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Hypersonic Flow Near the Forward Stagnation Point of a Blunt-Nosed Body of Revolution

By

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Summary. This paper is concerned primarily with the problem of flow in which the viscous layer is so thick that the flow field behind the shock wave is no longer distinctly divided into the viscous and inviscid regions. A theoretical method is presented on the basis of the simplifying assumptions that the flow is incompressible downstream of the shock wave and furthermore the radial distance r is nearly equal to the distance x measured along the surface from the stagnation point. These simplifications may be considered to lead to good approximation, so far as the ratio between the densities ahead of and just behind the shock wave is sufficiently small and the field under consideration is restricted to the vicinity of the nose. Then the method yields the solution of the Navier-Stokes equations, both of conditions at the shock wave and on the surface being satisfied. Due to the method, however, the solution can be evaluated only by the numerical integration. The alternative method is presented, yielding an analytical expression of the solution, but being applicable only to the case where the Reynolds number is large. The results due to both methods were found very close to each other down to the lower Reynolds number than anticipated.

The main results are summarized as follows:

(1) There exists a similar solution consistent with the conditions both at the shock wave and on the body surface.

(2) The surface pressure coefficient obeys the modified Newtonian law independently of the effect of viscosity.

(3) The distance of the shock wave from the body increases with $1/\sqrt{R_c}$ (R_c: Reynolds number based on the uniform velocity and radius of curvaturs of the shock wave) and the rate of its increase becomes small as R_c decreases.

(4) The skin friction increases almost linearly with $1/\sqrt{R_c}$, at least, over the range of R_c where the present analysis may be applied.

1. INTRODUCTION

The problem of a high speed flow past a blunt-nosed body has received a considerable amount of attention from a practical point of view, because the use of a blunt-nosed body is more profitable than that of a sharp-nosed body concerning surface heat-transfer rate near the nose. In the present paper we will consider a flow near the nose of a spherical body placed in a hypersonic flow. In this case the shock wave is detached from the body and the disturbed flow region occurs between the shock wave and the body. This disturbed region is termed "shock layer" because of its similarity to the boundary layer.

If the thickness of the viscous layer along the wall is thin enough to be ignored as compared with that of the shock layer, then the influence of the viscous layer on the disturbed flow field is so small that the entire flow field except the immediate vicinity of the body may be treated as an inviscid flow. Then the viscous layer can be analyzed within the framework of the boundary-layer approximation under the external condition obtained from the inviscid flow analysis. In fact, based upon this consideration, Mark [1] and Naruse [2] have investigated the effect of the vorticity induced by the bow shock wave on the viscous layer.

Evidently the above consideration is not appropriate to the case where the viscous layer becomes so thick that it plays a significant role in the shock layer, because the interaction phenomenon between the shock wave and the viscous layer becomes considerably complicated and the distinct division of the flow field into the inviscid and viscous regions is no loger permitted. This paper is mainly concerned with a theoretical approach to such a case of the flow around a spherical body placed in a hypersonic flow.

Let us assume that the shock wave itself is thin enough to be regarded as a discontinuous surface and that the effect of the viscosity may be ignored near the shock wave. Then the shock-wave condition may be prescribed by the usual shock relations. Moreover, the ratio, k, of the density ahead of the shock wave to that just behind it may be assumed to be small for the following reason: For the perfect gas of constant specific heats, the well-known Rankine-Hugoniot relations give

$$k=\frac{\gamma-1}{\gamma+1}\left[1+\frac{1}{(\gamma-1)M_\infty^2}\right],$$

where γ is the ratio of specific heats and M_∞, the Mach number of the undisturbed flow. It follows from the above equation that, for the diatomic gas with constant specific heats ($\gamma=1.4$), the value of k tends to 1/6 as M_∞ becomes infinitely large. In fact the temperature of the flow behind the shock wave increases so anomalously that such chemical reactions as dissociation and ionization occur there. According to the Moeckel's analysis [3] in which the effect of the dissociation of gas is taken into account, k may attain certain lower values than the extreme value of 1/6 for the diatomic perfect gas. For this reason the density ratio k may be considered to be small in a hypersonic flow.

According to Lighthill [4], the Mach number M_2 of the flow just behind the shock wave is given, under the strong shock-wave approximation by

$$M_2\simeq\sqrt{\frac{k^2+\cot^2\theta}{\gamma(1-k)k}},$$

where θ is the angle of inclination of the shock wave to the axis (see Fig. 1). Since in the vicinity of the nose $\theta\cong\pi/2$,

$$M_2\sim O(\sqrt{k})$$

there. Therefore, the disturbed flow region behind the shock wave may be assumed to be incompressible near the nose as applied in many literatures (see, for example, [1], [4], [5] and [6]). Furthermore, let us assume that the transport properties such as viscosity are constant.

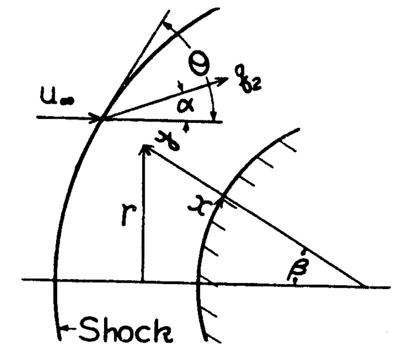

FIGURE 1.

As in the case of the axially symmetric boundary layer, it is convenient to introduce the coordinates (x, y) in which x is measured along the surface from the stagnation point and y perpendicular to it into the fluid. The curvilinear coordinates (x, y) are related near the nose to the cylindrical coordinates (r, z) in which r and z are the radial and axial distances, respectively, as follows:

$$\left.\begin{aligned} r &\simeq x(1+\kappa y), \\ z &\simeq y, \end{aligned}\right\} \tag{1}$$

where κ is the curvature of the body at the nose and its sign is taken positive when the body is convex to the oncoming flow. Let us denote the radius of curvature of the body at the nose by R_B, then

$$\kappa y = y/R_B,$$

its maximum value being δ/R_B where δ is the thickness of the shock layer. Since, as will be shown from the result, δ/R_B is of the order of magnitude of k, the term κy may be neglected as compared with unity. This approximation leads to

$$r \simeq x, \quad z \simeq y.$$

2. Fundamental Equations and Shock-Wave Conditions

On the basis of the assumptions made in the preceding section, the basic equations of motion governing the flow near the nose of axially symmetric body become [7]

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{1}{x}\frac{\partial u}{\partial x}-\frac{u}{x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{1}{x}\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial y^2}\right), \tag{3}$$

where u, v are the velocity components in the x- and y-axes, respectively, and p is the pressure, ρ the density and ν the kinematic viscosity. The continuity equation is

$$\frac{\partial(xu)}{\partial x}+\frac{\partial(xv)}{\partial y}=0. \tag{4}$$

Eliminating the terms concerning p from Eqs. (2) and (3), we get

$$u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}-\frac{u}{x}\omega=\nu\left[\frac{\partial^2\omega}{\partial x^2}+\frac{1}{x}\frac{\partial\omega}{\partial x}-\frac{\omega}{x^2}+\frac{\partial^2\omega}{\partial y^2}\right], \tag{5}$$

where ω denotes the vorticity defined by

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}. \tag{6}$$

Now we assume that u and v can be expressed in the form

$$\left.\begin{aligned} u&=xf'(y), \\ v&=-2f(y), \end{aligned}\right\} \tag{7}$$

so that the continuity equation (4) is satisfied. In general u will be expressed in a power series for small values of x, as follows:

$$u=c_0+c_1x+c_2x^2+\cdots.$$

Due to the symmetry of u the terms of even powers should be removed and, since $u=0$ at $x=0$, c_0 must vanish. Therefore, if only the leading term is retained, u takes just the form of Eqs. (7).

