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The Flow Near the Region of Vertex of Axially Symmetric Bodies at Supersonic Speeds

By

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Summary. An analytical method for axially symmetric supersonic flow involving shock wave is presented and applied to clarify the characteristic feature of the flow near the region of vertex of the pointed bodies of revolution with arbitrary geometry. The method used in the present approach is exact except for an assumption that the flow is inviscid. It is shown that the present method can be further applied, with a slight modification, to a circular-cone with such large semi-vertex angle that the flow behind the shock wave is partially subsonic.

Detailed examination reveals that in axially symmetric flow there exists a particular point which corresponds to Crocco point in plane flow and the flow characters near the region of vertex of the body are qualitatively quite the same as those of the plane flow.

It is concluded that the transition from attached to detached shock wave is continuous for circular-cones as well as for plane wedges.

Symbols

$(\bar{x}, \bar{r})$	non-dimensional cylindrical coordinates system normalized by length of the body
(x, r)	reduced coordinates system
$\tilde{r}$	strained radial coordinate
θ	conical variable defined by r/x
$\tilde{\theta}$	strained conical variable defined by $\tilde{r}/x$
$(\bar{u}, \bar{v})$	components of local velocity vector
$\bar{\rho}$	density
$\bar{p}$	pressure
(u, v)	reduced form of velocity components
ρ	reduced density
p	reduced pressure
M	free stream Mach number
β_s	initial shock wave angle
δ	initial semi-vertex angle of the body
τ	$\tan \beta_s$
ψ	stream function
ω	entropy function

$S(x)$	shape function of shock wave
C_p	pressure coefficient
K_s	initial shock wave curvature
K_b	initial body curvature
$l, m,$	coefficients in series expansion of shock shape
f, g h, i	functions in series expansion of stream function
F, G H, I	function in series expansion of density
κ	constant indicating the order of singularity

Subscripts:

s	value at shock wave
b	value on body surface
o	value on surface of basic cone
p	value at sonic point of basic cone
∞	value in free stream
$(\ )'$	derivative with respect to argument

1. Introduction

It is well known that most of supersonic flows past axially symmetric bodies with an attached shock wave can be solved numerically with required accuracy by use of method of characteristics. However, use can no longer be made of even the method of characteristics, when the body is so thick that the flow behind the attached shock wave is subsonic. For such flow patterns there seems to exist few previous work on general method of theoretical approach which is available.

Analytical approaches to supersonic flows near the region of vertex of axially symmetric bodies have already been developed by Van Dyke [1], Shen and Lin [2] and Karashima [3], etc. However, each of these approaches is approximate one and, hence, cannot be applied to such thick bodies with subsonic field behind the attached shock wave. In particular, Shen and Lin's analysis shows a result of a logarithmic singularity at the initial semi-vertex angle and the initial surface pressure gradient becomes infinite even for regular body shape.

For plane supersonic flows the approach to thick bodies with subsonic region behind the shock wave can be made comparatively easily. Busemann [4] gave a qualitative discussion on initial shock wave curvature by use of direction of spines in hodograph plane and quantitative arguments were made in detail by Guderley [5], Tamada [6] and Oguchi [7], etc. These approaches, although approximate, lead to a remarkable conclusion that the transition from attached to detached shock wave is continuous.

On the other hand, there does not seem to exist any available approach to axially symmetric flows except for the one proposed by Oguchi [8] for open-nosed axially symmetric bodies, since any conventional method applied to pointed bodies of revolution has a difficulty of a singularity at the initial semi-vertex angle.

Present paper has a purpose to give a general method of analytical approach to axially symmetric supersonic flows involving shock waves and to apply it to the flow near the region of pointed nose of the body. The method is exact except for an assumption that the flow is inviscid. Moreover, being based essentially upon the series-expansion method proposed initially by Van Dyke [1], the present method does not indicate such a singularity as is presented in conventional method.

2. Fundamental Equations and Boundary Conditions

Let the origin of a cylindrical coordinates system $(\bar{x}, \bar{r})$ be taken at the vertex of the body of revolution, $\bar{x}$-axis being aligned with the free stream direction, and $\bar{r}$-axis being normal to $\bar{x}$-axis (see Fig. 1). Introducing a transformation of variables such as

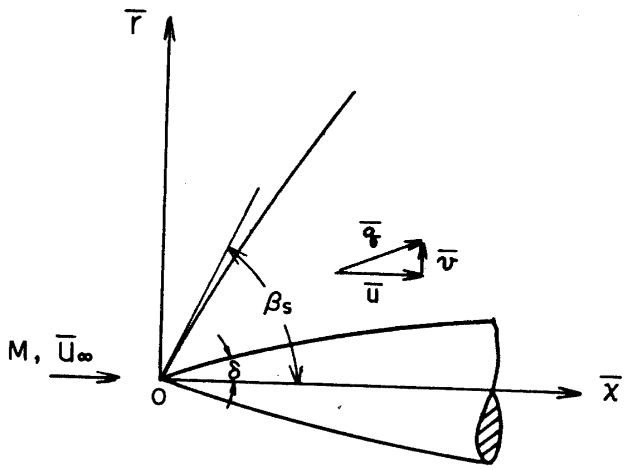

Fig. 1. Coordinates system.

$$\left.\begin{aligned}
&x=\bar{x}, \quad r=\frac{1}{\tau}\bar{r} && \tau=\tan\beta_s, \\
&\bar{u}=\bar{u}_\infty\{a+\tau^2 u(x, r)\}, && a=1+\frac{2}{(\gamma+1)M^2}, \\
&\bar{v}=\bar{u}_\infty \tau v(x, r), \\
&\bar{\rho}=\bar{\rho}_\infty \rho(x, r), \\
&\bar{p}=\bar{p}_\infty \gamma M^2 \tau^2 p(x, r),
\end{aligned}\right\} \qquad (2.1)$$

where β_s, $\bar{u}$, $\bar{v}$, $\bar{\rho}$, $\bar{p}$, M and γ denote initial shock wave angle, components of local velocity vector in $\bar{x}$- and $\bar{r}$-directions, density, pressure, free stream Mach number and ratio of specific heats, respectively, and rewriting equations of motion, then gives

$$
\left.
\begin{array}{ll}
\text{(continuity)} & \{\rho(a+\tau^2 u)\}_x+(\rho v)_r+\dfrac{\rho v}{r}=0, \\
(x\text{-momentum}) & (a+\tau^2 u)\,u_x+v u_r+\dfrac{1}{\rho}\,p_x=0, \\
(r\text{-momentum}) & (a+\tau^2 u)\,v_x+v v_r+\dfrac{1}{\rho}\,p_r=0, \\
\text{(entropy)} & (a+\tau^2 u)\left(\dfrac{p}{\rho^\gamma}\right)_x+v\left(\dfrac{p}{\rho^\gamma}\right)_r=0,
\end{array}
\right\} \tag{2.2}
$$

where subscripts denote differentiation.

Let shock wave shape and body be expressed, respectively, in the transformed coordinates system as

$$
r_s=S(x),
$$
$$
r_b=r_b(x),
$$

then, the boundary conditions along shock wave and on body surface are given, respectively, as

$$
\left.
\begin{array}{l}
u_s=-\dfrac{2}{\gamma+1}\,\dfrac{S'^2}{1+\tau^2 S'^2}, \\
v_s=\dfrac{2}{\gamma+1}\,\dfrac{S'}{1+\tau^2 S'^2}\left(1-\dfrac{1+\tau^2 S'^2}{M^2\tau^2 S'^2}\right), \\
\rho_s=\dfrac{(\gamma+1)M^2\tau^2 S'^2}{(\gamma-1)M^2\tau^2 S'^2+2(1+\tau^2 S'^2)}, \\
p_s=\dfrac{2\gamma M^2\tau^2 S'^2-(\gamma-1)(1+\tau^2 S'^2)}{\gamma(\gamma+1)M^2\tau^2(1+\tau^2 S'^2)},
\end{array}
\right\} \tag{2.3}
$$

$$
\text{(tangency)} \qquad v=(a+\tau^2 u)\frac{dr_b}{dx} \qquad \text{at} \quad r=r_b\,. \tag{2.4}
$$

Continuity equation may be accounted for by introducing a stream function $\psi(x, r)$,

$$
\psi_x=-r\rho v, \qquad \psi_r=r\rho(a+\tau^2 u). \tag{2.5}
$$

Then, entropy equation predicts that p/ρ^γ is a function of only ψ. Thus, it is convenient to define an entropy function ω as

$$
\omega(\psi)=\frac{p}{\rho^\gamma}. \tag{2,6}
$$

Elimination of u, v and p from momentum equations and energy equation by use of Eqs. (2.5) and (2.6) leads to the following simultaneous equations with respect to ψ and ρ, which are the fundamental equations to be solved in the present analysis;

$$
\left.
\begin{array}{l}
\psi_r^2\psi_{xx}-2\psi_x\psi_r\psi_{xr}+\psi_x^2\psi_{rr} \\
\qquad =r^2\{\gamma\omega\psi_r\rho^\gamma\rho_r+\omega'\psi_r^2\rho^{\gamma+1}+\tau^2(\gamma\omega\psi_x\rho^\gamma\rho_x+\omega'\psi_x^2\rho^{\gamma+1})\}, \\
\psi_r^2+\tau^2\psi_x^2+\dfrac{2\gamma}{\gamma-1}\,\tau^2\omega r^2\rho^{\gamma+1}=Kr^2\rho^2, \\
\omega'=\dfrac{d\omega}{d\psi}, \qquad K=1+\dfrac{2}{(\gamma-1)M^2}.
\end{array}
\right\} \tag{2.7}
$$

3. Flow Past Circular-Cones

As a simple example, consider a supersonic flow past a circular-cone with semi-vertex angle of δ. The flow is assumed to be supersonic everywhere for convention. Since the flow field downstream of the shock wave is conical, the stream function, and shock wave shape can be written, respectively, in the forms

$$\psi(x, r) = x^2 f(\theta), \qquad \rho(x, r) = F(\theta), \tag{3.1}$$

$$\theta = \frac{r}{x}, \tag{3.2}$$

$$r_s = S(x) = x. \tag{3.3}$$

The shock wave being straight, the flow downstream of it is irrotational and isentropic, so that the entropy function ω is constant everywhere between shock waves equal to ω_0 which is given as

$$\omega_0 = \frac{2\gamma M^2\tau^2 - (\gamma-1)(1+\tau^2)}{\gamma(\gamma+1)M^2\tau^2(1+\tau^2)} \left\{ \frac{(\gamma-1)M^2\tau^2 + 2(1+\tau^2)}{(\gamma+1)M^2\tau^2} \right\}^{\gamma}. \tag{3.4}$$

Substitution of Eq. (3.1) into Eq. (2.7) gives the following simultaneous equations for $f(\theta)$ and $F(\theta)$;

$$\left.\begin{aligned} &4f^2 f'' - 2ff'^2 = \gamma\omega_0\theta^2 F^{\gamma} F' \{f' - \tau^2\theta(2f - \theta f')\}, \\ &f'^2 + \tau^2(2f - \theta f')^2 + \frac{2\gamma}{\gamma-1}\tau^2\omega_0\theta^2 F^{\gamma+1} = K\theta^2 F^2. \end{aligned}\right\} \tag{3.5}$$

Shock wave location is given by $\theta = \theta_s = 1$ and the boundary conditions for f and F are expressed, respectively, by use of Eqs. (2.3) and (3.3) as

$$\left.\begin{aligned} &f(1) = \frac{1}{2}, \\ &f'(1) = \frac{\tau^2}{1+\tau^2} \frac{\{(\gamma+1) + (\gamma-1)\tau^2\}M^2 + 2(1+\tau^2)}{(\gamma-1)M^2\tau^2 + 2(1+\tau^2)}, \\ &F(1) = \frac{(\gamma+1)M^2\tau^2}{(\gamma-1)M^2\tau^2 + 2(1+\tau^2)}. \end{aligned}\right\} \tag{3.6}$$

Another condition is that the stream function vanishes on the cone surface $\theta = \theta_0$. Hence,

$$f(\theta_0) = 0. \tag{3.7}$$

Thus, semi-vertex angle δ of the cone is obtained by the relation

$$\tan\delta = \tau\theta_0. \tag{3.8}$$

It must be noted that Eq. (3.5) together with Eq. (3.6) gives an exact solution for circular-cones in a different sense from the conical theory proposed by Taylor and Maccoll [9].

