Aeronautical Research Institute, University of Tokyo
Report No. 348, September 1959

Theory of Plasticity for Small and Finite Deformations Based on Legitimate Concept of Strain*

By

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Summary. First it is illustrated that the strain, together with its increment, generally used in the theory of elasticity, which is specified by the change in the geometrical configuration, is not fitted for the description of plastic deformation. That the existing theory of plasticity, more precisely the incremental strain theory, has recourse to this strain and its increment, except in some special cases of deformation, is its essential inconsistency, which manifest itself more remarkably for the finite deformation. The object of the present investigation is to bring the theory of plasticity into a perfectly logical system covering the whole range of the small and finite deformations, by making some essential innovation on the concepts of strain and strain increment hitherto used.

In order to introduce a new concept of strain and its increment legitimate for the description of plastic deformation, we contemplated on the essential characters of plastic deformation, and deduced some fundamental conditions for them, as well as the stress, to satisfy. Basing on this apodictic reasoning, the strain increment at a deformed state is defined such that the deformed current state is at the same time an undeformed state. The strain for a deformed state is obtained as a result of integration of such strain increment along a given deformation path, and is shown to be dependent on the path, but not on the geometrical configuration directly. This strain is regarded as corresponding to the microstructural change of the material, and serves not only as the strain tensor itself for describing plastic deformation, but also as the strain history tensor specifying the history dependent state as anisotropy. Further, this strain is seen to be reduced to the so-called logarithmic strain for the special case of simple extension, and accordingly to be history dependent, generalized natural strain. The plastic deformation is thus seen to be a history dependent phenomenon in the duplicated sense, that is, first in the strain itself, and secondly in the stress-strain relationship.

The stress tensor is defined such that it gives for unit of area in the deformed state the actual force exerted through it. This stress is reduced to the so-called true stress for the special case of simple tension.

By basing on these definitions of strain increment and strain, together with that of stress, the principle of virtual work is shown to hold in the same form as for the small deformation over the whole range of small and finite deformations. In consequence of this, all the relations in this general case such as the equilibrium equations, the mechanical equations of state and others, are expressed also in the same form as for the small deformation. Thus the theory of plasticity, i.e. the incremental strain theory, being reorganized from the beginning, is extended quite naturally to the general case of small and finite deformations.

* Presented to the 32nd Annual Meeting JSME, April, 1955, the 5th Japan National Congress Appl. Mech., Sept., 1955 and the 33rd Annual Meeting JSME, April, 1956 in succession.

Errata

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Report No. 348 (Vol. 25, No. 8), 1959
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Page	Line	Wrong	Corrected
165	2	cannot known	cannot know
166	12	be seen later the···	be seen later that the···
167	Fig. 1a	$\boldsymbol{r}'+d\boldsymbol{r}\ (\overrightarrow{OQ'})$	$\boldsymbol{r}'+d\boldsymbol{r}'$
169	4 from the bottom	matter with coordinate system···	matter which coordinate system···
175	3	Roman index	Gothic index
176	Eq. (6.12)	$\Gamma^k_{ij}=g^{kr}\Gamma_{ij,k}$	$\Gamma^k_{ij}=g^{kr}\Gamma_{ij,r}$
181	2 from the bottom	not only of dx but of x.	not only of dx^i but of x^i.
186	the equation just above Eq. (8.22)	$\varepsilon_{33}=\gamma_1^2$	$\varepsilon_{33}=\gamma^2$
188	Fig. 8		the smaller vectors on the x^1 and x^2 axes are written $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ respectively
189	the line just above Eq. (8.31)	(8.9)	(8.29)
191	Eq. (8.39)	$\boldsymbol{e}_1=\frac{1}{l}\boldsymbol{e}^1$	$\boldsymbol{e}^1=\frac{1}{l}\boldsymbol{e}^1$
	Eq. (8.44)	$D\varepsilon_{12}-\varepsilon_{13}\left(\frac{Dl}{l}+\frac{Dm}{m}\right)=0$	$D\varepsilon_{12}-\varepsilon_{12}\left(\frac{Dl}{l}+\frac{Dm}{m}\right)=0$
195	the line just above Eq. (8.62)	the relation	the relations
201	4	···in Fig. 6,	···in Fig. 11,
202	15	$\boldsymbol{f}dS$ or $\boldsymbol{f}dS$	$\boldsymbol{f}dS$ or $\boldsymbol{f}dS$
203	Eq. (11.18)	$\sigma'_j=\sigma^i_j-\bar{\sigma}\delta^i_j$	$\sigma'^i_j=\sigma^i_j-\bar{\sigma}\delta^i_j$
205	Eq. (12.4)	$\sigma'^{22}=\sigma'_{22}=\sigma'^2_2=-\frac{1}{2}\sigma$	$\sigma'^{22}=\sigma'_{22}=\sigma'^2_2=-\frac{1}{3}\sigma$
212	Eq. (14.27)	$\nabla_j\sigma\ +\rho F=0$	$\nabla_j\sigma^{ij}+\rho F=0$
	Eq. (14.28)	$\nabla_j X\ +\sqrt{g}\,\rho F=0$	$\nabla_j X^{ij}+\sqrt{g}\,\rho F=0$
214	9 from Eq. (15.10)	*the fact it holds···*	*the fact that it holds···*
	Eq. (15.10″)	$\delta W_1=\int\sigma\ (\delta\varepsilon)\ dV$	$\delta W_1=\int\sigma^{ij}(\delta\varepsilon)_{ij}dV$
215	Eq. (16.4)	$DW=T\cdot\cdot\boldsymbol{DE}>DU-\boldsymbol{T}DS$	$DW=\boldsymbol{T}\cdot\cdot\boldsymbol{DE}>DU-TDS$
	foot note	trreversibility condition···	irreversibility condition···
216	4 from Eq. (16.8)	independent of···	independently of···
	Eq. (16.9″)	$(D\varepsilon)_{ij}=\frac{\partial f\sigma^{ij},\varepsilon_{ij}}{\partial\sigma^{ij}}D\lambda$	$(D\varepsilon)_{ij}=\frac{\partial f(\sigma^{ij},\varepsilon_{ij})}{\partial\sigma^{ij}}D\lambda$
218	Eq. (17.7″)	$D\lambda=\frac{Df(\sigma^{ij},\varepsilon_{ij})-H'(U)DQ}{H'(U)\sigma^{ij}\frac{\partial f(\sigma^{ij},\varepsilon_{ij})}{\partial\sigma^{ij}}}$	$D\lambda=\frac{Df(\sigma^{ij},\varepsilon_{ij})+H'(U)DQ}{H'(U)\sigma^{ij}\frac{\partial f(\sigma^{ij},\varepsilon_{ij})}{\partial\sigma^{ij}}}$
219	概要 1	慣用されている，歪増分	慣用されている歪，歪増分
	上から 10	歪増分の補足	歪増分の満足
	下から 5	有限変影	有限変形

1. INTRODUCTION

Of the two rival theories of plasticity, the so-called incremental strain and total strain theories, the former has come to be accepted, after not a few disputation, as that which holds in general for plastic deformation of metals. This conclusion seems to be due to rather physical considerations of plastic deformation than empirical. Namely, it is because the plastic deformation where the final state of stress (strain) is not determined by that of strain (stress) but by the deformation history up to the state is regarded as being possible to be described only by the incremental strain theory based on the mechanical equation of state of the form of a differential equation with respect to strain and stress.

What matters in this context is the concept of strain increment in the incremental strain theory and the strain as the result of its integration. Not to mention, the strain used in the theory of elasticity is that which is specified by the change in the geometrical configuration of the body from the uniquely determinable undeformed state but independent of the deformation path, and the strain increment is the increment of such strain. This is true in itself in view of the nature of the elastic deformation, but problem arises where the same concept of strain and strain increment was applied to the description of the plastic deformation whose nature is quite different from that of the elastic one. In fact, if it be done, various contradictions will be seen to appear. The most confutable among them is that which is exhibited in the finite plastic deformation of simple shear for an isotropic body. For the plastic deformation of an isotropic body, it is sure that the simple shear is caused by the shearing stress which has the constant principal direction kept at 45° with the slip direction of the simple shear, and further that the principal directions of strain must coincide with those of stress when the latter are kept constant during the deformation. Consequently, the principal direction of the simple shear produced by the shearing stress is needed to make the same angle 45° with the slip direction as that of the shearing stress does. We can find that this does not hold, if the strain for the deformation of the simple shear were assumed to be specified, as in the case of elastic deformation, by the change in the geometrical configuration produced by it, because the principal direction of such strain comes to make the angle still smaller than 45° with the slip direction as the deformation proceeds. *This fact is regarded as indicating that the strain reasonable for the description of plastic deformation should be some quantity other than that for elastic deformation.*

Now that the strain for the plastic deformation is as such, the problem that confront us is how to define it. The answer to this problem will be given, basing on critical considerations on the essential features of plastic deformation. The most essential of them is that the plastic deformation is the deformation due to the change in the mode of interconnection of particles constituting the material. Since the microscopic structure, as is represented in metals by the group pattern of dislocations which are effected by the change in the interatomic connection, is regarded as dependent on the deformation path, the strain representing the deformed state is also regarded as such. That is, *in plastic deformation, the strain is at the same time "the strain history"*. It is note worthy that this strain is such that it gives the logarithmic strain [*1*] particularly in the case of extension in a constant direction. It is needless to mention that the strain increment which leads to such strain, i.e. the strain history, is also one other than that for elastic deformation.

Though the differences between the strains and strain increments for elastic and plastic deformations geometrically identical are of small quantities of the second order for small deformation, they become finite for finite deformation. *After all, the plastic deformation is regarded as a hysteresis phenomenon in the double meaning, i.e. on the one hand in the non-holonomic character of the differential state equation and on the other hand in the dependence of the strain on the deformation history. Thus, by means of the introduction of the concept of strain history as strain, and its increment, the theory of plasticity will be seen to be not only formulated with both the physical and mathematical self-consistencies, but also extended quite naturally to the whole range of small and finite deformations.* Finally it must be remarked that *we can not further extend the plasticity theory to the case where the strain history phenomena such as anisotropy and the Bauschinger effect are present, without basing on the concept of strain and strain increment introduced in the present paper.* And such extended theory will be proposed in the paper which will be published in succession.

2. How to Define Plastict Strain, Strain Increment and Stress

It was deduced in the Introduction, taking as an example the case of the deformation due to simple shear, that the strain, hence the strain increment too, valid for the description of plastic deformation is quite other one than that reasonable for elastic deformation which can be specified by the difference of the geometrical configurations of the body before and after deformation. And further it was touched there in brief that the strain for plastic deformation, viewed from its essential character, is such that it depends on the deformation path from the undeformed to the deformed state*, and therefore it is regarded as a strain history. In order to proceed to the main subject of defining strain, to say more precisely strain increment, it is necessary for us to begin with confirming the validity of the above statement.

In general, a plastic deformation of a polycrystalline aggregate is realised as the result of a sequence of successive infinitesimal slips, by which the group pattern of dislocations in crystalls usually undergoes successive variation. For this reason, the group pattern of dislocations in the final state, even if its configuration may be geometrically identical, is different according to the process of slip, or the

* The term "state" is used to mean, as usual, what is specified by the geometrical configuration, that is, nominated, so to speak, geometrical state.

deformation history, up to the state. Not only the group pattern of dislocations, in the crystalline matrix, but also the microscopic structural change in the inter-crystalline amorphous layer will necessarily depend on the slip process. And what is important is that it is not the geometrical configuration of the body in deformed state but its microscopic structure such as the group pattern of dislocations what specifies the mechanical properties concerning plastic deformation. To say more precisely, the grade of work-hardening is regarded as dependent on the density of dislocations, and anisotropy and the Bauschinger effect on the way of their arrangement. The strain as a mechanical quantity should be such that it can represent the mechanical state which specifies the mechanical properties, accordingly is dependent on the deformation path up to the state. And for this purpose, the strain must be a quantity specified not by the geometrical configuration of the deformed state, but by the deformation path up to the state. Thus *it is concluded that the strain reasonable for the description of plastic deformation is something dependent on the deformation history, and such strain we will, in the present paper, denominate "plastic strain" or "strain history"*.

Here, it will be desirous to remark that, the elastic deformation being the deformation due to the change in the interatomic distances of the material, the deformed state, and therefore the strain representing it, is specified by its geometrical configuration.

That the plastic strain is dependent on the deformation history means that the quantity introduced primarily concerning plastic deformation is not the strain itself, but the strain increment from a current state t to a consecutive state $t+dt$, t being the time or some parameter representing the extent of deformation. Although it will be seen later that the origin of the parameter t is permitted to be taken at any state, now we may suppose it is tentatively chosen at the annealed state.

In order to deduce the definition of strain increment, the above condition alone is insufficient, and some other conditions, too, have to be taken into account. The first of them is that *the strain increment is a tensor.* This is because the mechanical equation of state for plastic deformation, just as the case for other physical laws, is to be expressed by a tensor equation of an invariant form, hence all the mechanical quantities such as strain increment, stress, etc., involved in it are needed to be tensors. *Since the strain increment is a tensor, it is regarded as one associated with the change in the metric, from t to $t+dt$, of the space which deforms in conformity with the body.*

The second requirement is that the strain increment from t to $t+dt$ should be measured, assuming the current state t as the state of no strain, i.e. as the state $t=0$. This is a matter of course in view of the nature of plastic deformation. That is, plastic deformation being caused by a sequence of slips which brings about the mere replacement of the atoms on one side of the slip plane with others, the deformed state is quite equivalent to the undeformed state, apart from the change in the plastic properties such as work-hardening and anisotropy, in respect that the atoms are situated at the potential trough of the same lattice field, unless there exist no external forces. This means that *the current deformed state t can also be regarded* *as the undeformed state $t=0$ with no strain, only the mechanical properties being different according to the state.* In fact, in plastic deformation, we cannot known in any way, from the mechanical consideration alone, the strain of a given state of a given material or its annealed virgin state, so that cannot but treat, in the mechanical theory, the given state as the state of strain zero, without having the complete knowledge about its previous history of deformation. It is required, however, in doing so, that the initial values of the mechanical properties are chosen as those where the strain is originated. To take the annealed state as the standard is necessary for the physical considerations of the process of plastic deformation, but not for its mechanics. The possibility for the origin of strain to be selected at any state will be illustrated later for the case of simple extension.

It was already shown that the elastic and plastic strains are distinguished in their dependency on the deformation path. And now it must further be remarked that they are distinguished from each other also in the point that the former is measured from the uniquely determinable undeformed state, contrasted with the latter whose origin is allowed to be chosen at any state, annealed or deformed.

From the preceeding statement, *it can be concluded that the quantity capable of being introduced primarily as for plastic deformation is not the strain, but the tensor of strain increment which is specified by the change in the metric, of the space deforming in conformity with the body, measured from the current state t to the state $t+dt$, assuming the state t as an unstrained state $t=0$.* In the next section, the strain increment will be seen to be defined apodictically from these conditions, for both small and finite deformations.

The plastic strain is obtained by integrating the strain increment thus defined along a certain path of deformation which is determined by the process of application of external loads. And it will be shown later both theoretically and practically that *the plastic strain thus obtained is,* as was deduced in the above from the microscopic viewpoint, *none other than a quantity called " strain history ",* which assume different values according to the path of deformation, even if the geometrical configuration in the final state is the same. Thus the notion that the strain depends on the deformation history, so that is at the same time the strain history, is seen to be justifiable from all of the physical and mathematical viewpoints, although it might appear somewhat strange for us who are accustomed to the use of the strain specified only by the change in the geometrical shape.