Substituting Eqs. (7) into Eq. (5) we obtain the ordinary differential equation

$$\nu f''''+2ff'''=0, \tag{8}$$

where the numbers of primes represent the order of differentiation. The integration of this equation leads to

$$\nu f'''+2ff''-f'^2+K^2=0, \tag{9}$$

where K^2 is an integral constant. For the three-dimensional case of the incompressible flow impinging on a wall perpendicular to it, the velocity components are also written in the form of Eqs. (7) and K^2 in Eq. (9) is found to be unity from the conditions at infinity [*8*],

$$f'=1, \quad f''=f'''=0.$$

For the present case, however, the disturbed flow region is bounded by wall and shock wave, and K^2 in Eq. (9) must be determined so as to be consistent with the conditions prescribed on the shock wave. The value of K^2 will be in general different from unity.

Next we proceed to obtain the conditions imposed on f at the shock wave. As mentioned before, our consideration is restricted to the case, in which the shock wave itself is thin enough to be regarded as a discontinuous surface and the viscous effect of the flow just behind it is negligibly small. Then we can obtain the flow condition just behind the shock wave by use of the usual shock relations. The conservations of the mass flow and of the velocity component tangential to the shock wave are specified by the following relations, respectively,

$$\rho_\infty u_\infty \sin\theta=\rho_2 q_2 \sin(\theta-\alpha), \tag{10}$$

$$u_\infty \cos\theta=q_2\cos(\theta-\alpha), \tag{11}$$

where subscripts ∞ and 2 stand for the quantities ahead of and just behind the shock wave, respectively, and q is the velocity, θ the inclination of the shock wave to the axis and α the turning angle of the flow direction across the shock wave (Fig. 1).

From the above relation we can express the velocity components u and v in terms of k, u_∞ and the radius of curvature, R_S, of the shock wave. From Eqs. (10) and (11)

$$q_2=\sqrt{(ku_\infty \sin\theta)^2+u_\infty^2\cos^2\theta}\,. \tag{12}$$

Since, for the flow near the nose under consideration, $\theta\simeq\pi/2$, $q_2\simeq ku_\infty$. The velocity component v_2 is given by $v_2\simeq -q_2$. Therefore, we get $v_2\simeq -ku_\infty$. From the definition the velocity component u_2 can be expressed in the form

$$u_2=q_2\cos(\theta-\alpha)\,. \tag{13}$$

Now, let R_S take the positive sign when the shock wave is convex to the oncoming flow, then

$$R_S=-1\Big/\frac{\partial\theta}{\partial s}\,,$$

where s is measured along the shock wave from the axis. Carrying out the limiting procedure for Eq. (11)

$$\lim_{s\to0}\frac{\cos(\theta-\alpha)}{s}=\lim_{s\to0}\frac{u_\infty}{q_2}\left(-\frac{\partial\theta}{\partial s}\right).$$

Using the above relation, we obtain from Eq. (13)

$$u_2=q_2\cos(\theta-\alpha)\simeq\frac{u_\infty s}{R_S}\simeq\frac{u_\infty x}{R_S}\,. \tag{14}$$

Also, as shown in Appendix, the vorticity ω induced by the shock wave is expressed in the form

$$\omega\simeq-\frac{(1-k)^2}{k}\,\frac{u_\infty x}{R_S^2}\,. \tag{15}$$

The above-obtained expressions of v_2, u_2 and ω just behind the shock wave near the nose are rewritten in terms of f by using Eqs. (7), as follows:

$$\left.\begin{aligned} &f(\delta)\simeq ku_\infty/2\,,\quad f'(\delta)\simeq u_\infty/R_S\,,\\ &f''(\delta)\simeq\frac{(1-k)^2}{k}\,\frac{u_\infty}{R_S^2}\,,\end{aligned}\right\} \tag{16}$$

where the shock-wave location is given by $y=\delta$. Here it is worth noting that, as seen from the derivation of the expression of vorticity, the above equations are valid only when the shock wave may be regarded as a discontinuous surface and furthermore the viscous effect may be neglected in the flow just behind the shock wave. Since the viscous effect is represented by the term $\nu f'''$ in Eq. (9), the avove argument leads to the conclusion that the relation

$$f'''\simeq0 \tag{17}$$

must be valid near the shock wave in order to ensure the validity of Eqs. (16). Then the function f should be expressed near the shock wave in a quardratic equation with respect to y, that is,

$$f(y) \simeq u_\infty \left[C_0 + C_1 \frac{y}{R_S} + C_2 \left(\frac{y}{R_S} \right)^2 \right], \tag{18}$$

where C_0, C_1 and C_2 are the constants to be determined by the shock-wave conditions (16). Substituting Eq. (18) into Eqs. (16), we get

$$\left. \begin{aligned} & C_0 + C_1 \frac{\delta}{R_S} + C_2 \left(\frac{\delta}{R_S} \right)^2 = \frac{k}{2}, \\ & C_1 \frac{\delta}{R_S} + 2C_2 \left(\frac{\delta}{R_S} \right)^2 = \frac{\delta}{R_S}, \\ & C_2 = \frac{(1-k)^2}{2k}, \end{aligned} \right\} \tag{19}$$

whence we obtain

$$\left. \begin{aligned} & C_0 = \frac{k}{2} - \frac{\delta}{R_S} + \frac{(1-k)^2}{2k} \left(\frac{\delta}{R_S} \right)^2, \\ & C_1 = 1 - \frac{(1-k)^2}{k} \frac{\delta}{R_S}, \\ & C_2 = \frac{(1-k)^2}{2k}. \end{aligned} \right\} \tag{20}$$

Substituting the external conditions (16) and (17) thus obtained into Eq. (9), we obtain

$$K = \frac{u_\infty}{R_S} \sqrt{k(2-k)}. \tag{21}$$

3. Mathematical Formulation and Method of Solution

It is now convenient to introduce the following transformation

$$\left. \begin{aligned} \sqrt{K\nu}\, \phi(\eta) &= f(y), \\ \eta &= \sqrt{K/\nu}\, y. \end{aligned} \right\} \tag{22}$$

Then Eqs. (8) and (9) are rewritten in the form

$$\phi'''' + 2\phi\phi''' = 0, \tag{23}$$

$$\phi''' + 2\phi\phi'' - \phi'^2 + 1 = 0, \tag{24}$$

respectively. It is evident from the analysis in the previous section that the integration of Eq. (23) and subsequent application of the shock conditions lead to Eq. (24). Eq. (24) is quite the same as the Homann's equation for the three-dimensional case of the incompressible stagnation flow. For the case of the incompressible stagnation flow the region considered covers the range, in terms of η, from zero to infinity. In the present problem, however, the region to be solved is confied within the range between the wall and the shock wave, whose location can be determined only after the solution has been obtained. Due to the above circumstances, a different approach from that to the incompressible stagnation flow problem becomes necessary.

