma_{k\infty})=0, \tag{13a}$$

$$\text{ii)}\quad s\ll1;\quad |\Gamma-s\Gamma_1-\gamma\mathbf{I}|\doteqdot(\gamma^2-\tfrac{6}{5}s^2)(\gamma^2-2s)\prod_{k=1}^{2n-2}(\gamma^2-\gamma_i^{(0)2})=0, \tag{13b}$$

where the subscript 0 and ∞ denote the values at $s=0$ and $s\to\infty$, respectively (see Table 1). Thus the solution for $1/s\ll1$ indicates free molecular flow solution and the solution for $s\ll1$ includes two different waves. If necessary, the solution for intermediate values of s in Eq. (12) can be integrated numerically and such result may converge to the analytical solution for $t\gg1$ as t increases.

3. Solution for $s\gg1$ ($t\gg1$)

In this section we seek a solution of $\tilde{X}$ for $s\ll1$, i.e., a solution of $\tilde{\boldsymbol{X}}$ for $t\gg1$. Let introduce a column vector $\tilde{Y}$ defined by

$$\tilde{\boldsymbol{X}}=(\mathbf{U}_0+s\mathbf{U}_1)\tilde{\boldsymbol{Y}}, \tag{14}$$

where $\mathbf{U}_0$ and $\mathbf{U}_1$ are the matrices of size $4n$. Substituting Eq. (14) into Eq. (8) and making a suitable choice for $\mathbf{U}_0$ and $\mathbf{U}_1$ (See Appendix A), we obtain

$$\frac{d\tilde{\boldsymbol{Y}}}{dx}=\left[\begin{pmatrix}\tilde{\Gamma} & \mathbf{0}\\ \mathbf{0} & \mathbf{W}\end{pmatrix}+O(s^2)\right]\tilde{\boldsymbol{Y}}, \tag{15}$$

where $\tilde{\Gamma}$ is the diagonal matrix of order $4n-4$ and $\boldsymbol{W}$ is the square matrix of size 4 [See Eq. (18)].

Thus, if we take into account the solution up to the order of s, Eq. (15) can be solved by dividing $\tilde{\boldsymbol{Y}}$ into $\tilde{\boldsymbol{Y}}_{4n-4}=(y_1, y_2, y_3,\dots, y_{4n-4})^t$ and $\tilde{\boldsymbol{Y}}_4=(y_{4n-3}, y_{4n-2}, y_{4n-1}, y_{4n})^t$.

A solution of $\tilde{\boldsymbol{Y}}_{4n-4}$ is obtained as (See Appendix A)

$$\tilde{\boldsymbol{Y}}_{4n-4}=\{\delta_{ij}\exp[-(\gamma_k^{(0)}+S\gamma_k^{(1)})x]\}\boldsymbol{p};\quad \boldsymbol{p}=(p_1, p_2, \cdots, p_{4n-4})^t; \tag{16}$$

A comparison of Eq. (15) with that of steady state problem [5], we find

$$\tilde{\boldsymbol{Y}}_4=(\tilde{N}, \tilde{U}, \tilde{\tau}, \tilde{\kappa})^t;\quad \tilde{\kappa}=\tfrac{4}{5}Pr\tilde{q}_x, \tag{17}$$

and Eq. (15) yields

$$\frac{d\tilde{\boldsymbol{Y}}_4}{dx}=\begin{bmatrix}0 & -2s & 0 & 1+Pr^{-1}s\\ -s & 0 & 0 & 0\\ 0 & 0 & 0 & -1-Pr^{-1}s\\ \tfrac{4}{5}Prs & 0 & -\tfrac{6}{5}Prs & 0\end{bmatrix}\tilde{\boldsymbol{Y}}_4. \tag{18}$$

The characteristic equation of Eq. (18) for $s\ll1$ yields following eigenvalues:

$$\pm\sqrt{6/5}\,s,\quad \pm\sqrt{2Prs}.$$

The eigenvalues with negative sign are corresponding to the phenomena in the vapor phase ($x>0$). Thus, a physically interesting solution of Eq. (18) yields

$$\tilde{Y}_4=\alpha_1 W_1 \exp(-\sqrt{6/5}\, sx)+\alpha_2 W_2 \exp(-\sqrt{2Prs}\, x), \tag{19}$$

where α_1 and α_2 are Pr arbitrary parameters and eigenvectors W_1 and W_2 corresponding to $-(6/5)^{1/2}s$ and $-(2Prs)^{1/2}$ are given by*

$$\begin{aligned} W_1&=\begin{bmatrix} 3/2 \\ \sqrt{15/8} \\ 1 \\ 0 \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\sqrt{6/5}\, s+O(s^{3/2}), \\ W_2&=\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}+\begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}\sqrt{s/2Pr}+\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}\frac{s}{Pr}+O(s^{3/2}). \end{aligned} \tag{20}$$