The definitions of plastic strain increment and strain will be found to be given with no ambiguity, basing on the underlying principles mentioned above. But it must be remarked here that these definitions are not yet given their final validity, until it would be confirmed that they, together with the definition of stress, constitute a reasonable mathematical theory of plastic deformation. We will now proceed to this last problem. It is sure that the stress, as well as the strain and others, must be a tensor. But from this condition alone it can not be defined uniquely, any tensors which have one to one correspondence with, and derivable from, the state of forces in the material element, will not be in conflict with this condition. We need further conditions for the stress tensor to be defined uniquely.

That is, the stress tensor is required to describe on one hand the equilibrium condition of the body, and on the other hand the mechanical equation of state, i.e. the process of plastic deformation, in a fashion reasonable as a mechanical theory. And for the latter purpose *it is further required for the plastic potential to exist, and accordingly for the stress and the strain increment tensors to satisfy, over the whole range of small and finite deformations, the principle of virtual work of the same form as that for the case of small deformation*

$$\begin{aligned} \text{Work} &= \int (\text{Stress tensor}) \, .. \, (\text{Strain increment tensor}) \\ &= \int \text{Spur}\,[(\text{Stress tensor})\,(\text{Strain increment tensor})] \qquad (2.1) \end{aligned}$$

where the dots ".." mean the double scalar product of two tensors. From this condition the stress tensor can be defined uniquely corresponding to the strain increment tensor which has already been given. And it will be seen later the stress thus obtained is what is most natural among those derivable from the state of forces. Thus the plastic strain increment, accordingly the plastic strain, and the stress defined in the line stated above are seen to be justifiable from all the viewpoints.

The reason we must pay such a special attention to the validity of the virtual work principle of the form (2.1) as the condition for the stress tensor is that, for finite deformation, it is not always fullfiled by the strain increment and the stress which seem legitimate. In fact, for the elastic finite deformation, it has been shown by F. D. Murnaghan [*2*] and also by the present author [*3*] that the stress defined in a natural way from the state of forces, together with the strain increment reasonable for the elastic deformation, does not satisfy the virtual work principle of the form (2.1). And in order for the principle to be satisfied, the stress, not the strain increment, is needed to be modified. For this reason the stress for the plastic deformation also should be examined about its validity in the light of the virtual work principle.

3. Definition of the Strain Increment Tensor

The deformation of a body is composed of the two parts, i.e. the elastic and the plastic deformations, which are to be described by the strain quite different in their mathematical definitions, as was mentioned in the preceding section. Hence, in the following, we will assume the body to be plastic-rigid, the elastic deformation being so small as to be negligible.

As was stated in the preceding section, the plastic strain increment should be defined as a tensor which is an invariant. Accordingly, it ought to have no effect substantially on its definition what coordinate system may be used. But in view both of the necessity of following one and the same material element, in such hysteresis phenomenon as plastic deformation in which the mechanical properties of the element depends on its deformation history, and of the fact that the subject of our investigation covers small and finite deformations, the so-called *Lagrangian* *method*, in which the coordinate system is considered to deform in conformity with the body, is regarded as most fitted. While this method has the merit to suffice for the theoretical purposes, it shows, on the other hand, the shortcomings, in dealing with practical problems, that it can provide no means to represent analytically the position of a material point in space, and that the coordinate system becomes deprived of its orthogonality and normality with the increase of deformation. And these defects will be supplemented by introducing the Eulerian method and others later on.

As was stated in the preceding section, in plastic deformation any state is permitted to be chosen as the undeformed state, i.e. the state $t=0$. Accordingly, in the following, we will measure t, assuming an arbitrary state, whether annealed or work-hardened, as the standard state $t=0$. As shown in Figs. 1a and 1b, we

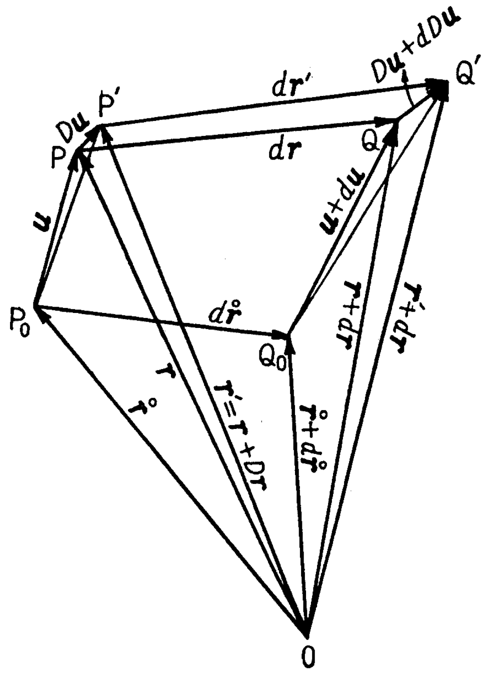

FIGURE 1a.

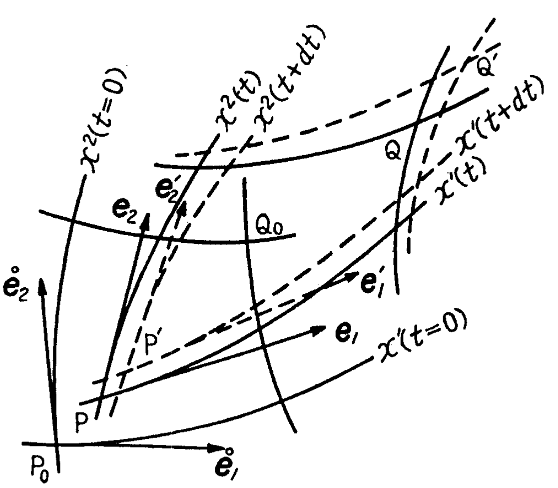

FIGURE 1b.

suppose that a material point represented by the Lagrangian coordinates* x^λ occupies the positions P_0, P and P′ successively, at $t=0$, t and $t+dt$, whose position vectors are $\mathring{\boldsymbol{r}}$, $\boldsymbol{r}$ and $\boldsymbol{r}'$ respectively. Then

$$\mathring{\boldsymbol{r}}=\boldsymbol{r}(x^1, x^2, x^3, 0)\,, \tag{3.1}$$

$$\boldsymbol{r}=\boldsymbol{r}(x^1, x^2, x^3, t)\,, \tag{3.2}$$

$$\boldsymbol{r}'=\boldsymbol{r}(x^1, x^2, x^3, t+dt)\,. \tag{3.3}$$

As mentioned in the preceding section, the quantity to be introduced primarily concerning plastic deformation is the strain increment tensor associated with the infinitesimal deformation from t to $t+dt$, when the state t was assumed as the undeformed state. Accordingly, what matters is the deformation from $\boldsymbol{r}$ to $\boldsymbol{r}'$,

* In the present paper, the Lagrangian coordinates, by which each material point is identified, are written as x^λ with Greek index representing 1, 2, 3.

referring to $\boldsymbol{r}$. Putting

$$D \equiv dt\frac{\partial}{\partial t}, \tag{3.4}$$

the variation of $\boldsymbol{r}$ and the displacement during dt, from t to $t+dt$, are written as $D\boldsymbol{r}$ and $D\boldsymbol{u}$ respectively. Hence

$$\boldsymbol{r}' = \boldsymbol{r} + D\boldsymbol{r} = \boldsymbol{r} + D\boldsymbol{u} \tag{3.5}$$

or

$$D\boldsymbol{r} = D\boldsymbol{u}. \tag{3.6}$$

Indicating the position of a material point adjacent to the material point P_0 at $t=0$ by Q_0, and its positions at t and $t+dt$ by Q and Q$'$ respectively, the relation between $\overrightarrow{PQ} = d\boldsymbol{r}$ and $\overrightarrow{P'Q'} = d\boldsymbol{r}'$ is given by

$$d\boldsymbol{r}' = d\boldsymbol{r} + d(D\boldsymbol{u}) \tag{3.7}$$

from (3.5), where

$$d \equiv dx^\lambda \partial_\lambda, \tag{3.8}$$

$$\partial_\lambda \equiv \frac{\partial}{\partial x^\lambda}. \tag{3.9}$$

Now introducing the base vectors

$$\boldsymbol{e}_\lambda = \partial_\lambda \boldsymbol{r}, \quad \boldsymbol{e}'_\lambda = \partial_\lambda \boldsymbol{r} \tag{3.10}$$

for the Lagrangian coordinate system at t and $t+dt$ respectively (Fig. 1b), we have

$$d\boldsymbol{r} = \boldsymbol{e}_\lambda dx^\lambda, \quad d\boldsymbol{r}' = \boldsymbol{e}'_\lambda dx^\lambda, \tag{3.11}$$

Accordingly, by means of (3.11) and

$$d(D\boldsymbol{u}) = \partial_\lambda(D\boldsymbol{u})dx^\lambda, \tag{3.12}$$

(3.7) is written

$$\boldsymbol{e}'_\lambda = \boldsymbol{e}_\lambda + \partial_\lambda(D\boldsymbol{u}). \tag{3.13}$$

Preparatory to the main subject of defining the strain increment tensor, we will now give recapitulation of the fundamental relations necessary for the subject. To begin with, the vectors $\boldsymbol{e}^\lambda$ reciprocal to $\boldsymbol{e}_\lambda$ are introduced by the relations

$$\boldsymbol{e}_\lambda \cdot \boldsymbol{e}^\mu = \delta^\mu_\lambda, \tag{3.14}$$

or by

$$\epsilon_{\lambda\mu\nu}\boldsymbol{e}^\nu = \frac{\boldsymbol{e}_\lambda \times \boldsymbol{e}_\mu}{[\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3]}, \quad \epsilon^{\lambda\mu\nu}\boldsymbol{e}_\nu = \frac{\boldsymbol{e}^\lambda \times \boldsymbol{e}^\mu}{[\boldsymbol{e}^1\boldsymbol{e}^2\boldsymbol{e}^3]}, \tag{3.15}$$

where $\epsilon_{\lambda\mu\nu}$ is the ϵ-system. Then the fundamental metric tensors are defined by

$$g_{\lambda\mu} = \boldsymbol{e}_\lambda \cdot \boldsymbol{e}_\mu, \quad g^{\lambda\mu} = \boldsymbol{e}^\lambda \cdot \boldsymbol{e}^\mu, \quad g^\lambda_\mu = \delta^\lambda_\mu \tag{3.16}$$

according to the co- and contra-variant and mixed components. Denoting the determinant consisting of the elements $g_{\lambda\mu}$ by g, we have the relations

$$g = |g_{\lambda\mu}|, \quad \frac{1}{g} = |g^{\lambda\mu}|, \tag{3.17}$$

$$\sqrt{g} = [\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3], \quad \frac{1}{\sqrt{g}} = [\boldsymbol{e}^1\boldsymbol{e}^2\boldsymbol{e}^3]. \tag{3.18}$$

And if the co-factors of $|g_{\lambda\mu}|$ with respect to $g_{\lambda\mu}$ are indicated by $G^{\mu\lambda}$, and those of $|g^{\lambda\mu}|$ with respect to $g^{\lambda\mu}$ by $G_{\mu\lambda}$, it follows that

$$g_{\lambda\mu}=gG_{\lambda\mu}, \quad g^{\lambda\mu}=\frac{G^{\lambda\mu}}{g}. \tag{3.19}$$

It will be needless to mention that $\boldsymbol{e}_\lambda$ and $\boldsymbol{e}^\lambda$ are transformed to each other by the relation

$$\boldsymbol{e}^\lambda=g^{\lambda\mu}\boldsymbol{e}_\mu, \quad \boldsymbol{e}_\lambda=g_{\lambda\mu}\boldsymbol{e}^\mu. \tag{3.20}$$

After these preliminaries, we will now go back to the main subject, and introduce the operator of gradient

$$\nabla\equiv\boldsymbol{e}^\lambda\partial_\lambda. \tag{3.21}$$

Then

$$d\equiv d\boldsymbol{r}\cdot\nabla \tag{3.22}$$

from (3.8), (3.11), (3.21) and (3.16). Hence

$$d(D\boldsymbol{u})=d\boldsymbol{r}\cdot\nabla(D\boldsymbol{u}), \tag{3.23}$$

and (3.7) is written

$$d\boldsymbol{r}'=d\boldsymbol{r}\cdot[\boldsymbol{I}+\nabla(D\boldsymbol{u})], \tag{3.24}$$

$\boldsymbol{I}$ representing the unit tensor

$$\boldsymbol{I}=\boldsymbol{e}_\lambda\boldsymbol{e}^\lambda=\boldsymbol{e}^\lambda\boldsymbol{e}_\lambda=g_{\lambda\mu}\boldsymbol{e}^\lambda\boldsymbol{e}^\mu=g^{\lambda\mu}\boldsymbol{e}_\lambda\boldsymbol{e}_\mu. \tag{3.25}$$

From (3.24), we obtain

$$(d\boldsymbol{r}')^2=d\boldsymbol{r}\cdot[\boldsymbol{I}+\nabla(D\boldsymbol{u})]\cdot[(D\boldsymbol{u})\nabla+\boldsymbol{I}]\cdot d\boldsymbol{r},$$

and therefore

$$(d\boldsymbol{r}')^2-(d\boldsymbol{r})^2=d\boldsymbol{r}\cdot[\nabla(D\boldsymbol{u})+(D\boldsymbol{u})\nabla]\cdot d\boldsymbol{r}, \tag{3.26}$$

neglecting the second order small quantities as regards $(D\boldsymbol{u})$. (3.26) shows that the change $(d\boldsymbol{r}')^2-(d\boldsymbol{r})^2$ in the metric, from t to $t+dt$, of the space deforming in conformity with the body is derived from the elementary vector $d\boldsymbol{r}$ of the space for t, by means of the symmetric tensor $\nabla(D\boldsymbol{u})+(D\boldsymbol{u})\nabla$, namely that the tensor $\nabla(D\boldsymbol{u})+(D\boldsymbol{u})\nabla$ satisfies the conditions for the plastic strain increment, deduced in the preceding section. *It is, therefore, justifiable to define the plastic strain increment tensor by*

$$D\boldsymbol{E}=\frac{1}{2}[\nabla(D\boldsymbol{u})+(D\boldsymbol{u})\nabla]. \tag{3.27}$$

By means of (3.27), (3.26) can be rewritten as

$$(d\boldsymbol{r}')^2-(d\boldsymbol{r})^2=d\boldsymbol{r}\cdot 2D\boldsymbol{E}\cdot d\boldsymbol{r}. \tag{3.28}$$

We have made use of the Lagrangian method in order to derive the definition (3.27) of the plastic strain increment $D\boldsymbol{E}$, but once it has been obtained as a tensor of the form (3.27), it does not matter with coordinate system may be used for its analytical expression. In the subsequent sections, we will give some expressions appropriate for practical purposes, as well as that for the Lagrangian coordinate system.

4. Analytical Expression of DE by Means of the Lagrangian Coordinate System

It is possible for any coordinate system to give analytical expression of the plastic strain increment DE obtained as a tensor, i.e. an invariant, in the preceding section. We will now begin with the case of the Lagrangian coordinate system which is most fitted for the theoretical purposes.

