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Stability Optimization of Laminated Composite Plates

By

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Summary: This paper is concerned with the optimum design of plates with orthotropic layers under axial compression and shear. The plates considered are the laminates of N orthotropic layers whose principal material axes coincide with the plate axes. Each layer is assumed to have the same thickness and an equal number of fibers in the direction of $+\alpha_i$ and $-\alpha_i$ with respect to the plate axis. The fiber directions which give the highest axial buckling stress and the highest shear buckling stress are found by utilizing a mathematical optimization technique for various aspect ratios of the plates. Inhomogeneity in the direction of the plate thickness (stacking sequence) is taken into account in this analysis.

NOMENCLATURE

a	= plate length in the x-direction (Fig. 1)
A_{ij}	= extensional stiffness of a laminated plate
b	= plate length in the y-direction (Fig. 1)
B_{ij}	= coupling stiffness of a laminated plate
D_{ij}	= bending stiffness of a laminated plate
E_1, E_2	= Young's moduli of a unidirectional composite parallel and transverse to the directions of the fibers, respectively
G_{12}	= shear modulus of a unidirectional composite
h	= thickness of each layer
H_{mn}	= Eqs. (17)
k	= the ratio of $\bar{N}_y$ to $\bar{N}_x$
m, n	= number of half waves in the x- and y-directions, respectively
N	= number of layers
$\bar{N}_x$, $\bar{N}_y$	= applied force per unit length in the x- and y-directions, respectively
$\bar{N}_{xy}$	= applied shear force per unit length
Q_{ij}	= reduced stiffness
$\bar{Q}_{ij}$	= transformed reduced stiffness
S_{mn}, T_{mn}, U_{mn}	= Eqs. (14)
T'_{ij}	= Eqs. (8)
u, v, w	= displacements in the x-, y- and z-directions, respectively
$\bar{u}_{mn}$, $\bar{v}_{mn}$, $\bar{w}_{mn}$	= displacement amplitudes defined by Eqs. (10)
V_{pq}	= Eq. (18)
α_i	= absolute value of fiber directions with respect to x in the i-th layer
α_{ij}	= Eqs. (12)
ν_{12}	= Poisson's ratio of a unidirectional composite
ϕ	= nondimensional critical buckling stress defined by Eq. (9)

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ψ = inverse of nondimensional critical buckling stress; defined by Eq. (16)

1. Introduction

Recently, filamentary composite materials have been suggested for primary structures of aircrafts and spacecrafts. The reason is mainly due to the weight savings which can be attained. There are many design criteria in applying the composite materials to structures. One of them is the buckling criterion. Several theoretical and experimental papers[1-7] have been published on the buckling of laminated composite plates under axial compression and shear. Most of them give their results only for such special cases as angle-ply or cross-ply plates. Therefore, we do not have enough information to design the laminated composite plates.

This paper will present a method to design the laminated plates (Fig. 1) with orthotropic layers under uniaxial or biaxial compression and shear. The design criterion is the buckling stress. Each layer of the plate is assumed to have the same thickness and an equal number of the same kind of fibers in the $+\alpha_i$ and $-\alpha_i$ directions with respect to the x coordinate in the same type of matrix. Therefore, each layer can be considered to be orthotropic. Inhomogeneity in the direction of the thickness of the plate (stacking sequence) is taken into account in the calculation.

The present problem is to find the fiber directions of all the layers that give the highest buckling stress and, therefore, is an unconstrained maximization problem. The objective function and the design variables are the critical buckling stress and the fiber directions respectively. Preassigned parameters are the material properties, the thickness of each layer, the number of layers, and the aspect ratio of the plates. The optimization technique used is Powell's method (conjugate direction technique). This method is one of the best ones to find the optimum without using the derivatives of the objective function.

2. Derivation of Buckling Differential Equations

Extensional, coupling and bending stiffness, which are expressed as A_{ij}, B_{ij}, and D_{ij} respectively, are first introduced. They are defined in terms of transformed reduced stiffness as follows.

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-t/2}^{t/2} \bar{Q}_{ij}(1, z, z^2)\,dz \tag{1}$$

The transformed reduced stiffnesses of each layer are calculated by the following equations.

$$\bar{Q}_{11} = Q_{11}\cos^4\alpha_i + 2(Q_{12} + 2Q_{66})\sin^2\alpha_i\cos^2\alpha_i + Q_{22}\sin^4\alpha_i \tag{2a}$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\alpha_i\cos^2\alpha_i + Q_{12}(\sin^4\alpha_i + \cos^4\alpha_i) \tag{2b}$$

$$\bar{Q}_{22} = Q_{11}\sin^4\alpha_i + 2(Q_{12} + 2Q_{66})\sin^2\alpha_i\cos^2\alpha_i + Q_{22}\cos^4\alpha_i \tag{2c}$$

$$\bar{Q}_{16} = 0 \tag{2d}$$

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$$\bar{Q}_{26} = 0 \tag{2e}$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\alpha_i \cos^2\alpha_i + Q_{66}(\sin^4\alpha_i + \cos^4\alpha_i) \tag{2f}$$

where Q_{ij}'s are reduced stiffnesses and are defined as

$$Q_{11} = E_1/(1 - \nu_{12}\nu_{21}) \tag{3a}$$

$$Q_{12} = \nu_{12}E_2/(1 - \nu_{12}\nu_{21}) \tag{3b}$$

$$Q_{22} = E_2/(1 - \nu_{12}\nu_{21}) \tag{3c}$$

$$Q_{66} = G_{12} \tag{3d}$$

In Eqs. (2) $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are equal to zero, because the number of fibers in the $+\alpha_i$ and $-\alpha_i$ direction are the same. Extensional, coupling and bending stiffnesses are calculated for the present case as