As a consequence Eq. (10) is written as

$$\tilde{X}=(\mathbf{U}_0+s\mathbf{U}_1)\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & W_1 W_2 \end{pmatrix}\tilde{p}(x)+\frac{1}{s}X_u; \tag{21}$$

$$\tilde{p}(x)=(p_1', p_2', \cdots, p_r', \alpha_1', \alpha_2')^t, \quad p_k'=p_k \exp[-(\gamma_k^{(0)}+s\gamma_k^{(1)})x],$$
$$\alpha_1'=\alpha_1 \exp(-\sqrt{6/5}\, sx), \quad \alpha_2'=\alpha_2 \exp(-\sqrt{2Prs}\, x),$$

where $\mathbf{I}$ is the unit matrix of size $4n-4$. If we take into account the boundary condition (9b), the arbitrary parameters p_k corresponding to the positive eigenvalues $[-\gamma_i(0)>0]$ should be zero and then we have

$$p_k=0 \quad (2n-1\leqq k\leqq 4n-4). \tag{22}$$

imposing the boundary condition (9a) and (22) on the solution (21), we obtain

$$\tilde{X}^+(0)=(\mathbf{U}_0+s\mathbf{U}_1)^+\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & W_1 W_2 \end{pmatrix}p(0)+\frac{1}{s}X_u^+=0, \tag{23}$$

where the superscript + denotes the upper half of a vector or a matrix.

Expanding W_i [See Eq. (20)], p, and α_i as

* Here, each vector is not normalized.

$$
\begin{aligned}
W_i &= W_i^{(0)} + \sqrt{s}\, W_i^{(1/2)} + s W_i^{(1)}, \\
p &= s^{-1} p^{(-1)} + s^{-1/2)} + p^{(0)} + \cdots, \\
\alpha_i &= s^{-1}\alpha_i^{(-1)} + s^{-1/2}\alpha_i^{(-1/2)} + \alpha_i^{(0)} + \cdots,
\end{aligned}
\tag{24}
$$

substituting these expansion forms (24) into Eq. (23), and equating each coefficient of $s^{i/2}$ (i is an integer) to zero, we can obtain $p^{(i)}$ and then $\alpha_1{}^{(i)}$ and $\alpha_2{}^{(i)}$:

$$
\begin{aligned}
\alpha_1^{(-1)} &= \frac{-\sqrt{8/15}\,\Delta p_u}{\sqrt{10/3}-(c_n+c_t)}, \\
\alpha_2^{(-1)} &= \frac{(-c_t+\sqrt{8/15})\Delta N_u + (c_n-\sqrt{6/5})\Delta t_u}{\sqrt{10/3}-(c_n+c_t)}, \\
\alpha_1^{(-1/2)} &= [(c_n+c_t)-2Pr(d_n+d_t)]\frac{\alpha_2^{(-1)}\alpha_1^{(-1)}}{\sqrt{2Pr}\,\Delta p_u}, \\
\alpha_2^{(-1/2)} &= [-c_n+\tfrac{3}{2}c_t+2Pr(d_n-\tfrac{3}{2}d_t)+2\sqrt{15/8}\,c_q]\frac{\alpha_2^{(-1)}\alpha_1^{(-1)}}{\sqrt{2Pr}\,\Delta p_u}, \\
&\cdots\cdots\cdots\cdots\cdots,
\end{aligned}
\tag{25}
$$

where c_n and c_t are the macroscopic jump coefficients of density and temperature due to evaporation and d_n and d_t are those due to heat transfer (See Appendix B). The coefficient c_q is a constant that is given in Appendix B.