4. Flow Past Convex Bodies of Revolution

Consider a supersonic flow past convex bodies of revolution with an attached shock wave. The flow behind the shock wave is assumed to be supersonic everywhere. In this case, the flow aft of the shock wave is considered to consist of a basic conical field upon which is superimposed a perturbation field due to body curvature, as was done by Van Dyke [1] and Karashima [3]. Although this can be done in such a way that the stream function ψ and density ρ are assumed to be expanded into power series of x with suitable coefficients built up by similar functions of the conical variable θ, a mathematical difficulty takes place that some of these similar functions for higher powers of x become non-analytic near the surface of the basic cone $\theta=\theta_0$, under which the solution does not exist. This mathematical difficulty together with the fact that the body surface lies under θ_0 for conventional convex bodies of revolution suggests that the solutions to ψ and ρ cannot be made beyond the singular point to the body surface. In order to avoid this difficulty, it is convenient to introduce a slightly strained radial coordinate $\tilde{r}$ such that the body surface is given by $\tilde{r}_b=\theta_0 x$.

Let the body surface be given by

$$r_b=\theta_0 x+\frac{1}{2}l\theta_1 x^2+\frac{1}{3}m\theta_2 x^3+\cdots , \tag{4.1}$$

and the corresponding shock wave by

$$r_s=x-\frac{1}{2}lx^2-\frac{1}{3}mx^3-\cdots , \tag{4.2}$$

then, the simplest choice for $\tilde{r}$ is

$$\tilde{r}=r-\left(\frac{1}{2}l\theta_1 x^2+\frac{1}{3}m\theta_2 x^3+\cdots\right), \tag{4.3}$$

and the strained conical variable $\tilde{\theta}$ is defined as

$$\tilde{\theta}=\frac{\tilde{r}}{x}=\theta-\left(\frac{1}{2}l\theta_1 x+\frac{1}{3}m\theta_2 x^2+\cdots\right). \tag{4.4}$$

Therefore, the body surface is expressed by use of the strained conical variable as

$$(\tilde{\theta})_{r=r_b}=\left(\frac{\tilde{r}}{x}\right)_{r=r_b}=\theta_0 . \tag{4.5}$$

The stream function and density function may be written, respectively, in the forms

$$\left.\begin{aligned}\psi(x, r)=x^2 f(\tilde{\theta})-lx^3\left\{g(\tilde{\theta})-\frac{1}{2}\theta_1 f'(\tilde{\theta})\right\}-x^4\Big[m\left\{h(\tilde{\theta})-\frac{1}{3}\theta_2 f'(\tilde{\theta})\right\}\\+l^2\left\{i(\tilde{\theta})+\frac{1}{2}\theta_1 g'(\tilde{\theta})-\frac{1}{8}\theta_1^2 f''(\tilde{\theta})\right\}\Big]-\cdots ,\end{aligned}\right\} \tag{4.6}$$

$$\rho(x, r)=F(\tilde{\theta})-lx\left\{G(\tilde{\theta})-\frac{1}{2}\theta_1F'(\tilde{\theta})\right\}-x^2\left[m\left\{H(\tilde{\theta})-\frac{1}{3}\theta_2F'(\tilde{\theta})\right\}\right.$$
$$\left.+l^2\left\{I(\tilde{\theta})+\frac{1}{2}\theta_1G'(\tilde{\theta})-\frac{1}{8}\theta_1^2F''(\tilde{\theta})\right\}\right]-\cdots .$$

In the same way the entropy function may be expanded into

$$\omega(\psi)=\omega_0(1-\omega_1 l\sqrt{2\psi}-\omega_2 m\psi-\cdots). \tag{4.7}$$

The conditions just behind the shock wave are found to be

$$\left.\begin{aligned}
&\left(\frac{\psi_r}{r}\right)_s=a_0+a_1lx+(a_1m+a_2l^2)x^2+\cdots ,\\
&\left(-\frac{\psi_x}{r}\right)_s=b_0+b_1lx+(b_1m+b_2l^2)x^2+\cdots ,\\
&\rho_s=\rho_0-\rho_1lx-(\rho_1m+\rho_2l^2)x^2+\cdots ,\\
&\omega_s=\omega_0\{1-\omega_1lx-(\omega_1m-\alpha l^2)x^2+\cdots\},
\end{aligned}\right\} \tag{4.8}$$

where

$$\left.\begin{aligned}
a_0&=\frac{\tau^2}{1+\tau^2}\frac{\{(\gamma+1)+(\gamma-1)\tau^2\}M^2+2(1+\tau^2)}{(\gamma-1)M^2\tau^2+2(1+\tau^2)},\\
b_0&=\frac{2}{1+\tau^2}\frac{M^2\tau^2-(1+\tau^2)}{(\gamma-1)M^2\tau^2+2(1+\tau^2)},\\
a_1&=-\left[\frac{4\tau^2}{1+\tau^2}\frac{\{(\gamma+1)+(\gamma-1)\tau^2\}M^2+2(1+\tau^2)}{\{(\gamma-1)M^2\tau^2+2(1+\tau^2)\}^2}\right.\\
&\qquad\left.-\frac{4\tau^2}{(1+\tau^2)^2}\frac{M^2\tau^2}{(\gamma-1)M^2\tau^2+2(1+\tau^2)}\right],\\
b_1&=-2\left[\frac{M^2\tau^2-(1+\tau^2)}{1+\tau^2}\frac{2-(\gamma-1)M^2\tau^2-2\tau^2}{\{(\gamma-1)M^2\tau^2+2(1+\tau^2)\}^2}\right.\\
&\qquad\left.+\frac{2}{(1+\tau^2)^2}\frac{M^2\tau^2}{(\gamma-1)M^2\tau^2+2(1+\tau^2)}\right],\\
a_2&=\frac{(\gamma+1)M^2\tau^2\{(\gamma-1)M^2\tau^2+2\tau^2\}\{3(\gamma-1)M^2\tau^2+6\tau^2-2\}}{(1+\tau^2)\{(\gamma-1)M^2\tau^2+2(1+\tau^2)\}^3}\\
&\quad-\frac{4(\gamma+1)M^2\tau^2\{(\gamma-1)M^2\tau^2+2\tau^2\}}{(1+\tau^2)^2\{(\gamma-1)M^2\tau^2+2(1+\tau^2)\}^2}\\
&\quad+\frac{\tau^2(1-3\tau^2)[\{(\gamma+1)+(\gamma-1)\tau^2\}M^2+2(1+\tau^2)]}{(1+\tau^2)^3\{(\gamma-1)M^2\tau^2+2(1+\tau^2)\}},\\
b_2&=-\frac{2(\gamma+1)M^2\tau^2\{3(\gamma-1)M^2\tau^2+6\tau^2-2\}}{(1+\tau^2)\{(\gamma-1)M^2\tau^2+2(1+\tau^2)\}^3}\\
&\quad+\frac{4(\gamma+1)M^2\tau^2(1-\tau^2)}{(1+\tau^2)^2\{(\gamma-1)M^2\tau^2+2(1+\tau^2)\}^2}\\
&\quad-\frac{2\tau^2(3-\tau^2)\{M^2\tau^2-(1+\tau^2)\}}{(1+\tau^2)^3\{(\gamma-1)M^2\tau^2+2(1+\tau^2)\}}
\end{aligned}\right\} \tag{4.9}$$

$$\left.\begin{aligned}
\rho_0&=\frac{(\gamma+1)M^2\tau^2}{(\gamma-1)M^2\tau^2+2(1+\tau^2)}\,,\\
\rho_1&=\frac{4(\gamma+1)M^2\tau^2}{\{(\gamma-1)M^2\tau^2+2(1+\tau^2)\}^2}\,,\\
\rho_2&=\frac{2(\gamma+1)M^2\tau^2\{3(\gamma-1)M^2\tau^2+6\tau^2-2\}}{\{(\gamma-1)M^2\tau^2+2(1+\tau^2)\}^3}\,,\\
\omega_1&=\frac{4\gamma(\gamma-1)\{M^2\tau^2-(1+\tau^2)\}^2}{(1+\tau^2)\{2\gamma M^2\tau^2-(\gamma-1)(1+\tau^2)\}\{(\gamma-1)M^2\tau^2+2(1+\tau^2)\}}\,,\\
\alpha&=\frac{2\gamma M^2\tau^2(1-3\tau^2)}{(1+\tau^2)^2\{2\gamma M^2\tau^2-(\gamma-)(1+\tau^2)\}}+\frac{6\gamma}{(\gamma-1)M^2\tau^2+2(1+\tau^2)}\\
&\quad-\frac{16\gamma^2M^2\tau^2}{(1+\tau^2)\{2\gamma M^2\tau^2-(\gamma-1)(1+\tau^2\}\{(\gamma-1)M^2\tau^2+2(1+\tau^2)\}}\\
&\quad+\frac{8\gamma(\gamma-1)}{\{(\gamma-1)M^2\tau^2+2(1+\tau^2)\}^2}\,,
\end{aligned}\right\}$$

and where ω_0 is given by Eq. (3.4). Since the stream function along the shock wave is given by $\psi_s=\frac{1}{2}r_s^2$, Eq. (4.6) can be expressed along the shock wave as

$$\omega(\psi_s)=\omega_s=\omega_0\{1-\omega_1 lx-\frac{1}{2}(\omega_2 m-\omega_1 l^2)x^2-\cdots\}\,. \tag{4.10}$$

Therefore, ω_2 is obtained by comparing Eq. (4.10) with Eq. (4.8) as

$$\omega_2=2\omega_1-(2\alpha-\omega_1)\frac{l^2}{m}\,. \tag{4.11}$$

Substituting Eqs. (4.6) and (4.7) into Eq. (2.7) and equating like powers of x yields for (f, F), (g, G), (h, H), (i, I), etc. the simultaneous ordinary differential equations, respectively. Equations for (f, F) are quite the same as is given by Eq. (3.5) as well as their boundary conditions corresponding to the basic conical flow.

Equation for g, which is obtained by eliminating G from the simultaneous equations for (g, G), and equation for G can be written, respectively, in the forms

$$Dg''=A+Bg+Cg'\,, \tag{4.12}$$

$$\bar{M}G=\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}g'+3\tau^2(2f-\tilde{\theta}f')g+N\,, \tag{4.13}$$

where

$$\left.\begin{aligned}
D&=\gamma\omega_0\tilde{\theta}^2\frac{F^\gamma}{\bar{M}}\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}^2-4f^2\,,\\
C&=\gamma\omega_0\tilde{\theta}^2\frac{F^\gamma\bar{M}'}{\bar{M}^2}\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}^2-8ff'-\gamma\omega_0\tilde{\theta}^2(1+\tau^2\tilde{\theta}^2)F^\gamma F'\\
&\quad-\gamma\omega_0\tilde{\theta}^2\frac{F^{\gamma-1}}{\bar{M}}\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}[F\{f''+3\tau^2(2f-\tilde{\theta}f')
\end{aligned}\right.$$

$$
\left.\begin{aligned}
&-\tau^2\tilde{\theta}(f'-\tilde{\theta}f'')\}+\gamma F'\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}],\\
B=\,&12ff''+3\tau^2\gamma\omega_0\tilde{\theta}^3F^\gamma F'-3\tau^2\gamma\omega_0\tilde{\theta}^2\frac{F^{\gamma-1}}{\overline{M}}[\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}\\
&\times\{F(f'-\tilde{\theta}f'')+\gamma F'(2f-\tilde{\theta}f')\}+\tau^2(2f-\tilde{\theta}f')^2F]\\
&+3\tau^2\gamma\omega_0\tilde{\theta}^2\frac{F^\gamma\overline{M}'}{\overline{M}^2}(2f-\tilde{\theta}f')\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\},\\
A=\,&-\gamma\omega_0\tilde{\theta}^2F^\gamma\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}\frac{d}{d\tilde{\theta}}\left(\frac{N}{\overline{M}}\right)\\
&-\gamma\omega_0\tilde{\theta}^2F^{\gamma-1}[\gamma F'\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}+\tau^2(2f-\tilde{\theta}f')F]\frac{N}{\overline{M}}\\
&-\gamma\omega_0\omega_1\tilde{\theta}^2\sqrt{2f}\,F^\gamma F'\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}\\
&-\omega_0\omega_1\tilde{\theta}^2\frac{F^{\gamma+1}}{\sqrt{2f}}\{f'^2+\tau^2(2f-\tilde{\theta}f')^2\},
\end{aligned}\right\} \tag{4.14}
$$

$$
\overline{M}=\overline{M}(\tilde{\theta})=\tilde{\theta}^2\left[KF-\frac{\gamma(\gamma+1)}{\gamma-1}\tau^2\omega_0F^\gamma\right], \tag{4.15}
$$

$$
N=N(\tilde{\theta})=\frac{\gamma}{\gamma-1}\tau^2\omega_0\omega_1\tilde{\theta}^2\sqrt{2f}\,F^{\gamma+1}. \tag{4.16}
$$

The boundary conditions for g and G are obtained from Eq. (4.8) as

$$
\left.\begin{aligned}
g(1)&=-\frac{1}{2}b_0,\\
g'(1)&=\frac{1}{2}\{f'(1)-f''(1)\}-a_1,\\
G(1)&=\rho_1-\frac{1}{2}F'(1).
\end{aligned}\right\} \tag{4.17}
$$

However, it must be noted that the final condition, $G(1)$, is automatically satisfied by Eq. (4.13), once g is determined from Eq. (4.12).