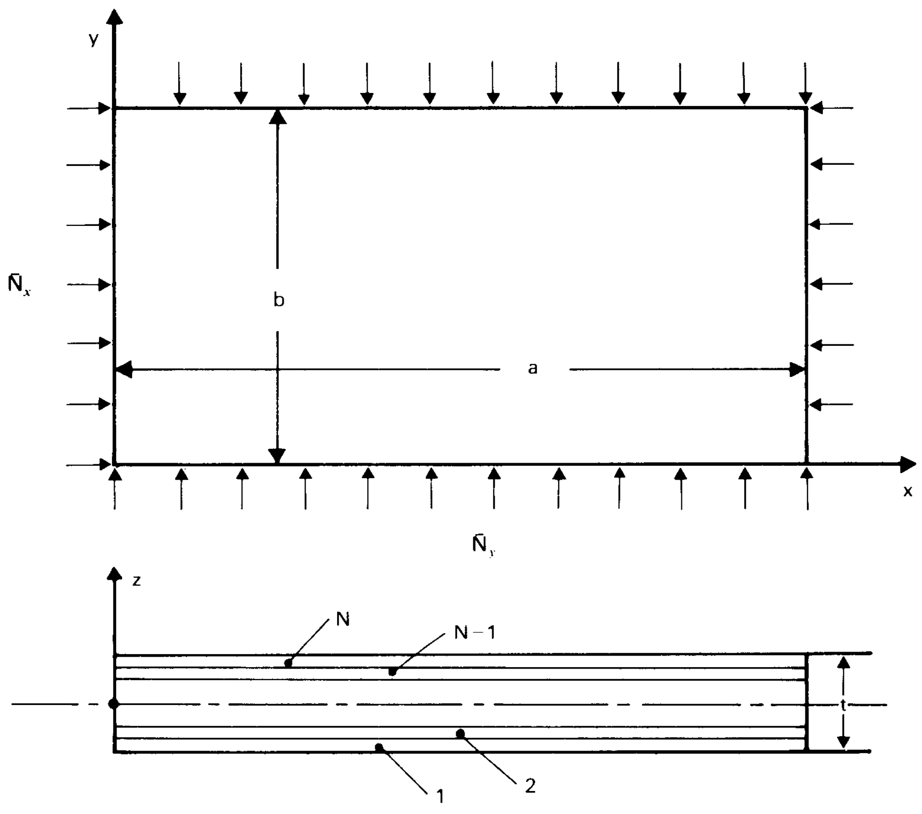

FIG. 1. Laminated plates with orthotropic layers.

$$A_{ij} = h\{(\bar{Q}_{ij})_1 + (\bar{Q}_{ij})_2 + \cdots + (\bar{Q}_{ij})_{N-1} + (\bar{Q}_{ij})_N\} \tag{4a}$$

$$\begin{aligned} 2B_{ij} = h^2[&(\bar{Q}_{ij})_1\{-N+1\} + (\bar{Q}_{ij})_2\{-N+3\} \\ &+ (\bar{Q}_{ij})_3\{-N+5\} + \cdots + (\bar{Q}_{ij})_{N-1}\{-N+(2N-3)\} \\ &+ (\bar{Q}_{ij})_N\{-N+(2N-1)\}] \end{aligned} \tag{4b}$$

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$$3D_{ij} = h^3\{(\bar{Q}_{ij})_1\left[\left\{-\frac{N}{2}+1\right\}^3 - \left\{-\frac{N}{2}\right\}^3\right]$$
$$+ (\bar{Q}_{ij})_2\left[\left\{-\frac{N}{2}+2\right\}^3 - \left\{-\frac{N}{2}+1\right\}^3\right] + \cdots$$
$$+ (\bar{Q}_{ij})_{N-1}\left[\left\{-\frac{N}{2}+(N-1)\right\}^3 - \left\{-\frac{N}{2}+(N-2)\right\}^3\right]$$
$$+ (\bar{Q}_{ij})_N\left[\left\{-\frac{N}{2}+N\right\}^3 - \left\{-\frac{N}{2}+(N-1)\right\}^3\right]\} \tag{4c}$$

where h is the thickness of each layer, N is the total number of the layers, and the subscript of $(\bar{Q}_{ij})$ is the number of each layer.

Whitney and Leissa [1] derived equilibrium equations for the general laminated plates. These equations are now simplified for the present problem as follows.

$$A_{11}u,_{xx} + A_{66}u,_{yy} + (A_{12} + A_{66})v,_{xy} - B_{11}w,_{xxx} - (B_{12} + 2B_{66})w,_{xyy} = 0 \tag{5a}$$

$$(A_{12} + A_{66})u,_{xy} + A_{66}v,_{xx} + A_{22}v,_{yy} - (B_{12} + 2B_{66})w,_{xxy} - B_{22}w,_{yyy} = 0 \tag{5b}$$

$$D_{11}w,_{xxxx} + 2(D_{12} + 2D_{66})w,_{xxyy} + D_{22}w,_{yyyy} - B_{11}u,_{xxx}$$
$$- (B_{12} + 2B_{66})u,_{xyy} - (B_{12} + 2B_{66})v,_{xxy}$$
$$- B_{22}v,_{yyy} + \bar{N}_x w,_{xx} + \bar{N}_y w,_{yy} - 2\bar{N}_{xy} w,_{xy} = 0 \tag{5c}$$

