Institute of Space and Aeronautical Science, University of Tokyo
Report No. 483, June 1972

Temperature Freezing in Spherical or Cylindrical Expansion into a Vacuum*

By

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Summary: The phenomenon of the temperature freezing in the source flow expansion into a vacuum and its dependence on the flow geometries are studied not only qualitatively but also quantitatively. A significant parameter which provides a measure for the occurrence of temperature freezing is introduced based on a dimensional analysis. Numerical analysis is worked out by means of the discrete ordinate method for three cases; a) spherical source flow expansion, b) cylindrical source flow expansion, and c) cylindrical source flow expansion of gases consisting of disc-like molecules. The criterion on the occurrence of temperature freezing, being obtained from the dimensional considerations, is examined from comparison with the results by numerical analyses.

I. INTRODUCTION

Free jet expansion from an orifice, especially flow along the jet axis, as pointed out by Sherman [*1*], can be represented by the spherical source flow which expands from a point source. Many experiments of the free jet expansion were carried out (for example, see References [*2*] and [*3*]). From those results it was found that the terminal Mach number measured takes a finite value which is much smaller than its isentropic expansion value. This fact means that in an actual expansion flow the temperature does not decrease so much as it does in the isentropic expansion. The analytical treatment for the source flow expansion is much easier compared with that for the free jet expansion. So, for the source flow expansion the behavior of expansion flow at infinity downstream (the radial distance $r \to \infty$) was studied from the kinetic view point [*4*], [*5*] and [*6*]. These analyses dealt with the gases consisted of the molecules which obey the "power-law" interaction potential, such as hard sphere molecules and Maxwell molecules. As the results, it was found that as $r \to \infty$ the temperature levels off to an asymptotic value which depends on the source Knusen number; this asymptotic value of the temperature is termed as "freezing temperature". The results by similar analyses for the cylindrical source flow expansion show that the gases in the cylindrical expansion flow indicate no "freezing temperature". It was pointed out in References [*5*] and

* This work contains a part of the doctoral thesis, which was completed at the Institute while the author was a post-graduate student of the University of Tokyo.

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[6] that such difference as above shown, between the spherical and cylindrical source flows, appears to come out from the difference in flow geometry. However, it should be noted that it depends on the interaction potential as well as te flow geometries whether the temperature freezing occurs or not, both because the temperature freezing occurs in the collisionless flow, and because the collision frequency depends on the interaction potential. We shall first obtain a physical picture concerning freezing phenomena based on a dimensional analysis, and introduce a significant parameter which measures the phenomenon of temperature freezing. Then the criterion of the temperature freezing will be clarified. Next we shall carry out a numerical analysis by means of the discrete ordinate method, which has previously been proposed by the author [7], for the three cases; a) spherical source flow expansion, b) cylindrical source flow expansion, and c) cylindrical source flow expansion of gases consisting of disc-like molecules, which have only two degrees of freedom of motion. Although in the last case c) the gases are fictitious from the physical point of view, some meaningful results may be expected to be derived from comparison with the case of real gases. Finally, the numerical results will be discussed and the criterion on the temperature freezing will be examined based on the numerical results.

II. Dimensional Analysis

The problem that we pose is illustrated in Fig. 1; we have a sphere and a cylinder of radius r^* from which gas is streaming with local velocity equal to the local speed of sound. The gas is allowed to expand radially, so that as $r\rightarrow\infty$ the density of gas will approach zero; we will specifically be interested in those conditions for which source flow realizes a supersonic expansion in the range $r>t^*$. The problem is one-dimensional in physical space in that the distribution function $f(r, \vec{v}; t)$ describing the state of the system depends on r. The Boltzmann equation with the B-G-K collision model is chosen as the basic equation.

In view of geometrical symmetry the basic equation can be written in spherical coordinates (r, θ, φ)

$$V_r\frac{\partial f}{\partial r}+\frac{V_\theta^2+V_\varphi^2}{r}\frac{\partial f}{\partial V_r}-\frac{V_rV_\theta}{r}\frac{\partial f}{\partial V_\theta}-\frac{V_rV_\varphi}{r}\frac{\partial f}{\partial V_\varphi}=\nu(F-f) \qquad (1)$$

and in cylindrical coordinates (r, θ, z)

$$V_r\frac{\partial f}{\partial r}+\frac{V_\theta^2}{r}\frac{\partial f}{\partial V_r}-\frac{V_rV_\theta}{r}\frac{\partial f}{\partial V_\theta}=\nu(F-f). \qquad (2)$$

Here F is the Maxwellian distribution pertinent to the number density n with mean velocity U and temperature T, i.e.,

$$F=n(2\pi RT)^{-3/2}\exp\{-(2RT)^{-1}[(V_r-U)^3+V_\theta^2+V_\varphi^2]\} \qquad (3)$$

where R is the gas constant. The collision frequency ν is given for gases obeying

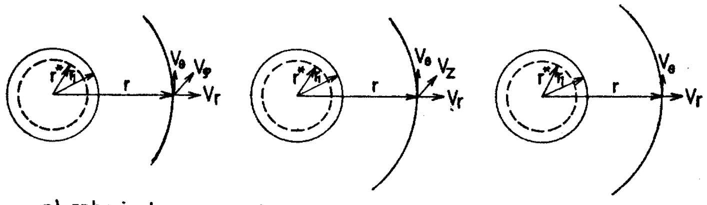

FIG. 1. Source flow expansion.
r^*: Sonic Radius
r_1: Reference Radius
V_i: Velocity Component

viscosity-temperature relation $\mu \propto T^s$ where the exponent s depends on the interaction potential, by $\nu = nkT/\mu$, where k is the Boltzmann constant.