Since the function f associated with the basic conical flow vanishes at the surface $\tilde{\theta}=\theta_0$, the coefficient A in differential equation for g becomes infinite at $\tilde{\theta}=\theta_0$. This indicates that the function g becomes non-analytic near the surface. On the other hand, the fact that f is analytic near the surface leads to possibility of a series expansion of f as

$$
f(\tilde{\theta})=(\tilde{\theta}-\theta_0)f'(\theta_0)+O[(\tilde{\theta}-\theta_0)^2]. \tag{4.18}
$$

It follows that near the surface the coefficients of the differential equation for g behave like

$$
\left.\begin{aligned}
&A\sim-A_0(\tilde{\theta}-\theta_0)^{-\frac{1}{2}},\\
&B\sim B_0,\\
&C\sim C_0,\\
&D\sim D_0,
\end{aligned}\right\} \tag{4.19}
$$

where

$$\left.\begin{aligned}
A_0 &= \frac{1}{\sqrt{2}}\omega_0\omega_1\theta_0^2(1+\tau^2\theta_0^2)f_0'^{\frac{3}{2}}F_0^{\gamma+1}\left\{1+\frac{\gamma^2}{\gamma-1}\tau^2\omega_0\theta_0^2\frac{F_0^\gamma}{\bar{M}_0}\right\},\\
B_0 &= -3\tau^2\gamma\omega_0\theta_0^3(1+\tau^2\theta_0^2)f_0'^2\frac{F_0^\gamma \bar{M}_0'}{\bar{M}_0^2}\\
&\quad -3\tau^2\gamma\omega_0\theta_0^2\frac{F_0^\gamma}{\bar{M}_0}\{f_0'(1+\tau^2\theta_0^2)(f_0'-\theta_0 f_0'')+\tau^2\theta_0^2 f_0'^2\},\\
C_0 &= \gamma\omega_0\theta_0^2(1+\tau^2\theta_0^2)^2 f_0'^2\frac{F_0^\gamma \bar{M}_0'}{\bar{M}_0^2}\\
&\quad -\gamma\omega_0\theta_0^2(1+\tau^2\theta_0^2)\frac{f_0'F_0^\gamma}{\bar{M}_0}\{f_0''(1+\tau^2\theta_0^2)-4\tau^2\theta_0 f_0'\},\\
D_0 &= \gamma\omega_0\theta_0^2(1+\tau^2\theta_0^2)^2 f_0'^2\frac{F_0^\gamma}{\bar{M}_0},
\end{aligned}\right\} \tag{4.20}$$

and where subscript 0 indicates the conditions at $\tilde{\theta}=\theta_0$. Although the point $\tilde{\theta}=\theta_0$ is a singular point of Eq. (4.12), it is an ordinary point of the homogeneous equation by deleting A. Therefore, the general solution of the homogeneous equation is analytic, while the particular solution of the non-homogeneous equation has 3/2-power branch point at $\tilde{\theta}=\theta_0$. The two unknown constants involved in the general solution of the homogeneous equation can be determined as the values of g and its first derivative at θ_0. Thus, the full solution of Eq. (4.12) can be obtained near $\tilde{\theta}=\theta_0$ as

$$\left.\begin{aligned}
g(\tilde{\theta}) &= -\frac{4}{3}\frac{A_0}{D_0}\tilde{\varepsilon}^{\frac{2}{3}}\left\{1+\frac{2}{5}\frac{D_0}{C_0}\tilde{\varepsilon}+\frac{4}{35}\left(\frac{C_0^2}{D_0^2}+\frac{B_0}{D_0}\right)\tilde{\varepsilon}^2+\cdots\right\}\\
&\quad +g(\theta_0)\left\{1+\frac{1}{2}\frac{B_0}{D_0}\tilde{\varepsilon}^2+\frac{1}{6}\frac{B_0}{D_0}\frac{C_0}{D_0}\tilde{\varepsilon}^3+\cdots\right\}\\
&\quad +g'(\theta_0)\tilde{\varepsilon}\left\{1+\frac{1}{2}\frac{C_0}{D_0}\tilde{\varepsilon}+\frac{1}{6}\left(\frac{C_0^2}{D_0^2}+\frac{B_0}{D_0}\right)\tilde{\varepsilon}^2+\cdots\right\},
\end{aligned}\right\} \tag{4.21}$$

where

$$\tilde{\varepsilon}=\tilde{\theta}-\theta_0\,. \tag{4.22}$$

The integration of Eq. (4.12) is carried out numerically step by step inward starting from the known values at $\tilde{\theta}=1$ and using the same intervals for $\tilde{\theta}$. The step by step solution is then joined at the two points in the very vicinity of $\tilde{\theta}=\theta_0$ with the series expansion of g about the singular point given by Eq. (4.21). Thus $g(\theta_0)$ and $g'(\theta_0)$ are determined and the full solution for g is obtained.

Differential equation for h, which results in eliminating H from the simultaneous equations for (h, H), and the equation for H are given, respectively, as

$$Dh''=J+Rh+Uh', \tag{4.23}$$

$$\bar{M}H=\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}h'+4\tau^2(2f-\tilde{\theta}f')h+P, \tag{4.24}$$

where

$$
\left.
\begin{aligned}
U=&\gamma\omega_0\tilde{\theta}^2\frac{F^{\gamma}\bar{M}'}{\bar{M}^2}\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}^2-12ff'-\gamma\omega_0\tilde{\theta}^2(1+\tau^2\tilde{\theta}^2)F^{\gamma}F'\\
&-\gamma\omega_0\tilde{\theta}^2\frac{F^{\gamma-1}}{\bar{M}}\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}[F\{f''-\tau^2\tilde{\theta}(f'-\tilde{\theta}f'')\\
&+5\tau^2(2f-\tilde{\theta}f')\}+\gamma F'\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}],\\
R=&4(f'^2+4ff''+\tau^2\gamma\omega_0\tilde{\theta}^3F^{\gamma}F')-4\tau^2\gamma\omega_0\tilde{\theta}^2\frac{F^{\gamma-1}}{\bar{M}}[\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}\\
&\times\{F(f'-\tilde{\theta}f'')+\gamma F'(2f-\tilde{\theta}f')\}+2\tau^2(2f-\tilde{\theta}f')^2F]\\
&+4\tau^2\gamma\omega_0\tilde{\theta}^2\frac{F^{\gamma}\bar{M}'}{\bar{M}^2}\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}(2f-\tilde{\theta}f'),\\
J=&-\gamma\omega_0\tilde{\theta}^2F^{\gamma}\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}\frac{d}{d\tilde{\theta}}\left(\frac{P}{\bar{M}}\right)\\
&-\gamma\omega_0\tilde{\theta}^2F^{\gamma-1}[\gamma F'\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}+2\tau^2(2f-\tilde{\theta}f')F]\frac{P}{\bar{M}}\\
&-2\omega_0\omega_1\tilde{\theta}^2F^{\gamma}[\gamma fF'\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}+F\{f'^2+\tau^2(2f-\tilde{\theta}f')^2\}]\\
P=&P(\tilde{\theta})=\frac{2\gamma}{\gamma-1}\tau^2\omega_0\omega_1\tilde{\theta}fF^{\gamma+1}.
\end{aligned}
\right\} \quad (4.25)
$$

The boundary conditions for h and H are obtained from Eq. (4.8) as

$$
\left.
\begin{aligned}
h(1)&=-\frac{1}{3}b_0,\\
h'(1)&=\frac{1}{3}\{f'(1)-f''(1)\}-a_1,\\
H(1)&=\rho_1-\frac{1}{3}F'(1),
\end{aligned}
\right\} \quad (4.26)
$$

and the final condition, $H(1)$, is automatically satisfied by Eq. (4.24), once h is determined from Eq. (4.23).

Equation for i, which is obtained by eliminating I from the simultaneous equations for (i, I), and the equation for I are expressed, respectively, as

$$
Di''=L+Ri+Ui', \quad (4.27)
$$

$$
\bar{M}I=\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}i'+4\tau^2(2f-\tilde{\theta}f')i+Q, \quad (4.28)
$$

where

$$
\begin{aligned}
L=&3(2f'gg'+2fg'^2-3f''g^2-4fgg'')\\
&-\omega_0\omega_1\tilde{\theta}^2\frac{g}{(2f)^{\frac{2}{3}}}F^{\gamma+1}\{f'^2+\tau^2(2f-\tilde{\theta}f')^2\}\\
&+\omega_0\omega_1\tilde{\theta}^2\frac{1}{\sqrt{2f}}[(\gamma+1)F^{\gamma}G\{f'^2+\tau^2(2f-\tilde{\theta}f')^2\}+2F^{\gamma+1}\{f'g'\\
&+\tau^2(2f-\tilde{\theta}f')(3g-\tilde{\theta}g')\}+\gamma F^{\gamma}F'g\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}]
\end{aligned}
$$

$$\left.\begin{aligned}
&+\gamma\omega_0\omega_1\tilde{\theta}^2\sqrt{2f}\,[\gamma F^{\gamma-1}F'G\{f'-\tau^2\tilde{\theta}^2(2f-\tilde{\theta}f')\}\\
&+F^{\gamma}F'\{g'-\tau^2\tilde{\theta}(3g-\tilde{\theta}g')\}+F^{\gamma}\{f'G'+\tau^2(2f-\tilde{\theta}f')(G-\tilde{\theta}G')\}]\\
&+\gamma\omega_0\tilde{\theta}^2\Big[\frac{1}{2}\gamma(\gamma-1)F^{\gamma-2}F'G^2\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}+\gamma F^{\gamma-1}F'G\\
&\times\{g'-\tau^2\tilde{\theta}(3g-\tilde{\theta}g')\}+\gamma F^{\gamma-1}G\{f'G'+\tau^2(2f-\tilde{\theta}f')(G-\tilde{\theta}G')\}\\
&+F^{\gamma}\{g'G'+\tau^2(3g-\tilde{\theta}g')(G-\tilde{\theta}G')\}]\\
&-\omega_0(2\alpha-\omega_1)\tilde{\theta}^2F^{\gamma}[\gamma fF'\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}+F\{f'^2+\tau^2(2f-\tilde{\theta}f')^2\}]\\
&-\gamma\omega_0\tilde{\theta}^2F^{\gamma}\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}\frac{d}{d\tilde{\theta}}\left(\frac{Q}{M}\right)\\
&-\gamma\omega_0\tilde{\theta}^2F^{\gamma-1}[\gamma F'\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}+2\tau^2(2f-\tilde{\theta}f')F]\frac{Q}{M}\,,\\
Q=Q(\tilde{\theta})&=\frac{1}{2}[K\tilde{\theta}^2G^2-g'^2-\tau^2(3g-\tilde{\theta}g')^2]-\frac{\gamma}{\gamma-1}\tau^2\omega_0\tilde{\theta}^2\Big[\omega_1\frac{g}{\sqrt{2f}}F^{\gamma+1}\\
&+(\gamma+1)\omega_1\sqrt{2f}\,F^{\gamma}G+(2\alpha-\omega_1)fF^{\gamma+1}+\frac{\gamma(\gamma+1)}{2}G^2F^{\gamma-1}\Big],
\end{aligned}\right\}\quad(4.29)$$

and the boundary conditions for i and I are obtained as

$$\left.\begin{aligned}
i(1)&=\frac{1}{4}\Big[b_2-a_2+\frac{1}{4}\{f'(1)-f''(1)+2g'(1)\}\Big],\\
i'(1)&=\frac{1}{2}\Big[\frac{1}{4}f'''(1)-\frac{1}{2}f''(1)+\frac{1}{2}f'(1)+g''(1)-g'(1)\Big],\\
I(1)&=\rho_2+\frac{1}{2}G'(1)+\frac{1}{8}F''(1),
\end{aligned}\right\}\quad(4.30)$$

where the final condition, $I(1)$, is automatically satisfied by Eq. (4.28), once i is known from Eq. (4.27).