The corresponding η-coordinate, η_S, to the shock-wave location $y=\delta$ is given from Eq. (22) by

$$\eta_S = [k(2-k)]^{1/4}\sqrt{R_e}\,\delta R_S\,, \tag{25}$$

where R_e is the Reynolds number referred to the undisturbed velocity u_∞ and the radius of curvature, R_S, of the shock wave—i.e.,

$$R_e = u_\infty R_S/\nu\,. \tag{26}$$

Since the function $f(y)$ can be expressed near the shock wave in the form of Eq. (18), the function ϕ takes the following form there

$$\phi(\eta) \simeq \frac{u_\infty}{\sqrt{K\nu}}\left[C_0 + C_1\frac{\eta}{L} + C_2\left(\frac{\eta}{L}\right)^2\right], \tag{27}$$

where L denotes the value of R_S measured in the η-coordinate—i.e.,

$$L = \sqrt{\frac{K}{\nu}}\,R_S\,,$$

or, using Eqs. (21) and (26),

$$L = [k(2-k)]^{1/4}\sqrt{R_e}\,. \tag{28}$$

K and L from Eqs. (21) and (28), respectively, are now substituted into Eq. (27), thus yielding

$$\phi(\eta) \simeq A\sqrt{R_e} + B\eta + C\eta^2/\sqrt{R_e}\,, \tag{29}$$

where

$$A = [k(2-k)]^{-1/4}C_0\,,$$
$$B = [k(2-k)]^{-1/2}C_1\,,$$
$$C = [k(2-k)]^{-3/4}C_2\,,$$

or, by using Eq. (20),

$$\left.\begin{aligned} A &= [k(2-k)]^{-1/4}\left[\frac{k}{2} + \frac{\delta}{R_S}\,\frac{(1-k)^2}{2k}\left(\frac{\delta}{R_S}\right)^2\right],\\ B &= [k(2-k)]^{-1/2}\left[1 - \frac{(1-k)^2}{k}\,\frac{\delta}{R_S}\right],\\ C &= [k(2-k)]^{-3/4}(1-k)^2/2k\,. \end{aligned}\right\} \tag{30}$$

Although, rigorously speaking, the viscous effect will never vanish at finite distance from the wall, it may be considered, from a physical point of view, that it becomes insignificant at the region far from the wall. Under the above consideration the asymptotic solution of Eqs. (8) or (9) for large y may be approximately representedy by the solution of Eq. (17), which results from disregarding the viscosity term. Therefore, the asymptotic solution of Eq. (23) or (24) can be written from the solution of $\phi''' \simeq 0$, in the form

$$\phi \simeq a + b\eta + c\eta^2\,, \tag{31}$$

where a, b and c are constants. In the present problem these constants should be determined so as to fulfill the conditions to be imposed at the shock wave ($\eta = \eta_S$) and then the function ϕ is found to be given by Eq. (29).

Since the asymptotic solution of Eqs. (23) or (24) for large η has the form of Eq. (31) as its most predominant part, generally it can be expressed in the form

$$\phi \simeq a+b\eta+c\eta^2+\varphi(\eta), \tag{32}$$

where $\varphi(\eta)$ represents a function of η only and is a first-order small quantity. Then substituting Eq. (32) into Eq. (23) and neglecting the cross-product term with respect to φ, we obtain

$$\varphi''''+2(a+b\eta+c\eta^2)\varphi'''=0. \tag{33}$$

Moreover, integrating this equation and applying the condition that φ tends to zero as η increases, we get

$$\varphi=d\int_\infty^{\eta}\int^{\eta'}\int^{\eta''}\exp\left[-2\left(a\eta'''+\frac{b}{2}\eta'''^2+\frac{c}{3}\eta'''^3\right)\right]d\eta' d\eta'' d\eta''', \tag{34}$$

where d is an integral constant. Thus the asymptotic solution of Eq. (23) or (24) for large η is given by Eq. (32) with φ of Eq. (34).

The foregoing discussion may lead to the conclusion that Eq. (31) is approximately valid near the shock wave $\eta=\eta_S$, only if the value of φ which represents the viscous effect is small enough to be neglected there. For this case only, the constants a, b and c can be determined, as mentioned before, from the shock-wave conditions and then Eq. (32) becomes

$$\phi=A\sqrt{R_e}+B\eta+B\eta+C\eta^2/\sqrt{R_e}+\varphi, \tag{35}$$

where φ is given by Eq. (34). Then, since $c=C/\sqrt{R_e}$ and, from Eqs. (30), $C>0$, we can see from Eq. (34) that φ rapidly decreases with increasing η.

Let us assume tentatively that, although the values of η_S still remain unknown, it is sufficiently large so that φ in Eq. (32) may be neglected as compared with the quardratic part, at the point $\eta=\eta_S$. Then it is evident that the values of φ for $\eta>\eta_S$ are also negligibly small. Hence, as seen from the form of Eq. (35), the condition

$$\phi''(\eta)\simeq 2C/\sqrt{R_e}=\text{const.} \tag{36}$$

should be imposed for large values of η. On the other hand the conditions at the wall are specified by

$$\phi=\phi'=0 \quad \text{at} \quad \eta=0. \tag{37}$$

The boundary conditions (36) and (37) are complete for the solution of the third-order differential equation (24).

We now proceed to the solution and, then, to the determination of the shock-wave location, $\eta=\eta_S$. Consider the case where the values of k and R_e are initially given. Then, since all the constants involved in Eq. (36) are known, the solution of Eq. (24) can be numerically determined. After the solution, the value of η_S (or from Eq. (25) the value of δ/R_S) remained as the only unknown can be determined from the second equation of Eqs. (30) by using the value of B evaluated from Eq. (35) to be the asymptotic value of $\phi'-\eta\phi''$ for large η. If the value of η_S thus obtained is large enough that the solution ϕ may be represented by Eq. (29) near the point $\eta=\eta_S$, then the solution may be accepted as a consistent one.

In the present problem, there exist four parameters k, R_e, δ/R_S and $\phi''(0)$, any two of which can be assumed independently. For the case where k and R_e are initially given, many iterations are required to solve Eq. (24) under the conditions (36) and (37). If, however, k and $\phi''(0)$ are initially given, then such a tedious labor is possibly avoided, because the solution can be determined by only the boundary conditions specified at the wall and, therefore, the solution can be readily found by step-by-step integration starting from the wall. The values of ϕ'' and $\phi'-\eta\phi''$ approach to the constant values of $2C/\sqrt{R_e}$ and B, respectively, with the increase of η. By using these values together with Eq. (30), the values of R_e and δ/R_S can be determined. Actual calculations have been carried out for this case. For the initial choice of $\phi''(0)=1.5$, 1.7 and 2.0, the results of the numerical integrations of Eq. (24) are shown in Figs. 2, 3 and 4, respectively. We can see from these figures that the values of ϕ'' and $\phi'-\eta\phi''$ rapidly approach to the respective asymptotic values as η increases. The results are presented in the upper lines of each columns of Table 1 and, $\phi''(0)$ and δ/R_S are plotted in Fig. 5 against the value of $1/\sqrt{R_e}$. The discussion on these results will be made in Section 5.