By integrating Eq. (1) over $(V_r, V_\theta, V_\varphi)$, using the weighting functions 1, V_r, $(V_r^2 + V_\theta^2 + V_\varphi^2)/2$, and by substituting F for f, we obtain the coservation equations for the case when $f = F$,

$$\begin{aligned} nUr^2 &= \text{const} \\ dU/dr &= -r^{-2}dP/dr \\ U^2/2 + 5RT/2 &= \text{const.} \end{aligned} \tag{4}$$

Let us introduce the following dimensionless variables referred to the quantities at the radius $r = r_1$,

$$r' = r/r_1,\ n' = n/n_1,\ U' = U/U_1,\ T' = T/T_1,\ P' = P/P_1,$$

where the subscript 1 denotes at $r = r_1$. Then the conservation equations (4) are rewritten as follows:

$$\begin{aligned} n'U'r'^2 &= 1 \\ dU'/dr' &= -(\gamma M_1^2)^{-1} r'^{-2} dP'/dr' \\ U'^2 + 5(\gamma M_1^2)^{-1} T' &= 1 + 5(\gamma M_1^2)^{-1}, \end{aligned}$$

where γ is the specific heat ratio and the Mach number $M_1 = U_1/(\gamma RT_1)^{1/2}$. The solutions of the above equations can be obtained as follows:

$$\begin{aligned} r'^{-4} &= T'^3(1 - T' + \gamma M_1^2/5)(5/\gamma M_1^2) \\ U' &= (1 - T' + \gamma M_1^2/5)(5/\gamma M_1^2)^{1/2} \\ n' &= 1/U'r'^2. \end{aligned}$$

If the temperature T' is much smaller than unity at $r' \gg 1$, then the temperature T', mean velocity U', and number density n' take the following asymptotic forms,

$$\begin{aligned} T' &\propto r'^{-4/3} \\ U' &= U'_\infty \\ n' &= r'^{-2}. \end{aligned} \tag{5}$$

When quite a similar procedure is applied to Eq. (2) for the cylindrical source flow expansion, we obtain the following asymptotic forms,

$$\begin{aligned} T' &\propto r'^{-2/3} \\ U' &= U'_\infty \\ n' &\propto r'^{-1}. \end{aligned} \tag{6}$$

For the disc-like molecules, which have only two degrees of freedom ($\vec{V}=(V_r, V_\theta)$), we have

$$\begin{aligned} T' &\propto r'^{-1} \\ U' &= U'_\infty \\ n' &\propto r'^{-1}. \end{aligned} \tag{7}$$

Let us introduce the following dimensionless distribution functions $f'(=n_1^{-1}V_1^3 f)$ and $F'(=n_1^{-1}V_1^3 F)$. Then the nondimensionalized collision term of Eqs. (1) and (2) becomes

$$J = A\nu'(F'-f'), \tag{8}$$

where

$$V_1=(2RT_1)^{1/2}, \qquad \nu'=n'T'/\mu'$$

and

$$F'=n'(\pi T')^{-3/2}\exp\left(-[(V'_r-U')^2+V'^2_\theta+V'^2_\varphi]/T'\right).$$

Here the nondimensionalized parameter A is

$$A=r_1\nu_1/V_1=r_1n_1kT_1/V_1\mu_1. \tag{9}$$

This A can be related to the Knudsen number $Kn_1=L_1/r_1$, where L_1 is the mean free path at reference radius. If the mean free path is given by

$$L_1=(16/5)\mu_1/[mn_1(2\pi RT_1)^{1/2}], \tag{10}$$

where m is the molecular mass, we have

$$A=\pi^{1/2}/2Kn_1. \tag{11}$$

We define the parameter A^* as the A pertinent to the quantities at the reference radius $r=r^*$, then

$$\begin{aligned} A/A^* &= (r_1n_1kT_1/V_1\mu_1)/(r^*n^*kT/V^*\mu^*) \\ &= \hat{r}_1\hat{n}_1\hat{T}_1/\hat{T}_1^{1/2}\hat{\mu}_1, \end{aligned} \tag{12}$$

where the hat means the dimensionless variables referred to the properties at the sonic radius $r=r^*$. If the gases are obeying the viscosity-temperature relation $\mu'\propto T'^s$ where the exponent s depends on the interaction potential, then the ratio can be rewritten as

$$A/A^*=\hat{r}_1\hat{n}_1\hat{T}_1^{1/2-s}. \tag{13}$$

Using the asymptotic form ($n^1 \propto r'^{-\alpha}$, $T' \propto r'^{-\beta}$) obtained for isentropic cases before, the above equation becomes

$$A/A^*=\hat{r}_1\hat{r}_1^{-\alpha}\hat{r}_1^{-\beta(1/2-s)}=\hat{r}_1^{\delta}, \tag{14}$$

$$\delta=1-\alpha+s\beta-\beta/2. \tag{15}$$

The parameter α, β, and s are determined for both the molecular model and flow geometry fixed; these values are summarized in Table 1.