It is clear that the function h is analytic everywhere, while the function i is non-analytic near the surface $\tilde{\theta}=\theta_0$. The same mathematical procedure as was made or $g(\tilde{\theta})$ is, therefore, applied to clarify the behaviour of i near the singular point, indicating that the particular solution of the non-homogeneous equation for i has 1/2-power branch point at $\theta=\theta_0$. However, it seems to be convenient to represent the full solution of i near the singular point in the form of not $i(\tilde{\theta})$ itself but the combination $i(\tilde{\theta})+g'(\tilde{\theta})\frac{g_0}{f_0'}$. Hence, the series employed is

$$\left.\begin{aligned}
i(\tilde{\theta})+g'(\tilde{\theta})\frac{g_0}{f_0'}&=-\frac{\sqrt{2}\omega_1}{3\gamma}f_0'^{\frac{1}{2}}g_0\tilde{\varepsilon}^{\frac{3}{2}}(\chi_0+2\chi_1\tilde{\varepsilon}+\cdots)\\
&+i(\theta_0)\left(1+\frac{1}{2}\bar{\lambda}\tilde{\varepsilon}^2+\frac{1}{6}\bar{\lambda}\bar{\mu}\tilde{\varepsilon}^3+\cdots\right)\\
&+\sigma\tilde{\varepsilon}\left\{1+\frac{1}{2}\bar{\mu}\tilde{\varepsilon}+\frac{1}{6}(\bar{\lambda}+\bar{\mu}^2)\tilde{\varepsilon}^2+\cdots\right\}
\end{aligned}\right\}\quad(4.31)$$

$$
\left.\begin{aligned}
&+\frac{1}{2}\bar{\nu}\bar{\lambda}\tilde{\varepsilon}^{2}\left\{1+\frac{1}{3}\bar{\mu}\tilde{\varepsilon}+\frac{1}{12}(\bar{\lambda}+\bar{\mu}^{2})\tilde{\varepsilon}^{2}+\cdots\right\}\\
&+\frac{g_0^2}{f_0'}\lambda\tilde{\varepsilon}\left\{1+\frac{1}{2}\mu\tilde{\varepsilon}+\frac{1}{6}(\lambda+\mu^{2})\tilde{\varepsilon}^{2}+\cdots\right\}\\
&+\frac{g_0g_0'}{f_0'}\left\{1+\mu\tilde{\varepsilon}+\frac{1}{2}(\lambda+\mu^{2})\tilde{\varepsilon}^{2}+\cdots\right\}
\end{aligned}\right\}
$$

where

$$
\left.\begin{aligned}
&\lambda=\frac{D_0}{B_0}, \qquad \mu=\frac{C_0}{D_0}, \qquad \bar{\lambda}=\frac{R_0}{D_0}, \qquad \bar{\mu}=\frac{U_0}{D_0}, \qquad \bar{\nu}=\frac{L_2}{R_0},\\
&\chi_0=\frac{L_1}{L_0}-2(\bar{\mu}-\mu), \qquad \chi_1=\frac{1}{5}\left[\frac{L_1}{L_0}\bar{\mu}-2\bar{\mu}^2-2\bar{\lambda}\right]+\frac{2}{5}(\lambda+\mu^2),
\end{aligned}\right\} \tag{4.32}
$$

and where

$$
\left.\begin{aligned}
U_0=&\,\gamma\omega_0\theta_0^2(1+\tau^2\theta_0^2)^2f_0'^2\frac{F_0^\gamma\bar{M}_0'}{\bar{M}_0^2}\\
&-\gamma\omega_0\theta_0^2(1+\tau^2\theta_0^2)\frac{f_0'F_0^\gamma}{\bar{M}_0}\{(1+\tau^2\theta_0^2)f_0''-6\tau^2\theta_0f_0'\},\\
R_0=&\,4f_0'^2-4\tau^2\gamma\omega_0\theta_0^3(1+\tau^2\theta_0^2)f_0'^2\frac{F_0^\gamma\bar{M}_0'}{\bar{M}_0^2}\\
&-4\tau^2\gamma\omega_0\theta_0^2\frac{f_0'F_0^\gamma}{\bar{M}_0}\{(1+\tau^2\theta_0^2)(f_0'-\theta_0f_0'')+2\tau^2\theta_0^2f_0'\},\\
L_0=&-\frac{1}{2\sqrt{2}}\omega_0\omega_1\theta_0^2(1+\tau^2\tau_0^2)f_0'^{\frac{1}{2}}g_0F_0^{\gamma+1}-\gamma\omega_0\theta_0^2(1+\tau^2\theta_0^2)\frac{f_0'F_0^\gamma}{\bar{M}_0}Q_3,\\
L_1=&\,\frac{1}{\sqrt{2}}\omega_0\omega_1\theta_0^2f_0'^{\frac{1}{2}}F_0^\gamma[(\gamma+1)(1+\tau^2\theta_0^2)f_0'\dot{G}_0+2F_0\{g_0'-\tau^2\theta_0(3g_0-\theta_0g_0')\}]\\
&+\gamma\omega_0\theta_0^2F^{\gamma-1}\phi_1[\gamma f_0'G_0(1+\tau^2\theta_0^2)+F_0\{g_0'-\tau^2\theta_0(3g_0-\theta_0g_0')\}]\\
&-\gamma\omega_0\theta_0^2(1+\tau^2\theta_0^2)\frac{f_0'F_0^\gamma}{\bar{M}_0}(Q_4\bar{M}_0-Q_1\bar{M}_0')\\
&-\gamma\omega_0\theta_0^2f_0'F^{\gamma-1}\{\gamma F_0'(1+\tau^2\theta_0^2)-2\tau^2\theta_0F_0\}\frac{Q_1}{\bar{M}_0},\\
L_2=&\,6f_0'g_0g_0'-9f_0''g_0^2+\sqrt{2}\,\gamma\omega_0\omega_1\theta_0^2(1+\tau^2\theta_0^2)f_0'^{\frac{3}{2}}F_0^\gamma\phi_1\\
&-\gamma\omega_0\theta_0^2F_0^{\gamma-1}[\gamma\tau^2\theta_0f_0'G_0^2-\gamma(1+\tau^2\theta_0^2)f_0'G_0\phi_2-\tau^2F_0(3g_0-\theta_0g_0')G_0\\
&-F_0\{g_0'-\tau^2\theta_0(3g_0-\theta_0g_0')\}\phi_2]-\omega_0(2\alpha-\omega_1)\theta_0^2(1+\tau^2\theta_0^2)f_0'^2F_0^{\gamma+1}\\
&-\gamma\omega_0\theta_0^2(1+\tau^2\theta_0^2)\frac{f_0'F_0^\gamma}{\bar{M}^2}(\bar{M}_0Q_5-\bar{M}_0'Q_2)+2\tau^2\gamma\omega_0\theta_0^3f_0'F_0^\gamma\frac{Q_2}{\bar{M}_0},\\
Q_1=&-\frac{1}{\sqrt{2}}\frac{\gamma}{\gamma-1}\tau^2\omega_0\omega_1\theta_0^2f_0'^{-\frac{1}{2}}g_0F_0^{\gamma+1},\\
Q_2=&\,\frac{1}{2}\{K\theta_0^2G_0^2-g_0'^2-\tau^2(3g_0-\theta_0g_0')^2\}-\frac{\gamma^2(\gamma+1)}{2(\gamma-1)}\tau^2\omega_0\theta_0^2G_0^2F_0^{\gamma-1},
\end{aligned}\right\} \tag{4.33}
$$

$$Q_3 = -\frac{1}{2} Q_1$$

$$Q_4 = K\theta_0^2 G_0 \phi_1 + \{g_0' - \tau^2\theta_0(3g_0 - \theta_0 g_0')\} \frac{A_0}{D_0} - \frac{\sqrt{2}\gamma}{\gamma - 1} \tau^2 \omega_0 \omega_1 \theta_0 f_0'^{-\frac{1}{2}} g_0 F_0^{\gamma+1}$$
$$- \frac{\gamma^2(\gamma+1)}{\gamma-1} \tau^2 \omega_0 \theta_0^2 G_0 \phi_1 F_0^{\gamma-1}$$
$$- \frac{1}{\sqrt{2}} \frac{\gamma}{\gamma-1} \tau^2 \omega_0 \omega_1 \theta_0^2 f_0'^{-\frac{1}{2}} F_0^{\gamma} \{g_0' F_0 - (\gamma+1) f_0' G_0\},$$

$$Q_5 = K\theta_0 G_0^2 + K\theta_0^2 G_0 \phi_2 - \{g_0' - \tau^2\theta_0(3g_0 - \theta_0 g_0')\} \left(\frac{B_0}{D_0} g_0 + \frac{C_0}{D_0} g_0'\right)$$
$$- 2\tau^2(3g_0 - \theta_0 g_0') g_0' - \frac{\gamma^2(\gamma+1)}{\gamma-1} \tau^2 \omega_0 \theta_0 G_0^2 F_0^{\gamma+1}$$
$$- \frac{\sqrt{2}\,\gamma(\gamma+1)}{\gamma-1} \tau^2 \omega_0 \omega_1 \theta_0^2 f_0'^{\frac{1}{2}} F_0^{\gamma} \phi_1 - \frac{\gamma}{\gamma-1} \tau^2 \omega_0 \theta_0^2 F_0^{\gamma-1} \{\gamma(\gamma+1) G_0 \phi_2$$
$$+ (2\alpha - \omega_1) f_0' F_0^2\},$$

$$\phi_1 = -(1+\tau^2\theta_0^2) \frac{A_0}{D_0} \frac{f_0'}{M_0} + \frac{1}{\sqrt{2}} \frac{\gamma}{\gamma-1} \tau^2 \omega_0 \omega_1 \theta_0^2 \frac{f_0'^{\frac{1}{2}} F_0^{\gamma+1}}{M^0},$$

$$\phi_2 = \frac{1}{M_0} \left[(1+\tau^2\theta_0^2) f_0' \left(\frac{B_0}{D_0} g_0 + \frac{C_0}{D_0} g_0'\right) + \{(1+\tau^2\theta_0^2) f_0'' - 3\tau^2\theta_0 f_0'\} g_0'\right.$$
$$\left. + 3\tau^2(f_0' - \theta_0 f_0'') g_0\right] - \frac{f_0' M_0'}{M_0^2} \{(1+\tau^2\theta_0^2) g_0' - 3\tau^2 \theta_0 g_0\}.$$

The unknown constant σ in Eq. (4.31) can be obtained from the equation

$$i'(\theta_0) + g''(\theta_0) \frac{g_0}{f_0'} = \sigma + \frac{g_0}{f_0'} (\lambda g_0 + \mu g_0'). \tag{4.34}$$

The differential equations for h and i are also integrated numerically in the same procedure as outlined previously. Higher terms in series expansions of the stream function and density can also be obtained in quite the same way as outlined above, if necessary.