In this case, ∇ and Du are needed to be expressed by (3.21), i.e.

$$\nabla \equiv \boldsymbol{e}^{\lambda}\partial_{\lambda} \tag{4.1}$$

and

$$\boldsymbol{Du}=(Du)_{\lambda}\boldsymbol{e}^{\lambda}\,, \tag{4.2}$$

respectively, referring to $\boldsymbol{e}^{\lambda}$. It must be remarked here that the components $(Du)_{\lambda}$ of $\boldsymbol{Du}$ are distinguished from the increments Du_{λ} of the components of $\boldsymbol{Du}$, because $\boldsymbol{e}^{\lambda}$ depends on t. Substituting (4.1) and (4.2) into (3.27), and then applying

$$\partial_{\lambda}\boldsymbol{e}^{\mu}=-\Gamma^{\mu}_{\lambda\alpha}\boldsymbol{e}^{\alpha}\,, \tag{4.3}$$

$$\left.\begin{aligned} \Gamma^{\nu}_{\lambda\mu}&=g^{\nu\alpha}\Gamma_{\lambda\mu,\alpha}\,,\\ \Gamma_{\lambda\mu,\nu}&=\frac{1}{2}(\partial_{\lambda}g_{\mu\nu}+\partial_{\mu}g_{\nu\lambda}-\partial_{\nu}g_{\lambda\mu})\,, \end{aligned}\right\} \tag{4.4}$$

we obtain the result

$$\boldsymbol{DE}=\frac{1}{2}[\nabla_{\lambda}(Du)_{\mu}+\nabla_{\mu}(Du)_{\lambda}]\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}\,, \tag{4.5}$$

where $\nabla_{\lambda}(Du)_{\mu}$ means the convariant derivative of $(Du)_{\mu}$

$$\nabla_{\lambda}(Du)_{\mu}=\partial_{\lambda}(Du)_{\mu}-(Du)_{\alpha}\Gamma^{\alpha}_{\lambda\mu}\,. \tag{4.6}$$

Indicating, therefore, the covariant components of DE by $(D\varepsilon)_{\lambda\mu}$, (4.5) is rewritten as

$$\left.\begin{aligned} \boldsymbol{DE}&=(D\varepsilon)_{\lambda\mu}\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}\,,\\ (D\varepsilon)_{\lambda\mu}&=\frac{1}{2}[\nabla_{\lambda}(Du)_{\mu}+\nabla_{\mu}(Du)_{\lambda}]\,. \end{aligned}\right\} \tag{4.7}$$

The second equation (4.7) is the analytical expression of the components $(D\varepsilon)_{\lambda\mu}$ of DE by means of the Lagrangian coordinate system, and what is important is that *the basic tensors to which the components* $(D\varepsilon)_{\lambda\mu}$ *are referred are those for the state t,* $\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}$, *but not those for the state* $t=0$, $\mathring{\boldsymbol{e}}^{\lambda}\mathring{\boldsymbol{e}}^{\mu}$. In connection with this, special attention must be paid to the fact that the components $(D\varepsilon)_{\lambda\mu}$ of DE are other than the increments $D\varepsilon_{\lambda\mu}$ of the components $\varepsilon_{\lambda\mu}$ of DE, as will be shown later, for the reason of the dependence of $\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}$ on t. Although the expression (4.7) is fitted for the theoretical treatment of the problem, it is not so for the practical purposes because of the non-orthogonality of $\boldsymbol{e}^{\lambda}$.

As an alternative to (4.7), we will now introduce its expression by means of metric tensor, which is more geometrical.

Substituting (3.11) and the first equation (4.7) into (3.28), i.e.

$$(d\boldsymbol{r}')^2-(d\boldsymbol{r})^2=d\boldsymbol{r}\cdot 2D\boldsymbol{E}\cdot d\boldsymbol{r}\,, \tag{4.8}$$

and then introducing the metric tensor

$$g'_{\lambda\mu}=\boldsymbol{e}'_\lambda\cdot\boldsymbol{e}'_\mu \tag{4.9}$$

for the Lagrangian coordinate system at the state $t+dt$, we obtain the relation

$$(g'_{\lambda\mu}-g_{\lambda\mu})dx^\lambda dx^\mu=2(D\varepsilon)_{\lambda\mu}dx^\lambda dx^\mu\,,$$

so that

$$(D\varepsilon)_{\lambda\mu}=\frac{1}{2}(g'_{\lambda\mu}-g_{\lambda\mu})\,, \tag{4.10}$$

$g_{\lambda\mu}$ and $g'_{\lambda\mu}$ being the metric tensors at t and $t+dt$ respectively. (4.10) is otherwise written as

$$(D\varepsilon)_{\lambda\mu}=\frac{1}{2}Dg_{\lambda\mu}\,, \tag{4.11}$$

and consequently, $D\boldsymbol{E}$ as

$$\left.\begin{aligned} D\boldsymbol{E}&=(D\varepsilon)_{\lambda\mu}\boldsymbol{e}^\lambda\boldsymbol{e}^\mu\,,\\ (D\varepsilon)_{\lambda\mu}&=\frac{1}{2}Dg_{\lambda\mu}\,, \end{aligned}\right\} \tag{4.12}$$

referring also to the basic tensors $\boldsymbol{e}^\lambda\boldsymbol{e}^\mu$ dependent on t. This expression by means of the metric tensor serves to obtain $D\boldsymbol{E}$ practically, when the change in the geometrical configuration of the body is given.

The first equation (4.12) can be written as

$$D\boldsymbol{E}=\frac{1}{2}(g'_{\lambda\mu}\boldsymbol{e}^\lambda\boldsymbol{e}^\mu-\boldsymbol{I}) \tag{4.13}$$

by means of (4.10) and (3.25). In this result, it is seen that the state t is represented by the unit tensor $\boldsymbol{I}$. *This fact is consistent with the principle, established in the first place when deriving* $D\boldsymbol{E}$, *that the* $D\boldsymbol{E}$ *is the strain increment tensor measured assuming the state* t *as an undeformed state.*

Although the expression (4.7) of $D\boldsymbol{E}$ has been derived from the definition (3.27), it is also derivable from (4.10) as follows. That is, operating ∂_λ to the both sides of (4.2), and applying (4.3) and (4.6), we have

$$\partial_\lambda(D\boldsymbol{u})=[\nabla_\lambda(Du)_\mu]\boldsymbol{e}^\mu\,, \tag{4.14}$$

so that, from (3.13),

$$\boldsymbol{e}'_\lambda=\boldsymbol{e}_\lambda+[\nabla_\lambda(Du)_\alpha]\boldsymbol{e}^\alpha\,. \tag{4.15}$$

Substituting (4.15) into (4.9), and then applying (3.16), we can express (4.10) in terms of $(Du)_\lambda$, and the result is easily found to be the same as (4.7).

The components considered in the above being covariant, the contravariant and mixed components are obtained from the covariant one given in (4.7) and (4.12), by the relations

$$\left.\begin{aligned} (D\varepsilon)^{\lambda\mu}&=g^{\lambda\alpha}g^{\mu\beta}(D\varepsilon)_{\alpha\beta}\,,\\ (D\varepsilon)^\lambda_\mu=(D\varepsilon)^\lambda_{\cdot\mu}&=(D\varepsilon)_\mu^{\cdot\lambda}=g^{\lambda\alpha}(D\varepsilon)_{\alpha\mu}\,. \end{aligned}\right\} \tag{4.16}$$

5. Analytical Expression of DE by Means of Eulerian Coordinate System

The Lagrangian method used in the preceding section has the defects both of being impossible to identify analytically the position of a material point in space and of making the analytical expression of the mechanical quanties as DE complicated because of the non-conservation of orthogonality of the coordinate system, although it is well-grounded for the theoretical purposes. And to obviate these defects, we need to use the *Eulerian method* by means of a space fixed coordinate system. In the following, we will give the analytical expression of DE by this method.

As against the Lagrangian coordinates x^λ ($\lambda=1, 2, 3$), we will now indicate the Eulerian coordinates by x^i with Italic index, say i, representing $\dot{1}, \dot{2}, \dot{3}$. Then the transposition of each material point is expressed by the relation

$$x^i = x^i(x^1, x^2, x^3, t)\,, \tag{5.1}$$

which can be solved reciprocally to give the equations

$$x^\lambda = x^\lambda(x^{\dot{1}}, x^{\dot{2}}, x^{\dot{3}}, t)\,, \tag{5.2}$$

provided

$$|\partial_\lambda x^i| \neq 0\,.$$

The Eulerian coordinates x^i of a material point may be coincident or not with its Lagrangian coordinates x^λ for $t=0$.

As tensor maintains its form in all coordinate systems, introducing the base vectors for the Eulerian coordinate system

$$\boldsymbol{e}_i(x^1, x^2, x^3, t) = \partial_i \boldsymbol{r}(x^1, x^2, x^3, t) \tag{5.3}$$

or simply

$$\boldsymbol{e}_i = \partial_i \boldsymbol{r} \tag{5.4}$$

pertaining to the material point x^λ in the state t, and the base vectors $\boldsymbol{e}^i$ reciprocal to $\boldsymbol{e}_i$, so that defined by

$$\boldsymbol{e}^i \cdot \boldsymbol{e}_j = \delta^i_j\,, \tag{5.5}$$

we have, likewise to the case of the Lagrangian coordinate system, the expressions

$$\nabla \equiv \boldsymbol{e}^i \partial_i \tag{5.6}$$

$$D\boldsymbol{u} = (Du)_i \boldsymbol{e}^i\,, \tag{5.7}$$

$$\left.\begin{aligned} D\boldsymbol{E} &= (D\varepsilon)_{ij} \boldsymbol{e}^i \boldsymbol{e}^j\,, \\ (D\varepsilon)_{ij} &= \frac{1}{2}[\nabla_i (Du)_j + \nabla_j (Du)_i]\,, \end{aligned}\right\} \tag{5.8}$$

where

$$\nabla_i (Du)_j = \partial_i (Du)_j - (Du)_r \Gamma^r_{ij}\,, \tag{5.9}$$

$$\left.\begin{aligned} \Gamma^k_{ij} &= g^{kr} \Gamma_{ij,r}\,, \\ \Gamma_{ij,k} &= \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij})\,. \end{aligned}\right\} \tag{5.10}$$

$$\left.\begin{aligned} g_{ij}&=\boldsymbol{e}_i\cdot\boldsymbol{e}_j=gG_{ij}\,,\\ g^{ij}&=\boldsymbol{e}^i\cdot\boldsymbol{e}^j=\frac{1}{g}G^{ij}\,, \end{aligned}\right\} \tag{5.11}$$

$$g=|g_{ij}|\,,\quad \sqrt{g}=[\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3]\,, \tag{5.12}$$

G^{ji} indicating the cofactor of $|g_{ij}|$ for the element g_{ij}, and G_{ji} the cofactor of $|g^{ij}|$ for the element g^{ij}.

When the Eulerian coordinate system is especially rectangular Cartesian, all Γ'^{i}_{s} vanishing, the second equation (5.8) is written

$$(D\varepsilon)_{ij}=\frac{1}{2}[\partial_i(Du)_j+\partial_j(Du)_i]\,. \tag{5.13}$$

The expressions for the case of the cylindrical and spherical coordinate systems also are easily obtained, and the results are seen to have a form somewhat different from that usually found in many text books. This is because the basic tensors to which the components $(D\varepsilon)_{ij}$ are referred are those resulting from the natural reference frame, but not from the normalized one.

The components $(D\varepsilon)_{ij}$ expressed for the Eulerian coordinate system are related to those for the Lagrangian coordinate system $(D\varepsilon)_{\lambda\mu}$ through (5.1) or (5.2). If we put the derivatives of (5.1) and (5.2) with respect to x^λ and x^i as

$$A^i_\lambda\equiv\partial_\lambda x^i\,,\qquad A^\lambda_i\equiv\partial_i x^\lambda\,, \tag{5.14}$$

respectively, then

$$A^i_\lambda A^\lambda_j=\delta^i_j\,,\qquad A^\lambda_i A^i_\mu=\delta^\lambda_\mu\,, \tag{5.15}$$

that is, A^i_λ is the normalized cofactor for the element A^λ_i of the determinant $|A^\lambda_i|$, and vice versa. By virtue of (5.14), we have

$$\partial_i=A^\lambda_i\partial_\lambda\,,\qquad \partial_\lambda=A^i_\lambda\partial_i\,, \tag{5.16}$$

$$\boldsymbol{e}_i=A^\lambda_i\boldsymbol{e}_\lambda\,,\qquad \boldsymbol{e}_\lambda=A^i_\lambda\boldsymbol{e}_i\,, \tag{5.17}$$

$$\boldsymbol{e}^i=A^i_\lambda\boldsymbol{e}^\lambda\,,\qquad \boldsymbol{e}^\lambda=A^\lambda_i\boldsymbol{e}^i\,. \tag{5.18}$$

That the gradient ∇ is written as (5.6) as well as (3.21) is seen to be based on (5.16) and (5.18). Owing to (5.17) and (5.18), we obtain

$$\left.\begin{aligned} (D\varepsilon)_{ij}&=A^\lambda_i A^\mu_j(D\varepsilon)_{\lambda\mu}\,,\\ (D\varepsilon)_{\lambda\mu}&=A^i_\lambda A^j_\mu(D\varepsilon)_{ij}\,. \end{aligned}\right\} \tag{5.19}$$

The contravariant and mixed components are obtained by

$$(D\varepsilon)^{ij}=g^{ir}g^{js}(D\varepsilon)_{rs}\,,$$

$$(D\varepsilon)^i_j=(D\varepsilon)^i_{\cdot j}=(D\varepsilon)_j^{\cdot i}=g^{ir}(D\varepsilon)_{rj}$$

from the covariant components $(D\varepsilon)_{ij}$, as in the case of the Lagrangian coordinate system.

6. Local Coordinate System and the Expression of *DE* by Means of It

In the preceding section, we have introduced the Eulerian method in order to remove the demerit of the Lagrangian method that not only it can not describe the position in space of a material point, but also the orthogonality of the base vectors is not conserved. But even in the Eulerian method, there remains the unfitness that the change in the components of *DE*, more generally of any tensor, is effected by not only the change in the tensor itself, but also that in the basic tensors which is due to the transposition of the material point under consideration. The Cartesian Eulerian coordinate system is the only case in which the reference frame does not change with position.