From the definition we have

$$A/A^*=Kn^*/Kn_1, \tag{16}$$

and using Eq. (14),

$$Kn_1=(A/A)Kn^*=\hat{r}_1^{-\delta}Kn^*. \tag{17}$$

The behavior of Kn_1 as $\hat{r}_1\to\infty$ is subject to the sign of δ (plus, minus, and zero), namely,

i) for cases $\delta<0$

$$Kn_1=\hat{r}_1^{-\delta}Kn^*\to\infty. \quad \text{(free molecular limit)}$$

ii) for cases $\delta=0$

$$Kn_1=\hat{r}_1^{-\delta}Kn^*\to Kn^*.$$

iii) for cases $\delta>0$

$$Kn_1=\hat{r}_1^{-\delta}Kn^*\to 0. \quad \text{(collision dominated)}$$

It follows from above relations that when δ is positive or zero ($\delta\geq 0$) the collision terms in Eqs. (1) and (2) play an important role still at infinity downstream $r\to\infty$. On the other hand, for cases when δ is negative ($\delta<0$) the effect of collision terms in Eqs. (1) and becomes negligibly small as $r\to\infty$. In case i) the

TABLE 1. Values of parameters α, β, δ, and s.

Source type	Molecule type	s	α	β	δ
Spherical	Maxwell	1	2	4/3	−1/3
	hard sphere	1/2	2	4/3	−1
Cylindrical	Maxwell	1	1	2/3	1/3
	hard sphere	1/2	1	2/3	0
Cylindrical	Maxwell (disc)	1	1	1	1/2
	hard sphere (disc)	1/2	1	1	0

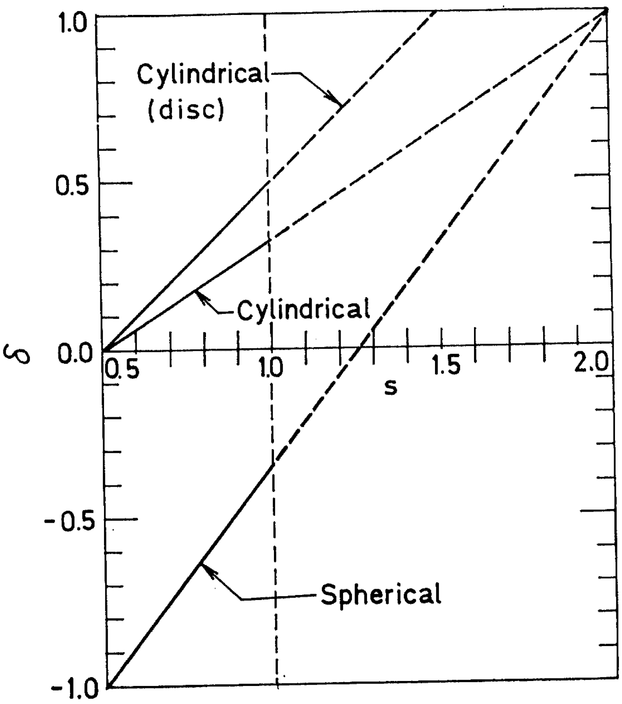

FIG. 2. Values of parameter δ vs s.

expansion flow becomes collisionless as $r \to \infty$, and the temperature of gas must level off and "freezing" occurs. In cases ii) and iii) such phenomenon will not be expected to occur.

The values of δ are given by Eq. (15), and plotted against s in Fig. 2. In the case of cylindrical source flow, as can be seen from Fig. 2, the parameter δ is always non-negative for $s \geq 0.5$. On the other hand, in the case of spherical source fllow the parameter δ is non-negative for $s \geq 1.25$, while it is negative for $s < 1.25$. For inert gases the values of exponent s are ranged between 0.5 (hard sphere molecules) and 1.0 (Maxwell molecules). Therefore, for the case of spherical source flow expansion, the temperature of gas will always freeze as $r \to \infty$, while for the cylindrical source flow expansion, of either three-dimensional or two-dimensional (disc-like) molecular gases, the temperature of gas will not freeze even at the limit $r \to \infty$.

III. NUMERICAL ANALYSIS

The basic equations (1) and (2) are rewritten in terms of the dimensionless variables in spherical and cylindrical coordinates, respectively, as follows:

$$V'_r \frac{\partial f'}{\partial r'} + \frac{V_\theta'^2 + V_\varphi'^2}{r'} \frac{\partial f'}{\partial V'_r} - \frac{V'_r V'_\theta}{r'} \frac{\partial f'}{\partial V'_\theta} - \frac{V'_r V'_\varphi}{r'} \frac{\partial f'}{\partial V'_\varphi} = A\nu'(F' - f') \quad (1')$$

$$V'_r \frac{\partial f'}{\partial r'} + \frac{V_\theta'^2}{r'} \frac{\partial f'}{\partial V'_r} - \frac{V'_r V'_\theta}{r'} \frac{\partial f'}{V'_\theta} = A\nu'(F' - f'). \quad (2')$$