TABLE 1.

k	0.10			0.15			0.20		
$\phi''(0)$	1.5	1.7	2.0	1.5	1.7	2.0	1.5	1.7	2.0
$\sqrt{R_e}$	103.0 102.0	51.8 51.2	30.5 30.3	45.9 45.7	23.2 23.0	— —	24.4 24.1	— —	— —
$\frac{\delta}{R_S}$	0.0769 0.0768	0.0822 0.0824	0.0878 0.0880	0.113 0.113	0.124 0.124	— —	0.150 0.151	— —	— —
η_S	5.20 5.19	2.81 2.79	1.77 1.76	3.76 3.75	2.08 2.07	— —	2.83 2.81	— —	— —

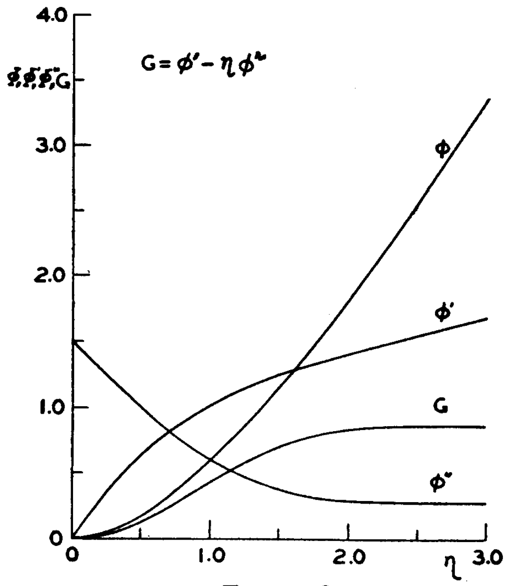

FIGURE 2.

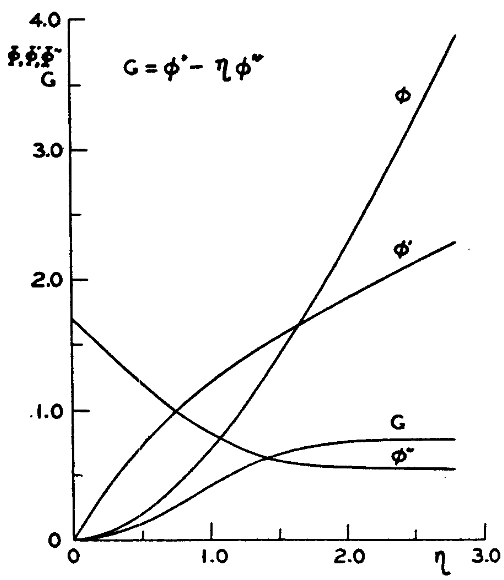

FIGURE 3.

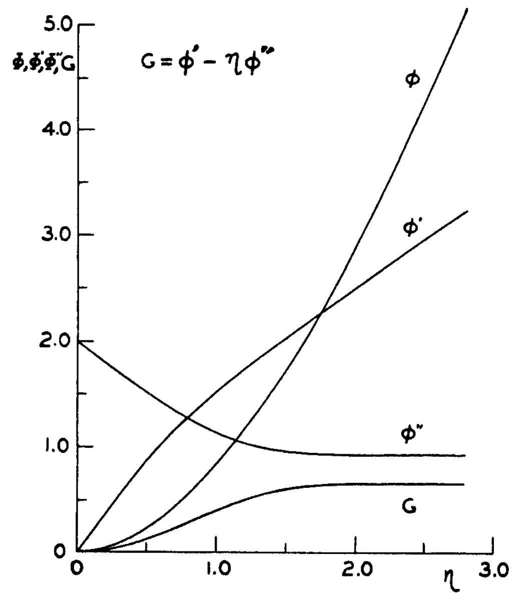

FIGURE 4.

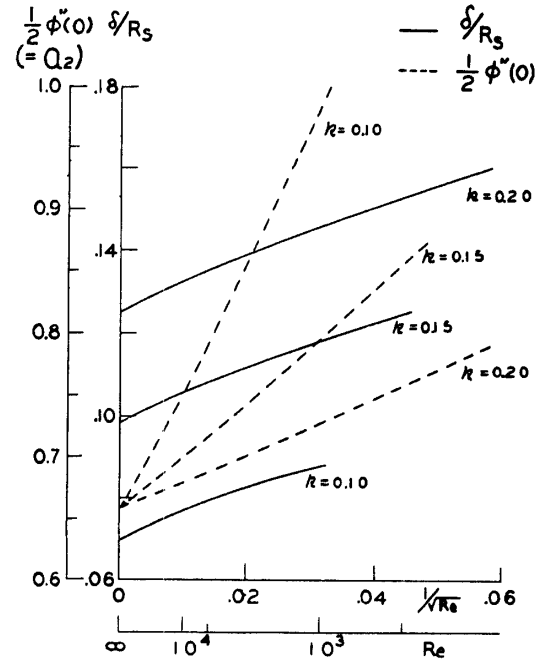

FIGURE 5.

4. Analytical Method of Solution for Case Where Viscous Layer is Thin

The method of solution presented in the preceding section is applicable even to the case where the viscous layer is considerably thich, so far as the viscous effect may be negligibly small near the shock wave. However, this method is not convenient for an analytical examination of the behavior of flow.

For this reason, we put forward in the present section an alternative, in which the solution may be analytically expressed. First, consider the extreme case where R_e is infinitely large. In this case the viscous layer may be considered infinitely thin and hence the solution of Eq. (9) is given by Eq. (18) everywhere in the disturbed region. Since $v=0$ at the wall, from Eqs. (7) $f(0)=0$. Hence C_0 in Eq. (18) vanishes. Then, from Eqs. (20), δ/R_S, C_1 and C_2 are obtained in terms of k as follows:

$$\left.\begin{aligned} C_1 &= \sqrt{1-(1-k)^2} \simeq \sqrt{k(2-k)}\,, \\ C_2 &= \frac{(1-k)^2}{2k}\,, \\ \frac{\delta}{R_S} &= \frac{k\{1-\sqrt{1-(1-k)^2}\}}{(1-k)^2} \simeq \frac{k\{1-\sqrt{k(2-k)}\}}{(1-k)^2}\,. \end{aligned}\right\} \qquad (38)$$