Substituting Eqs. (24), (25), and (20) into Eq. (19) and carrying out the inverse transformation [See Eq. (12)], we obtain the original function of the solution,

$$
\begin{aligned}
(N, U, \tau, \kappa)^t = &\{\alpha_1^{(-1)} W_1^{(0)} \mathrm{H}(t-\sqrt{6/5}\,x) + \alpha_1^{(-1/2)} W_1^{(0)} [\pi(t-\sqrt{6/5}\,x)]^{-1/2} \\
&+ \alpha^{(0)} W_1^{(0)} \delta(t-\sqrt{6/5}\,x) + \cdots\} \\
&+ \left\{\alpha_2^{(-1)} W_2^{(0)} \operatorname{erfc}\left(\frac{\sqrt{Pr}\,x}{\sqrt{2t}}\right) + \left[\alpha_2^{(-1/2)} W_2^{(0)} + \alpha_2^{(-1)} \frac{1}{\sqrt{2}} W_2^{(1/2)}\right]\right. \\
&\times (\pi t)^{-1/2} \exp\left(-\frac{Prx^2}{2t}\right) + \left[\alpha_2^{(0)} W_2^{(0)} + \alpha_2^{(-1/2)} \frac{1}{\sqrt{2}} W_2^{(1/2)} + \alpha_2^{(-1)} W_2^{(1)}\right] \\
&\left.\times \left(\frac{Pr}{2\pi t}\right)^{1/2} \left(\frac{x}{t}\right) \exp\left(-\frac{Prx^2}{2t}\right) + \cdots\right\} + (\Delta N_u, 0, \Delta t_u, 0)^t,
\end{aligned}
\tag{26}
$$

where $H(z)$ is the step function [$H(z)=1$, $z<0$ and $H(z)=0$, $z>0$], $\delta(z)$ the Dirac's delta function, and

$$
\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-z^2) dz.
$$

The contents of the first brace in the righthand side of Eq. (26) express discontinuous waves which propagate with the sound speed $(5/6)^{1/2}$; the first term in the brace is the step function, the second term expresses a relaxation of pressure wave, the third term is the delta function (this is the derivative of the first term), and subsequent terms* in the brace include following functions:

$$\frac{1}{\sqrt{\pi}}\frac{d^n}{dt^n}(t-\sqrt{6/5}\,x)^{-1/2}, \quad \frac{d^n}{dt^n}\delta(t-\sqrt{6/5}\,x).$$

Thus in the region, $t-(6/5)^{1/2}x>1$, the first term remains and it gives a uniform pressure p_2 and a constant flow velocity U_2.

On the other hand, the contents of the second brace express dissipating waves (contact surface); the second and the third terms in the brace decay repidly as the time proceedes and the subsequent terms** include following functions:

$$\frac{d^n}{dt^n}\operatorname{erfc}\left(\frac{\sqrt{Pr}\,x}{\sqrt{2t}}\right), \quad \frac{d^n}{dt^n}\left[(\pi t)^{-1/2}\exp\left(-\frac{Prx^2}{2t}\right)\right];$$

these terms decay as fast as the second or the third term. The last term in the righthand side of Eq. (26) evidently expresses the undisturbed vapor ahead the evaporation wave.

As a consequence, the behavior of vapor for $t>1$ and $|t-(6/5)^{1/2}x|>1$ can be given by

$$\begin{pmatrix} N \\ U \\ \tau \end{pmatrix} = \alpha_1^{(-1)}\begin{pmatrix} 3/2 \\ \sqrt{15/8} \\ 1 \end{pmatrix}\mathrm{H}(t-\sqrt{6/5}\,x)+\alpha_2^{-1}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\operatorname{erfc}\left(\frac{\sqrt{Pr}\,x}{\sqrt{2t}}\right)+\begin{pmatrix} \Delta N_u \\ 0 \\ \Delta t_u \end{pmatrix}. \tag{27}$$

Equation (27) yields two uniform flow region behind the evaporation wave;

(i) $1 \ll x \ll t^{1/2}$:

$$N_\infty = c_n U_\infty, \quad \tau_\infty = c_t U_\infty, \quad U_\infty = \frac{-\Delta p_u}{\sqrt{10/3}-(c_n+c_t)}, \quad \kappa_\infty = 0, \tag{28a}$$

(ii) $t^{1/2} \ll x \ll (5/6)^{1/2} t$:

$$\begin{aligned} N_2 &= \left[\sqrt{6/5}-(\sqrt{10/3}-c_n-c_t)\frac{\Delta N_u}{\Delta p_u}\right]U_\infty, \\ \tau_2 &= \left[\sqrt{8/15}-(\sqrt{10/3}-c_n-c_t)\frac{\Delta t_u}{\Delta p_u}\right]U_\infty, \\ U_2 &= U_\infty, \quad \kappa_2 = 0. \end{aligned} \tag{28b}$$