In the present analysis three cases will be treated; a) spherical source flow expansion, b) cylindrical source flow expansion, c) cylindrical source flow expansion of gases consisting of disc-like molecules. Although in the last case the gases, which have only two degrees of freedom of motion, are fictitious from the physical point of view, some meaningful results may be expected to be derived from comparison with the case a) or b). The macroscopic moments are obtained as follows:

$$\left.\begin{aligned}
n' &= \int\int\int_{-\infty}^{\infty} f' dV'_\theta dV'_\varphi V'_r \\
U' &= \int\int\int_{-\infty}^{\infty} V'_r f' dV'_\theta dV'_\varphi dV'_r \\
T' &= (2/3)\int\int\int_{-\infty}^{\infty} [(V'_r - S_1U')^2 + V'^2_\theta + V'^2_\varphi] f' dV'_\theta dV'_\varphi dV'_r \\
&\qquad\qquad \text{(spherical)} \\
T' &= (2/3)\int\int\int_{-\infty}^{\infty} [(V'_r - S_1U')^2 + V'^2_\theta + V'^2_z] f' dV'_\theta dV'_z dV'_r \\
&\qquad\qquad \text{(cylindrical)} \\
T' &= \int\int\int_{-\infty}^{\infty} [(V'_r - S_1U')^2 + V'^2_\theta] f' dV' dV'_r \\
&\qquad\qquad \text{(cylindrical, disc-like molecules)}
\end{aligned}\right\} \tag{18}$$

Detail of the analysis is discussed in Ref. [7].

a) *Spherical source flow expansion*

By integrating Eq. (1′) over (V'_θ, V'_φ), using weighting functions r'^2 and $r'^4(V'^2_\theta + V'^2_\varphi)$, we obtain

$$V'_r \frac{\partial g'}{\partial r'} + \frac{1}{r'^3}\frac{\partial h'}{\partial V'_r} = A\nu'(G' - g') \tag{19}$$

$$V'_r \frac{\partial h'}{\partial r'} + \frac{2T'_p}{r'}\frac{\partial h'}{\partial V'_r} = A\nu'(H' - h') \tag{20}$$

where g' and h' are modified distribution function [8]

$$g' = r'^2\int\int_{-\infty}^{\infty} f' dV'_\theta dV'_\varphi, \qquad h' = r'^4\int\int_{-\infty}^{\infty} (V'^2_\theta + V'^2_\varphi) f' dV'_\theta dV'_\varphi. \tag{21}$$

These functions were first applied by Chu [9] in analyzing the unsteady plane shock problem, and afterwards by other investigators [10]–[13] in analyzing various rarefied gas dynamic problem. The functions G' and H' are the Maxwellian distribution functions:

$$G' = n'(\pi T')^{-1/2} \cdot \exp[-(V'_r - S_1U')^2/T'] \cdot r'^2, \qquad H' = G'T'r'^2.$$

In the derivation of Eq. (20), we assumed the following relation

$$\frac{\partial}{\partial r'_r}\left[\int\int_{-\infty}^{\infty} (V'^2_\theta + V'^2_\varphi)^2 f' dV'_\theta dV'_\varphi\right] = 2T'_p\left(\frac{\partial h'}{\partial V'_r}\right), \tag{22}$$

where T'_p is the perpendicular temperature defined by

$$T'_p = \int\int\int_{-\infty}^{\infty} (V'^2_\theta + V'^2_\varphi) f' dV'_\theta dV'_\varphi dV'_r. \tag{23}$$

b) *Cylindrical source flow expansion*

By integrating Eq. (2′) over (V'_θ, V'_z), using the weighting functions r', $r'^3 V'^2_\theta$, and $r' V'^2_z$, we obtain

$$V'_r \frac{\partial g'}{\partial r'} + \frac{1}{2r'^2} \frac{\partial h'}{\partial V'_r} = A\nu'(G' - g') \tag{24}$$

$$V'_r \frac{\partial h'}{\partial r'} + \frac{3T'_\theta}{2r'} \frac{\partial h'}{\partial V'_r} = A\nu'(H' - h') \tag{25}$$

$$V'_r \frac{\partial i'}{\partial r'} + \frac{T'_z}{2r'} \frac{\partial h'}{\partial V'_r} = A\nu'(I' - i'). \tag{26}$$

Here g', h' and i' are the modified distribution functions similar to those of the spherical case;

$$g' = r' \int\int_{-\infty}^{\infty} f' dV'_\theta dV'_z, \qquad h' = r'^3 \int\int_{-\infty}^{\infty} V'^2_\theta f' dV'_\theta dV'_z,$$

$$i' = r' \int\int_{-\infty}^{\infty} V'^2_z f' dV'_\theta dV'_z. \tag{27}$$

The functions G', H' and I' are Maxwellian distributions

$$G' = n'(\pi T')^{-1/2} \exp\left[-(V'_r - S_1 U')^2 / T'\right] \cdot r',$$
$$H' = G' T' r'^2, \qquad I' = G'.$$