3. Calculation of Axial Buckling Stress

For the present case $\bar{N}_{xy}$ in Eq. (5c) is equal to zero, and the buckling deformations are assumed as

$$u = \bar{u}\cos(m\pi x/a)\sin(n\pi y/b) \tag{6a}$$

$$v = \bar{v}\sin(m\pi x/a)\cos(n\pi y/b) \tag{6b}$$

$$w = \bar{w}\sin(m\pi x/a)\sin(n\pi y/b) \tag{6c}$$

These deformations satisfy the simply supported boundary conditions at $x=0$, a and $y=0$, b (S2 of Ref. 9). Substituting Eqs. (6) into Eqs. (5) and letting $\bar{N}_y = k\bar{N}_x$ give the following buckling formula.

$$\frac{12\bar{N}_x b^2}{\pi^2 t^3 Q_{22}} = \frac{12(b/a)^2}{\pi^4 t^3 Q_{22}\{m^2 + kn^2(a/b)^2\}}\left[T'_{33} + \frac{2T'_{12}T'_{23}T'_{13} - T'_{22}T'^2_{13} - T'_{11}T'^2_{23}}{T'_{11}T'_{22} - T'^2_{12}}\right] \tag{7}$$

where

$$T'_{11} = A_{11}m^2\pi^2 + A_{66}n^2\pi^2(a/b)^2 \tag{8a}$$

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$$T'_{12} = (A_{12} + A_{66})mn\pi^2(a/b) \tag{8b}$$

$$T'_{13} = B_{11}m^3\pi^3 + (B_{12}+2B_{66})mn^2\pi^3(a/b)^2 \tag{8c}$$

$$T'_{22} = A_{66}m^2\pi^2 + A_{22}n^2\pi^2(a/b)^2 \tag{8d}$$

$$T'_{23} = (B_{12} + 2B_{66})m^2n\pi^3(a/b)+B_{22}n^3\pi^3(a/b)^3 \tag{8e}$$

$$T'_{33} = D_{11}m^4\pi^4 + 2(D_{12}+2D_{66})m^2n^2\pi^4(a/b)^2 + D_{22}n^4\pi^4(a/b)^4 \tag{8f}$$

To get the critical buckling stress the smallest value of Eq. (7) must be found by a searching procedure involving integer values of m and n. The critical buckling stress is denoted by $(\bar{N}_x)_{cr}/t$ and a new notation ϕ is introduced.

$$\phi = 12(\bar{N}_x)_{cr}b^2/(\pi^2t^3Q_{22}) \tag{9}$$

For isotropic plates ϕ is equal to 4, when $a/b=1$ and $k=0$.

4. Calculation of Shear Buckling Stress

For the present case $\bar{N}_x$ and $\bar{N}_y$ are equal to zero, and the plates are assumed to be simply supported (S2 of Almroth) at four edges. The following deflection function satisfy the boundary conditions.

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \bar{u}_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{10a}$$

$$v = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \bar{v}_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{10b}$$

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \bar{w}_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{10c}$$

Substitution of Eqs. (10) into Eqs. (5a) and (5b) yields:

$$a\alpha_{11}\bar{u}_{mn} + a\alpha_{12}\bar{v}_{mn} - \pi\alpha_{13}\bar{w}_{mn} = 0 \tag{11a}$$

$$a\alpha_{12}\bar{u}_{mn} + a\alpha_{22}\bar{v}_{mn} - \pi\alpha_{23}\bar{w}_{mn} = 0 \tag{11b}$$

where

$$\alpha_{11} = A_{11}m^2 + A_{66}n^2R^2 \tag{12a}$$

$$\alpha_{12} = (A_{12} + A_{66})mnR \tag{12b}$$

$$\alpha_{13} = B_{11}m^3 + (B_{12} + 2B_{66})mn^2R \tag{12c}$$

$$\alpha_{22} = A_{66}m^2 + A_{22}n^2R^2 \tag{12d}$$

$$\alpha_{23} = (B_{12} + 2B_{66})m^2nR + B_{22}n^3R^3 \tag{12e}$$

In the above expressions R is the aspect ratio a/b of the plates. From Eqs. (11a) and

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(11b) $\bar{u}_{mn}$ and $\bar{v}_{mn}$ can be expressed in terms of $\bar{w}_{mn}$.

$$\bar{u}_{mn} = \frac{\pi}{a}\frac{S_{mn}}{U_{mn}}\bar{w}_{mn} \tag{13a}$$

$$\bar{v}_{mn} = \frac{\pi}{a}\frac{T_{mn}}{U_{mn}}\bar{w}_{mn} \tag{13b}$$

where

$$S_{mn} = \alpha_{13}\alpha_{22} - \alpha_{12}\alpha_{23} \tag{14a}$$

$$T_{mn} = \alpha_{11}\alpha_{23} - \alpha_{12}\alpha_{13} \tag{14b}$$

$$U_{mn} = \alpha_{11}\alpha_{22} - \alpha_{12}{}^2 \tag{14c}$$

Now Eqs. (13) are substituted into Eq. (5c) and Galerkin's method is applied. Then we get the following expression.

$$\Psi\bar{w}_{pq} - \frac{pq}{V_{pq}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} H_{mn}\bar{w}_{mn} = 0 \tag{15}$$

where p and q are the number of half waves in the x- and y-directions respectively; Ψ, H_{mn} and V_{pq} are defined as:

$$\Psi = \frac{1}{\bar{N}_{xy}}\frac{\pi^4}{32a^2R} \tag{16}$$

$$H_{mn} = \frac{m}{p^2 - m^2}\frac{n}{q^2 - n^2} \quad \text{when } p + m\text{: odd and } q + n\text{: odd} \tag{17a}$$

$$= 0 \quad \text{when } p + m\text{: even or } q + n\text{: even} \tag{17b}$$

$$\begin{aligned} V_{pq} = {} & \{D_{11}p^4 + 2(D_{12} + 2D_{66})p^2q^2R^2 + D_{22}q^4R^4\} \\ & - \{B_{11}p^3 + (B_{12} + 2B_{66})pq^2R^2\}S_{pq}/U_{pq} \\ & - \{(B_{12} + 2B_{66})p^2qR + B_{22}q^3R^3\}(T_{pq}/U_{pq}) \end{aligned} \tag{18}$$

Eq. (15) is a system of homogeneous linear equations in $\bar{w}_{mn}$. This system can be divided into two groups, one containing $\bar{w}_{mn}$ for which $m+n$ are odd and the other for which $m+n$ are even. Two buckling forces are obtained from these two groups and the lower one corresponds to the critical buckling force of the laminated plates.