The dimensionless evaporation rate U_∞ in Eq. (28a) shows a good agreement with the one obtained in the previous paper [3]***,

* The subsequent terms are the original functions of $s^{i/2}\exp(-\sqrt{6/5}sx)$ where i is a positive integer.
** The subsequent terms are the original functions of $s^{i/2}\exp(-\sqrt{2Prs}x)$ where i is a positive integer.
*** In Eq. (29) the mistyped error in Ref. 3 is corrected ($2\gamma \to \sqrt{2\gamma}$).

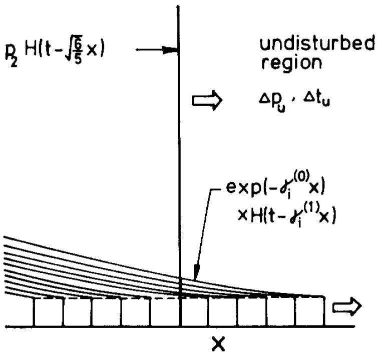

Fig. 3. Schematic drawing of an evaporation wave.

$$\frac{\dot{m}}{\rho_s c_{ms}}=\frac{-\Delta p_u}{\sqrt{2\gamma}+\dfrac{9\sqrt{\pi}}{8}}, \tag{29}$$

where γ is the ratio of the specific heats.

The kinetic part of the solution (21) include a term

$$\frac{1}{s}p_i^{(-1)}\exp\left[-(\gamma_i^{(0)}+s\gamma_i^{(1)})x\right]$$

and the original function of this term is given by

$$p_i^{(-1)}\exp\left[-\gamma_i^{(0)}x\right]\mathrm{H}(t-\gamma_i^{(1)}x).$$

In accordance with the value of $\gamma_i^{(1)}$ ($i=1$ to $2n-2$) (See Table 1) such waves make a dispersive wave front (See Fig. 3) but the waves decay rapidly as they proceede: Eventually a single evaporation wave (shock wave) that propagates with the speed of sound remains. Consequently the kinetic part of the solution in the Kundsen layer $[x\approx 0(1)]$ approaches the solution of the steady state problem,

$$\begin{bmatrix} N \\ U \\ \tau \\ \kappa \end{bmatrix}_k = \begin{bmatrix} X_1^t \\ X_2^t \\ X_3^t \\ X_4^t \end{bmatrix}^* \sum_{k=1}^{2n-2} p_k^{(-1)} U_k \exp(-\gamma_i^{(0)}x). \tag{30}$$

Thus the quasisteady solution of the evaporation problem is given by

* In Eq. (30) X_2 and X_4 are the vectorial forms of c_x and $c_x(c^2-5/2)$, respectively.

$$\begin{bmatrix} N \\ U \\ \tau \\ \kappa \end{bmatrix}_{qs} = \text{Eq. (27)} + \text{Eq. (30)}. \tag{31}$$

4. Discussion

In the previous section a transient evaporation problem was analyzed by use of a kinetic model equation: The vapor motions were revealed and the resultant evaporation rate was obtained. The speed of the evaporation wave, the evaporation rate, and uniform flow conditions are little dependent upon the Prandtl number Pr and the scaling factor β, i.e., upon the model of kinetic equation. On the other hand, the behavior of the contact surface depend rightly on Pr and on β through the scales of x and t.

Within the restriction of the linearization of the problem, present results are valid for a positive value of Δp_u, i.e., for the condensation problem. Even if $\Delta p_u > 0$, the uniform flow condition with the subscript ∞ [See Eq. (28)] is uniquely determined by the condensation rate $U_\infty (<0)$, while in the steady state condensation problem such uniqueness is not found [8]. So, the uniform flow adjacent to the Kundsen layer in the transient consensation problem may be a particular case of the steady state problem.