Moreover, we assume the following relations

$$\frac{\partial}{\partial V'_r}\left[\int\int_{-\infty}^{\infty} V'^4_\theta f' dV'_\theta dV'_z\right] = (3/2) T'_\theta \left(\frac{\partial h'}{\partial V'_r}\right), \tag{28}$$

$$\frac{\partial}{\partial V'_r}\left[\int\int_{-\infty}^{\infty} V'^2_z V'^2_\theta f' dV'_\theta dV'_z\right] = (1/2) T'_z \left(\frac{\partial h'}{\partial V'_r}\right), \tag{29}$$

where the temperature components T'_θ and T'_z are written, respectively, as follows:

$$T'_\theta = 2\int\int\int_{-\infty}^{\infty} V'^2_\theta f' dV'_\theta dV'_z dV'_r, \qquad T'_z = 2\int\int\int_{-\infty}^{\infty} V'^2_z f' dV'_\theta dV'_z dV'_r. \tag{30}$$

c) *Cylindrical source flow expansion (disc-like molecules)*

By integrating Eq. (2′) over (V'_θ), using the weighting functions r' and $r'^3 V'^2_\theta$, we obtain

$$V'_r\frac{\partial g'}{\partial r}+\frac{1}{r'^3}\frac{\partial h'}{\partial V'_r}=A\nu'(G'-g'), \tag{31}$$

$$V'_r\frac{\partial h}{\partial r'}+\frac{3T'_\theta}{2r'}\frac{\partial h'}{\partial V'_r}=A\nu'(H'-h'). \tag{32}$$

Here g' and h' are modified distribution functions similar to those of the spherical case:

$$g'=r'\int_{-\infty}^{\infty}f'dV'_\theta, \qquad h'=r'^3\int_{-\infty}^{\infty}V'^2_\theta f'dV'_\theta. \tag{33}$$

G' and H' are Maxwellian distributions,

$$G'=n'(\pi T')^{-1/2}\exp\left[-(V'_r-S_1U')^2/T'\right]\cdot r', \qquad H'=G'T'r'^2.$$

Moreover, we used the assumption

$$\frac{\partial}{\partial V'_r}\left[\int_{-\infty}^{\infty}V'^4_\theta f'dV'_\theta\right]=(3/2)T'_\theta\left(\frac{\partial h'}{\partial V'_r}\right) \tag{34}$$

with the perpendicular temperature T'_θ defined by

$$T'_\theta=2\int\int_{-\infty}^{\infty}V'^2f'dV'_\theta dV'_r. \tag{35}$$

It should be noted that the assumption Eqs. (22), (28), (29), and (34) made, respectively, for cases a), b), and c) are exactly valid for the equilibrium distribution. The ratios of the second term to the first term in all equations (20), (25), (26) and (32) are inversely propotional to the square of the flow Mach number. Therefore, in these equations, the second term becomes negligibly small compared with the first term at far downstream where the flow Mach number is very large. With the above fact we expect that the neglection of the second term causes no significant error for the solutions as a whole. We now summarize the governing equations as follows:

$$V'_r\frac{\partial q'}{\partial r'}+\Delta_{q'}=A\nu'(Q'-q'), \tag{36}$$

where $q'=g', h'$ or i' and $Q',=G', H'$. Here $\Delta_{q'}$ represents the second term of the governing equations.

Our considerations are confined to the case when the flow is in equilibrium from the sonic radius to a moderate distance downstream. We now choose a certain radius $r=r_1$ within that region, as the reference. On the other hand, the density vanishes at infinity downstream. The boundary conditions are thus specified as follows:

$$r'=1, \qquad q'=Q'_1=(\pi)^{1/2}\exp\left[-(V'_r-S_1)^2\right]. \tag{37}$$

Following the discrete ordinate method which has already been developed for the analyses of several rarefied flow problem (for example, see Reference [12], the velocity space V'_r is represented by finite discrete velocity points, say V_n ($n=1, 2, 3, \cdots, m$). The application of the method reduces the set of governing integro-differential equation (36) to a set of ordinary differential equations. That is, the resulting differential equations are

$$V'_n \frac{dq'_n}{dr'} + \Delta_{q'_n} = A\nu'(Q'_n - q'_n) \qquad (n=1, 2, 3, \cdots, m) \tag{38}$$

where q'_n, Q'_n, and $\Delta_{q'_n}$ represent q', Q', and $\Delta_{q'}$ evaluated at the discrete velocity points $V'_n(n=1, 2, 3, \cdots, m)$, respectively.

In the discrete ordinate scheme, the velocity V'_n acts only as parametric variables. Therefore, the equation (38) can be solved by applying an ordinary difference scheme to physical space r'. The reduced distribution q'_n is conveniently divided into a a two-sided one: $q'^{\pm}_n$ for $V'_n \gtrless 0$. The difference form of Eq. (38), for q'^{+}_n becomes

$$V'_n \frac{q'_n(r') - q'_n(r' - \Delta r')}{\Delta r'} + \Delta_{q'_n}(r') = A\nu'(r')(Q'_n(r') - q'_n(r')), \tag{39}$$

where $\Delta r'$ is radial increment. Rearranging Eq. (39), we obtain

$$q'_n(r') = \frac{(V'_n/\Delta r')q'_n(r' - \Delta r') + A\nu'(r')Q'_n(r') + \Delta_{q'_n}(r')}{(V'_n/\Delta r') + A\nu'(r')}. \tag{40}$$

Remembering the assumption that there exists an equilibrium region near the sonic radius, we may choose the reference radius $r=r_1$ where the flow is moderately supersonic. In the actual calculations, the reference radius r_1 was chosen such that the flow Mach number there is ~ 2.0. For such a case the contribution of q'^{-}_n (for $V'_n < 0$) is much smaller compared with that of q'^{+}_n even in the vicinity of the reference radius. In the far downstream region, evidently the contribution of q'^{-}_n becomes negligibly small. In view of the aforementioned facts, q'^{-}_n was not taken into accounts in the present paper.