A computer program to solve Eq. (15) was written, and was checked for the case of isotropic plates by comparing with the results obtained by Stein and Neff [*10*].

5. Method of Optimization

The problem is to find the fiber directions which give the maximum critical buckling stress without any constraints. Therefore we can apply one of the unconstrained optimization techniques. Since the objective function for the case of axial compression is the rather complicated function of the design variables and the objective function for

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TABLE 1. Material properties

Property	Boron/Epoxy
E_1	2.11×10^4 kg/mm^2 (30×10^6 psi)
E_2	2.11×10^3 kg/mm^2 (3×10^6 psi)
ν_{12}	0.3
G_{12}	7.03×10^2 kg/mm^2 (1×10^6 psi)

TABLE 2. Optimum fiber directions for 3-layered plates ($a/b = 1$, $k = 0$) under axial compression

Case		Fiber directions (in degree)			Critical stress	Number of half waves	
		α_1	α_2	α_3	ϕ	m	n
1	S	0.0	0.0	0.0	12.921	1	1
	F	45.0	45.0	45.0	22.000	1	1
2	S	0.0	90.0	0.0	12.921	1	1
	F	45.0	135.0	45.0	22.000	1	1
3	S	45.0	0.0	45.0	21.664	1	1
	F	45.0	0.0	45.0	21.664	1	1
4	S	0.0	45.0	0.0	13.258	1	1
	F	45.0	44.9	45.0	22.000	1	1
5	S	30.0	30.0	30.0	19.730	1	1
	F	45.0	45.2	45.0	22.000	1	1
6	S	90.0	0.0	90.0	9.671	2	1
	F	135.0	45.0	135.0	22.000	1	1
7	S	45.0	45.0	45.0	22.000	1	1
	F	45.0	45.0	45.0	22.000	1	1

S: Starting values, F: Final optimum values.

the case of shear is not obtained explicitly, optimization methods without using the derivatives are suitable for solving the problem. Powell's method [*11*] (conjugate direction technique) is selected for use, since it is one of the best methods to find the optimum without using the derivatives [*12*]. This method is intuitively explained by Fox [*13*] as follows: "Given that the function has been minimized once in each of the coordinate directions and then in the associated pattern direction, discard one of the coordinate directions in favor of the pattern direction for inclusion in the next minimizations, since this is likely to be a better direction than the discarded coordinate direction. After the next cycle of minimizations, generate a new pattern direction and again replace one of the coordinate directions."

Powell's method is now applied to find the maximum value of ϕ and the minimum

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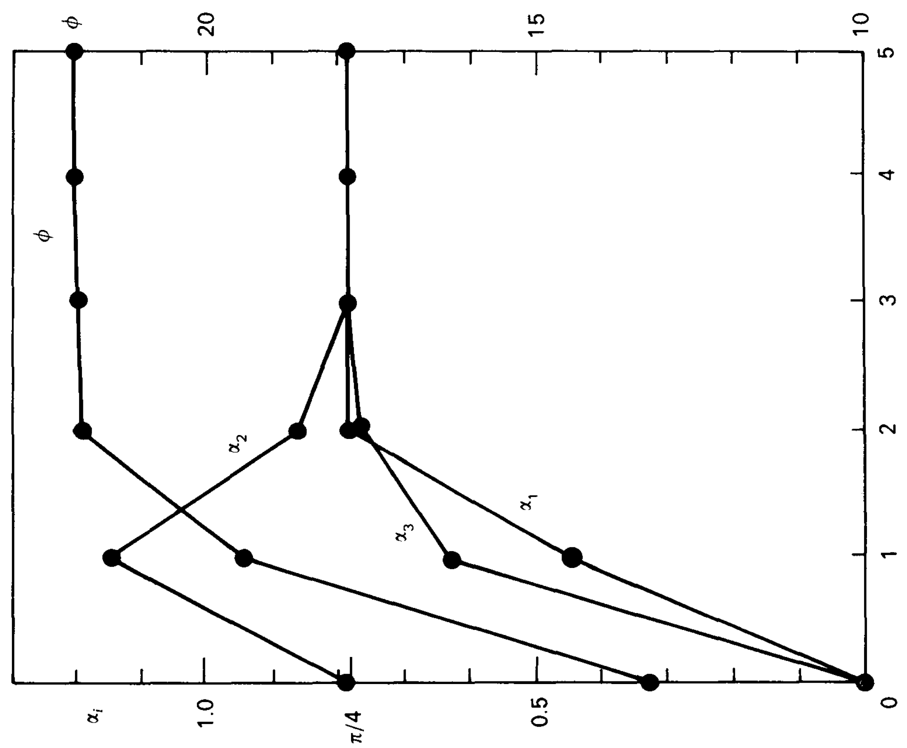

FIG. 2. Variation of α_i and ϕ with number of iterations (Case 4 of Table 2).