These results are all the same as those found by Li and Geiger [5]. On the other hand, although the viscous layer itself vanishes at the limit as R_e becomes infinitely large, we can see that there exists the solution of ϕ formally. Substituting δ/R_S of Eqs. (38) in Eqs. (30), we obtain

$$A=0, \quad B=1\,.$$

Then Eq. (29) becomes

$$\phi(\eta)=\lim_{\substack{A\to 0\\ R_e\to\infty}} A\sqrt{R_e}+\eta\,.$$

The limit in the right hand side may be expected to tend to a finite value. Indeed, as shown later, this limiting value is found to be -0.5576. Since, from Eq. (25), $\eta_S\to\infty$ as $R_e\to\infty$, the region under consideration spreads over, in terms of η, from zero to infinity. Therefore, the external condition is given by

$$\phi\to\text{const.}+\eta\,, \tag{39}$$

or

$$\phi'\to 1 \quad\text{as}\quad \eta\to\infty\,.$$

This condition is quite the same as for the three-dimensional case of the incompressible stagnation flow impinging on a wall perpendicular to it. Furthermore, Eq. (24) and the conditions at the wall are also of the same form for both cases. Thus ϕ for an extreme case where the Reynolds number becomes infinitely large, is quite identical with that for the case of the stagnation flow which has already been solved by Homann [*8*]. To make it sure we can see from Fig. 5 that the value of ϕ'' tends closely to Homann's value $\phi''(0)=0.667$ at the limit as $R_e\to\infty$. According to the result of Homann,

$$\phi\to -0.5576+\eta$$

as $\eta\to\infty$. Hence the value of the constant in Eq. (39) is found to be -0.5576.

Comparing Eq. (32) with Eq. (35), we can obtain the solution ϕ in the form

$$\phi=A\sqrt{R_e}+B\eta+C\eta^2/\sqrt{R_e}+D\int_{\infty}^{\eta}\int_{\infty}^{\eta'}\int_{\infty}^{\eta''} e^{-F(\eta''')}d\eta' d\eta'' d\eta''' \tag{40}$$

for large η. Here D is an integral constant and

$$F(\eta)=2(A\sqrt{R_e}+B\eta/2+C\eta^2/3\sqrt{R_e})\eta\,. \tag{41}$$

The differentiations of Eq. (40) lead to

$$\phi' =B+2C\eta/\sqrt{R_e}+D\int_{\infty}^{\eta}\int_{\infty}^{\eta'} e^{-F(\eta'')}d\eta' d\eta''\,, \tag{42}$$

$$\phi''=2C/\sqrt{R_e}+D\int_{\infty}^{\eta} e^{-F(\eta')}d\eta'\,. \tag{43}$$

On the other hand the solution valid near the wall is assumed in the form

$$\phi=\sum_{n=0}^{\infty}a_n\eta^n\,. \tag{44}$$

Substituting ϕ of Eq. (44) in Eq. (24) and using the conditions (37), the recurrence formula for a_ns will be found and then the coefficient a_2 is retained as the only unknown among all coefficients. This was carried out by Homann [*8*] and the expressions of a_ns from a_0 to a_{25} are presented in his paper. In the present analysis his results are used. From Eq. (44) we get

$$\phi' = \sum_{n=0}^{\infty}(n+1)a_{n+1}\eta^n, \tag{45}$$

$$\phi'' = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\eta^n. \tag{46}$$

Now let us select a certain appropriate point, say η_0, in the intermediate region in which both expressions for ϕ are valid. Then equating ϕ, ϕ' and ϕ'' from the asymptotic solution to those from the series solution, respectively, at this point, we have

$$\left.\begin{aligned}
&\sum_{n=0}^{\infty} a_n\eta_0^n = A\sqrt{R_e} + B\eta_0 + C\eta_0^2/\sqrt{R_e} + D\int_{\infty}^{\eta_0}\int^{\eta'}\int^{\eta''} e^{-F(\eta''')}d\eta' d\eta'' d\eta'', \\
&\sum_{n=0}^{\infty} a_{n+1}(n+1)\eta_0^n = B + 2C\eta_0/\sqrt{R_e} + D\int_{\infty}^{\eta_0}\int^{\eta'} e^{-F(\eta'')}d\eta' d\eta'', \\
&\sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2)\eta_0^n = 2C/\sqrt{R_e} + D\int_{\infty}^{\eta_0} e^{-F(\eta')}d\eta'.
\end{aligned}\right\} \tag{47}$$

As pointed out previously, there exist in the present problem four parameters k, R_e, δ/R_S and $\phi''(0)$. Since, from Eq. (44), $a_2=\phi''(0)/2$, we may take a_2 instead of $\phi''(0)$. Then, if two of the parameters are initially given, we can determine from Eqs. (47) the remaining two parameters and the unknown coefficient D. In general, a considerable amount of laborious calculations will be required for their determination. For the case where the Reynolds number R_e is large, however, the above-mentioned procedure becomes feasible under some approximations. This will be shown in the following.

First, let us consider an integral

$$I = \int_{\infty}^{\eta_0} e^{-F(\eta')}d\eta',$$

in which F is given by Eq. (41). As seen from the numerical results obtained in the preceding section, the sign of C in Eq. (41) as well as of c may be considered positive. Therefore, the integrand $\exp(-F)$ is much more predominant near the point $\eta=\eta_0$ in the region $\eta_0<\eta<\infty$, so that the value of the integral I depends mostly upon the behavior of F near the point $\eta=\eta_0$. It is suggested from the behavior of F in the extreme case where $R_e\to\infty$ that the function F given by Eq. (41) may be approximately represented only by the second term when R_e is large—i.e.,

$$F(\eta) \simeq B\eta^2. \tag{48}$$

This approximation was already made by Homann for the three-dimensional case of an incompressible stagnation flow and satisfactory results were obtained.

Using the above approximation,

$$I \simeq \int_{\infty}^{\eta_0} e^{-B\eta'^2}d\eta'. \tag{49}$$

Integrating Eq. (49) by parts, we get

$$I \simeq \int_\infty^{\eta_0} e^{-B\eta'^2} d\eta' = \int_0^{\eta_0} e^{-B\eta'^2} d\eta' - \int_0^\infty e^{-B\eta'^2} d\eta' = \frac{\sqrt{\pi}}{2\sqrt{B}} [\Phi(\sqrt{B}\,\eta_0) - 1] \,, \tag{50}$$

where Φ represents the error integral defined by

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \,.$$

In the similar manner the other integrals involved in Eq. (47) become after the same simplification

$$\begin{aligned}
\int_\infty^{\eta_0}\int_\infty^{\eta'} e^{-F(\eta'')} d\eta' d\eta'' &\simeq \eta_0 \int_\infty^{\eta_0} e^{-B\eta'^2} d\eta - \int_\infty^{\eta_0} \eta' e^{-B\eta'^2} d\eta' \\
&= \eta_0 \int_\infty^{\eta_0} e^{-B\eta'^2} d\eta' + \frac{1}{2B} e^{-B\eta_0^2} \\
&= \frac{\sqrt{\pi}}{2\sqrt{B}} \eta_0 [\Phi(\sqrt{B}\,\eta_0) - 1] + \frac{1}{2B} e^{-B\eta_0} \,,
\end{aligned} \tag{51}$$