In order to avoid such inexpediency as seen in the Lagrangian or the Eulerian method, we will now introduce other one which is regarded as a modification of the Lagrangian method and is more favourable to treat the plastic deformation in which any material element is needed to be followed. Namely, we consider an orthogonal, generally curvilinear, coordinate system with respect to which the coordinates of a particle at the initial unstrained state $t=0$, whether it may be actually annealed or work-hardened, are represented by $x^{\mathfrak{i}}$ with Gothic index, say $\mathfrak{i}$, representing **1, 2, 3**. This coordinate system may be coincident or not with the Eulerian system or with the Lagrangian one for $t=0$. Then we suppose that each portion of the coordinate system in the neighbourhood of a material point is transported with the point with no distortion of the configuration which it had in the state $t=0$. By doing so, not only the coordinates of each particle but also the form of the coordinate system, hence the reference frame too, belonging to and in the neighbourhood of the particle, is maintained during the deformation. This can be said in other words such that each material point possesses a *local coordinate system* which is attached to, and is convected with, it and which particularly for the state $t=0$ being connected with each other with no dislocation, constitutes, as a whole, the prescribed orthogonal curvilinear coordinate system. Consequently, in the deformed state t, our coordinate system is non-holonomic, individual local coordinate systems being disjoined with each other. This is shown in Fig. 2 schematically. In the Figure, P_0 and Q_0 are the positions of two adjacent material points in the undeformed initial state $t=0$, P and Q those in the deformed state t, and the orthogonal parametric curves through P_0 and Q_0 are converted to the oblique ones through P and Q represented by the full curves, the latter representing the Lagrangian coordinate system for the state t resulting from the former, i.e. the coordinate system $x^{\mathfrak{i}}$ for $t=0$. The local coordinate systems attached to the two material points are coincident, for the state $t=0$, with the full curves through P_0 and Q_0, but for the state t they are represented by the dotted curves through P and Q which are obtained by the parallel displacement of the curves through P_0 and Q_0 and have dislocations as shown in the figure. By dint of the merit that the reference frame, hence its orthogonality too, is kept unchanged during deformation for the same material element, the method of local

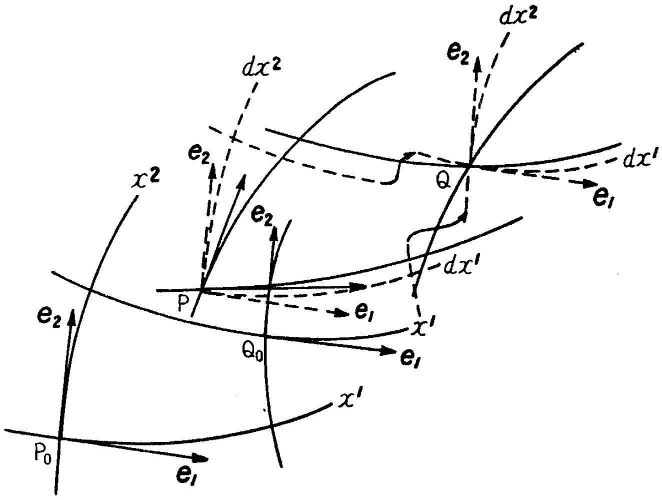

FIGURE 2.

coordinate system is considered to more than make up for the demerit of its being non-holonomic.

Since we denoted by x^i with Roman index the coordinates of a material point in the state $t=0$ referred to an orthogonal curvilinear coordinate system, into which the local corrdinate systems are compounded by getting together for the state $t=0$, the local coordinates of any material point with respect, and near, to a material point x^i are represented by dx^i for any state t. Now by writing

$$\partial_i \equiv \frac{\partial}{\partial x^i}, \tag{6.1}$$

the base vectors $\mathring{\boldsymbol{e}}_i$ and $\mathring{\boldsymbol{e}}^i$ for the state $t=0$ are defined by

$$\mathring{\boldsymbol{e}}_i = \partial_i \mathring{\boldsymbol{r}} \tag{6.2}$$

and

$$\mathring{\boldsymbol{e}}_i \cdot \mathring{\boldsymbol{e}}^j = \delta_i^j, \tag{6.3}$$

and the base vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}^i$ for the state t by

$$\boldsymbol{e}_i = \partial_i \boldsymbol{r} \tag{6.4}$$

and

$$\boldsymbol{e}_i \cdot \boldsymbol{e}^j = \delta_i^j. \tag{6.5}$$

Then, it is evident that

$$\boldsymbol{e}_i = \mathring{\boldsymbol{e}}_i, \quad \boldsymbol{e}^i = \mathring{\boldsymbol{e}}^i, \tag{6.6}$$

hence

$$D\boldsymbol{e}_i = 0, \quad D\boldsymbol{e}^i = 0 \tag{6.7}$$

for the same material point. This means that the reference frame belonging to a material point performs parallel translation with the transposition of the point.

According to the method of local coordinate system, we can put

$$D\boldsymbol{u} = (Du)_i \boldsymbol{e}^i \tag{6.8}$$

and

$$\nabla = \boldsymbol{e}^i \partial_i \tag{6.9}$$

instead of (4.1) and (4.2), and therefore obtain, from (3.27),

$$\left.\begin{aligned} \boldsymbol{DE} &= (D\varepsilon)_{ij}\boldsymbol{e}^i\boldsymbol{e}^j\,, \\ (D\varepsilon)_{ij} &= \frac{1}{2}[\nabla_i(Du)_j + \nabla_j(Du)_i]\,, \end{aligned}\right\} \tag{6.10}$$

where

$$\nabla_i(Du)_j = \partial_i(Du)_j - (Du)_r\Gamma^r_{ij}\,, \tag{6.11}$$

$$\left.\begin{aligned} \Gamma^k_{ij} &= g^{kr}\Gamma_{ij,k}\,, \\ \Gamma_{ij,k} &= \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij})\,, \end{aligned}\right\} \tag{6.12}$$

$$g_{ij} = \boldsymbol{e}_i\cdot\boldsymbol{e}_j\,, \qquad g^{ij} = \boldsymbol{e}^i\cdot\boldsymbol{e}^j\,. \tag{6.13}$$

If, in particular, the local coordinate system is rectangular Cartesian, then

$$(D\varepsilon)_{ij} = \frac{1}{2}[\partial_i(Du)_j + \partial_j(Du)_i]\,. \tag{6.14}$$

Now, we will try to get the expression of $\boldsymbol{DE}$ available for practical purposes by means of the method of local coordinate system, when the process of change in the geometrical configuration of the material element is given. For this purpose, we take the Lagrangian coordinate system x^λ for the state $t=0$ so as to coincide with the local coordinate system x^i for the same state $t=0$. Then the base vectors for the Lagrangian coordinate system for $t=0$, defined by

$$\mathring{\boldsymbol{e}}_\lambda = \partial_\lambda \mathring{\boldsymbol{r}}\,, \tag{6.15}$$

$$\mathring{\boldsymbol{e}}_\lambda\cdot\mathring{\boldsymbol{e}}^\mu = \delta^\mu_\lambda \tag{6.16}$$

are identical to those for the local coordinate system for any state t respectively, i.e.

$$\left.\begin{aligned} \mathring{\boldsymbol{e}}_\iota &= \mathring{\boldsymbol{e}}_i = \boldsymbol{e}_i\,, \\ \mathring{\boldsymbol{e}}^\iota &= \mathring{\boldsymbol{e}}^i = \boldsymbol{e}^i \end{aligned}\right\} \tag{6.17}$$

for the corresponding value of ι and $\boldsymbol{i}$.

We suppose that the geometrical configuration of the material element at the current state t referred to that at the state $t=0$, i.e. the relation

$$\boldsymbol{e}_\lambda = c^\kappa_\lambda \mathring{\boldsymbol{e}}_\kappa \tag{6.18}$$

is given, then substituting (6.17) into (6.18), and putting

$$c^\kappa_\lambda = c^k_\lambda \tag{6.19}$$

for the corresponding value of κ and $\boldsymbol{k}$, we have

$$\boldsymbol{e}_\lambda = c^k_\lambda \boldsymbol{e}_k\,. \tag{6.20}$$

Since c^κ_μ in (6.18) is specified by the change in the geometrical configuration, c^k_λ in (6.20), which is identical to c^κ_λ, is also the case. But, on the other hand, c^k_λ being the matrix which transforms the local reference frame $\boldsymbol{e}_l$ into the Lagrangian reference frame $\boldsymbol{e}_\lambda$ for the same material point at t, it is expressed also by

$$c^j_\lambda = \partial_\lambda x^j\,. \tag{6.21}$$

The matrix c^λ_i inverse to c^i_λ given by (6.21) is defined by

$$c^\lambda_i = \partial_i x^\lambda\,, \tag{6.21'}$$

or by

$$c_\lambda^r c_r^\mu = \delta_\lambda^\mu, \quad c_i^\rho c_\rho^j = \delta_i^j, \tag{6.22}$$

and it follows that

$$\boldsymbol{e}_i = c_i^\lambda \boldsymbol{e}_\lambda, \tag{6.20'}$$

$$\boldsymbol{e}^\lambda = c_i^\lambda \boldsymbol{e}^i, \quad \boldsymbol{e}^i = c_\lambda^i \boldsymbol{e}^\lambda. \tag{6.23}$$

We have from (3.16) and (6.20)

$$g_{\lambda\mu} = c_\lambda^r c_\mu^s g_{rs}, \tag{6.24}$$

where g_{ij} has been defined by the first equation (6.13), and is, together with g^{ij}, equivalent, on account of (6.17), to the metric tensors for the local coordinate system at $t=0$

$$\mathring{g}_{ij} = \mathring{\boldsymbol{e}}_i \cdot \mathring{\boldsymbol{e}}_j, \quad \mathring{g}^{ij} = \mathring{\boldsymbol{e}}^i \cdot \mathring{\boldsymbol{e}}^j \tag{6.25}$$

or also to those for the Lagrangian coordinate system at $t=0$

$$\mathring{g}_{\lambda\mu} = \mathring{\boldsymbol{e}}_\lambda \cdot \mathring{\boldsymbol{e}}_\mu, \quad \mathring{g}^{\lambda\mu} = \mathring{\boldsymbol{e}}^\lambda \cdot \mathring{\boldsymbol{e}}^\mu. \tag{6.26}$$

We have from (4.12) and (6.24)

$$(D\varepsilon)_{\lambda\mu} = \frac{1}{2} D(c_\lambda^r b_\mu^s) g_{rs} \tag{6.27}$$

because g_{rs} is independent of t. Since

$$(D\varepsilon)_{ij} = c_i^\lambda c_j^\mu (D\varepsilon)_{\lambda\mu}, \quad (D\varepsilon)_{\lambda\mu} = c_\lambda^i c_\mu^j (D\varepsilon)_{ij} \tag{6.28}$$

by virtue of (6.20), we obtain

$$\left.\begin{aligned} \boldsymbol{DE} &= (D\varepsilon)_{ij} \boldsymbol{e}^i \boldsymbol{e}^j, \\ (D\varepsilon)_{ij} &= \frac{1}{2} c_i^\lambda c_j^\mu D(c_\lambda^r c_\mu^s) g_{rs} \end{aligned}\right\} \tag{6.29}$$

from (6.27) and (6.28). This is the expression of $\boldsymbol{DE}$ by means of the local coordinate system, convenient for the practical purposes.

Basing on the covariant components considered so far, the contravariant and mixed components are obtained by

$$\left.\begin{aligned} (D\varepsilon)^{ij} &= g^{ik} g^{jl} (D\varepsilon)_{kl}, \\ (D\varepsilon)_j^i &= (D\varepsilon)^i{}_{.j} = (D\varepsilon)_j{}^{.i} = g^{ir} (D\varepsilon)_{rj} \end{aligned}\right\} \tag{6.30}$$

similarly to the cases of other methods.

7. Plastic Strain or Strain History Tensor

It will be needless to mention that, if the strain increment tensor $\boldsymbol{DE}$ defined in Section 3 is integrated along a given path of deformation for a certain material element, we get the plastic strain tensor

$$\boldsymbol{E} = \int_0^t \boldsymbol{DE} \tag{7.1}$$

for the same element. That this tensor $\boldsymbol{E}$ depends on the deformation path, and therefore is none other than the strain history tensor mentioned in Sections 1 and 2, will be shown thereafter in general, and then by some examples.

For practical purposes, it is convenient to perform the integration (7.1) by the method of local coordinate system. But, the results being the same by any method, we will now begin the problem with the Lagrangian method which has theoretical coherency. In this case $\boldsymbol{DE}$ is given by (4.7) or (4.12), i.e. by

$$\left.\begin{aligned} \boldsymbol{DE}&=(D\varepsilon)_{\lambda\mu}\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}\,, \\ (D\varepsilon)_{\lambda\mu}&=\frac{1}{2}Dg_{\lambda\mu}=\frac{1}{2}[\nabla_{\lambda}(Du)_{\mu}+\nabla_{\mu}(Du)_{\lambda}]\,, \end{aligned}\right\} \tag{7.2}$$

and not only $(D\varepsilon)_{\lambda\mu}$ but also $\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}$ varies with t, and consequently (7.1) is not expected to be easily integrated. Provided, however, that the integration (7.1) could have been carried out in any way to give the result

$$\boldsymbol{E}=\varepsilon_{\lambda\mu}\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}\,, \tag{7.3}$$

we have, from (7.2) and (7.3),

$$D(\varepsilon_{\lambda\mu}\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu})=(D\varepsilon)_{\lambda\mu}\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}$$

hence

$$D\varepsilon_{\lambda\mu}\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}+\varepsilon_{\lambda\mu}(\boldsymbol{D}\boldsymbol{e}^{\lambda})\boldsymbol{e}^{\mu}+\varepsilon_{\lambda\mu}\boldsymbol{e}^{\lambda}\boldsymbol{D}\boldsymbol{e}^{\mu}=(D\varepsilon)_{\lambda\mu}\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}\,. \tag{7.4}$$

In order to calculate $\boldsymbol{De}^{\lambda}$ in (7.4), we must begin with $\boldsymbol{De}_{\lambda}$. The reference frame for the state $t+dt$ being $\boldsymbol{e}_{\lambda}+\boldsymbol{De}_{\lambda}$ as well as $\partial_{\lambda}(\boldsymbol{r}+\boldsymbol{Dr})$, we have

$$\boldsymbol{e}_{\lambda}+\boldsymbol{De}_{\lambda}=\partial_{\lambda}(\boldsymbol{r}+\boldsymbol{Dr})\,,$$

hence

$$\boldsymbol{De}_{\lambda}=\partial_{\lambda}\boldsymbol{Dr}\,. \tag{7.5}$$

On the other hand, from (3.10),

$$\boldsymbol{De}_{\lambda}=D\partial_{\lambda}\boldsymbol{r}\,. \tag{7.6}$$

That the order of the operators D and ∂_{λ} is commutable is seen by comparing (7.5) with (7.6). Applying (3.6) and (4.14) to (7.5), we obtain

$$\boldsymbol{De}_{\lambda}=[\nabla_{\lambda}(Du)_{\kappa}]\boldsymbol{e}^{\kappa}\,. \tag{7.7}$$

By applying this result (7.7) to the relation

$$(\boldsymbol{De}^{\mu})\cdot\boldsymbol{e}_{\rho}=-\boldsymbol{e}^{\mu}\cdot(\boldsymbol{De}_{\rho})$$

which is obtained by operating D to (3.14), we have

$$(\boldsymbol{De}^{\mu})\cdot\boldsymbol{e}_{\rho}=-\boldsymbol{e}^{\mu}\cdot[\nabla_{\rho}(Du)_{\kappa}]\boldsymbol{e}^{\kappa}=-g^{\mu\kappa}\nabla_{\rho}(Du)_{\kappa}\,,$$

so that the relation aimed at

$$\boldsymbol{De}^{\mu}=-g^{\mu\kappa}\nabla_{\rho}(Du)_{\kappa}\boldsymbol{e}^{\rho}\,. \tag{7.8}$$

Substituting (7.8) into (7.4), *we obtain the simultaneous differential equations*

$$D\varepsilon_{\lambda\mu}-g^{\kappa\rho}[\varepsilon_{\kappa\mu}\nabla_{\lambda}(Du)_{\rho}+\varepsilon_{\lambda\kappa}\nabla_{\mu}(Du)_{\rho}]=(D\varepsilon)_{\lambda\mu} \tag{7.9}$$

with respect to the components $\varepsilon_{\lambda\mu}$. We mentioned previously that $(D\varepsilon)_{\lambda\mu}$ should be distinguished from $D\varepsilon_{\lambda\mu}$, and the difference between them is seen to be clarified now. By virtue of (7.9), the symmetrical structure of $\boldsymbol{E}$, i.e.

$$\varepsilon_{\lambda\mu}=\varepsilon_{\mu\lambda} \tag{7.10}$$

is deduced from that of $\boldsymbol{DE}$, i.e.

$$(D\varepsilon)_{\lambda\mu}=(D\varepsilon)_{\mu\lambda}$$

which is evident from the definition (3.27). In consequence of this, (7.9) is reduced to the six simultaneous differential equations with respect to the six independent components $\varepsilon_{\lambda\mu}$.