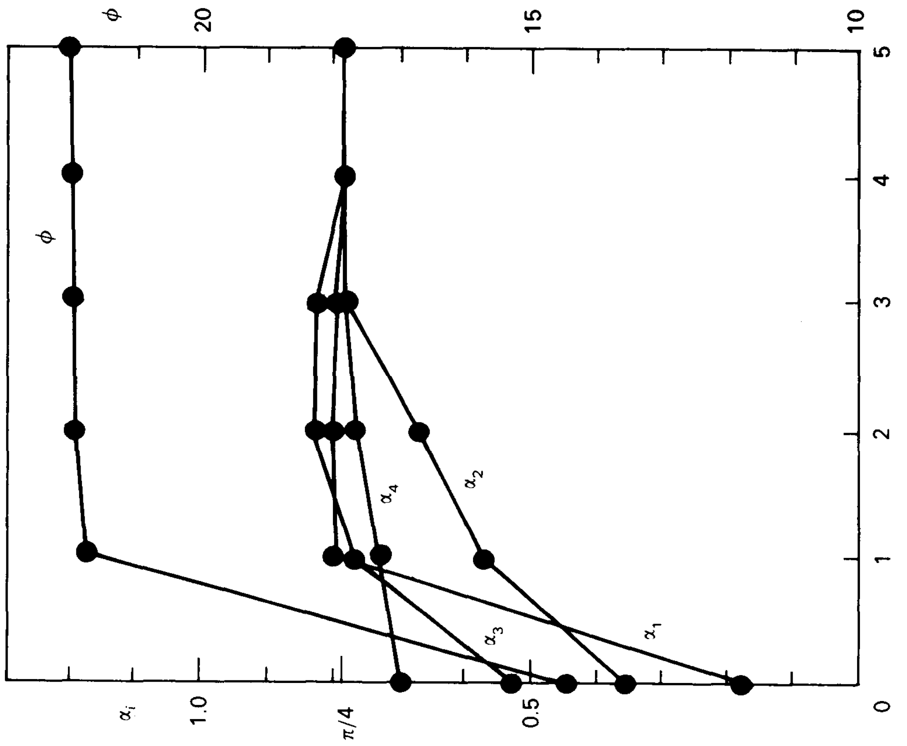

FIG. 3. Variation of α_i and ϕ with number of iterations (Case 8 of Table 3).

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TABLE 3. Optimum fiber directions for 4-layered plates ($a/b=1$, $k=0$) under axial compression

Case		Fiber directions (in degree)				Critical stress	Number of half waves	
		α_1	α_2	α_3	α_4	ϕ	m	n
1	S	0.0	0.0	0.0	0.0	12.921	1	1
	F	45.0	45.1	45.0	45.0	22.000	1	1
2	S	30.0	30.0	30.0	30.0	19.730	1	1
	F	45.0	45.1	45.1	45.0	22.000	1	1
3	S	45.0	45.0	45.0	45.0	22.000	1	1
	F	45.0	45.0	45.0	45.0	22.000	1	1
4	S	90.0	90.0	90.0	90.0	8.421	2	1
	F	135.0	135.0	135.0	135.0	22.000	1	1
5	S	90.0	0.0	0.0	90.0	12.640	2	1
	F	135.0	45.0	45.0	135.0	22.000	1	1
6	S	45.0	0.0	0.0	45.0	20.865	1	1
	F	45.0	45.1	45.0	45.0	22.000	1	1
7	S	0.0	45.0	45.0	0.0	14.056	1	1
	F	45.0	45.0	44.9	45.0	22.000	1	1
8	S	10.0	20.0	30.0	40.0	14.431	1	1
	F	45.0	45.0	44.9	45.0	22.000	1	1

S: Starting values, F: Final optimum values.

value of ψ. Starting values of fiber directions (α_1, α_2 $\cdots$, α_n) are necessary to begin the calculation, and the new fiber directions which give the higher buckling stress are obtained after each iteration. Powell's method requires that the objective function be unimodal. But we do not know if the function is unimodal or not. Therefore, trials with several starting points are desirable.

6. NUMERICAL RESULTS FOR THE CASE OF AXIAL BUCKLING

Numerical calculations were made for the laminated plates with three, four and six layers. The plates considered have the various aspect ratios and are under uniaxial or biaxial compression. Seven or eight different combinations of fiber directions are used to start the calculation. Computer code developed by Powell was combined to the code written for the present problem.

Convergence limits for all the design variables were set to be equal to 0.1° and maximum step size multiplier [*14*] in single variable search was set to be equal to 10.0. The materials considered are Boron/Epoxy and the properties [*8*] are shown in Table 1. The thickness of each layer is assumed to be 0.254 mm (0.01 in.). The buckling formula

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TABLE 4. Optimum fiber directions for 6-layered plates ($a/b=1$, $k=0$) under axial compression

Case		Fiber directions (in degree)						Critical stress	Number of half waves	
		α_1	α_2	α_3	α_4	α_5	α_6	ϕ	m	n
1	S	0.0	0.0	0.0	0.0	0.0	0.0	12.921	1	1
	F	45.0	45.1	45.1	44.9	45.1	45.0	22.000	1	1
2	S	30.0	30.0	30.0	30.0	30.0	30.0	19.730	1	1
	F	45.0	45.0	45.0	44.9	45.0	45.0	22.000	1	1
3	S	45.0	45.0	45.0	45.0	45.0	45.0	22.000	1	1
	F	45.0	45.0	45.0	45.0	45.0	45.0	22.000	1	1
4	S	90.0	90.0	90.0	90.0	90.0	90.0	8.421	2	1
	F	135.0	135.0	135.0	135.1	134.9	135.0	22.000	1	1
5	S	90.0	0.0	90.0	0.0	90.0	0.0	12.272	1	1
	F	135.0	45.0	135.4	44.8	135.0	45.0	22.000	1	1
6	S	45.0	45.0	0.0	0.0	45.0	45.0	21.664	1	1
	F	45.0	45.0	45.0	45.1	44.9	45.0	22.000	1	1
7	S	0.0	0.0	45.0	45.0	0.0	0.0	13.258	1	1
	F	44.9	44.9	47.4	45.4	45.1	45.1	21.999	1	1
8	S	10.0	20.0	30.0	40.0	50.0	60.0	11.711	1	1
	F	45.0	45.0	44.8	45.3	44.6	44.9	22.000	1	1

S: Starting values, F: Final optimum values.

is a function of half waves in the x- and y-directions. Therefore, the numbers of half waves in the x- and y-directions were varied from 1 to 10 and from 1 to 5, respectively, to get the buckling stress for the assigned fiber directions.