and

$$\begin{aligned}
\int_\infty^{\eta_0}\int_\infty^{\eta'}\int_\infty^{\eta''} e^{-F(\eta''')} d\eta' d\eta'' d\eta''' &\simeq \int_\infty^{\eta_0} \eta' \int_\infty^{\eta'} e^{-B\eta''^2} d\eta' d\eta'' + \frac{1}{2B} \int_\infty^{\eta_0} e^{-B\eta'^2} d\eta' \\
&= \frac{\eta_0^2}{2} \int_\infty^{\eta_0} e^{-B\eta'^2} d\eta' - \frac{1}{2} \int_\infty^{\eta_0} \eta'^2 e^{-B\eta'^2} d\eta' \\
&= \left(\frac{1}{2B} + \frac{\eta_0^2}{2}\right) \int_\infty^{\eta_0} e^{-B\eta'^2} d\eta' + \frac{\eta_0}{4B} e^{-B\eta_0^2} - \frac{1}{4B} \int_\infty^{\eta_0} e^{-B\eta'^2} d\eta' \\
&= \left(\frac{1}{4B} + \frac{\eta_0^2}{2}\right) \int_\infty^{\eta_0} e^{-B\eta'^2} d\eta' + \frac{\eta_0}{4B} e^{-B\eta_0^2} \\
&= \left(\frac{1}{4B} + \frac{\eta_0^2}{2}\right) \frac{\sqrt{\pi}}{2\sqrt{B}} [\Phi(\sqrt{B}\,\eta_0) - 1] + \frac{\eta_0}{4B} e^{-B\eta_0^2} \,.
\end{aligned} \tag{52}$$

By using Eqs. (49) to (52), Eqs. (47) become

$$\left.\begin{aligned}
\sum_{n=0}^\infty a_n \eta_0^n &= A\sqrt{R_e} + B\eta_0 + \frac{C\eta_0^2}{\sqrt{R_e}} \\
&\quad + \frac{D}{2B}\left[\left(\frac{1}{2} + B\eta_0^2\right) \frac{\sqrt{\pi}}{2\sqrt{B}} \left[\Phi(\sqrt{B}\eta_0 - 1) + \frac{\eta_0}{2} e^{-B\eta_0^2}\right]\right], \\
\sum_{n=0}^\infty (n+1) a_{n+1} \eta_0^n &= B + \frac{2C\eta_0}{\sqrt{R_e}} + \frac{D}{2B} [\sqrt{\pi B}\eta_0 \{\Phi\sqrt{B}\eta_0) - 1\} + e^{-B\eta_0^2}] \,, \\
\sum_{n=0}^\infty (n+1)(n+2) a_{n+2} \eta_0^n &= \frac{2C}{\sqrt{R_e}} + \frac{D}{2\sqrt{B}} [\Phi(\sqrt{B}\eta_0) - 1] \,.
\end{aligned}\right\} \tag{53}$$

Thus, for the case where R_e is large, Eqs. (47) have been reduced to Eqs. (53).

The joining of the asymptotic solution with the series solution is now made by the choice of $\eta_0 = 1.5$ so that both solutions are valid at $\eta = \eta_0$. Actual calculations have been effected for the case where the values of k and $a_2 (= \phi''(0)/2)$ are initially given. Then the other parameters R_e and δ/R_S (or η_S) and the unknown

D were determined by solving Eqs. (53). Here it is worth noting that the results thus obtained are acceptable only when the value of η_S is found to be $\eta_S > \eta_0$.

The main results are shown in lower lines of each columns in Table 1. The approximation assumed in the above analysis is valid only where R_e is large. Nevertheless, the results indicate the surprisingly excellent agreement with the previous ones up to the considerably low Reynolds number. This seems mainly due to the fact that, as the Reynolds number R_e decreases, the above approximation made in the present analysis certainly becomes inadequate, while the terms involving the coefficient D in Eqs. (53) become insignificant. Indeed, the magnitude of the deviation of ϕ from Eq. (29), which represents the term involving D in Eqs. (40), becomes small with the increase of $\phi''(0)$ or with the decrease of R_e, as seen from Figs. 2, 3 and 4.

5. Pressure Distribution on Surface and Skin Friction

So far, we were not concerned with the pressure p, one of the important flow variables. Since the expression of v is a function of y only, Eq. (3) becomes

$$v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial y^2} .$$

Integrating this equation with respect to y, we get

$$\frac{p}{\rho} + \frac{1}{2}v^2 - \nu\frac{\partial v}{\partial y} = F(x) + \text{const.}, \tag{54}$$

where $F(x)$ denotes a function of x only. Substituting u and v of Eqs. (7) and p/ρ of Eq. (54) in Eq. (2), we obtain

$$dF/dx = -x(f'^2 - 2ff'' - vf''') .$$

From Eq. (9) we obtain

$$f'^2 - 2ff'' - \nu f''' = K^2 .$$

Hence

$$dF/dx = -xK^2 ,$$

and, from the integration,

$$F = -\frac{1}{2}K^2 x + \text{const.} .$$

With the above form of F, Eq. (54) becomes

$$\frac{p}{\rho} + \frac{1}{2}K^2 x^2 + \frac{1}{2}v^2 - \nu\frac{\partial v}{\partial y} = \text{const.} . \tag{55}$$

The constant in the right hand side of Eq. (55) will be determined from the condition just behined the shock wave. The relations of the momentum across the shock wave are

$$p_2 + \rho_2 v_2^2 \simeq \rho_\infty u_\infty^2 \sin^2\theta . \tag{56}$$

Since, from the definition of v_2, $v_2 \simeq -q_2 \sin(\theta - \alpha)$, we have from Eq. (8)

$$v_2 \simeq -ku_\infty \sin\theta .$$

Putting the above form of v_2 into Eq. (56),

$$p_2/\rho_2 \simeq k(1-k)u_\infty^2 \sin^2\theta .$$

Recalling the previous assumption that the viscous effect is small enough to be neglected just behind the shock wave, we may ignore the term $\nu\partial v/\partial y$ when Eq. (55) is applied to the station. Then, the substitution of the above-obtained v_2 and p_2/ρ_2 in Eq. (55) yields

$$\frac{p}{\rho}+\frac{1}{2}x^2K^2+\frac{1}{2}v^2-\nu\frac{\partial v}{\partial y}=\frac{p_2}{\rho_2}+\frac{1}{2}K^2x_S^2+\frac{1}{2}v_2^2$$
$$\simeq \frac{k(2-k)u_\infty^2}{2}\left[\sin^2\theta-\frac{x_S^2}{R_S^2}\right],$$

where x_S is the length measured along the shock wave from the point on the axis. Since, in the vicinity of the nose under consideration, $\sin\theta \simeq 1-(x_S/R_S)^2/2$, we obtain

$$\frac{p_2}{\rho_2}+\frac{1}{2}x_S^2K^2+\frac{1}{2}v_2^2 \simeq \frac{k(2-k)u_\infty^2}{2}.$$

Hence

$$\frac{p}{\rho}+\frac{1}{2}x^2K^2+\frac{1}{2}v^2-\nu\frac{\partial v}{\partial y} \simeq \frac{k(2-k)}{2}u_\infty^2. \tag{57}$$