A solution $\varepsilon_{\lambda\mu}$ *of* (7.9), *satisfying the initial condition* $\varepsilon_{\lambda\mu}=0$ $(t=0)$, *defines a strain* $\boldsymbol{E}$ *at each deformed state t. And since the coefficients* $\nabla_\lambda(Du)_\rho$, $\nabla_\mu(Du)_\rho$ *and* $(D\varepsilon)_{\lambda\mu}$ *in the equations, being specified by the infinitesimal deformation from t to t+dt, depends on the deformation path, the solution too is considered to depend on it in general.* Namely, if we denote the strain obtained by integrating (7.9) from the initial state O to the deformed state C along a deformation path A, shown in

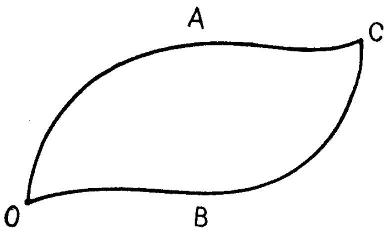

FIGURE 3.

Fig. 3, by $\boldsymbol{E}$ (OAC), and that obtained for the same state along another path B by $\boldsymbol{E}$ (OBC), then generally

$$\boldsymbol{E}\,(\mathrm{OAC}) \neq \boldsymbol{E}\,(\mathrm{OBC})\,. \tag{7.11}$$

This can also be expressed such that the strain after a cycle of deformation OACBO is completed does not vanish, e.g.

$$\oint D\boldsymbol{E} \neq 0\,. \tag{7.12}$$

Hereupon we are required to remember what has been stated in Section 2 concerning the essential character of plastic deformation. To repeat the statement briefly, plastic deformation is the deformation due to the change in the mode of the interatomic connection, therefore accompanied with that in the microscopic structure of the material such as the group pattern of dislocations, caused by a sequence of successive slips. In consequence of this the final state of the same geometrical configuration reached by plastic deformation is distinguished in its microscopic structure, hence in the mechanical properties as anisotropy specified by it, according to the process of slip, i.e. the deformation path up to the state. Accordingly, in plastic deformation, to specify the geometrical configuration is not to specify the state of the material, but the state in the true sense of the word is identified by the microscopic structure which depends on the deformation path. *If this true state of material specified by the micro-structural state we now denominate "mechanical state", as against the "geometrical state" frequently represented by the term "state", the state quantity (or variable) representing the mechanical state should also be path dependent, and hence is supposed quite naturally to be the strain* $\boldsymbol{E}$ *obtained as dependent on the deformation path, as shown in (7.11), as a solution of (7.9).* In this meaning we can designate the strain $\boldsymbol{E}$ for a mechanical state the "*strain history*", or the "*plastic strain*" in contrast with the "elastic strain" for the geometrical state.

The plastic strain $\boldsymbol{E}$ is generally different, as shown in (7.11), for the same geometrical state, whereas if we wish to make the plastic strains equal for the two deformation paths A and B as shown in Fig. 4, the final geometrical state should be chosen differently from each other, as shown by C_1 and C_2 in the Figure.

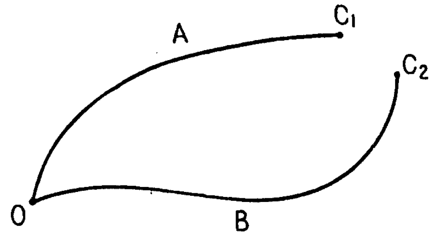

FIGURE 4.

Thus the conjecture for the plastic strain to be dependent on the deformation path, made from the physical view-point of plastic deformation in Sections 1 and 2, is now seen to be realized with mathematical reliability. This accordance between the physical and mathematical stand points is considered to afford a firm foundation to our present theory of plasticity.

As against the Lagrangian method, the strain history tensor $\boldsymbol{E}$ is seen to be obtained more easily by the method of local coordinate system previously introduced. In this case $D\boldsymbol{E}$ is given by (6.10) or (6.29), i.e. by

$$\left.\begin{aligned} &D\boldsymbol{E}=(D\varepsilon)_{ij}\boldsymbol{e}^i\boldsymbol{e}^j\,, \\ &(D\varepsilon)_{ij}=\frac{1}{2}[\nabla_i(Du)_j+\nabla_j(Du)_i]\,, \\ \text{or}\quad &(D\varepsilon)_{ij}=\frac{1}{2}c_i^\lambda c_j^\mu D(c_\lambda^r c_\mu^s)g_{rs}\,. \end{aligned}\right\} \tag{7.13}$$

If the integration (7.1) were obtained in any way, as

$$\boldsymbol{E}=\varepsilon_{ij}\boldsymbol{e}^i\boldsymbol{e}^j\,, \tag{7.14}$$

then

$$D\varepsilon_{ij}=(D\varepsilon)_{ij}\,, \tag{7.15}$$

because the basic tensors $\boldsymbol{e}^i\boldsymbol{e}^j$ are independent of t. That is, *according to the method of local coordinate system, the components of the strain increment tensor are equal to the increment of the strain components. It is the same as in the case of the Lagrangian method that the solution of the differential equations (7.15) for the same geometrical state depends on the deformation path up to the state, because $(D\varepsilon)_{ij}$ in (7.15) is given by the path dependent quantities $\nabla_i(Du)_j$, $Dc_\lambda^r,\cdots$, as shown in (7.13).*

It will be needless to mention that $\boldsymbol{E}=\varepsilon_{\lambda\mu}\boldsymbol{e}^\lambda\boldsymbol{e}^\mu$ solved from (7.9) and $\boldsymbol{E}=\varepsilon_{ij}\boldsymbol{e}^i\boldsymbol{e}^j$ from (7.15) is the same for the same deformation path, i.e. for the same mechanical state, differing only in their expressions. The components are related to each other by

$$\left.\begin{aligned} \varepsilon_{\lambda\mu}&=c_\lambda^i c_\mu^j\varepsilon_{ij}\,, \\ \varepsilon_{ij}&=c_i^\lambda c_j^\mu\varepsilon_{\lambda\mu}\,, \end{aligned}\right\} \tag{7.16}$$

as in the case of (6.28), where c_i^λ and c_λ^i are given by (6.20) and (6.22).

The contravariant and mixed components of $\boldsymbol{E}$ are given by

$$\left.\begin{aligned} \varepsilon^{\lambda\mu} &= g^{\lambda\kappa} g^{\mu\rho} \varepsilon_{\kappa\rho}\,, \\ \varepsilon_\lambda^\mu = \varepsilon_\lambda^{\cdot\mu} = \varepsilon^\mu_{\cdot\lambda} &= g^{\mu\kappa} \varepsilon_{\lambda\kappa}\,, \end{aligned}\right\} \tag{7.17}$$

$$\left.\begin{aligned} \varepsilon^{ij} &= g^{ir} g^{js} \varepsilon_{rs}\,, \\ \varepsilon_i^j = \varepsilon_i^{\cdot j} = \varepsilon^j_{\cdot i} &= g^{jr} \varepsilon_{is}\,, \end{aligned}\right\} \tag{7.18}$$

as in the case of DE. When the local coordinate system is rectangular Cartesian, the distinction among ε^{ij}, ε_i^j and ε^{ij} disappears.

8. Some Examples of DE and E

We will now apply the results of our general theory concerning DE and E obtained so far to the cases of extension, simple shear, their combination and so on. The results thus obtained will serve not only on considering the strain of circular tubes but also for a more complete understanding of the general theory of strain.

(1) *Combination of Simple Extension and Simple Shear*

In order to consider the combined extension-torsion of a circular tube, we introduce the Lagrangian coordinate system and the local one, both of which, as shown in Fig. 5a, being rectangular Cartesian at the initial state $t=0$, have the origin on the wall of the tube and the x^1, x^2, x^3 and x^1, x^2, x^3 axes coincident with the r, θ, z directions respectively at the point. Hereafter since we will consider only uniform deformations of the tube, we can take account, in case of the local coordinate, not only of dx but of x. Although, at $t=0$ both the x^λ and the x^i axes coincide with the above mentioned rectangular Cartesian axes, generally the

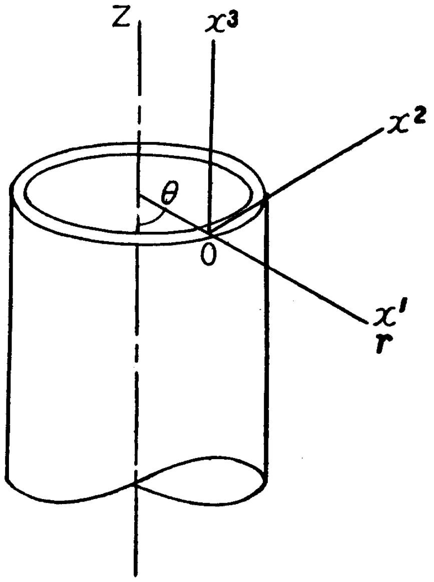

FIGURE 5a.

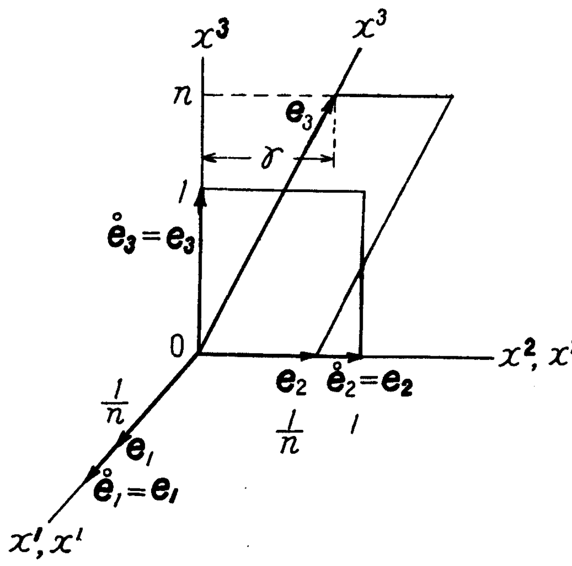

FIGURE 5b.

former become oblique, as the deformation increases, the latter remaining as it was. If we assume the thickness of the tube to be so small that the deformation may be regarded as uniform over the thickness, the combined extension-torsion of the tube of Fig. 5a is replaced, as shown in Fig. 5b, by the combination of extension in the x^3 direction and simple shear in the x^2, x^3 plane of a plane plate lying in the plane.

Accordingly the combined extension-torsion of the tube carried out in any way to give a configuration extended by n times in the axial direction and twisted such that a generator become a helix tilted by the angle arc tan (γ/n) with the axis is represented by the deformation, as shown in Fig. 5b, of the unit cube having as the intersecting edges the Lagrangian or local reference frame for $t=0$, $\mathring{\boldsymbol{e}}_1=\boldsymbol{e}_1$, $\mathring{\boldsymbol{e}}_2=\boldsymbol{e}_2$ and $\mathring{\boldsymbol{e}}_3=\boldsymbol{e}_3$, to a parallelepiped with the edges equal to the Lagrangian reference frame for t, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. From Fig. 5b, we have

$$\boldsymbol{e}_1=\frac{\boldsymbol{e}_1}{\sqrt{n}}, \quad \boldsymbol{e}_2=\frac{\boldsymbol{e}_2}{\sqrt{n}}, \quad \boldsymbol{e}_3=n\boldsymbol{e}_3+\gamma\boldsymbol{e}_2\,, \tag{8.1}$$

as a special case of (6.20), assuming the material to the incompressible. Hence, corresponding to (6.20′) and (6.23),

$$\boldsymbol{e}_1=\sqrt{n}\,\boldsymbol{e}_1\,, \quad \boldsymbol{e}_2=\sqrt{n}\,\boldsymbol{e}_2\,, \quad \boldsymbol{e}_3=\frac{1}{n}\boldsymbol{e}_3-\frac{\gamma}{\sqrt{n}}\boldsymbol{e}_2\,, \tag{8.1′}$$

and

$$\boldsymbol{e}^1=\sqrt{n}\,\boldsymbol{e}^1\,, \quad \boldsymbol{e}^2=\sqrt{n}\,\boldsymbol{e}^2-\frac{\gamma}{\sqrt{n}}\boldsymbol{e}^3\,, \quad \boldsymbol{e}^3=\frac{1}{n}\boldsymbol{e}^3\,, \tag{8.2}$$

$$\boldsymbol{e}^1=\frac{\boldsymbol{e}^1}{\sqrt{n}}, \quad \boldsymbol{e}^2=\frac{1}{\sqrt{n}}\boldsymbol{e}^2+\gamma\boldsymbol{e}^3, \quad \boldsymbol{e}^3=n\boldsymbol{e}^3\,. \tag{8.2′}$$

$\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, being the basis of the local coördinate system coincident with the rectangular Cartesian one at $t=0$, constitute a system of mutually perpendicular unit vectors, accordingly so with $\boldsymbol{e}^1$, $\boldsymbol{e}^2$ and $\boldsymbol{e}^3$. Hence we have in this case $\boldsymbol{e}_i\cdot\boldsymbol{e}_j=\delta_{ij}$ and $\boldsymbol{e}^i\cdot\boldsymbol{e}^j=\delta^{ij}$, so that from (3.16)

$$\left.\begin{aligned} &g_{11}=\frac{1}{n}, \quad g_{22}=\frac{1}{n}, \quad g_{33}=n^2+\gamma^2, \\ &g_{23}=\frac{\gamma}{\sqrt{n}}, \quad g_{31}=g_{12}=0\,, \end{aligned}\right\} \tag{8.3}$$

$$\left.\begin{aligned} &g^{11}=n\,, \quad g^{22}=\frac{n^2+\gamma^2}{n}, \quad g^{33}=\frac{1}{n^2}, \\ &g^{23}=-\frac{\gamma}{n^{3/2}}, \quad g^{31}=g^{12}=0\,. \end{aligned}\right\} \tag{8.3′}$$

Thus from (4.12), we get the components of DE

$$\left.\begin{aligned} &(D\varepsilon)_{11}=-\frac{Dn}{2n^2}, \quad (D\varepsilon)_{22}=-\frac{Dn}{2n^2}, \quad (D\varepsilon)_{33}=nDn+\gamma D\gamma\,, \\ &(D\varepsilon)_{23}=\frac{D\gamma}{2\sqrt{n}}-\frac{\gamma Dn}{4n^{3/2}}, \quad (D\varepsilon)_{31}=(D\varepsilon)_{12}=0 \end{aligned}\right\} \tag{8.4}$$

referred to the basic tensors $e^\lambda e^\mu$ for the Lagrangian coordinate system at the state t.