The results for three- and four-layered plates with $a/b=1$ and $k=0$ are presented in Table 2 and Table 3, respectively. These two tables show that the results obtained do not depend on the starting values of fiber directions except in Case 4 of Table 2. This case shows that the calculation converged to local maximum. To show the numerical convergence two examples are shown in Figs. 2 and 3 for Case 4 of three-layered plates and for Case 8 of four-layered plates, respectively. In these figures the abscissa is the number of iteration. Each iteration includes many function evaluations. From these figures and tables it can be concluded that this method of finding the best fiber directions works well. Then, the method was applied to six-layered plates and some of the obtained results are shown in Tables 4, 5 and 6. Table 4 is for the case of $a/b=1$ and $k=0$. This table shows that the present method also works well for six-layered plates. Almost all the fiber directions obtained are close to 45°, but some of the directions are not close to 45° because of the slow convergence of the numerical calculations. Table 5 is for the case of $a/b=0.5$ and $k=0.5$. The table shows that the final critical buckling

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TABLE 5. Optimum fiber directions for 6-layered plates ($a/b=0.5$, $k=0.5$) under axial compression

Case		Fiber directions (in degree)						Critical stress	Number of half waves	
		α_1	α_2	α_3	α_4	α_5	α_6	ϕ	m	n
1	S	0.0	0.0	0.0	0.0	0.0	0.0	34.457	1	2
	F	13.2	0.0	0.0	0.2	15.9	0.0	37.011	1	1
2	S	30.0	30.0	30.0	30.0	30.0	30.0	32.633	1	1
	F	3.0	11.7	35.7	−21.9	0.1	11.6	36.966	1	1
3	S	45.0	45.0	45.0	45.0	45.0	45.0	26.016	1	1
	F	4.0	10.0	36.1	−2.0	−21.8	2.1	36.934	1	1
4	S	90.0	90.0	90.0	90.0	90.0	90.0	7.486	1	1
	F	186.3	179.3	90.5	180.0	189.3	167.9	36.492	1	1
5	S	90.0	0.0	90.0	0.0	90.0	0.0	21.294	1	1
	F	173.2	0.2	146.6	0.1	157.2	1.8	36.919	1	1
6	S	45.0	45.0	0.0	0.0	45.0	45.0	26.441	1	1
	F	0.0	2.4	138.7	0.4	−25.3	2.4	36.796	1	1
7	S	0.0	0.0	45.0	45.0	0.0	0.0	35.353	1	1
	F	10.7	−0.9	36.4	39.1	76.9	4.9	36.909	1	1
8	S	10.0	20.0	30.0	40.0	50.0	60.0	18.566	1	1
	F	−7.9	−3.8	22.1	−0.5	−7.4	12.3	37.024	1	1

S: Starting values, F: Final optimum values.

stresses obtained for the different starting values are almost the same but the corresponding fiber directions are not the same. This may be due to the fact that the objective function is not unimodal for this case. Table 6 is for the case of $a/b=1.0$ and $k=0.5$. Summary of the numerical results is given in Table 7. In this table the rows with an asterisk show that the final results obtained depend on the starting values and the values shown correspond to the highest critical buckling stress obtained among 8 cases. In these cases the critical buckling stresses obtained are not much different from each other, but the fiber directions highly depnd on the starting values. The fiber directions in the rows without an asterisk have no decimal, because almost all the directions obtained for seven or eight cases are close to the values shown.

The computer used was IBM 360/67 and average cpu time to calculate eight cases for a six-layered plate under $k=0$ was 158 seconds.

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TABLE 6. Optimum fiber directions for 6-layered plates ($a/b=1$, $k=0.5$) under axial compression

Case		Fiber directions (in degree)						Critical stress	Number of half waves	
		α_1	α_2	α_3	α_4	α_5	α_6	ϕ	m	n
1	S	0.0	0.0	0.0	0.0	0.0	0.0	8.614	1	1
	F	45.0	45.0	.45.0	44.8	45.0	45.0	14.667	1	1
2	S	30.0	30.0	30.0	30.0	30.0	30.0	13.154	1	1
	F	45.0	45.0	45.3	45.1	44.9	45.0	14.667	1	1
3	S	45.0	45.0	45.0	45.0	45.0	45.0	14.667	1	1
	F	45.0	45.0	45.0	45.0	45.0	45.0	14.667	1	1
4	S	90.0	90.0	90.0	90.0	90.0	90.0	7.486	2	1
	F	135.1	134.9	134.9	135.0	135.0	135.0	14.667	1	1
5	S	90.0	0.0	90.0	0.0	90.0	0.0	8.182	1	1
	F	134.9	44.8	134.9	44.9	134.9	45.0	14.667	1	1
6	S	45.0	45.0	0.0	0.0	45.0	45.0	14.442	1	1
	F	45.0	45.1	44.7	44.9	45.1	45.0	14.667	1	1
7	S	0.0	0.0	45.0	45.0	0.0	0.0	8.838	1	1
	F	45.0	45.0	45.1	45.1	45.1	45.0	14.667	1	1
8	S	10.0	20.0	30.0	40.0	50.0	60.0	7.807	1	1
	F	45.0	45.1	44.7	45.3	45.0	45.0	14.667	1	1

S: Starting values, F: Final optimum values.