Now that the stream function is determined, the body shape is obtained from the tangency condition that ϕ must vanish along the body surface $\tilde{\theta} = \theta_0$.

$$\phi(x, \theta_0) = 0 \tag{4.35}$$

Thus, the radius of the body is obtained by use of Eqs. (4.6) and (4.35) as

$$r_b = \theta_0 x + \frac{g_0}{f_0'} l x^2 + \left\{m \frac{h_0}{f_0'} + l^2 \left(\frac{i_0}{f_0'} + \frac{g_0 g_0'}{f_0'^2} - \frac{1}{2} \frac{g_0^2}{f_0'^3} f_0''\right)\right\} x^3 + \cdots. \tag{4.36}$$

The unknown constants, l, m, in Eq. (4.36) are determined from the given body shape. Also, the surface density distribution is given as

$$\left.\begin{aligned} \rho_b = \rho(x, \theta_0) = F_0 - lxG_0 - x^2 \Big[mH_0 + l^2 \Big\{ \Big(I_0 + \frac{g_0}{f_0'} G_0' \Big) \\ - \frac{1}{f_0'} \Big(i_0 + \frac{g_0 g_0'}{f_0'} \Big) - \frac{1}{2} \Big(\frac{g_0}{f_0'} \Big)^2 \Big(F_0'' - \frac{f_0''}{f_0'} \Big) \Big\} \Big] - \cdots, \end{aligned}\right\} \tag{4.37}$$

where

$$\left.\begin{aligned}\bar{M}_0\left(I_0+\frac{g_0}{f_0'}G_0'\right)=&(1+\tau^2\theta_0^2)f_0'\left(i_0'+g_0''\frac{g_0}{f_0'}\right)-4\tau^2\theta_0 f_0' i_0\\&+\frac{1}{2}\{K\theta_0^2G_0^2-g_0'^2-\tau^2(3g_0-\theta_0 g_0')^2\}\\&-\frac{\gamma^2(\gamma+1)}{2(\gamma-1)}\tau^2\omega_0\theta_0^2G_0^2F_0^{\gamma-1}\\&+\{(1+\tau^2\theta_0^2)f_0''-3\tau^2\theta_0 f_0'\}\frac{g_0 g_0'}{f_0'}\\&+3\tau^2(f_0'-\theta_0 f_0'')\frac{g_0^2}{f_0'}\\&-g_0\frac{\bar{M}_0'}{\bar{M}_0}\{(1+\tau^2\theta_0^2)g_0^2-3\tau^2\theta_0 g_0\}\,.\end{aligned}\right\}\tag{4.38}$$

Initial radio of shock to body curvature and initial gradient of surface pressure are expressed, respectively, as

$$\frac{K_s}{K_b}=-\frac{f_0'}{2g_0}\left(\frac{1+\tau^2\theta_0^2}{1+\tau^2}\right)^{\frac{3}{2}},\tag{4.39}$$

$$\frac{1}{K_b}\frac{dC_p}{dx}=-\frac{\tau\gamma\omega_0(1+\tau^2\theta_0^2)^{\frac{3}{2}}{}'_0 f_0' F_0^{\gamma-1}G_0}{g_0}\tag{4.40}$$

where

$$G_0=\frac{f_0'}{\bar{M}_0}\{(1+\tau^2\theta_0^2)g_0'-3\tau^2\theta_0 g_0\}\,.\tag{4.41}$$

5. ADDITIONAL DISCUSSIONS AND RESULTS

In the last section a general approach to supersonic flows past axially symmetric bodies with an attached shock wave has been presented, assuming that the flow downstream of the shock wave consists of a basic conical field upon which is superimposed a perturbation field due to body curvature. The present choice of conical field as the basic field is quite rigorous so long as the flow remains to be supersonic everywhere, while it seems to be unrealistic for such a flow pattern that the body is so thick that the flow downstream of the shock wave is partially subsonic, since the flow past a finite cone with very large semi-vertex angle is no longer conical and is modified by a perturbation due to rotationality caused by a curved shock wave.

In spite of this circumstance the present approach may still be applicable further to the case of partially subsonic flow downstream of the shock wave, if the body has an analytic shape such as given by Eq. (4.1). The reason for this is in that the flow downstream of a shock wave attached to an infinite cone still remains to be conical even in the subsonic pattern and the choice of either conical field, as the basic field, appropriate to an infinite cone or any other known field does not degenerate the essential feature of the approach to the problem under consideration. In this sense, the present choice of the basic conical field is only conventional.

However, it must be noted that the present choice is compatible with only an infinite cone in the case of subsonic flow pattern and, consequently, the solution to a finite cone cannot be obtained only be vanishing l, m, ... in Eq. (4.6). In this sense, the approach to supersonic flow past a finite cone with subsonic flow pattern may be considered to be an isolated problem, which will be discussed in detail in the next section.

The flow behaviour near the region of vertex of axially symmetric bodies is of the most interest, which can be clarified by detailed arguments on solutions of the simultaneous equations for (g, G). In the case of supersonic flow everywhere, the numerical integration of Eq. (4.12) can be easily carried out step by step inward from the shock wave without any difficulty except for a singularity at point $\tilde{\theta}=\theta_0$, while in the case of partially subsonic flow downstream of the shock wave another mathematical difficulty takes place at a point $\tilde{\theta}=\theta_p$, at which $\overline{M}(\tilde{\theta}_p)$ given by Eq. (4.15) vanishes. This indicates that all coefficients in differential equatoin for g, which are given by Eq. (4.14), become infinite at $\tilde{\theta}=\theta_p$. Since the sonic density F_p corresponding to the basic infinite cone is expressed by

$$F_p^{\gamma-1}=\frac{(\gamma-1)K}{\gamma(\gamma+1)\tau^2\omega_0}, \tag{5.1}$$

the point $\tilde{\theta}=\theta_p$ has a physical meaning of sonic point appropriate to the basic infinite cone.

Although the function g defined by Eq. (4.12) seems apparently to become non-analytic near the point $\tilde{\theta}=\theta_p$, it is still analytic essentially because of Eq. (4.13). This curiosity clearly arises from the procedure of eliminating G from the simultaneous for (g, G). Therefore, the differential equation for g can be expressed near the point $\tilde{\theta}=\theta_p$ as

$$\{f'-\tau^2\tilde{\theta}(2f-\tilde{\theta}f')\}g'+3\tau^2(2f-\tilde{\theta}f')g+N=0, \tag{5.2}$$

which is analytic.

Differentiation of the above equation with respect to $\tilde{\theta}$ then gives

$$\alpha_0 g''+\alpha_1 g'+\alpha_2 g=\alpha_3, \tag{5.3}$$

where

$$\left.\begin{aligned}
&\alpha_0=f'_p-\tau^2\theta_p(2f_p-\theta_p f'_p),\\
&\alpha_1=f''_p-\tau^2\theta_p(f'_p-\theta_p f''_p)+2\tau^2(2f_p-\theta_p f'_p),\\
&\alpha_2=3\tau^2(f'_p-\theta_p f''_p),\\
&\alpha_3=-N'(\theta_p),
\end{aligned}\right\} \tag{5.4}$$

and where subscript p denotes the conditions at $\tilde{\theta}=\theta_p$. Since $\tilde{\theta}=\theta_p$ is a regular point of Eq. (5.3), a series expansion of g is possible in the vicinity of the point, in which two unknown constants involved in the general solution of the homogeneous equation can be determined as the values of g and its first derivative at $\tilde{\theta}=\theta_p$. Thus, the series employed is

$$\left.\begin{aligned} g(\tilde{\theta}) &= g(\theta_p)\left(1-\frac{1}{2}\frac{\alpha_2}{\alpha_0}\tilde{\eta}^2+\frac{1}{6}\frac{\alpha_1}{\alpha_0}\frac{\alpha_2}{\alpha_0}\tilde{\eta}^3+\cdots\right) \\ &+g'(\theta_p)\tilde{\eta}\left\{1+\frac{1}{2}\frac{\alpha_1}{\alpha_0}\tilde{\eta}+\frac{1}{6}\left(\frac{\alpha_1^2}{\alpha_0^2}-\frac{\alpha_2}{\alpha_0}\right)\tilde{\eta}^2+\cdots\right\} \\ &+\frac{1}{2}\tilde{\eta}^2\left(\frac{\alpha_3}{\alpha_0}-\frac{1}{3}\frac{\alpha_1}{\alpha_0}\frac{\alpha_3}{\alpha_0}\tilde{\eta}-\cdots\right), \end{aligned}\right\} \tag{5.5}$$

where

$$\tilde{\eta}=\tilde{\theta}-\theta_p. \tag{5.6}$$

The numerical integration of the function g is carried out by use of the following procedure. The solution of step by step integration of Eq. (4.12), which starts inward from the known values at $\tilde{\theta}=1$, is joined at the two points in the very vicinity of the regular point $\tilde{\theta}=\theta_p$ with the series expansion of g, Eq. (5.5). Once $g(\theta_p)$ and $g'(\theta_p)$ are thus determined, the integration of Eq. (4.12) is then continued again step by step, starting further inward from the known values at a point in the very aft vicinity of $\tilde{\theta}=\theta_p$. The jointing procedure of the step by step solution of g thus obtained with the series expansion of g about the singular point at $\tilde{\theta}=\theta_0$ is quite the same as has been already mentioned in the last section.

In Fig. 2 are presented the results of numerical computation for g and G at Mach number of 2, in which β_s denotes initial shock wave angle. The interval of step by step integration of Eq. (4.12) was taken to be 0.05. As is seen in the figure, the value of g_0 decreases monotonously with increase of β_s. It must be noticed that

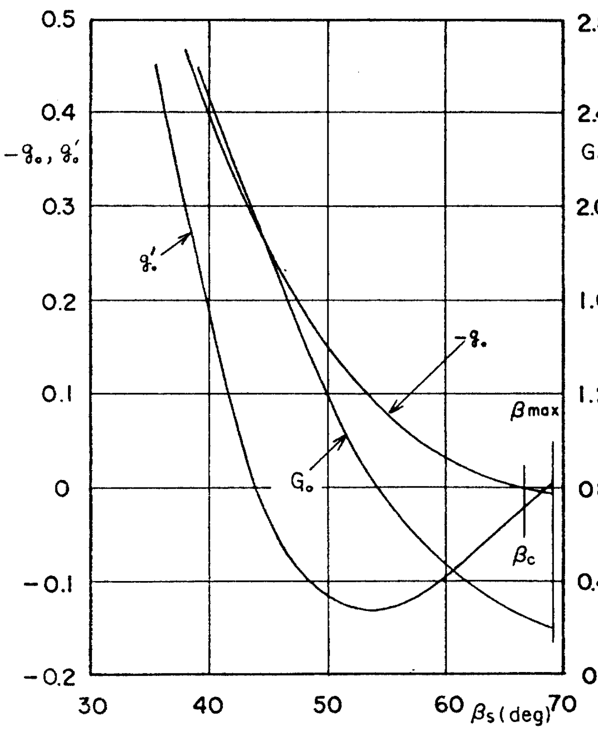

FIG. 2. Variable of g_0, g_0' and G_0 with initial shock wave angle. $M=2$.

there exists a particular value of initial shock wave angle, β_c, for which g_0 vanishes and, consequently, the second term in right-hand side of Eq. (4.36) becomes zero.

For any initial shock wave angle smaller than β_c, if the body has an analytic shape and finite curvature at its vertex, the unknown constant, l, in Eq. (4.36), which corresponds to the initial shock wave curvature, can be obtained definitely. At point $\beta_s=\beta_c$, however, l becomes indefinite because of $g_0=0$, thus indicating that no inviscid solution satisfying the given tangency condition exists, if the body has finite curvature at its vertex. In spite of this circumstance a certain physical solution still seems to be capable of existing by taking the viscous effect into consideration, since the displacement thickness of the boundary layer may distort the effective shape of the boundary streamline associated with the outer inviscid flow into ogival shape with cusp at its vertex, for which a mathematical solution is only available even at $\beta_s=\beta_c$, as is seen in Eq. (4.36).

In the case of plane flow the same mathematical difficulty as has been just predicted in axially symmetric flow occurs at the well known Crocco point, on which many authors have already discussed in detail by use of the shock polar in hodograph plane. In the present approach, therefore, such a point as gives $g_0=0$ is to be called 'Crocco point in axially symmetric flow'.

When β_s exceeds Crocco point but is smaller than its maximum value, β_{max}, a regular inviscid solution is again valid mathematically but giving the reversed sign of l because g_0 is positive in this range. This clearly indicates a physical meaning that, in order to have an attached shock wave at the vertex of the axially symmetric bodies, the shock wave must have a concave shape to the free stream if the body is convex. This result, however, is a consequence of the analytical approach used in the development and seems to be physically trivial, because, in the range between β_c and β_{max}, the flow phenomena near the region of vertex of the axially symmetric body depend strongly upon the shock wave shape but are less sensitive to the body shape. These circumstances may be obviously confirmed by the subsequent qualitative discussions.

The fact that Eq. (4.36) defines the initial curvature of the boundary streamline suggests, together with the flow conditions just at the vertex of the body, a possibility to draw in hodograph plane a shock polar and spines for axially symmetric flow analogously to two-dimensional ones. However ,it must be particularly noticed that the axially symmetric shock polar thus described indicates downstream conditions only at the intersection of the shock wave with the body boundary, while two-dimensional one is valid just downstream of the shock wave throughout. From the results shown in Fig. 2, it can be easily found that the spines must change the direction of their beginning from inwards to outwards as the maximum flow deflection angle is approached through Crocco point, at which the initial direction of the spine is parallel to the flow deflection angle. Fig. 3 illustrates a qualitative pattern of the axially symmetric shock polar, where q, q^* and δ denote resultant velocity, sonic speed and flow deflection angle at the intersection, respectively. These flow characteristics concerning the axially symmetric shock polar are qualitatively quite the same as those of the two-dimensional one.