Following the difference scheme specified by Eq. (40), the evaluation for the reduced distributions q'_n is advanced step by step, starting from the reference radius. At each step, however, a number of iterations are required to reach a solution. Suppose that we know all the values of quantities at the radius $r' - \Delta r'$. As the initial estimate of the quantities on the right hand side of Eq. (40), their respective values at the previous step are employed. Using the zeroth iterate q'^{+}_n thus evaluated, the new macroscopic moments can be determined by applying the ten-point Gauss-Hermite quadrature to the integration of Eq. (18). Then, all the terms on the right hand side of Eq. (40) is re-evaluated. Such a procedure is iterated until a satisfactory convergence has been assured for all velocity points; normally, three or four iterations were sufficient to fulfil, for both number density and mean velocity, the convergence criteria that the departure from the previous iterate must be less than 10^{-6} times their respective values. The accuracy of the calculation was estimated by examining the constancy of mass flux. The error in the step by step calculation accumulates as proceeding toward the downstream. The actual computation was stopped at the point, beyond which the value of mass flux indicates a departure more than 1% from that at the reference radius. This implies that the maximum error in mass flux is less than 1%. The spatial region, thus computed with the increment being $\Delta r'=r'/100$, was covered from $r/r^*=r_1/r^*$ to $r/r^*=10^4$, where r^* is the sonic radius.

IV. Numerical Results and Discussion

The numerical examples are listed in Table 2. In case a) the source Knudsen number Kn_D is employed, being defined as $Kn_D=L_0/D$ where D is the orifice diameter, from which an equivalent source blows out, and L_0 the stagnation mean free path. In cases b) and c) the Knudsen number Kn^* is employed, being defined as $Kn^*=L_0/r^*$ where the radius r^* is the sonic radius; by Hamel and Willis it is related to the orifice diameter $r^*=0.680\times D$. The results will be summarized in the following.

a) *spherical source flow expansion*

In Fig. 3 is shown the temperature for the source Knudsen number $Kn_D=0.001$ for both cases of hard sphere and Maxwell molecules, respectively. Actual calculations are carried out from the reference radius (see Table 2) to $r/r^*\sim 1.0\times 10^4$. In that figure the broken lines show the level of the terminal temperature or freezing temperature. It can be seen that the phenomenon of temperature freezing apparently occurs for both hard sphere and Maxwell molecules. It can also be seen that, as expected from the data shown in Table 1, for Maxwell

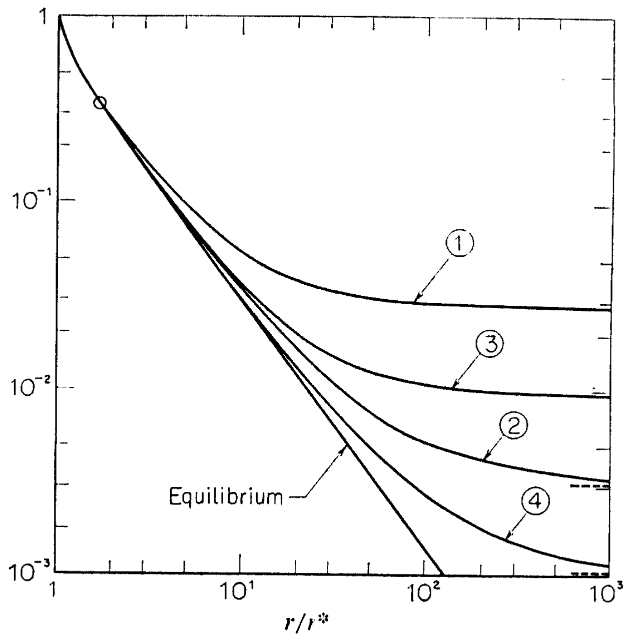

Fig. 3. Temperature distributions (spherical); ① T_r/T^*, ③ T_p/T^* (hard sphere), ② T_r/T^*, ④ T_p/T^* (Maxwell).

TABLE 2. Numerical examples.

Case number		Flow geometry	Molecule type	$Kn_D=L_0/D$	$Kn^*=L_0/r^*$	M_1	r_1/r^*
A	1	Spherical	Maxwell	0.001		3	1.732
	2	Spherical	hard sphere	0.001		3	1.732
B	3	Cylindrical	Maxwell		0.05	3	3.000
	4	Cylindrical	Maxwell		0.005	3	3.000
	5	Cylindrical	hard sphere		0.05	3	3.000
	6	Cylindrical	hard sphere		0.005	3	3.000
C	7	Cylindrical	Maxwell (disc)		0.05	2	1.414
	8	Cylindrical	Maxwell (disc)		0.005	2	1.414
	9	Cylindrical	hard sphere (disc)		0.05	2	1.414
	10	Cylindrical	hard sphere (disc)		0.005	2	1.414

molecules the temperature freezing occurs at farther downstream than for hard sphere molecules. In all the cases the perpendicular temperature almost coincides with their isentropic values over a wide range of the expansion. However, it should be noted that at far downstream the perpendicular temperature decreases in inversely propotional to the radius r, being different from that in isentropic expansion. Comparison with the result of Eq. (7) for the disc-like molecules suggests that in the spherical source flow the internal degree of freedom of molecules degenerates to 2 as in disc-like molecules, at the far downstream. Further detailed discussions should be referred to Reference [7].