The components of DE referred to the basic tensors $e^i e^j$ for the local coordinate system can be obtained by substituting (8.4) into (6.28), in which c_i^λ is given by the coefficients in (8.1′), as

$$\left.\begin{aligned} &(D\varepsilon)_{11}=(D\varepsilon)_{22}=-\frac{Dn}{2n}, \quad (D\varepsilon)_{33}=\frac{Dn}{n}, \\ &(D\varepsilon)_{23}=\frac{\gamma}{4n^2}Dn+\frac{1}{2n}, \quad (D\varepsilon)_{31}=(D\varepsilon)_{12}=0. \end{aligned}\right\} \tag{8.5}$$

Now that the plastic strain increment has been calculated as (8.4) and (8.5), the plastic strain should be obtained by integrating (7.9) or (7.15) using (8.4) or (8.5). We will now begin with the case of Lagrangian method. In case of the combined extension-torsion, we can put

$$\left.\begin{aligned} &\partial_1(Du)_2=\partial_1(Du)_3=0, \\ &\partial_2(Du)_1=\partial_2(Du)_3=0, \\ &\partial_3(Du)_1=0, \end{aligned}\right\} \tag{8.6}$$

so that, by virtue of (4.7),

$$\left.\begin{aligned} &\partial_1(Du)_1=(D\varepsilon)_{11}, \quad \partial_2(Du)_2=(D\varepsilon)_{22}, \quad \partial_3(Du)_3=(D\varepsilon)_{33}, \\ &\partial_3(Du)_2=\partial_3(Du)_2+\partial_2(Du)_3=2(D\varepsilon)_{23}, \end{aligned}\right\} \tag{8.7}$$

because in this case $\nabla_\lambda(Du)_\mu$ in (4.7) equals $\partial_\lambda(Du)_\mu$. Substituting (8.7) into (7.9), and then applying (8.3′) and (8.4) to the result, we obtain the following differential equations with regard to the components $\varepsilon_{\lambda\mu}$:

$$\left.\begin{aligned} &D\varepsilon_{11}+\frac{Dn}{n}\varepsilon_{11}=-\frac{Dn}{2n^2}, \\ &D\varepsilon_{22}+\frac{(n^2+\gamma^2)Dn}{n^3}\varepsilon_{22}-\frac{\gamma Dn}{n^{7/2}}\varepsilon_{23}=-\frac{Dn}{2n^2}, \\ &D\varepsilon_{33}-\left(\frac{2}{n}+\frac{\gamma^2}{n^3}\right)Dn\varepsilon_{33}+\left[\left(\frac{3\gamma}{\sqrt{n}}+\frac{\gamma^3}{n^{5/2}}\right)Dn+2\sqrt{n}\,D\gamma\right]\varepsilon_{23} \\ &\qquad =nDn+\gamma D\gamma, \\ &D\varepsilon_{23}-\frac{Dn}{2n}\varepsilon_{23}+\frac{1}{2}\left[\left(\frac{3\gamma}{\sqrt{n}}-\frac{\gamma^3}{n^{5/2}}\right)Dn+2\sqrt{n}\,D\gamma\right]\varepsilon_{22}-\frac{\gamma Dn}{2n^{7/2}}\varepsilon_{33} \\ &\qquad =\frac{D\gamma}{2\sqrt{n}}-\frac{\gamma Dn}{4n^{3/2}}, \\ &D\varepsilon_{31}-\frac{1}{2}\left(\frac{1}{n}+\frac{\gamma^2}{n^3}\right)Dn\varepsilon_{31}+\frac{1}{2}\left[\left(\frac{3\gamma}{\sqrt{n}}+\frac{\gamma^3}{n^{5/2}}\right)Dn+2\sqrt{n}\,D\gamma\right]\varepsilon_{21}=0, \\ &D\varepsilon_{12}+\left(\frac{1}{n}+\frac{\gamma^2}{2n^3}\right)Dn\varepsilon_{12}-\frac{\gamma Dn}{2n^{7/2}}\varepsilon_{31}=0. \end{aligned}\right\} \tag{8.8}$$

These are the results by the Lagrangian method.

In case of the local coordinate system, the differential equations are given by (7.15), so that from (8.5)

$$\left.\begin{aligned} &D\varepsilon_{11}=D\varepsilon_{22}=-\frac{Dn}{2n}, \quad D\varepsilon_{33}=\frac{Dn}{n}, \\ &D\varepsilon_{23}=\frac{\gamma}{4n^2}Dn+\frac{1}{2n}D\gamma, \\ &D\varepsilon_{31}=D\varepsilon_{12}=0. \end{aligned}\right\} \tag{8.9}$$

It is seen that the equations by the method of the local coordinate system are much simpler than those by the Lagrangian method, and are fitted for the practical use.

If the deformation path, that is, the process of variation of the parameters n and γ, is given, the differential equations (8.8) and (8.9) can be integrated to yield the results $\boldsymbol{E}=\varepsilon_{\lambda\mu}\boldsymbol{e}^\lambda\boldsymbol{e}^\mu$ and $\boldsymbol{E}=\varepsilon_{ij}\boldsymbol{e}^i\boldsymbol{e}^j$ respectively. As a matter of course, both the results should be the same for the same deformation path, only their expressions being different. To show this actually for some examples will serve not only for certifying the self-consistency of our present theory but also for deeper insight of it. In the following we will apply the above results to some special cases and give the expressions by the both methods, which will be seen to be identical.

(1.1) *Simple Extension*

Since, in this case, we can put

$$\gamma=0, \tag{8.10}$$

(8.4) reduces to

$$\left.\begin{aligned} &(D\varepsilon)_{11}=-\frac{Dn}{2n^2}, \quad (D\varepsilon)_{22}=-\frac{Dn}{2n^2}, \quad (D\varepsilon)_{33}=nDn, \\ &(D\varepsilon)_{23}=(D\varepsilon)_{31}=(D\varepsilon)_{12}=0, \end{aligned}\right\} \tag{8.11}$$

and (8.8) and (8.9) to

$$\left.\begin{aligned} &\frac{D\varepsilon_{11}}{Dn}-\frac{\varepsilon_{11}}{n}=-\frac{1}{2n^2}, \quad &&\frac{D\varepsilon_{23}}{Dn}-\frac{\varepsilon_{23}}{2n}=0, \\ &\frac{D\varepsilon_{22}}{Dn}-\frac{\varepsilon_{22}}{n}=-\frac{1}{2n^2}, \quad &&\frac{D\varepsilon_{12}}{Dn}-\frac{\varepsilon_{12}}{2n}=0, \\ &\frac{D\varepsilon_{33}}{Dn}-\frac{\varepsilon_{33}}{n}=n, \quad &&\frac{D\varepsilon_{12}}{Dn}-\frac{\varepsilon_{12}}{2n}=0, \end{aligned}\right\} \tag{8.12}$$

and

$$\left.\begin{aligned} &D\varepsilon_{11}=D\varepsilon_{22}=-\frac{Dn}{2n}, \quad D\varepsilon_{33}=\frac{Dn}{n}, \\ &D\varepsilon_{23}=D\varepsilon_{31}=D\varepsilon_{12}=0 \end{aligned}\right\} \tag{8.13}$$

respectively.

The equations (8.12) can easily be integrated along the path of simple extension, namely with respect to n, from 0 to n, under the initial condition

$$\varepsilon_{\lambda\mu}=0 \quad \text{at} \quad n=1 \ (t=0), \tag{8.14}$$

to give the result

$$\left.\begin{aligned} &\varepsilon_{11}=\varepsilon_{22}=-\frac{1}{2n}\log n\,, \\ &\varepsilon_{33}=n^2\log n\,, \\ &\varepsilon_{23}=\varepsilon_{31}=\varepsilon_{12}=0\,. \end{aligned}\right\} \tag{8.15}$$

On the other hand, (8.13) gives the result

$$\left.\begin{aligned} &\varepsilon_{11}=\varepsilon_{22}=-\frac{1}{2}\log n\,, \quad \varepsilon_{23}=\log n\,, \\ &\varepsilon_{23}=\varepsilon_{31}=\varepsilon_{12}=0\,, \end{aligned}\right\} \tag{8.16}$$

for the same deformation path and initial condition. What must be noted here is that (8.15) and (8.16) are the components of the same strain of the simple extension referred to the basic tensors $\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}$ and $\boldsymbol{e}^{i}\boldsymbol{e}^{j}$ respectively, and therefore are derivable from each other by (7.16) whose coefficients read as those of (8.1) or (8.1′) in which γ is put equal to zero. It is clear from (8.15) and (8.16) how the expression of the strain components depends on the method used.

The components ε_{ij} obtained in (8.16) are seen to be the logarithmic strain itself introduced by P. Ludwik (1909) [*1*]. This shows that *the logarithmic strain of ordinary use is none other than the extensional plastic strain referred to the local coordinate system,* i.e. that *the plastic strain or the strain history tensor* $\boldsymbol{E}$ *introduced in the present paper is a generalization of the logarithmic strain for extension.* That the logarithmic strain, together with the true stress, is quite reasonable for describing the plastic deformation of finite extension will be shown later.

(1.2) *Simple Shear* (*Torsion of Thin Circular Tubes*)

Since, in this case, we may set

$$n = 1 \tag{8.17}$$

as shown in Fig. 6, (8.4), (8.8) and (8.9) are written

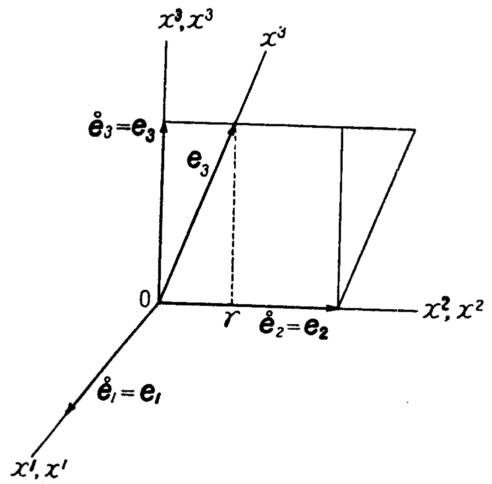

FIGURE 6.

$$\left.\begin{aligned} &(D\varepsilon)_{33}=\gamma D\gamma\,, \quad (D\varepsilon)_{23}=\frac{1}{2}D\gamma\,, \\ &(D\varepsilon)_{11}=(D\varepsilon)_{22}=(D\varepsilon)_{31}=(D\varepsilon)_{12}=0\,, \end{aligned}\right\} \tag{8.18}$$

$$\left.\begin{array}{l} D\varepsilon_{11}=0\,, \quad D\varepsilon_{22}=0\,, \quad D\varepsilon_{33}-2\varepsilon_{23}D\gamma=\gamma D\gamma\,, \\ D\varepsilon_{23}=\dfrac{1}{2}D\gamma\,, \quad D\varepsilon_{31}-\varepsilon_{21}D\gamma=0\,, \quad D\varepsilon_{12}=0\,, \end{array}\right\} \tag{8.19}$$

and

$$\left.\begin{array}{l} D\varepsilon_{23}=\dfrac{1}{2}D\gamma\,, \\ \text{the other } D\varepsilon'_{ij}s=0 \end{array}\right\} \tag{8.20}$$

respectively.

Integrating the first, the second, the forth and the sixth quation (8.19) with respect to γ from 0 to γ, under the initial condition

$$\varepsilon_{\lambda\mu}=0 \quad \text{at} \quad \gamma=0 \ (t=0)\,, \tag{8.21}$$

we obtain

$$\varepsilon_{11}=0\,, \quad \varepsilon_{22}=0\,, \quad \varepsilon_{23}=\frac{1}{2}\gamma\,, \quad \varepsilon_{12}=0\,.$$

Hence the third and the fifth equation give

$$\varepsilon_{33}=\gamma_1^2\,, \qquad \varepsilon_{31}=0$$

under the same initial condition. To sum up, the results are

$$\left.\begin{array}{l} \varepsilon_{23}=\dfrac{1}{2}\gamma\,, \qquad \varepsilon_{33}=\gamma^2\,, \\ \text{the other } \varepsilon'_{\lambda\mu}s=0\,. \end{array}\right\} \tag{8.22}$$

The components ε_{ij} for the local coordinate system can be obtained by applying the second relation (7.16) to (8.22), or by integrating (8.20), in the form

$$\varepsilon_{23}=\frac{1}{2}\gamma\,, \quad \text{the other } \varepsilon'_{ij}s=0\,. \tag{8.23}$$

It can be seen that the normal component of $\boldsymbol{E}$ with respect to the x^3 or x^3 direction is present or not according as the method of analysis is Lagrangian or of local coordinate, and further that *the expression of* $\boldsymbol{E}$ *corresponding to the so-called logarithmic strain in case of simple extension is given by* (*8.23*).

(1.3) *Simple Shear after Simple Extension*

As shown in Fig. 7, we suppose that circular tubes are first extended from $n=1$ to $n=n_0$ under the condition $\gamma=0$, and then twisted from $\gamma=0$ to $\gamma=\gamma_1$, under the condition $n=n_0$. The strain components $\varepsilon_{\lambda\mu}$ just after the tubes are extended to the state $n=n_0$ are

$$\left.\begin{array}{c} \varepsilon_{11}=\varepsilon_{22}=-\dfrac{1}{2n_0}\log n_0\,, \qquad \varepsilon_{33}=n_0^2\log n_0\,, \\ \varepsilon_{23}=\varepsilon_{31}=\varepsilon_{12}=0 \end{array}\right\} \tag{8.24}$$

according to (8.15).

Then for the second process of deformation, i.e. the process of twisting from $\gamma=0$ to $\gamma=\gamma_1$, keeping n in the constant n_0, the differential equations (8.8) are written

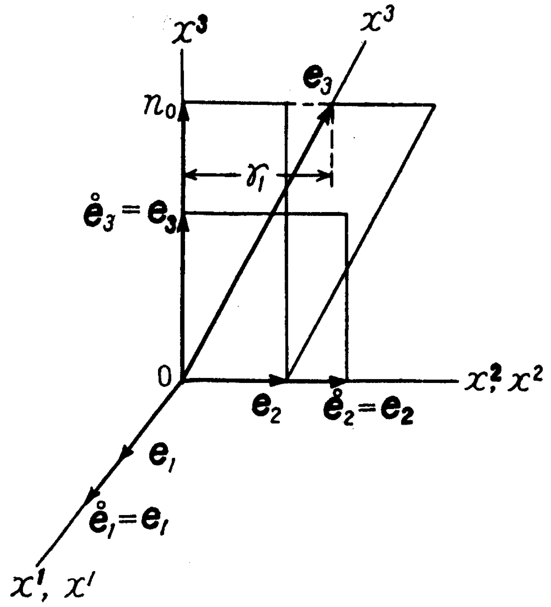

FIGURE 7.

$$\left.\begin{aligned} &D\varepsilon_{11}=0\,, \qquad D\varepsilon_{22}=0\,,\\ &D\varepsilon_{33}-2\varepsilon_{23}\sqrt{n_0}\,D\gamma=\gamma D\gamma\,,\\ &D\varepsilon_{23}-\varepsilon_{22}\sqrt{n_0}\,D\gamma=\frac{1}{2\sqrt{n_0}}D\gamma\,,\\ &D\varepsilon_{31}=D\varepsilon_{12}=0\,. \end{aligned}\right\} \tag{8.25}$$

The first and second equations (8.25) show that ε_{11} and ε_{22} are kept constant and equal to the right-hand sides of the first and second equations (8.24) respectively. Similarly, ε_{31} and ε_{12} are both kept equal to zero. Consequently, substituting ε_{22} of (8.24) into the forth equation (8.25) and then integrating from $\gamma=0$ to $\gamma=\gamma_1$ under the initial condition

$$\varepsilon_{23}=0 \quad \text{at} \quad \gamma=0\,,$$

it follows that

$$\varepsilon_{23}=\frac{1}{2\sqrt{n_0}}\,(1-\log n_0)\gamma_1\,.$$

Again substituting the ε_{23} thus obtained into the third equation (8.25), and then integrating from $\gamma=0$ to $\gamma=\gamma_1$ under the initial condition

$$\varepsilon_{33}=n_0^2\log n_0 \quad \text{at} \quad \gamma=0\,,$$

we get

$$\varepsilon_{33}=(2-\log n_0)\frac{\gamma_1^2}{2}+n_0^2\log n_0\,.$$

The results are summarized as follows:

$$\left.\begin{aligned} &\varepsilon_{11}=\varepsilon_{22}=-\frac{1}{2n_0}\log n_0\,,\\ &\varepsilon_{33}=(2-\log n_0)\frac{\gamma_1^2}{2}+n_0^2\log n_0\,, \end{aligned}\right\} \tag{8.26}$$

$$\left.\begin{aligned}\varepsilon_{23}&=\frac{1}{2\sqrt{n_0}}(1-\log n_0)\gamma_1\,,\\ \varepsilon_{31}&=\varepsilon_{12}=0\,.\end{aligned}\right\}$$

It is seen in (8.26) that ε_{23} becomes negative when $n_0>2.73\cdots$, and its absolute value increases with γ_1 and that ε_{33} is also possible to become negative when $n_0>(2.73\cdots)^2$ for large γ_1. These strange results that the shear and normal components of strain begin to decrease as the deformation increases are considered to be due to the fact that the basic tensors $\boldsymbol{e}^\lambda\boldsymbol{e}^\mu$ to which these components are referred, vary with deformation. These apparent contradiction will be seen to be solved by considering their correspondence with the stress introduced later on.