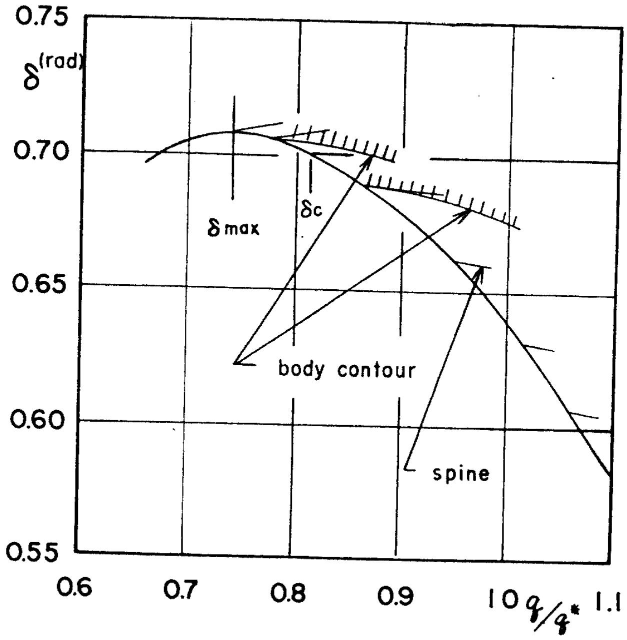

FIG. 3. Axially symmetric shock polar applicable at only intersection. $M=2$.

When the shock polar is applied to the convex body with finite curvature at its vertex, it is easily known that convex curvature of the shock wave causes no difficulty at the intersection and the solution near the vertex is regular so long as the spines are only inward bound, since, along a convex surface, the flow behind an attached shock wave must be accelerated irrespective of its having initially either supersonic or subsonic speed and, therefore, the body contour is always inward bound. At Crocco point, however, a contradiction takes place that the spine is radial at the intersection, while the body contour still remains to be inward bound. This difficulty suddenly terminates the regular characteristics of the flow and requires a singularity to deflect a streamline rapidly towards the body contour in order to have an attached shock wave. When the flow deflection angle exceeds Crocco point, the same contradiction in boundary conditions as at Crocco point still remains in hodograph plane and requires a singularity, although mathematical soluton may be regular for concave curvature of the shock wave. Therefore, the regular solution may be physically trivial in this range. The singularity required to deflect a streamline suddenly towards the body contour must have the same character as that of a finite cone after Crocco point is passed.

In Figs. 4 and 5 are shown, respectively, the initial ratio of shock to body curvature and initial gradient of surface pressure for axially symmetric body at Mach number of 2. As is seen in Fig. 4, the curvature ratio increases as β_s grows and becomes infinite at Crocco point. The result shown by a dotted line in the range of shock wave angle between β_c and β_{max} is physically trivial, although it may be reasonable mathematically. Qualitatively the same trend is shown by the initial gradient of surface pressure, as is seen in Fig. 5.

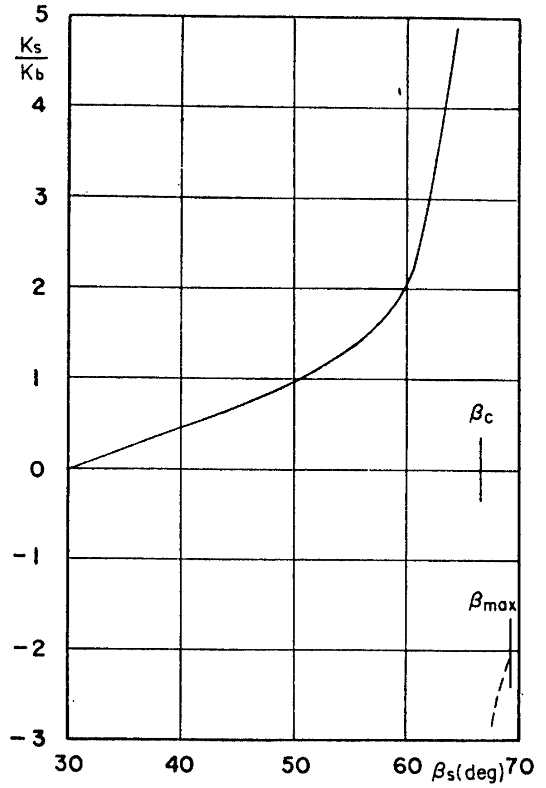

FIG. 4. Initial ratio of shock to body curvature for axially symmetric flow. $M=2$.

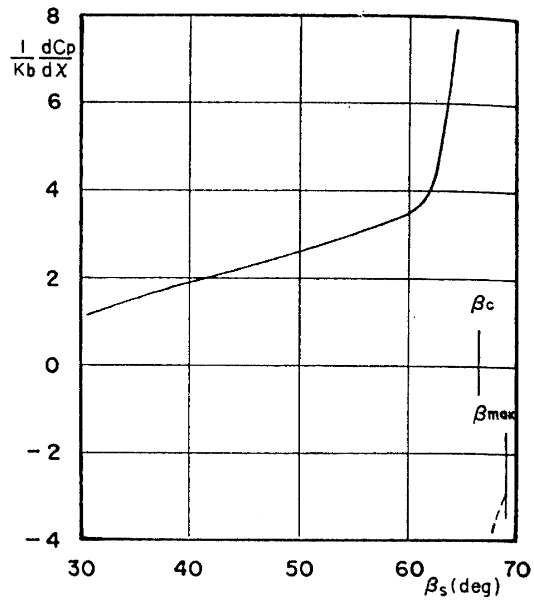

FIG. 5. Initial gradient of surface pressure for axially symmetric flow. $M=2$.

6. A Finite Cone with Subsonic Flow Pattern

In case that the flow past a finite cone is supersonic everywhere, there exists a conical solution which can be predicted by Eq. (3.5). However, if the cone angle grows so large that the flow downstream of the attached shock wave becomes partially subsonic, the shock wave may be curved because the flow along the cone surface must be accelerated to arrive at sonic speed just at the shoulder. Such a flow can be analysed in such a way that the flow downstream of the shock wave is assumed to consist of a basic conical field given by Eq. (3.5) upon which is superimposed a perturbation field due to curved shock wave. This perturbation field, however, seems to have to involve a singularity at the vertex of the cone for the following reasons.

When the shock polar shown in Fig. 3 is applied to the problem under consideration, it is readily evident that a contradiction in initial directions between spine and body contour at the intersection occurs at any angle of the cone with subsonic flow pattern except for the Crocco point, at which the spine has parallel direction to the body contour. This particular cone angle corresponding to Crecco point is, therefore, the only case that is not singular. The overdetermination in boundary conditions, of course, takes place at one point, the intersection, and causes there a singularity so as to deflect a streamline suddenly towards the body contour. These characters of the subsonic flow pattern past a finite cone are qualitatively the same as those of the plane finite wedge.

Thus, the shock wave shape, stream function, density and entropy function may be, respectively, assumed to have the forms

$$
\left.\begin{aligned}
r_s &= x - \frac{1}{\kappa+1} l x^{\kappa+1} - \cdots\cdots , \\
\psi &= x^2 f(\theta) - l x^{\kappa+2} g(\theta) - \cdots\cdots , \\
\rho &= F(\theta) - l x^{\kappa} G(\theta) - \cdots\cdots , \\
\omega &= \omega_0 \{1 - \omega_1 l (2\psi)^{\frac{\kappa}{2}} - \cdots\cdots\} ,
\end{aligned}\right\} \quad (6.1)
$$

$$\theta = \frac{x}{r}$$

where l, $\kappa, \cdots$ are the unknown constants to be determined by the given boundary conditions. It must be noted that κ indicates the order of singularity at the vertex of the cone and is assumed to be positive.

Substituting Eq. (6.1) into Eq. (2.7) and equating like powers of x yields for (f, F), (g, G), ... the simultaneous equations, respectively. Equations for (f, F) are quite the same that is given by Eq. (3.5) as well as their boundary conditions, which are given by Eq. (3.6). Equation for g, which is obtained by eliminating G from the simultaneous equations for (g, G), and equation for G can be written, respectively, in the forms

$$\bar{D} g'' = \bar{A} + \bar{B} g + \bar{C} g' , \quad (6.2)$$

$$\bar{M} G = \{f' - \tau^2\theta(2f - \theta f')\} g' + (\kappa+2)\tau^2(2f - \theta f') g + \bar{N} , \quad (6.3)$$

where

$$
\left.\begin{aligned}
\bar{D} &= \gamma\omega_0 \frac{\theta^2 F^{\gamma}}{\bar{M}} \{f' - \tau^2\theta(2f - \theta f')\}^2 - 4f^2 \doteq D(\theta) , \\
\bar{C} &= \gamma\omega_0 \frac{\theta^2 F^{\gamma}}{\bar{M}} \frac{\bar{M}'}{\bar{M}} \{f' - \tau^2\theta(2f - \theta f')\}^2 - 4(\kappa+1) f f' - \gamma\omega_0\theta^2(1 + \tau^2\theta^2) F^{\gamma} F' \\
&\quad - \gamma\omega_0 \frac{\theta^2 F^{\gamma-1}}{\bar{M}} \{f' - \tau^2\theta(2f - \theta f')\} [F \{f'' + (2\kappa+1)\tau^2(2f - \theta f') \\
&\quad - \tau^2\theta(f' - \theta f'')\} + \gamma F' \{f' - \tau^2\theta(2f - \theta f')\}] , \\
\bar{B} &= (\kappa+2)(\kappa-1) f'^2 + 4(\kappa+2) f f'' + (\kappa+2)\tau^2\gamma\omega_0\theta^3 F^{\gamma} F' \\
&\quad - (\kappa+2)\tau^2\gamma\omega_0 \frac{\theta^2 F^{\gamma-1}}{\bar{M}} [\{f' - \tau^2\theta(2f - \theta f')\} \{F(f' - \theta f'') \\
&\quad + \gamma F'(2f - \theta f')\} + \kappa\tau^2(2f - \theta f')^2 F] \\
&\quad + (\kappa+2)\tau^2\gamma\omega_0 \frac{\theta^2 F^{\gamma}}{\bar{M}} \frac{\bar{M}'}{\bar{M}} (2f - \theta f') \{f' - \tau^2\theta(2f - \theta f')\} , \\
A &= -\gamma\omega_0\theta^2 F^{\gamma} \{f' - \tau^2\theta(2f - \theta f')\} \frac{d\theta}{d} \left(\frac{\bar{N}}{\bar{M}}\right) \\
&\quad - \gamma\omega_0\theta^2 F^{\gamma-1} [\gamma F' \{f' - \tau^2\theta(2f - \theta f')\} + \kappa\tau^2(2f - \theta f') F] \frac{\bar{N}}{\bar{M}}
\end{aligned}\right\} \quad (6.4)
$$

$$
\left.\begin{aligned}
&-\gamma\omega_0\omega_1\theta^2(2f)^{\frac{\kappa}{2}}F^{\gamma}F'\{f'-\tau^2\theta(2f-\theta f')\}\\
&-\kappa\omega_0\omega_1\theta^2(2f)^{\frac{\kappa-2}{2}}F^{\gamma+1}\{f'^2+\tau^2(2f-\theta f')^2\},\\
\bar{N}=&\frac{1}{\gamma-1}\tau^2\omega_0\omega_1\theta^2F^{\gamma+1}(2f)^{\frac{\kappa}{2}}.
\end{aligned}\right\}
$$

The boundary conditions for g and G are obtained from Eq. (4.8) as

$$
\left.\begin{aligned}
g(1)&=-\frac{1}{\kappa+1}b_0,\\
g'(1)&=-a_1+\frac{1}{\kappa+1}\{f'(1)-f''(1)\},\\
G(1)&=\rho_1-\frac{1}{\kappa+1}F'(1),
\end{aligned}\right\}\quad(6.5)
$$

where the final condition, $G(1)$, is automatically satisfied by Eq. (6.3), once g is known from Eq. (6.2).