b) *cylindrical source flow expansion*

In Figs. 4 and 5 are shown the temperature and its three components denoted by T, T_r, T_θ and T_z, respectively, for hard sphere and Maxwell molecules. In the case of Maxwell molecules, the temperature and its three components deviate slightly from that of isentropic case and all of them decrease as $r\rightarrow\infty$ in inversely propotional to the radius r with a certain constant power close to the isentropic value 2/3. When the source Knudsen number $Kn^*\ll 1$ they almost coincide with that of the isentropic case, and above mentioned power approaches to 2/3. These fact can be expected from the discussion in Sec. II. That is, for the case of Maxwell molecules the role of the collision term in Eq. (2) becomes more predominant than the remaining terms, as one proceeds downstream. On the contrary, for the case of the hard sphere molecules the temperature and its components deviate largely from that of isentropic value even if the Knudsen number $Kn^*\ll 1$, and no temperature freezing occurs anywhere even at infinity downstream. However, for the case of the hard sphere molecules with $\delta=0$ the temperature indicates somewhat peculiar behavior; the rate of temperature decrease appears to become smaller as r increases.

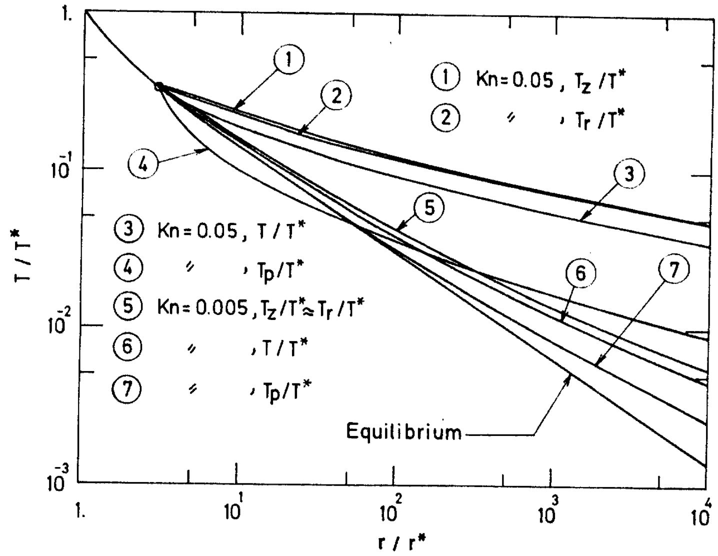

FIG. 4. Temperature distributions for hard sphere molecules (cylindrical expansion)

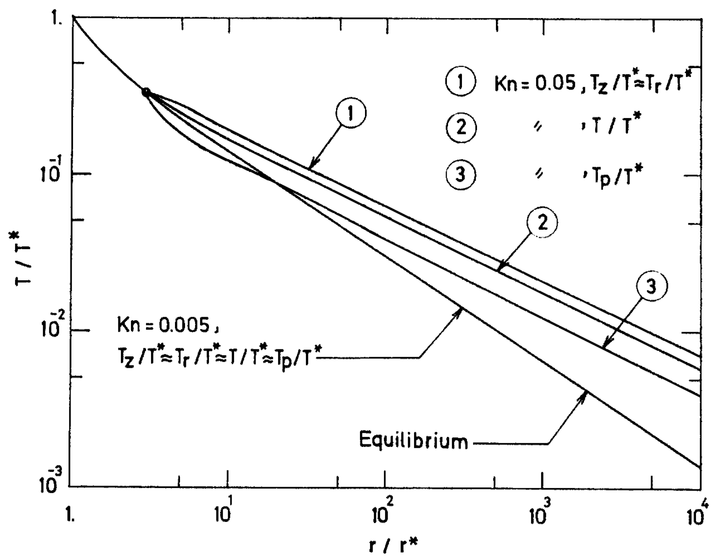

FIG. 5. Temperature distributions for Maxwell molecules (cylindrical expansion).

In Fig. 6 is shown the comparison with the results of Bird [*14*]. The radial temperature T is nearly equal to the axial temperature T_z. This agrees qualitatively with the predictions by Edwards and Cheng [*5*] and by Hamel and Willis [*6*].

c) *cylindrical source flow expansion of gases consisting of disc-like molecules*

In Figs. 7 and 8 are shown the temperature and its two components, respectively, for the hard sphere and Maxwell molecules. The behavior of the temperature shows a qualitative agreement with that for the case of cylindrical source flow expansion, and no temperature freezing occurs also. From comparison of the case b) with the case c), it is found that in cylindrical source flow the molecular motion in the axial direction, along which flow variables are unchanged, has no