7. NUMERICAL RESULTS FOR THE CASE OF SHEAR BUCKLING

Powell's computer code was rearranged into a code for the calculation of the shear buckling stress. The material considered is Boron/Epoxy (Table 1). The thickness of each layer is assumed to be 0.254 mm (0.01 in). The buckling stresses were calculated by taking $m=1\sim 3$, $n=1\sim 3$ for $R=1$; and $m=1\sim 5$, $n=1\sim 5$ for $R=1.5$ and 3. The numerical errors of calculated buckling stresses for $R=1$ and 1.5 are less than 3% and the error for $R=3$ is 11%, when all fibers are in the direction of 90° with respect to the x-axis. Therefore, the obtained results for $R=1$ and 1.5 are accurate enough, but the results for $R=3$ may not be accurate enough.

Numerical calculations were first made for the case of three-layered plates with $R=1$ and an example of the numerical convergence is shown in Figure 4. In this figure the abscissa is the number of iterations, and each iteration includes many function evaluations. Then calculations were made for the case of six-layered plates with $R=1$, 1.5 and 3. The results for these cases are presented in Tables 8, 9 and 10. Asterisks in these tables indicate that the numerical calculation was stopped because of the fact that a maximum change in a single variable search did not alter the objective function value.

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TABLE 7. Summary of optimum fiber directions for the case of axial compression

No. of layers	k	a/b	α_1	α_2	α_3	α_4	α_5	α_6	ϕ
3	0	1.0	45°	45°	45°				22.000
4	0	1.0	45°	45°	45°	45°			22.000
6	0	0.5	0°	0°	0°	0°	0°	0°	42.171
6	0	0.8	38°	38°	38°	38°	38°	38°	23.154
6	0	1.0	45°	45°	45°	45°	45°	45°	22.000
*6	0	1.25	49.9°	51.0°	48.6°	48.8°	51.0°	49.9°	23.116
6	0	2.0	45°	45°	45°	45°	45°	45°	22.000
*6	0.5	0.5	7.9°	3.8°	22.1°	0.5°	7.4°	12.3°	37.024
6	0.5	1.0	45°	45°	45°	45°	45°	45°	14.667
*6	0.5	2.0	67.1°	56.4°	56.2°	55.5°	64.0°	61.4°	12.556
6	1.0	1.0	45°	45°	45°	45°	45°	45°	11.000
*6	1.0	2.0	71.6°	68.1°	77.5°	61.2°	71.1°	74.1°	8.051

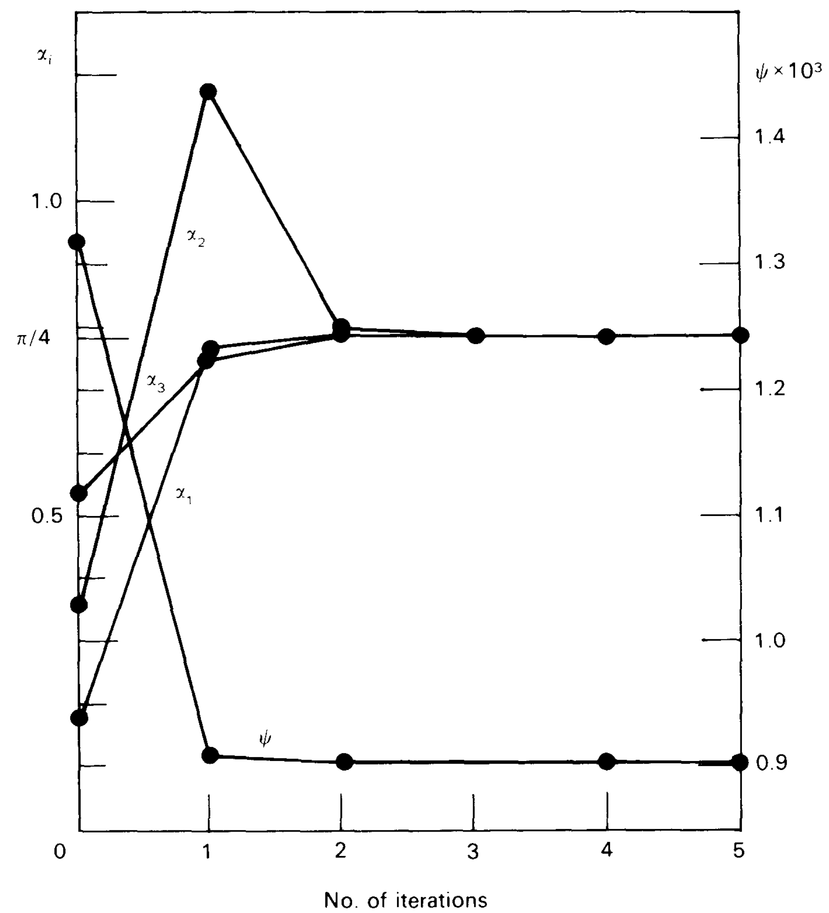

FIG. 4. Variation of α_i and ψ with number of iterations.