It is clear that the function $\bar{M}(\theta)$ vanishes at point $\theta=\theta_p$, which indicates sonic surface appropriate to the basic conical field. The mathematical difficulty in integration of Eq. (6.2), which occurs at this point, can be treated in the same way as has been already mentioned in the last section by considering an analytical continuation of step by step solution of Eq. (6.2) with a series expansion of g about the point $\theta=\theta_p$. Thus, the series employed is

$$
\left.\begin{aligned}
g(\theta)=g(\theta_p)&\left\{1-\frac{1}{2}\frac{\bar{\alpha}_2}{\bar{\alpha}_0}\eta^2+\frac{1}{6}\frac{\bar{\alpha}_1}{\bar{\alpha}_0}\frac{\bar{\alpha}_2}{\bar{\alpha}_0}\eta^3+\cdots\cdot\right\}\\
&+g'(\theta_p)\eta\left\{1-\frac{1}{2}\frac{\bar{\alpha}_1}{\bar{\alpha}_0}\eta+\frac{1}{6}\left(\frac{\bar{\alpha}_1^2}{\bar{\alpha}_0^2}-\frac{\bar{\alpha}_2}{\bar{\alpha}_0}\right)\eta^2+\cdots\cdot\right\}\\
&+\frac{1}{2}\eta^2\left\{\frac{\bar{\alpha}_3}{\bar{\alpha}_0}-\frac{1}{3}\frac{\bar{\alpha}_1}{\bar{\alpha}_0}\frac{\bar{\alpha}_3}{\bar{\alpha}_0}\eta-\cdots\cdot\right\}.
\end{aligned}\right\}\quad(6.6)
$$

$$
\eta=\theta-\theta_p,
$$

where

$$
\left.\begin{aligned}
&\bar{\alpha}_0-f'_p-\tau^2\theta_p(2f_p-\theta_pf'_p),\\
&\bar{\alpha}_1=f''_p-\tau^2\theta_p(f'_p-\theta f''_p)+(\kappa+1)\tau^2(2f_p-\theta_pf'_p)\\
&\alpha_2=(\kappa+2)\tau^2(f'_p-\theta_pf''_p)\\
&\alpha_3=-N'(\theta_p)
\end{aligned}\right\}\quad(6.7)
$$

and where subscript p denotes the conditions at $\theta=\theta_p$.

Since the coefficient $\bar{A}$ in differential equation for g, Eq. (6.2), involves the term $(2f)^{\frac{\kappa-2}{2}}$, the function g becomes non-analytic near the surface of the cone $\theta=\theta_0$ if $\kappa<2$, while it is analytic if $\kappa>2$, because f vanishes at $\theta=\theta_0$. The singular behaviour of g for $\kappa<2$ can be treated in quite the same way as has been already mentioned in Section 4 by making a series expansion of g about the singular point, which is given by ($\kappa<2$)

$$\left.\begin{aligned} g(\theta)=&-\frac{4}{\kappa(\kappa+2)}\frac{\bar{A}_0}{\bar{D}_0}\varepsilon^{\frac{\kappa}{2}+1}\left\{1+\frac{2}{\kappa+4}\frac{\bar{C}_0}{\bar{D}_0}\varepsilon\right.\\ &\left.+\frac{4}{(\kappa+6)(\kappa+4)}\left(\frac{\bar{C}_0^2}{\bar{D}_0^2}+\frac{\bar{B}_0}{\bar{D}_0}\right)\varepsilon^2+\cdots\right\}\\ &+g(\theta_0)\left\{1+\frac{1}{2}\frac{\bar{B}_0}{\bar{D}_0}\varepsilon^2+\frac{1}{6}\frac{\bar{B}_0}{\bar{D}_0}\frac{\bar{C}_0}{\bar{D}_0}\varepsilon^3+\cdots\right\}\\ &+g'(\theta_0)\varepsilon\left\{1+\frac{1}{2}\frac{\bar{C}_0}{\bar{D}_0}\varepsilon+\frac{1}{6}\left(\frac{\bar{C}_0^2}{\bar{D}_0^2}+\frac{\bar{B}_0}{\bar{D}_0}\right)\varepsilon^2+\cdots\right\}, \end{aligned}\right\} \tag{6.8}$$

$$\varepsilon=\theta-\theta_0,$$

where

$$\left.\begin{aligned} \bar{D}_0=&\gamma\omega_0\theta_0^2(1+\tau^2\theta_0^2)^2 f_0'^2\frac{F_0^\gamma}{\bar{M}_0},\\ \bar{C}_0=&\gamma\omega_0\theta_0^2(1+\tau^2\theta_0^2)^2 f_0'^2\frac{F_0^\gamma\bar{M}_0'}{\bar{M}_0^2}-\gamma\omega_0\theta_0^2(1+\tau^2\theta_0^2)\frac{f_0'F_0^\gamma}{\bar{M}_0}\{f_0''(1+\tau^2\theta_0^2)\\ &-(2\kappa+2)\tau^2\theta_0 f_0'\},\\ \bar{B}_0=&(\kappa+2)(\kappa-1)f_0'^2-(\kappa+2)\tau^2\gamma\omega_0\theta_0^3(1+\tau^2\theta_0^2)f_0'^2\frac{F_0^\gamma\bar{M}_0'}{\bar{M}_0^2}\\ &-(\kappa+2)\tau^2\gamma\omega_0\theta_0^2\frac{f_0'F_0^\gamma}{\bar{M}_0}\{(1+\tau^2\theta_0^2)(f_0'-\theta_0 f_0'')+\kappa\tau^2\theta_0^2 f_0'\},\\ \bar{A}_0=&2^{\frac{\kappa-2}{2}}\kappa\omega_0\omega_1\theta_0^2(1+\tau^2\theta_0^2)f_0'^{\frac{\kappa+2}{2}}F_0^{\gamma+1}\left\{1+\frac{\gamma^2}{\gamma-1}\tau^2\omega_0\theta_0^2\frac{F_0^\gamma}{\bar{M}_0}\right\} \end{aligned}\right\} \tag{6.9}$$

Here an attention must be paid to the fact that the unknown constant κ is involved in both the differential equation for g and its boundary conditions at $\theta=1$, and, consequently, the integration of Eq. (6.2) cannot be made readily. However, from the tangency condition that ϕ must vanish along the cone surface, it is easily known that g must vanish at $\theta=\theta_0$ because of $f(\theta_0)=0$, that is

$$g(\theta_0;\ \kappa)=0. \tag{6.10}$$

Therefore, the unknown constant κ can be determined from the condition given above.

Since κ is initially unknown, the integration of Eq. (6.2) must be carried out, by use of a trial and error method, by assuming several proper values of κ, although it may be very laborious. The analytical continuations at points θ_p and θ_0 are carried out in the same way as mentioned previously. By use of κ thus determined, the integration of Eq. (6.2) is repeated again once more and the values of $g'(\theta_0)$ is obtained. From Eq. (6.3) $G(\theta_0)$ can be expressed as

$$G_0=G(\theta_0)=(1+\tau^2\theta_0^2)\frac{f_0'g_0'}{\bar{M}_0}. \tag{6.11}$$

The another unknown constant l can be determined from the condition that the flow must be accelerated to arrive at sonic speed just at the shoulder ($x=1$) of the cone, that is

$$l=\frac{F_0-\rho^*}{G_0} \tag{6.12}$$

where

$$\rho^*=\rho(\theta_0;\ x=1)=\left[\frac{(\gamma-1)K}{\gamma(\gamma+1)\tau^2\omega_0}\right]^{\frac{1}{\gamma-1}}. \tag{6.13}$$

In Fig. 6 is presented the variation of κ with initial shock wave angle β_s at Mach number of 2. As is seen in the figure, κ decreases with increase of β_s and becomes of unity at Crocco point, β_c. In the range of shock wave angle below β_c, the initial shock wave curvature is found to be zero, while it is infinite in the range between β_c and β_{max}, thus indicating that only the flow with Crocco point at the vertex has normal acceleration and, therefore, a regular attached shock wave with finite curvature throughout. These trends are quite the same as have been already found in the problem of a plane finite wedge with subsonic flow pattern.

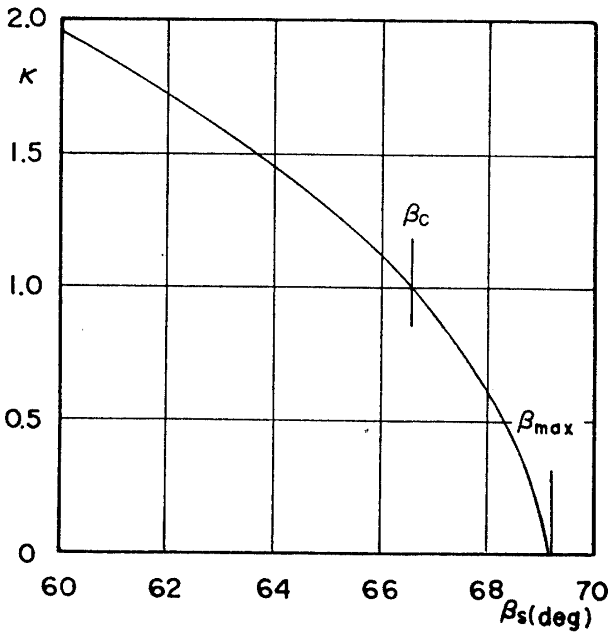

FIG. 6. Variation of with initial shock was angle. $M=2$.

Fig. 7 sohws the theoretical results of shock wave shape and sonic line for a finite cone with semi-vertex angle of 38.05° and at Mach number of 2. A dotted line in the figure indicates the shock wave shape obtained from Taylor-Maccoll's conical theory. So long as the shock wave shape is concerned, deviation from the conical theory is not so large, as is seen in the figure. In Fig. 8 is presented a surface pressure distribution on the cone calculated under the same conditions as in Fig. 7. Since κ is 1.960 in this case, the singularity at the vertex is not so strong and the initial gradient of surface pressure is zero.

When shock wave angle reaches β_{max}, the assumption of the series expansions used in the present development becomes inadequate because κ vanishes. In such a limiting case, since the spine of shock polar giving the initial direction of the

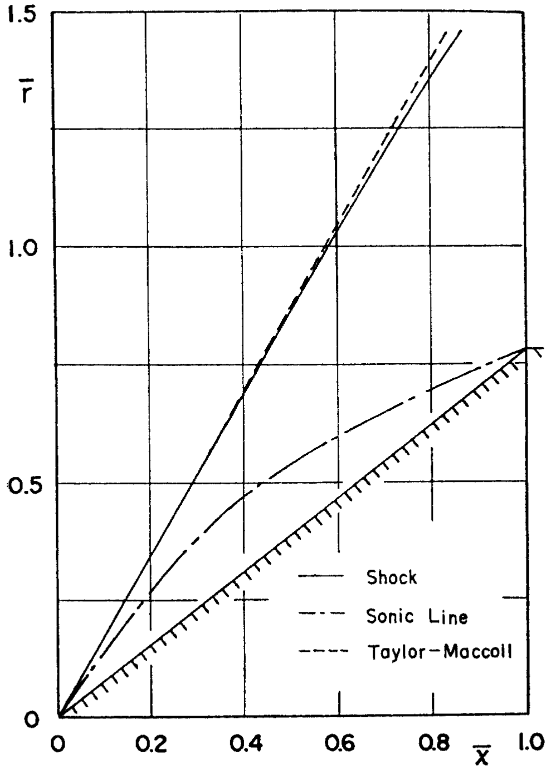

FIG. 7. Shock wave shape and sonic line for a finite cone. $M=2$, $\delta=38.05°$.

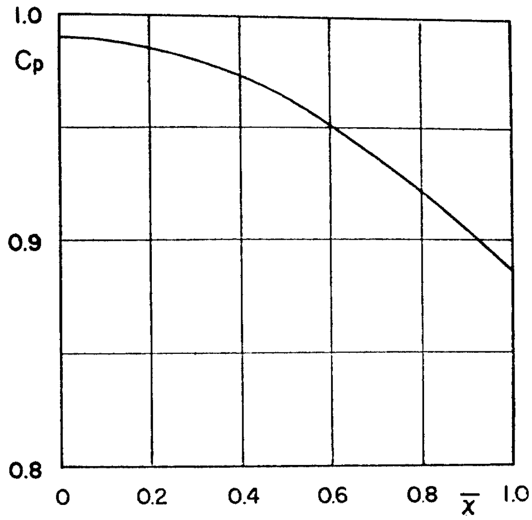

FIG. 8. Surface pressure distribution on a finite cone. $M=2$, $\delta=38.05°$.

boundary stramline prescribes an outward directed beginning at the intersection with the body boundary, it will be easily found that the flow near the vertex also requires a singularity to deflect a streamline suddenly towards the body contour. This characteristic feature of the streamline pattern in the velocity plane corresponds qualitatively to the incompressible concave corner, which is avoided by the streamlines. The change of velocity in the physical plane is, therefore, rapid near the vertex of the cone, indicating that the initial gradient of physical properties is infinite along the cone surface.

This circumstance implicitly suggests that the flow must have a logarithmic singularity at the vertex of the cone. Although there is no rigorous proof for this statement, it seems to be reasonable and consistent with transition process from attached to detached shock wave, since the vertex of the cone must be a stagnation point after detachment of the shock wave. This may be further confirmed by the fact that Tamada [6] and Oguchi [7] succeeded in clarifying the limiting characteristics of flow past a plane finite wedge by assuming a logarithmic singularity at the tip of the wedge.