Now, we are able to obtain the pressure distribution on the surface by using Eq. (57). On the surface $u=v=0$, thus yielding $f(0)=f'(0)=0$. Since, from Eq. (7), $\partial v/\partial y=2f'(y)$, $\partial v/\partial y$ vanishes there. Applying the above conditions to Eq. (57), we have on the surface

$$\left(\frac{p}{\rho}\right)_{\text{sur.}} \simeq \frac{k(2-k)u_\infty^2}{2}\left(1-\frac{x^2}{R_S^2}\right),$$

where the symbol $(\ \ )_{\text{sur.}}$ represents the value on the surface. Let us denote the radius of curvature of the body by R_B, then for a spherical body $R_S \simeq R_B$. Therefore, we obtain the following formula

$$\left(\frac{p}{\rho}\right)_{\text{sur.}} \simeq \frac{k(2-k)u_\infty^2}{2}\left(1-\frac{x^2}{R_B^2}\right), \tag{58}$$

or, since $(1-x^2/R_B^2) \simeq \cos^2\beta$,

$$\left(\frac{p}{\rho}\right)_{\text{sur.}} \simeq \frac{k(2-k)u_\infty^2}{2}\cos^2\beta . \tag{59}$$

By using Eq. (58) or (59) the pressure coefficient defined by

$$C_p=(p-p_\infty)\Big/\frac{1}{2}\rho_\infty u_\infty^2$$

becomes for the surface pressure

$$(C_p)_{\text{sur.}} \simeq \frac{p}{\rho_\infty u_\infty^2/2} \simeq (2-k)\left(1-\frac{x^2}{R_B^2}\right), \tag{60}$$

or,

$$(C_p)_{\text{sur.}} \simeq (2-k)\cos^2\beta .$$

They are quite the same as those obtained by Li and Geiger [5] under the assumption that the flow is inviscid ranging from the shock wave over to the wall. Here it is worth noting that Eq. (60) is also identical with the result by the modified Newtonian law put forward by Lees [9]. This law leads to an excellent agreement with an experimental data [10], [12], though derived on the only empirical basis, as pointed out by Lighthill [4]. We have now provided for it the theoretical basis even for cases where the viscous effect plays a significant role.

The experimental data [10], [11] and [12] of the surface pressure on a sphere are presented in Fig. 6 for the comparison with the theoretical values given by Eq. (60). Even for the flow of the low Mach number, in which the approximations made in the present analysis can not always be expected to be valid, the good agreement between them has been found.

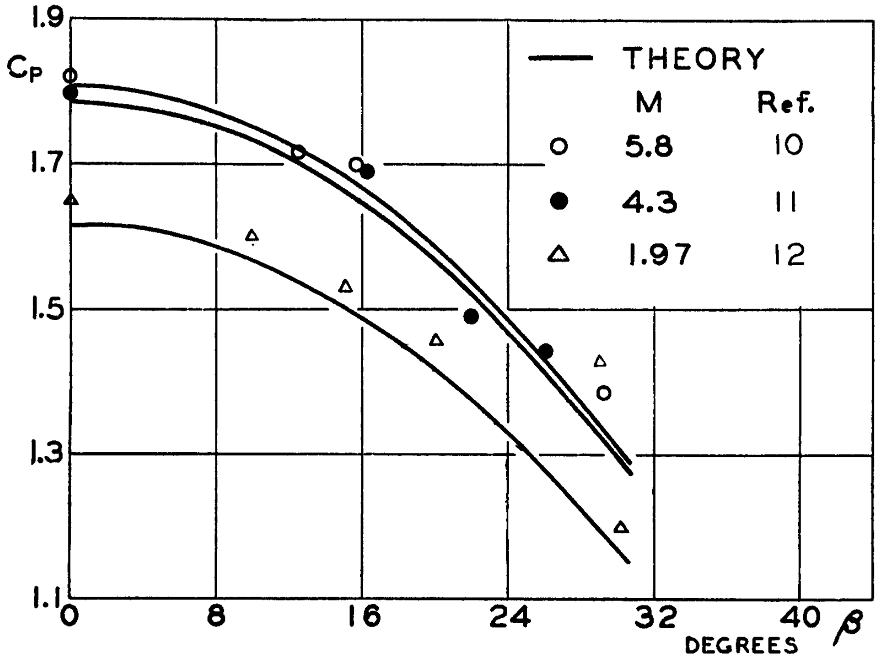

FIGURE 6.

We next consider the skin friction τ_w, which has a close correlation not only to the drag of the body, but also to the surface heat-transfer rate. From the definition

$$\tau_w=\left(\mu\frac{\partial u}{\partial y}\right)_w,$$

where μ is the viscosity coefficient and the symbol $(\ )_w$ denotes the values at the wall. According to the present approximation, the viscosity coefficient μ may be considered as constant throughout the shock layer, as will be shown below.

First, let us show that the change in pressure as well as in density is very small throughout the shock layer near the nose. Here we restrict our considerations to the case in which the viscous effect is negligibly small just behind the shock wave. Hence, we obtain

$$\left| v\frac{\partial v}{\partial y} \right| \gg \left| \nu\frac{\partial^2 v}{\partial y^2} \right|, \tag{61}$$

or, by using Eqs. (7),

$$|2ff'| \gg |\nu f''|.$$

This condition can be rewritten, by using Eqs. (16), as

$$1 \gg \frac{1}{R_e k^2}. \tag{62}$$

It follows from the above arguments that the present analysis is applicable only to the case where the condition (62) is valid. It can be confirmed for all of the cases shown in Table 1 that the condition (62) is almost satisfied. As an example, for the case of $k=0.1$, the values of R_e are found to be larger than 10^3 and thus the values of $1/R_e k^2$ are smaller than 1/10. On the basis of the condition (62) let us consider the order of magnitude of each term in Eq. (3). The terms v and $\partial v/\partial y$ take just behind the shock wave the maximum values of $-ku_\infty$ and $\nu f'(\delta)$ ($=u_\infty^2/R_e$ from Eqs. (16)), respectively. Therefore, we obtain the following inequalities:

$$k^2u_\infty^2 \geq v^2,$$

$$\frac{u_\infty^2}{R_e} \geq \nu\frac{\partial v}{\partial y}.$$

From the condition (62) we obtain the following relations

$$O(k^2u_\infty^2) \gg O\left(\nu\frac{\partial v}{\partial y}\right).$$

Due to the above estimate, we obtain

$$\frac{p}{\rho} \simeq \frac{k}{2}u_\infty^2 + O(k^2u_\infty^2) + O(x^2).$$

It follows from this equation that the change in pressure may be assumed to be sufficiently small and therefore so is the change in temperature as well as viscosity coefficient. Thus the skin friction can be written approximately as

$$\tau_w \simeq \mu\left(\frac{\partial u}{\partial y}\right)_w,$$

where μ is the value of the viscosity coefficient behind the shock wave and it is a constant over the disturbed region near the nose. Then, from Eqs. (7), (22) and (21) we obtain successively

$$\begin{aligned}\tau_w &\simeq \mu x f''(0) = x(\mu\rho)^{1/2}K^{3/2}\phi''(0)\\ &= x[k(2-k)]^{3/4}(\mu\rho)^{1/2}\left(\frac{u_\infty}{R_S}\right)^{3/2}\phi''(0).\end{aligned}$$

As shown previously, $\phi''(0)$ increases almost linearly with $1/\sqrt{R_e}$ (see Fig. 5). Therefore the skin friction also increases with $1/\sqrt{R_e}$ in the same trend as $\phi''(0)$, provided that the other parameters are held fixed.