Applying (7.16) to (8.26), or integrating (8.9) along the same deformation path of twisting after extension as above, we obtain the strain components referred to the local coordinate system

$$\left.\begin{aligned}\varepsilon_{11}&=\varepsilon_{22}=-\frac{1}{2}\log n_0\,,\qquad \varepsilon_{33}=\log n_0\,,\\ \varepsilon_{23}&=\frac{\gamma_1}{2n_0}\,,\qquad \varepsilon_{31}=\varepsilon_{12}=0\,.\end{aligned}\right\}\tag{8.27}$$

As against the result (8.26) by the Lagrangian method, (8.27) has a plausible form. By comparing (8.27) with (8.23) we can see that *the shear components* ε_{23} *in the both cases are equal, when the generator of the tube before deformation makes equal angle with the axis of the tube after deformation, but not when the value of* γ *is equal.*

(1.4) *Simple Extension after Simple Shear*

We suppose, as shown in Fig. 8, that after being twisted from $\gamma=0$ to $\gamma=\gamma_0$ under the condition $n=1$, the tube is extended in the axial direction from $n=1$ to $n=n_1$. The strain components $\varepsilon_{\lambda\mu}$ just after the twisting are

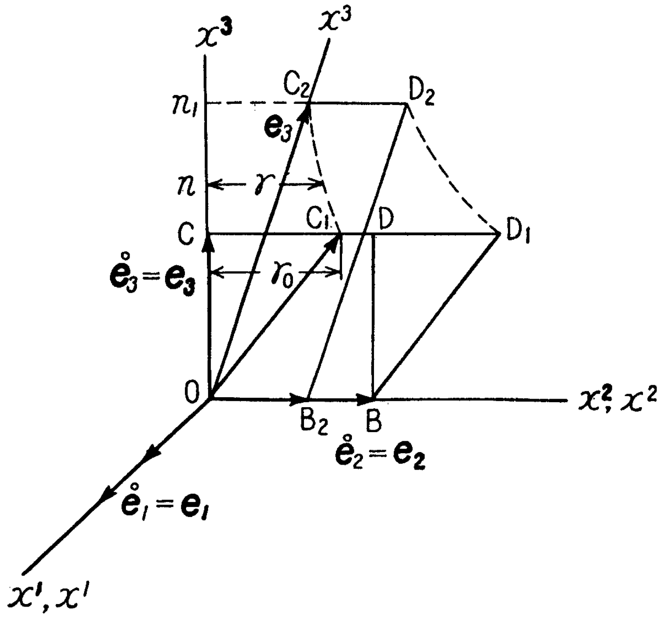

FIGURE 8.

$$\varepsilon_{23}=\frac{1}{2}\gamma_0, \quad \varepsilon_{33}=\gamma_0^2, \quad \text{the other } \varepsilon'_{\lambda\mu}s=0 \tag{8.28}$$

according to (8.22).

In the stage of simple extension subsequent to the simple shear, the material points situated at C_1 and D_1 in Fig. 8 just after the simple shear are seen to trace the paths C_1C_2 and D_1D_2 respectively such that

$$\frac{1}{\sqrt{n}}=\frac{\gamma}{\gamma_0}, \quad 1\leqq n\leqq n_0. \tag{8.29}$$

Accordingly, if we denote by n_1 and γ_1 the values which n and γ assume in the final state, the relation

$$\gamma_1 = \frac{\gamma_0}{\sqrt{n_1}} \tag{8.30}$$

holds between them. The differential equations with respect to $\varepsilon_{\lambda\mu}$ for this process of deformation of simple extension are obtained from (8.8) and (8.9) as

$$\left.\begin{aligned}
&D\varepsilon_{11}+\varepsilon_{11}\frac{Dn}{n}=-\frac{Dn}{2n^2},\\
&D\varepsilon_{22}+\varepsilon_{22}\left(\frac{1}{n}+\frac{\gamma_0^2}{n^4}\right)Dn-\varepsilon_{23}\frac{\gamma_0}{n^4}Dn=-\frac{1}{2}\frac{Dn}{n^2},\\
&D\varepsilon_{33}-2\varepsilon_{33}\left(\frac{1}{n}+\frac{\gamma_0^2}{2n^4}\right)Dn+2\varepsilon_{23}\left(2\frac{\gamma_0}{n}+\frac{\gamma_0^3}{2n^4}\right)Dn=\left(n-\frac{\gamma_0^2}{2n^2}\right)Dn,\\
&D\varepsilon_{23}-\varepsilon_{23}\frac{Dn}{2n}-\varepsilon_{33}\frac{\gamma_0}{2n^4}Dn+\varepsilon_{22}\left(2\frac{\gamma_0}{n}+\frac{\gamma_0^3}{2n^4}\right)Dn=-\frac{1}{2}\frac{\gamma_0}{n^2}Dn,\\
&D\varepsilon_{31}-\varepsilon_{31}\left(\frac{1}{n}+\frac{\gamma_0^2}{2n^4}\right)Dn+\varepsilon_{21}\frac{\gamma_0}{n}Dn=0,\\
&D\varepsilon_{12}+\varepsilon_{12}\left(\frac{1}{n}+\frac{\gamma_0^2}{2n^4}\right)Dn-\varepsilon_{31}\frac{\gamma_0}{2n^4}Dn=0.
\end{aligned}\right\} \tag{8.31}$$

But these equations can not be easily integrated.

On the other hand, if we have recourse to the method of local coordinate, we can perform the integration with no difficulty. The strain components ε_{ij} just after the simple shear (torsion) from $\gamma=0$ to $\gamma=\gamma_0$ under the condition $n=1$ are

$$\varepsilon_{23}=\frac{1}{2}\gamma_0, \quad \text{the other } \varepsilon'_{ij}s=0 \tag{8.32}$$

according to (8.23). Then for the stage of simple extension, we may integrate (8.9) under the condition (8.29) and have the result

$$\left.\begin{aligned}
&\varepsilon_{11}=\varepsilon_{22}=-\frac{1}{2}\log n_1, && \varepsilon_{33}=\log n_1,\\
&\varepsilon_{23}=\frac{1}{2}\gamma_0=\frac{1}{2}\sqrt{n_1}\,\gamma_1, && \varepsilon_{31}=\varepsilon_{12}=0.
\end{aligned}\right\} \tag{8.33}$$

Comparing (8.33) with (8.23), one can see that *the shear component ε_{23} is not influenced by the extension after the simple shear,* and further by comparing it with

(8.27) that *the component* ε_{23} *is* $\dfrac{\gamma_1}{2n_1}$ *or* $\dfrac{1}{2}\sqrt{n_1}\gamma_1$, *according as the same final state* (n_1, γ_1) *is reached by the simple shear after the simple extension or the inverse process.*

The components $\varepsilon_{\lambda\mu}$ referred to the Lagrangian coordinate system are obtained, by applying (7.16) to (8.33), in the form

$$\left.\begin{aligned}
\varepsilon_{11}&=\varepsilon_{22}=-\frac{1}{2n_1}\log n_1\\
\varepsilon_{33}&=n_1^2\log n_1-\frac{\gamma_0^2}{2n_1}\log n_1+\sqrt{n_1}\,\gamma_0^2\\
&=n_1^2\log n_1-\frac{1}{2}\gamma_1^2\log n_1+n_1^{3/2}\gamma_1^2\,,\\
\varepsilon_{23}&=-\frac{\gamma_0}{2n_1}\log n_1+\frac{1}{2}\gamma_0\sqrt{n_1}\\
&=-\frac{\gamma_1}{2\sqrt{n_1}}\log n_1+\frac{1}{2}\gamma_1 n_1\,,\\
\varepsilon_{31}&=\varepsilon_{12}=0\,.
\end{aligned}\right\} \tag{8.34}$$

This is the result which would be obtained directly from (8.31), when integrated in any way. It is the same for this case too that some of the strain components decrease in a certain case with the increase of deformation; but this is not a contradiction, being due to the change in the basic tensors.

(1.5) *Coincident Extensin and Torsion in a Constant Ratio*

We will next consider, as an example, the case in which extension and torsion take place coincidently in the constant ratio k such that

$$\gamma=k(n-1)\,,\qquad k=\text{const.} \tag{8.35}$$

The differential equation (8.8) in this case are also so complicated that they are difficult to be integrated.

Having recourse to the method of local coordinate, the differential equations (8.9) are easily integrated, under the condition (8.35), from $n=1$ to n, to give the result

$$\left.\begin{aligned}
&\varepsilon_{11}=\varepsilon_{22}=-\frac{1}{2}\log n\,,\qquad \varepsilon_{33}=\log n\,,\\
&\varepsilon_{23}=\frac{3}{4}k\log n+\frac{k}{4}\left(\frac{1}{n}-1\right),\\
&\varepsilon_{31}=\varepsilon_{12}=0\,,
\end{aligned}\right\} \tag{8.36}$$

The components $\varepsilon_{\lambda\mu}$ are obtained from (8.36) in quite the same way as in the preceding article and therefore may not be put down.

(2) *Tri-axial Extension*

We suppose, as shown in Fig. 9, that a body is extended (or contracted) in three mutually perpendicular directions fixed to it by l, m and n times. Where

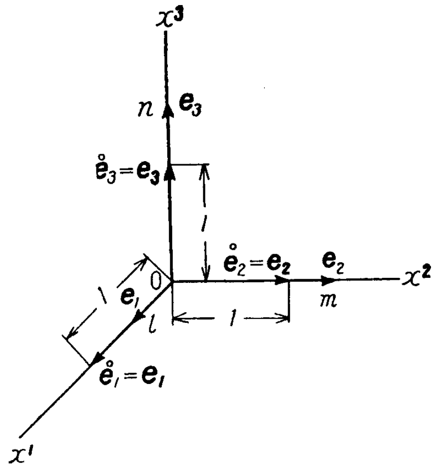

FIGURE 9.

$$lmn = 1\,, \tag{8.37}$$

because plastic deformation is incompressible. If we take these directions coincident with x^1, x^2 and x^3 axes of the rectangular Cartesian local coordinate system, (6.20) is read as

$$\boldsymbol{e}_1=l\boldsymbol{e}_1\,,\quad \boldsymbol{e}_2=m\boldsymbol{e}_2\,,\quad \boldsymbol{e}_3=n\boldsymbol{e}_3\,, \tag{8.38}$$

hence (6.23) as

$$\boldsymbol{e}_1=\frac{1}{l}\boldsymbol{e}^1\,,\quad \boldsymbol{e}^2=\frac{1}{m}\boldsymbol{e}^2\,,\quad \boldsymbol{e}^3=\frac{1}{n}\boldsymbol{e}^3\,. \tag{8.39}$$

Accordingly,

$$g_{11}=l^2\,,\quad g_{22}=m^2\,,\quad g_{32}=n^2\,,\quad g_{23}=g_{31}=g_{12}=0\,, \tag{8.40}$$

$$g^{11}=\frac{1}{l^2}\,,\quad g^{22}=\frac{1}{m^2}\,,\quad g^{33}=\frac{1}{n^2}\,,\quad g^{23}=g^{31}=g^{12}=0\,, \tag{8.41}$$

$$\left.\begin{aligned}(D\varepsilon)_{11}=lDl\,,\quad (D\varepsilon)_{22}=mDm\,,\quad (D\varepsilon)_{33}=nDn\,,\\ (D\varepsilon)_{23}=(D\varepsilon)_{31}=(D\varepsilon)_{12}=0\,,\end{aligned}\right\} \tag{8.42}$$

so that by applying the first equation (6.28) to (8.42),

$$\left.\begin{aligned}(D\varepsilon)_{11}=\frac{Dl}{l}\,,\quad (D\varepsilon)_{22}=\frac{Dm}{m}\,,\quad (D\varepsilon)_{33}=\frac{Dn}{n}\,,\\ (D\varepsilon)_{23}=(D\varepsilon)_{31}=(D\varepsilon)_{12}=0\,.\end{aligned}\right\} \tag{8.43}$$

From (8.41), (8.42) and the relations

$$(D\varepsilon)_{11}=\nabla_1(Du)_1\,,\quad (D\varepsilon)_{22}=\nabla_2(Du)_2\,,\quad (D\varepsilon)_{33}=\nabla_3(Du)_3\,,$$
$$(D\varepsilon)_{23}=(D\varepsilon)_{31}=(D\varepsilon)_{12}=0\,,$$

which hold in this case, the differential equations (7.9) are obtained in the form

$$\left.\begin{aligned}&D\varepsilon_{11}-2\varepsilon_{11}\frac{Dl}{l}=lDl\,, && D\varepsilon_{23}-\varepsilon_{23}\Big(\frac{Dm}{m}+\frac{Dn}{n}\Big)=0\,,\\ &D\varepsilon_{22}-2\varepsilon_{22}\frac{Dm}{m}=mDm\,, && D\varepsilon_{31}-\varepsilon_{31}\Big(\frac{Dn}{n}+\frac{Dl}{l}\Big)=0\,,\\ &D\varepsilon_{33}-2\varepsilon_{33}\frac{Dn}{n}=nDn\,, && D\varepsilon_{12}-\varepsilon_{13}\Big(\frac{Dl}{l}+\frac{Dm}{m}\Big)=0\,.\end{aligned}\right\} \tag{8.44}$$

On the other hand, the differential equations (7.15) referred to the local coordinate system are written by (8.43) as

$$\left.\begin{array}{c} D\varepsilon_{11}=\dfrac{Dl}{l}, \quad D\varepsilon_{22}=\dfrac{Dm}{m}, \quad D\varepsilon_{33}=\dfrac{Dn}{n}, \\ D\varepsilon_{23}=D\varepsilon_{31}=D\varepsilon_{12}=0 . \end{array}\right\} \tag{8.45}$$

Integrating (8.44) for the process of extension in each of the x^1, x^2 and x^3 directions, we have

$$\left.\begin{array}{c} \varepsilon_{11}=l^2 \log l, \quad \varepsilon_{22}=m^2\log m, \quad \varepsilon_{33}=n^2 \log n, \\ \varepsilon_{23}=\varepsilon_{31}=\varepsilon_{12}=0 . \end{array}\right\} \tag{8.46}$$

For the same process of deformation, (8.45) gives

$$\left.\begin{array}{c} \varepsilon_{11}=\log l, \quad \varepsilon_{22}=\log m, \quad \varepsilon_{33}=\log n, \\ \varepsilon_{23}=\varepsilon_{31}=\varepsilon_{12}=0 . \end{array}\right\} \tag{8.47}$$

Here, it is needless to say that (8.37) holds between l, m and n, and also that (8.46) and (8.47) are derivable from each other by (7.16).