References

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Appendix A

The inverse matrix of $\mathbf{U}_0+s\mathbf{U}_1$ for $s\ll 1$ can be given by*

$$(\mathbf{U}_0+s\mathbf{U}_1)^{-1}=\mathbf{U}_0^{-1}+(-\mathbf{U}_0^{-1}\mathbf{U}_1\mathbf{U}_0^{-1})s+(-\mathbf{U}_0^{-1}\mathbf{U}_1)^2\mathbf{U}_0^{-1}s^2+\cdots, \tag{A-1}$$

where we assume that $\mathbf{U}_0$ is a regular matrix. Substituting Eq. (14) into Eq. (8) and using the relation (A-1), we obtain

$$\frac{d\tilde{Y}}{dx}=[\mathbf{U}_0^{-1}\Gamma\mathbf{U}_0^{-1}+(\mathbf{U}_0^{-1}\Gamma\mathbf{U}_1-\mathbf{U}_0^{-1}\mathbf{U}_1\mathbf{U}_0^{-1}\Gamma\mathbf{U}_0-\mathbf{U}_0^{-1}\Gamma_1\mathbf{U}_0)s+\cdots]\tilde{Y} \tag{A-2}$$

If we choose

$$\left.\begin{aligned}&\gamma_{ij}^{(0)}=\delta_{ij}\gamma_i^{(0)}+(\delta_{i4n-3}-\delta_{i4n-1})\delta_{j4n};\quad \mathbf{U}_0^{-1}\Gamma\mathbf{U}_0=\{\gamma_{ij}^{(0)}\},\\ &(\gamma_i^{(0)}-\gamma_j^{(0)})\tilde{u}_{ij}=\gamma_{ij}^{(1)}\quad (i\neq j),\quad \tilde{u}_{ii}=0,\\ &\tilde{u}_{ij}=0\quad (i\geqq 4n-3 \text{ and } j\geqq 4n-3);\\ &\mathbf{U}_0^{-1}\mathbf{U}_1=\{\tilde{u}_{ij}\},\quad \mathbf{U}_0^{-1}\Gamma_1\mathbf{U}_0=\{\gamma_{ij}^{(1)}\},\end{aligned}\right. \tag{A-3}$$

where $\gamma_i^{(0)}$ is the eigenvalue of Eq. (11) for $s=0$ (i.e., $\mathbf{U}_0$ is the matrix of solution for steady state problem), Eq. (A-2) yields

$$\frac{d\tilde{Y}}{dx}=\left[\begin{pmatrix}\tilde{\Gamma} & \mathbf{0}\\ \mathbf{0} & \mathbf{W}\end{pmatrix}+O(s^2)\right]\tilde{Y}, \tag{15}$$

where

$$\begin{aligned}&\tilde{\Gamma}=\{(\gamma_i^{(0)}+\gamma_i^{(1)})\delta_{ij}\};\quad \gamma_i^{(1)}=\gamma_{ii}^{(1)}\quad (1\leqq i\leqq 4n-4),\\ &\mathbf{W}=\{\delta_{j4n}(\delta_{i4n-3}-\delta_{i4n-1})+s\gamma_{ij}^{(1)}\}\quad (4n-3\leqq i, j\leqq 4n),\end{aligned}$$

and δ_{ij} is the Kronecker's delta.

Appendix B

Conveniently Eq. (23) is rewritten as

$$\tilde{\boldsymbol{X}}^+(0)=(\mathbf{K}_0+\sqrt{s}\,\mathbf{K}_{1/2}+s\boldsymbol{K}_1)\sum_{i=-2}^{\infty}s^{i/2}\boldsymbol{p}^{(i)}(0)+s^{-1}\boldsymbol{X}_u^+, \tag{A-4}$$

with

* If $1/s\ll 1$, we have another expansion form, $(U_0+sU_1)^{-1}=s^{-1}U_1^{-1}+s^{-2}(-U_1^{-1}U_0U^{-1})+\cdots$

$$\mathbf{K}_0=(U_1^{(0)+}, U_2^{(0)+}, \cdots, U_{2n-2}^{(0)+}, \Omega_1^{(0)}, \Omega_2^{(0)}),$$
$$\mathbf{K}_{1/2}=(\mathbf{0}, \mathbf{0}, \cdots, \Omega_1^{(1/2)}, \Omega_2^{(1/2)}),$$
$$\mathbf{K}_1=(U_1^{(1)+}, U_2^{(1)+}, \cdots, U_{2n-2}^{(1)+}, \Omega_1^{(1)}, \Omega_2^{(1)}),$$
$$\Omega_1^{(0)}=\tfrac{5}{2}X_1^+ +\sqrt{15/8}\,X_2^+ +X_3^+ -X_1^+, \quad \Omega_2^{(0)}=X_3^+ -X_1^+,$$
$$\Omega_1^{(1/2)}=0, \quad \Omega_2^{(1/2)}=-\frac{1}{\sqrt{2Pr}}(-X_2^+ +2X_4^+),$$
$$\Omega_1^{(1)}=\sqrt{6/5}\,X_4^+, \quad \Omega_2^{(1)}=-\frac{1}{pr}X_4^+,$$

where $\mathbf{U}_i^{(0)}$ and $\mathbf{U}_i^{(1)}$ are the ith columns of the matrix $\mathbf{U}_0$ and $\mathbf{U}_1$, respectively and X_2 and X_4 are the vectorial forms of c_x and $c_x(c^2-5/2)$, respectively.