CONCLUDING REMARKS

The two important assumptions are made in the present analysis, that is, (1) the flow behind the shock wave is incompressible and (2) in the vicinity of the nose of the body, $r\simeq x$. First, we will discuss the latter assumption in more detail than done in the introduction of the paper.

Using the symbols shown in Fig. 1, the following relations are derived from the simple geometrical consideration

$$dr=(1+\kappa y)\cos\beta\cdot dx\,|_{y=\text{const.}},$$
$$dr=\sin\beta\cdot dy\,|_{x=\text{const.}}.$$

These can be written in the form

$$\frac{\partial r}{\partial x}=(1+\kappa y)\cos\beta,$$
$$\frac{\partial r}{\partial y}=\sin\beta.$$

The assumption $r\simeq x$ is equivalent to that $\partial r/\partial x\simeq 1$ and $\partial r/\partial y\simeq 0$. This means that the conditions $1+\kappa y\simeq 1$ and $\beta\simeq 0$ must be valid in order to ensure the assumption $r\simeq x$. The relation $\beta\simeq 0$ is approximately satisfied so far as the vicinity of the nose is concerned. Moreover, the validity of the condition $1+\kappa y\simeq 1$ is evidently ensured by the fulfilment of the condition $\delta/R_S\ll 1$.

Therefore, we can assume in the present analysis that $x\simeq r$ only if the condition $\delta/R_S\ll 1$ is satisfied. It will be shown in the following that this condition is equivalent to that $k\ll 1$. For an extreme case as $R_e\to\infty$, as seen from the last of Eqs. (38).

$$(\delta/R_S)_{R_e\to\infty}=O(k).$$

In fact, the values of $(\delta/R_S)_{R_e\to\infty}$ obtained for the cases of $k=0.1$, 0.15 and 0.20 were confirmed to be identical with those obtained from the last of Eqs. (38) (Fig. 5). As seen from Fig. 5, we may consider

$$\delta/R_S=O(k)$$

over the range of R_e, to which the present analysis is applicable. As mentioned before, the value of k is, in a hypersonic flow, small as compared with unity. The foregoing arguments lead to the conclusion that the assumption $r\simeq x$ can be made, when the flow field under consideration is near the nose. The assumption that the flow behind the shock wave is incompressible near the nose can also be made when the value of k is small, for the reason mentioned in Section 1. Therefore the simple assumptions made in the present analysis are based on the fact that the value of k is small.

In the present paper the flow near the nose of a spherical body has been analyzed taking the effect of the viscosity into account. Summarizing, it is concluded that (1) there exists a similar solution consistent with the conditions both on the shock wave and on the body surface, (2) the surface pressure coefficient obeys the modified Newtonian law independently of viscosity effect, (3) the distance of the shock wave from the body increases with $1/\sqrt{R_e}$ and the rate of its increase tends to be small as R_e becomes small, and, (4) the skin friction τ_w increases almost linearly with $1/\sqrt{R_e}$, at least, within the range of R_e under consideration.

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Appendix

The vorticity induced by the curved shock wave in a hypersonic flow

Eq. (56) becomes approximately

$$p_2 \simeq (1-k)\rho_\infty u_\infty^2 \sin^2\theta$$
$$= (1-k)\rho_\infty u_{\infty n}^2\,, \qquad \text{A-1}$$

where $u_{\infty n}$ is the velocity component normal to the shock wave—i.e.,

$$u_{\infty n} = u_\infty \sin\theta\,.$$

From the conservation of energy across the shock wave

$$\frac{1}{2}u_{\infty n}^2 \simeq i_2 + \frac{1}{2}u_{2n}^2\,.$$

where i_2 is the enthalpy per unit mass of the fluid behind the shock wave and u_{2n} the velocity component normal to the shock wave just behind it. Hence

$$i_2 \simeq (1-k^2)u_{\infty n}^2/2\,. \qquad \text{A-2}$$

The value of $u_{\infty n}$ changes along the shock wave when it is curved, and therefore so do the enthalpy i_2, entropy s_2 and pressure p_2 behind the shock wave. The change in entropy s_2 is given by

$$T_2 ds_2 = di_2 - dp_2/\rho_2\,,$$

where T denotes the temperature. Combining Eq. A–1 and Eq. A–2, we obtain after simplification

$$T_2 ds_2 \simeq (1-k)^2 u_{\infty n} du_{\infty n}\,. \qquad \text{A-3}$$

Neglecting the viscosity terms, Eqs. (2) and (3) can be written in the following form

$$\boldsymbol{\omega}\times\boldsymbol{v} + \nabla\left(\frac{1}{2}q^2\right) + \frac{1}{\rho}\nabla p = 0\,,$$

where $\boldsymbol{v}$ is the velocity vector and $\boldsymbol{\omega}(0, v, \omega)$ the vorticity vector. This can be rewritten by using the energy equation, as

$$\boldsymbol{\omega}\times\boldsymbol{v} = \nabla i - \frac{1}{\rho}\nabla p = T\nabla s\,.$$

Let us choose x' in the direction which is tangential to the shock wave at a point on it in a meridian plane and in which the value of θ decreases, and y' is the direction normal to it into the uniform flow. Let us denote the velocity components in the x' and y' directions by u' and v', respectively. Then from the above equation we get

$$\omega v_2' = -T_2 \frac{ds_2}{dx'}\,.$$

Hence

$$\omega = -\frac{1}{v_2'} T_2 \frac{ds_2}{du_{\infty n}} \frac{du_{\infty n}}{dx'}\,. \qquad \text{A-4}$$

Since $\partial\theta/\partial x' = -1/R_S$ and $u_2' = u_\infty \cos\theta$, we obtain

$$\frac{\partial u_{\infty n}}{\partial x'} = \frac{du_{\infty n}}{d\theta} \frac{\partial \theta}{\partial x'} = -\frac{u_2'}{R_S}.$$

By using this relation with $ds_2/du_{\infty n}$ from Eq. A–3, Eq. A–4 becomes

$$\omega \simeq -\frac{(1-k)^2}{k} \frac{u_2'}{R_S}.$$

Noting $u_2' \simeq u_2$, the above equation becomes

$$\omega \simeq -\frac{(1-k)^2}{k} \frac{u_2}{R_S}.$$

Consequently the substitution of u_2 of Eq. (14) leads to Eq. (15). For the three-dimensional curved shock wave, Lighthill [*4*] derived the general expression of the relation between the vorticity and the radius of curvature of the shock wave.