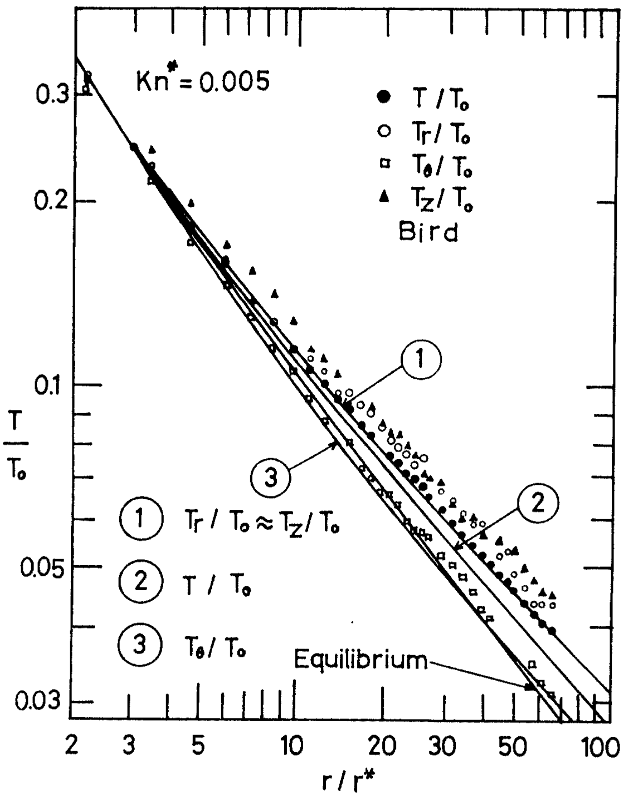

FIG. 6. Comparison with the result of Bird [14] for hard sphere molecules (cylindrical expansion).

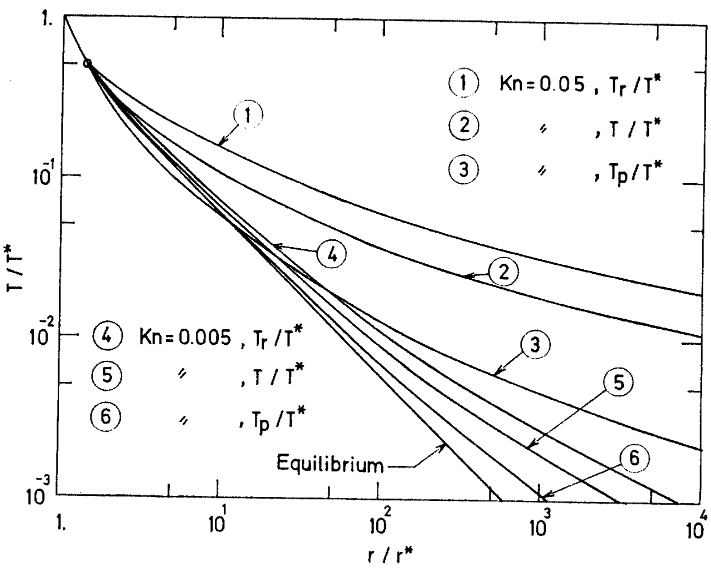

FIG. 7. Temperature distributions for hard sphere molecules (cylindrical expansion, disc-like molecules).

significant contribution to the temperature freezing. In conclusion the characteristic behavior of the temperature in the source flow expansion into a vacuum has been clarified not only qualitatively but also quantitatively. Furthermore the criterion on the temperature freezing in the source flow expansion has been specified

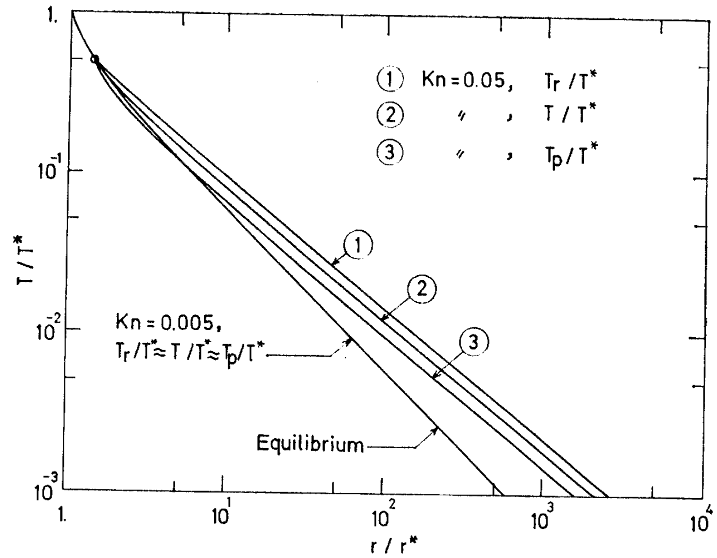

FIG. 8. Temperature distributions for Maxwell molecules (cylindrical expansion, disc-like molecules).

in terms of the dimensionless parameter δ; in any flow geometries, the temperature freezing occurs only when this parameter δ is negative ($\delta<0$), while it does not occur for non-negative ($\delta\geq 0$), for either spherical or cylindrical source flow. This has been confirmed from the results of the numerical analysis based on the discrete ordinate method.

ACKNOWLEDGEMENTS

The author wishes to express his sincere gratitude to Professor Hakuro Oguchi for valuable discussion and constant support during the cource of this investigation.

The numerical calculations of this work were carried out by the use of HITAC 5020F digital computer of the Data Processing Center of the Institute and partly by the use of FACOM 230-60 digital computer of the Computer Center of Nagoya University.

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March 31, 1972 and in revised form April 26, 1972

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