Equating all coefficients of $s^{i/2}$ to zero (i is a positive integer), we obtain

$$\begin{aligned} p_{(0)}^{(-1)} &= -\mathbf{K}_0^{-1}X_u^+, \\ p_{(0)}^{(-1)} &= -\mathbf{K}_0^{-1}(-X_2^+ +2X_4^+)\frac{\alpha_2^{(-1)}}{\sqrt{2pr}}, \\ p_{(0)}^{(0)} &= -\mathbf{K}_0^{-1}\left[(-X_2^+ +2X_4^+)\frac{\alpha_2^{(-1/2)}}{\sqrt{2Pr}}+\mathbf{K}_1 p_{(0)}^{(-1)}\right], \cdots. \end{aligned} \tag{A-5}$$

Let define a determinant Δ_0 by

$$\Delta_0=\det(U_1^{(0)+}, U_2^{(0)+}, \cdots, U_{2n-2}^{(0)+}, X_1^+, X_3^+),$$

and a determinant $\Delta_i(Z)$ by replacing the ith vector in the Δ_0 with a vector Z. By using these definitions, we have the following relations:

$$\frac{\Delta_{2n-1}(X_2^+)}{\Delta_0}=-c_n, \qquad \frac{\Delta_{2n}(X_2^+)}{\Delta_0}=-c_t,$$
$$\frac{\Delta_{2n-1}(X_4^+)}{\Delta_0}=-Pr\,dn_n, \qquad \frac{\Delta_{2n}(X_4^+)}{\Delta_0}=-Pr\,d_t,$$
$$c_n=\begin{cases}-1.6853 & (\beta=1)\\ -1.6778 & (\beta=2/3),\end{cases} \qquad c_t=\begin{cases}-0.4467 & (\beta=1)\\ -0.4438 & (\beta=2/3),\end{cases}$$
$$d_n=\begin{cases}0.7443/Pr & (\beta=1),\\ 0.7467/Pr & (\beta=2/3),\end{cases} \qquad d_t=\begin{cases}-1.3027/Pr & (\beta=1)\\ -1.3016/Pr & (\beta=2/3).\end{cases}$$

The determinant of $\mathbf{K}_0$ yields

$$\det(\mathbf{K}_0)=\Delta_0\sqrt{15/8}(\sqrt{10/3}-c_n-c_t),$$

and then $\alpha_1^{(-1)}$ and $\alpha_2^{(-1)}$ are given by

$$\alpha_1^{(-1)} = \frac{1}{\Delta_0} \Delta_{2n-1}(\boldsymbol{X}_1^+ \varDelta N_u + \boldsymbol{X}_3^+ \varDelta t_u) = \frac{-\sqrt{15/8}\,\varDelta p_u}{\sqrt{10/3} - c_n - c_t},$$

$$\alpha_2^{(-2)} = \frac{1}{\Delta_0} \Delta_{2n}(\boldsymbol{X}_1^+ \varDelta N_u + \boldsymbol{X}_3^+ \varDelta t_u) = \frac{(-c_t + \sqrt{8/15})\varDelta N_u + (c_n - \sqrt{6/5})\varDelta t_u}{\sqrt{10/3} - c_n - c_t}.$$

The coefficient c_q *is given by*

$$c_q = \frac{1}{\Delta_0} \det(\boldsymbol{U}_1^{(0)+}, \boldsymbol{U}_2^{(0)+}, \cdots, \boldsymbol{U}_{2n-2}^{(0)+}, \boldsymbol{X}_2^+, \boldsymbol{X}_4^+) = \begin{cases} 3.7923 & (\beta = 1,\ Pr = 2/3) \\ 3.7736 & (\beta = Pr = 2/3) \\ 2.5282 & (\beta = Pr = 1)\,. \end{cases}$$