Although the flow past a limiting cone must have a singularity at the vertex, the singular behaviour of the flow properties is restricted to the very vicinity of the vertex and the most of flow acceleration takes place near the shoulder. Thus, the shock wave shape, stream function and density may be assumed, respectively, to have forms

$$\left.\begin{aligned} r_s &= x + l\zeta x(\log dx)^{-1} + \cdots \\ \psi &= x^2 f(\theta) + l x^2 (\log dx)^{-1} g(\theta) + \cdots \\ \rho &= F(\theta) + l(\log dx)^{-1} G(\theta) + \cdots \\ d &= \frac{1}{\pi}\tan\beta_{\max} \end{aligned}\right\} \tag{6.14}$$

where l and ζ are unknown constants to be determined from the boundary conditions. In order to simplify the analysis it is assumed that the entropy function ω is constant everywhere equal to ω_0. Substitution of Eq. (6.14) into Eq. (2.7) yields for (f, F), (g, G), $\cdots$ the simultaneous equations, respectively. Equations for (f, F) are quite the same that is given by Eq. (3.5) as well as their boundary conditions.

Equation for g, which is obtained by eliminating G from simultaneous equations for (g, G), and equation for G can be written, respectively, in the forms

$$\tilde{D}g'' = \tilde{B}g + \tilde{C}g', \tag{6.15}$$

$$\bar{M}G = \{f' - \tau^2\theta(2f - \theta f')\}g' + 2\tau^2(2f - \theta f')g, \tag{6.16}$$

where

$$\left.\begin{aligned} \tilde{B} &= -2f'^2 + 8ff'' + 2\tau^2\gamma\omega_0\theta^3 F^\gamma F' \\ &\quad -2\tau^2\gamma\omega_0\theta^2 \frac{F^{\gamma-1}}{\bar{M}}\{f' - \tau^2\theta(2f - \theta f')\}\{F(f' - \theta f'') + \gamma F'(2f - \theta f')\} \\ &\quad +2\tau^2\gamma\omega_0\theta^2 \frac{F^\gamma \bar{M}'}{\bar{M}^2}(2f - \theta f')\{f' - \tau^2\theta(2f - \theta f')\}, \\ \tilde{C} &= -4ff' - \gamma\omega_0\theta^2(1 + \tau^2\theta^2)F^\gamma F' + \gamma\omega_0\theta^2 \frac{F^\gamma \bar{M}'}{\bar{M}^2}\{f' - \tau^2\theta(2f - \theta f')\}^2 \\ &\quad -\gamma\omega_0\theta^2 \frac{F^{\gamma-1}}{\bar{M}}\{f' - \tau^2\theta(2f - \theta f')\}[F\{f'' + \tau^2(2f - \theta f') \\ &\quad -\tau^2\theta(f' - \theta f'')\} + \gamma F'\{f' - \tau^2\theta(2f - \theta f')\}], \end{aligned}\right\} \tag{6.17}$$

$$\tilde{D} = \bar{D}$$

Boundary conditions for g and G are obtained from Eq. (4.8) as

$$\left.\begin{aligned} g(1) &= -\zeta b_0, \\ g'(1) &= -\zeta\{a_1 - f'(1) + f''(1)\}, \\ G(1) &= \zeta\{\rho_1 - F'(1)\}, \end{aligned}\right\} \tag{6.18}$$

where the final condition, $G(1)$, is automatically satisfied, once g is determined from Eq. (6.15).

Since the unknown constant ζ is involved in the boundary conditions for g, integration of Eq. (6.15) must be carried out, using a trial error method, by assuming initially several suitable values of ζ. It is so determined as to satisfy the tangency condition that the function g must vanish at $\theta = \theta_0$, that is

$$g(\theta_0; \zeta) = 0, \tag{6.19}$$

The another constant l is determined from the sonic condition just at the shoulder ($x=1$) of the cone such as

$$l=\frac{(\rho^*-F_0)\log d}{G_0}, \tag{6.20}$$

where G_0 and ρ^* are given by Eqs. (6.11) and (6.13), respectively.

Fig. 9 shows theoretical shapes of shock wave and sonic line on limiting cone at Mach number of 2, in which experimental shock wave shape is also presented for comparison. The experiment was carried out by use of a cone-cylinder with semi-vertex angle of 40.57° and 40 mmϕ in base diameter. Reynolds number of the model referred to cone length is 5.82×10^5. Fig. 10 shows a schlieren photograph of the limiting flow field past the model at $M=2$. So long as the shock wave shape is concerned, the agreement between present theory and experiment is good near the vertex. Slight deviation of the theoretical result from the experiment near the shoulder may be due to the approximation made in the present development. It may be an interesting result that the sonic line impinges on the shoulder from slight downstream of it, as is seen in Fig. 9.

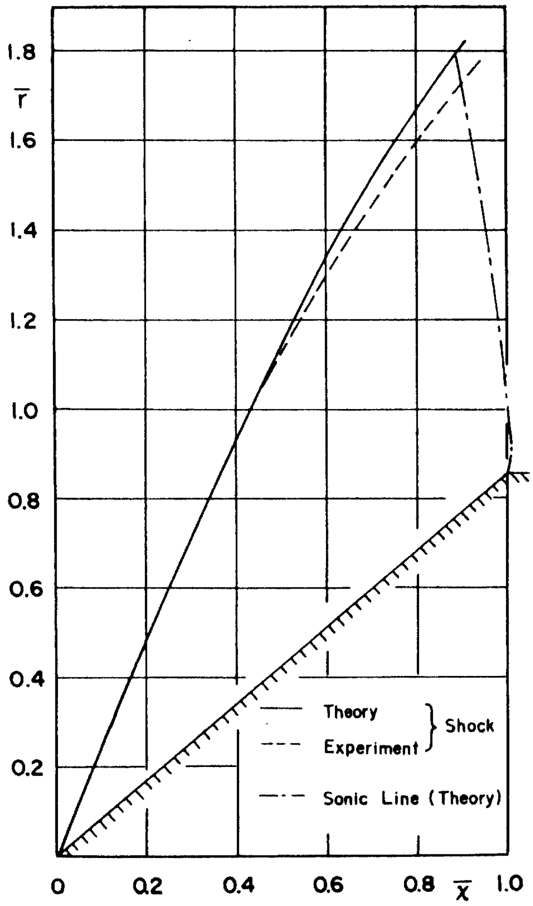

FIG. 9. Shock wave shape and sonic line for a limiting cone. $M=2$, $\delta=40.57°$.

FIG. 10. Schlieren photograph of a limiting flow field. $M=2$, $\delta=40.57°$.

In Fig. 11 is presented a theoretical surface pressure distribution on the limiting cone together with experimental results for comparison. The agreement between theory and experiment is fairly good. Fig. 12 shows pressure drag coefficient of a limiting cone with semi-vertex angle of 25° together with experimental data proposed by Solomon [10] for comparison, where the drag coefficient is defined by

$$C_D=2\int_0^1 xC_p dx . \tag{6.21}$$

The detachment Mach number in this case is 1.3277 and the two Solomon's data on left side in the figure indicate drag coefficients in the case of shock detachment. Therefore, it may be confirmed from Fig. 12 that the transition from attached to detached shock wave is continuous in axially symmetric flow as well as in plane flow.

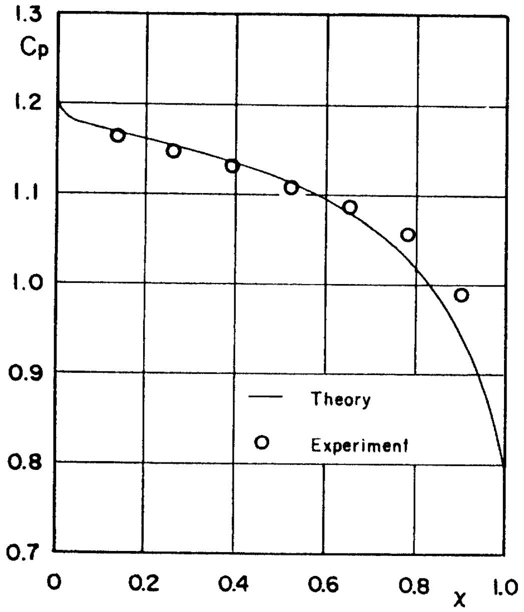

FIG. 11. Surface pressure distribution on a limiting $M=2$, $\delta=40.57°$.

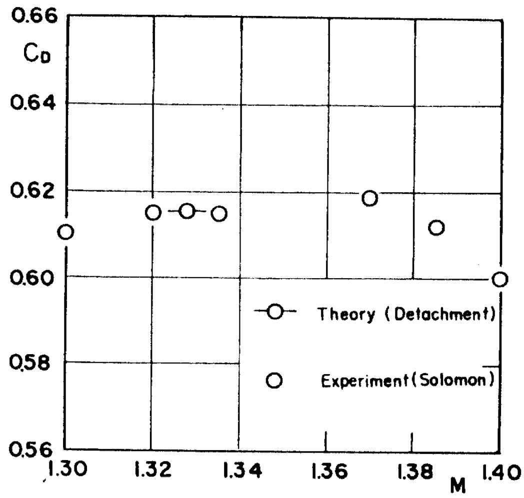

FIG. 12. Variation of drag coefficient of a finite cone with Mach number. $\delta=25°$

7. COMPLEMENTAL ARGUMENTS

When the shock polar illustrated in Fig. 3 is applied to a finite cone with subsonic flow pattern, it will be easily found from the contradiction in initial directions between spine and body contour at the intersection that the singularity required at the vertex must change its type as the maximum flow deflection angle is approached beyond Crocco point.

In the range of flow deflection angle smaller than Crocco point, the spines prescribe an inward bound, so that the singularity required to deflect a streamline suddenly to a more outward direction corresponds to the incompressible convex corner, where streamlines are crowded. However, the real flow in the physical plane will have an appreciable area, indicated by the number of streamlines, where velocity is almost uniform. This characteristic feature of the flow pattern clearly indicates that the flow near the vertex is almost undisturbed by the singularity. This, in turn, gives a physical evidence to the theoretical result of $\kappa > 1$ proposed by the present development. Therefore, such a singularity may be called as 'supersonic type'.

When growing cone angle goes beyond Crocco point, the direction of spines at the intersection changes from inward to outward and the corresponding singularity, from a convex corner to a concave corner which is avoided by the streamlines. The change of velocity in the physical plane is, therefore, rapid near the vertex and the initial gradient of flow properties becomes infinite. Although this flow pattern may be cnofirmed by the theoretical result of $\kappa < 1$ in this range of cone angle, another physical evidence must be taken into consideration that the singular behaviour of flow properties is restricted to the very vicinity of the vertex and most of flow acceleration occurs near the shoulder, indicating that the change of flow velocity is fairly rapid near the shoulder but is not singular. A similar flow pattern to this being given by the transonic flow past a finite cone, the singularity appropriate to this range of cone angle may be, therefore, called as 'transonic type'.

From the theoretical point of view, although the form of singularity, x^{κ}, assumed in the present development, Eq. (6.1), may be reasonable for the supersonic type $(\kappa > 1)$, it might be inadequate to represent a flow pattern of the transonic type $(\kappa < 1)$, since it does not seem to be capable of predicting accurately a physical evidence of rapid acceleration of flow near the shoulder. However, considering a continuous change of the real flow phenomena from supersonic type to transonic type and finally up to the shock detachment as growing cone angle goes beyond Crocco point, the present theory might be reasonably improved by replacing x^{κ} in Eq. (6.1) by $x^{\kappa}(\log dx)^{\kappa-1}$ in the range of flow deflection angle beyond Crocco point, where $d = \frac{1}{\pi} \tan \beta_s$.

In order to result in continuous transition from attached to detached shock wave, the appearance of a transonic type singularity in the range of flow deflection angle between δ_c and δ_{max} is necessary and, therefore, it may be recognized as a transient process of preparation for detachment of the shock wave.

8. Conclusion

A general method of analytical approach has been presented for axially symmetric supersonic flow involving shock wave and applied to clarify the flow characters near the region of vertex of pointed bodies of revolution with finite curvature. It was found that there exists a characteristic semi-vertex angle of the body for which inviscid solution cannot satisfy the given tangency condition when initial body curvature is finite.

Detailed examination revealed that this flow character is quite analogous to that arises at the well known Crocco point in plane flow. In the range between Crocco point and maximum semi-vertex angle, it was found that the real flow requires a singularity at the vertex to deflect the boundary streamline suddenly towards the body contour, although there may still exist a regular solution mathematically, which is physically trivial.

The present theory was shown to be applicable further to the problems of a finite cone with subsonic flow pattern and indicated a noticeable result that the transition from attached to detached shock wave is continuous in axially symmetric flow as well as in plane flow.

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