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TABLE 8. Optimum fiber directions for 6-layered plates ($a/b=1$) under shear

Case		Fiber directions (in degree) α_1	α_2	α_3	α_4	α_5	α_6	$\psi \times 10^4$
1	S	0.0	0.0	0.0	0.0	0.0	0.0	1.9753
	F	45.6	0.0	0.2	0.1	0.0	43.7	1.2708
2	S	30.0	30.0	30.0	30.0	30.0	30.0	1.2665
	F	45.2	44.9	45.0	45.1	45.1	45.1	1.1283
3	S	45.0	45.0	45.0	45.0	45.0	45.0	1.1283
	F	45.0	45.0	45.0	45.0	45.0	45.0	1.1283
4	S	90.0	0.0	90.0	0.0	90.0	0.0	1.9011
	F	135.0	45.0	134.8	44.4	134.9	45.0	1.1283
5	S	45.0	45.0	0.0	0.0	45.0	45.0	1.1431
	F			*				
6	S	0.0	0.0	45.0	45.0	0.0	0.0	1.9103
	F	44.9	45.3	45.4	45.3	45.2	45.0	1.1283
7	S	10.0	20.0	30.0	40.0	50.0	60.0	2.0520
	F	45.1	45.1	44.3	45.1	45.0	45.0	1.1283
8	S	0.0	0.0	45.0	45.0	90.0	90.0	3.4567
	F			*				
9	S	90.0	90.0	90.0	90.0	90.0	90.0	1.9753
	F			*				
10	S	60.0	60.0	60.0	60.0	60.0	60.0	1.2665
	F	45.1	45.0	43.8	45.0	45.1	44.9	1.1283

S: Starting values, F: Final optimum values.

The computer used for the present case was IBM 3033 and cpu time to obtain Table 9 was 1087 seconds.

8. CONCLUSIONS

A method to find the best fiber directions of the laminated plates under axial compression and shear has been presented in this paper. From Table 7 it can be said that the best fiber angles in all layers are 45° for the plate with $a/b=1$ and 2 under uniaxial compression. For the plates with $a/b=0.5$, 0.8 and 1.25 under $k=0$ the best angles in all layers are 0°, 38° and 50° respectively. It is interesting to note that the best fiber directions for the case of $k=0$ is the same in all layers, even if the stacking sequence is taken into account. For the plates under $k=0$, simple conclusions cannot be obtained. From Tables 8, 9 and 10 the following conclusion can be obtained for the

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TABLE 9. Optimum fiber directions for 6-layered plates ($a/b = 1.5$) under shear

Case		Fiber directions (in degree)						$\psi \times 10^5$
		α_1	α_2	α_3	α_4	α_5	α_6	
1	S	0.0	0.0	0.0	0.0	0.0	0.0	12.279
	F	54.6	54.5	54.4	54.3	54.6	54.5	4.4804
2	S	30.0	30.0	30.0	30.0	30.0	30.0	6.0322
	F	54.7	54.4	54.6	54.6	54.7	54.5	4.4805
3	S	45.0	45.0	45.0	45.0	45.0	45.0	4.6801
	F	54.5	54.7	54.2	54.3	54.8	54.4	4.4805
4	S	90.0	0.0	90.0	0.0	90.0	0.0	6.8648
	F			*				
5	S	45.0	45.0	0.0	0.0	45.0	45.0	4.7749
	F			*				
6	S	0.0	0.0	45.0	45.0	0.0	0.0	11.516
	F	54.5	54.5	55.2	55.1	54.4	54.7	4.4805
7	S	10.0	20.0	30.0	40.0	50.0	60.0	8.3499
	F	54.6	54.3	54.6	54.1	54.6	54.5	4.4805
8	S	0.0	0.0	45.0	45.0	90.0	90.0	12.486
	F	54.4	54.7	54.1	54.6	125.6	125.5	4.4805
9	S	90.0	90.0	90.0	90.0	90.0	90.0	6.7886
	F	126.0	125.4	90.3	90.4	124.9	126.3	4.5004
10	S	60.0	60.0	60.0	60.0	60.0	60.0	4.5425
	F	54.6	54.4	54.3	54.1	54.4	54.6	4.4805

S: Starting values, F: Final optimum values.

case of shear buckling. An angle-ply laminate gives the highest shear buckling stress, even if almost complete freedom is given in the selection of fiber directions. The best fiber directions for the cases $R=1$, 1.5 and 3 are 45°, 55° and 60° respectively. These angles are equal to the ones obtained by Housner-Stein for the case of angle-ply laminated plates.

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TABLE 10. Optimum fiber directions for 6-layered plates ($a/b=3$) under shear

Case		Fiber directions (in degree) α_1	α_2	α_3	α_4	α_5	α_6	$\psi\times 10^6$
1	S	0.0	0.0	0.0	0.0	0.0	0.0	23.473
	F			*				
2	S	30.0	30.0	30.0	30.0	30.0	30.0	11.261
	F	60.6	61.4	61.5	66.9	60.8	59.6	6.5480
3	S	45.0	45.0	45.0	45.0	45.0	45.0	7.5115
	F	61.2	58.5	62.3	62.7	61.0	59.6	6.5484
4	S	90.0	0.0	90.0	0.0	90.0	0.0	10.015
	F	90.0	35.4	90.0	0.0	90.0	0.0	9.6774
5	S	45.0	45.0	0.0	0.0	45.0	45.0	7.6926
	F	60.6	60.6	58.5	57.9	60.7	60.5	6.5469
6	S	0.0	0.0	45.0	45.0	0.0	0.0	21.521
	F	60.7	60.5	62.3	66.9	59.8	60.2	6.5477
7	S	10.0	20.0	30.0	40.0	50.0	60.0	13.175
	F	60.9	60.8	60.1	69.2	59.8	60.0	6.5484
8	S	0.0	0.0	45.0	45.0	90.0	90.0	18.204
	F	59.3	121.0	88.2	58.4	116.8	119.5	6.5558
9	S	90.0	90.0	90.0	90.0	90.0	90.0	8.2017
	F	90.0	90.0	90.0	90.0	90.0	90.0	8.2017
10	S	60.0	60.0	60.0	60.0	60.0	60.0	6.5478
	F	60.7	60.0	60.0	60.0	60.8	60.4	6.5476

S: Starting values, F: Final optimum values.

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