The results (8.46) and (8.47) do not depend on the order of integration with respect to l, m and n. This means that *the plastic strain for the case of extensions in three perpendicular directions is the same, being independent of the order of the extensions, if the geometrical configuration of the final state is the same.* This is an important point in which the present case is different from that of combined extension-torsion.

(2.1) *Pure Shear (As the Resultant of Extension and Contraction in Two Orthogonal Directions)*

As shown in Fig. 10, we consider the pure shear in which the body is extended by n times in the x^3 direction and contracted by $1/n$ times in the x^2 direction,

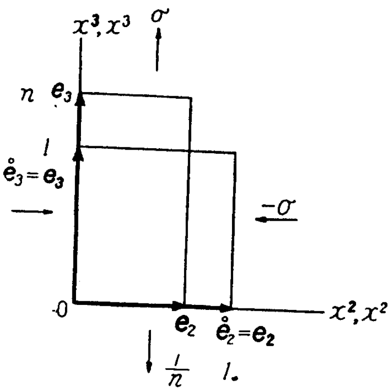

FIGURE 10.

being neither extended nor contracted in the x^1 direction. Since we can put, in this case, for l, m and n in the preceding section as

$$l=1, \quad m=\frac{1}{n}, \tag{8.48}$$

(8.46) and (8.47) are written

$$\left.\begin{array}{c}\varepsilon_{22}=-\dfrac{1}{n^2}\log n\,, \qquad \varepsilon_{33}=n^2\log n\,,\\ \text{the other}\quad \varepsilon'_{\lambda\mu}s=0\,,\end{array}\right\}\tag{8.49}$$

and

$$\left.\begin{array}{c}\varepsilon_{22}=-\log n\,, \qquad \varepsilon_{33}=\log n\,,\\ \text{the other}\quad \varepsilon'_{\gamma}s=0\,,\end{array}\right\}\tag{8.50}$$

respectively.

(3) *Pure Shear* (*As the Resultant of Simple Shears in Two Orthogonal Directions*)

As against the case of Article (2.1) in which the coordinate axes were taken coincident with the principal directions of the strain due to pure shear, here we will consider, as shown in Fig. 11, the case where the coordinate axes are rotated

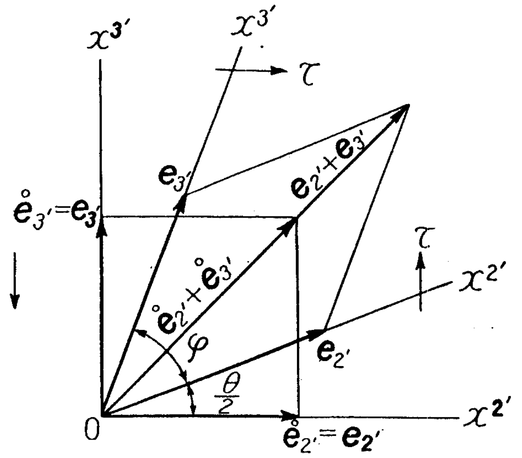

FIGURE 11.

by 45° from the principal directions, i.e. where they are taken parallel to the directions of the shear stresses. Since our plastic strain is a tensor, its components in the present case would be obtained from those in the preceding Article by some transformation formula of tensor components, but now we know nothing about it for such finite deformation as is treated in the present paper. For this reason, and also for the purpose of giving the present theory certification from various angles, we will begin with the direct calculation of the strain components referred to the rotated axes.

As shown in Fig. 11, we represent by $x^{2'}$ and $x^{3'}$ the axes of the rectangular Cartesian local coordinate system rotated by 45° from those in Fig. 10, and suppose that the square having two adjacent sides $\boldsymbol{e}_{2'}$ and $\boldsymbol{e}_{3'}$ is converted to the parallelogram with the sides $\boldsymbol{e}_{2'}$ and $\boldsymbol{e}_{3'}$. Then

$$\left.\begin{array}{l}\boldsymbol{e}_{1'}=\boldsymbol{e}_{1'}\,,\\ \boldsymbol{e}_{2'}=\alpha\boldsymbol{e}_{2'}+\beta\boldsymbol{e}_{3'}\,,\\ \boldsymbol{e}_{3'}=\beta\boldsymbol{e}_{2'}+\alpha\boldsymbol{e}_{3'}\,,\end{array}\right\}\tag{8.51}$$

because the relation of $\boldsymbol{e}_{2'}$ to $\boldsymbol{e}_{2'}$ and $\boldsymbol{e}_{3'}$ is the same as that of $\boldsymbol{e}_{3'}$ to $\boldsymbol{e}_{3'}$ and $\boldsymbol{e}_{2'}$, where α and β are parameters representing the extent of deformation, which will be related later on to the elongation in the principal direction. Applying (3.15) to (8.51) and taking account of the incompressibility condition $[\boldsymbol{e}_{1'}\boldsymbol{e}_{2'}\boldsymbol{e}_{3'}]=1$, we get

$$\left.\begin{aligned} \boldsymbol{e}^{1'}&=(\alpha^2-\beta^2)\boldsymbol{e}^{1'}\,, \\ \boldsymbol{e}^{2'}&=\alpha\boldsymbol{e}^{2'}-\beta\boldsymbol{e}^{3'}\,, \\ \boldsymbol{e}^{3'}&=-\beta\boldsymbol{e}^{2'}+\alpha\boldsymbol{e}^{3'}\,. \end{aligned}\right\} \tag{8.52}$$

We have from (8.51)

$$\left.\begin{aligned} &g_{1'1'}=1\,, \qquad g_{2'2'}=g_{3'3'}=\alpha^2+\beta^2\,, \\ &g_{2'3'}=2\alpha\beta\,, \qquad g_{3'1'}=g_{1'2'}=0\,, \end{aligned}\right\} \tag{8.53}$$

hence

$$\left.\begin{aligned} (D\varepsilon)_{2'2'}=(D\varepsilon)_{3'3'}=\frac{1}{2}D(\alpha^2+\beta^2)\,, \\ (D\varepsilon)_{2'3'}=D(\alpha\beta)\,, \quad (D\varepsilon)_{1'1'}=(D\varepsilon)_{3'1'}=(D\varepsilon)_{1'2'}=0\,. \end{aligned}\right\} \tag{8.54}$$

Applying to (8.54) the first equation (6.28) whose coefficients c_i^j are read as those of (8.51), we can obtain $(D\varepsilon)_{ij}$ and therefore the differential equations (7.15) in the form

$$\left.\begin{aligned} D\varepsilon_{2'2'}=D\varepsilon_{3'3'}=(\alpha^2+\beta^2)\frac{1}{2}D(\alpha^2+\beta^2)-2\alpha\beta D(\alpha\beta)\,, \\ D\varepsilon_{2'3'}=-\alpha\beta D(\alpha^2+\beta^2)+(\alpha^2+\beta^2)D(\alpha\beta)\,, \\ D\varepsilon_{1'1'}=D\varepsilon_{3'1'}=D\varepsilon_{1'2'}=0\,. \end{aligned}\right\} \tag{8.55}$$

Since the parallelogram in Fig. 11 is brought about by the pure shear from the initial square, their areas are equal to each other, and the relation

$$\boldsymbol{e}^{1'}=\boldsymbol{e}_{2'}\times\boldsymbol{e}_{3'}=\boldsymbol{e}_{2'}\times\boldsymbol{e}_{3'}=\boldsymbol{e}^{1}$$

holds. From this relation and the first equation (8.52), we have

$$\alpha^2-\beta^2=1\,. \tag{8.56}$$

On the other hand, we suppose that the magnification in the principal direction, i.e. the diagonal direction of the square, due to the pure shear is n, then

$$\boldsymbol{e}_{2'}+\boldsymbol{e}_{3'}=n(\boldsymbol{e}_{2'}+\boldsymbol{e}_{3'})\,,$$

and hence substituting (8.51) into this equation, we have

$$\alpha+\beta=n\,. \tag{8.57}$$

Consequently, from (8.56) and (8.57),

$$\alpha=\frac{1}{2}\left(n+\frac{1}{n}\right), \qquad \beta=\frac{1}{2}\left(n-\frac{1}{n}\right). \tag{8.58}$$

Substituting (8.58) into (8.55), we get

$$D\varepsilon_{2'3'}=\frac{Dn}{n}, \qquad \text{the other } D\varepsilon_{i'j'}\text{s}=0\,, \tag{8.59}$$

hence by integration

$$\varepsilon_{2'3'}=\log n\,, \qquad \text{the other } \varepsilon_{i'j'}\text{s}=0\,. \tag{8.60}$$

It deserves special attention that *the transformation relation between the components (8.50) and (8.60) for the case of pure shear of finite magnitude is the same as that for small pure shear,* as shown in Fig. 12a.

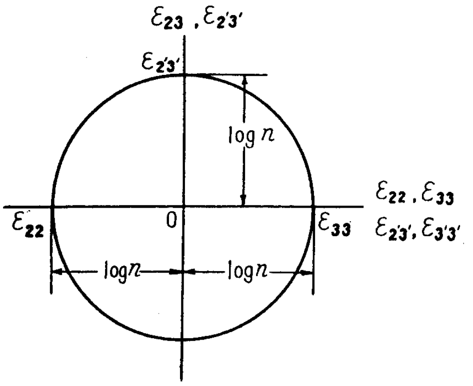

FIGURE 12a.

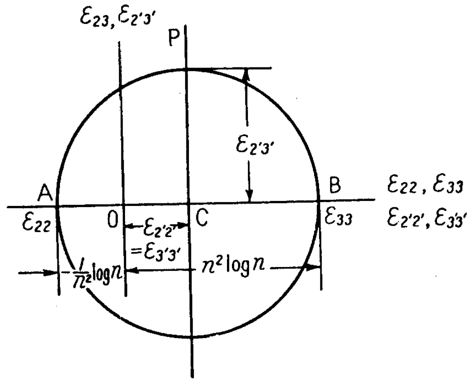

FIGURE 12b.

By applying to (8.60) the first equation (7.16) whose coefficients c_λ^i are given by those in the inverse relation of (8.52), we have the components $\varepsilon_{\lambda'\mu'}$ for the Lagrangian coordinate system

$$\left.\begin{aligned} \varepsilon_{2'2'}=\varepsilon_{3'3'}&=\frac{1}{2}\left(n^2-\frac{1}{n^2}\right)\log n\,, \\ \varepsilon_{2'3'}&=\frac{1}{2}\left(n^2+\frac{1}{n^2}\right)\log n\,, \\ &\text{the other } \varepsilon_{\lambda'\mu'}'s=0\,. \end{aligned}\right\} \tag{8.61}$$

It is seen by comparing (8.61) with (8.49) that the relation

$$\left.\begin{aligned} \varepsilon_{2'2'}=\varepsilon_{3'3'}&=\frac{1}{2}(\varepsilon_{22}+\varepsilon_{33})\,, \\ \varepsilon_{2'3'}&=\frac{1}{2}(\varepsilon_{33}-\varepsilon_{22})\,, \end{aligned}\right\} \tag{8.62}$$

hold between the components before and after the coordinate axes were rotated by 45°. This is regarded as illustrating that, *in case of the Lagrangian method too, the same transformation formula as that for small deformation, as shown by the Mohr's representation in Fig. 12b, is also valid for large deformation.* This fact illustrated for the case of pure shear, concerning the transformation of the components of $\boldsymbol{E}$, will be proved later on generally.

If we denote the angle between $\boldsymbol{e}_{2'}$ and $\boldsymbol{e}_{3'}$ in Fig. 11 by φ, and that between $\boldsymbol{e}_{2'}$ and $\boldsymbol{e}_{2'}$ by $\theta/2$, then the relations

$$\tan\frac{\varphi}{2}=\frac{1}{n^2}\,, \tag{8.63}$$

$$n=\sqrt[4]{\frac{1+\sin\theta}{1-\sin\theta}} \tag{8.64}$$

hold, and accordingly (8.60) is written in the form

$$\varepsilon_{2'3'}=\frac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta}, \quad \text{the other } \varepsilon_{i'j'}\text{s}=0\,. \tag{8.65}$$

It will be needless to mention that the strains for the pure shear and for the simple shear are the same for small deformation when the changes in the angle between the axes perpendicular to each other and tilted by 45° with the principal directions are the same. But we can clearly see by the comparison of (8.23) and (8.65) that these strains become different for the same change in the angle, as the deformations proceed.

9. Illustration of the Fact that $\boldsymbol{E}$ is the Strain History Tensor

In Section 7, we have already deduced, by a general reasoning, that the plastic strain $\boldsymbol{E}$ obtained as a solution of the differential equations (7.9) or (7.15) is not specified by the final geometrical configuration, but depends on the deformation path along which they are integrated, and therefore is a strain history tensor. This conclusion is one of the most import starting points as well as a conclusion which decides the essential character of our present theory, and so its validity seems to be worthy of being examined carefully by examples.

First we will consider the case of combined extension-torsion treated in the preceding section. For example, the plastic strain $\boldsymbol{E}_{E,T}$ when the body takes the configuration specified by (n, γ) by a simple shear after a simple extension has the components

$$\left.\begin{aligned} &\varepsilon_{11}=\varepsilon_{22}=-\frac{1}{2}\log n\,, \qquad \varepsilon_{33}=\log n\,,\\ &\varepsilon_{23}=\frac{\gamma}{2n}\,, \qquad \varepsilon_{31}=\varepsilon_{12}=0\,, \end{aligned}\right\} \tag{9.1}$$

according to (8.27), and the strain $\boldsymbol{E}_{T,E}$ for the same configuration (n, γ) brought about by a simple extension after a simple shear is

$$\left.\begin{aligned} &\varepsilon_{11}=\varepsilon_{22}=-\frac{1}{2}\log n\,, \qquad \varepsilon_{33}=\log n\,,\\ &\varepsilon_{23}=\frac{1}{2}\sqrt{n}\,\gamma\,, \qquad \varepsilon_{31}=\varepsilon_{12}=0\,, \end{aligned}\right\} \tag{9.2}$$

according to (8.33). It is seen that $\boldsymbol{E}_{T,E}$ and $\boldsymbol{E}_{E,T}$ are distinguished by the component ε_{23}, namely that

$$\boldsymbol{E}_{T,E}-\boldsymbol{E}_{E,T}=\frac{1}{2}\gamma\left(\sqrt{n}-\frac{1}{n}\right)(\boldsymbol{e}^2\boldsymbol{e}^3+\boldsymbol{e}^3\boldsymbol{e}^2)\,. \tag{9.3}$$

Expressed otherwise, the plastic strain of a material element, when it performs the cycle, torsion extension-reversed torsion-contraction, to assume again the initial shape, does not vanish but has the value

$$\boldsymbol{E}_{T,E,\bar{T},\bar{E}}=\frac{1}{2}\gamma\left(\sqrt{n}-\frac{1}{n}\right)(\boldsymbol{e}^2\boldsymbol{e}^3+\boldsymbol{e}^3\boldsymbol{e}^2)\,. \tag{9.4}$$

This result (9.3) or (9.4) is considered to be due to the very fact that the micro-structural state of the body as represented, in metals, by the group pattern of dislocations, differs according as it is extended after twisting or twisted after extension. In fact, it is clear that by the twisting process in the former case the slip, and therefore the micro-structual change due to it, of the amount $\frac{1}{2}\gamma\sqrt{n}$ is produced, while for that in the latter case it is $\frac{1}{2}\left(\frac{\gamma}{n}\right)$. If, on the other hand, the plastic strain, and hence the micro-structural state, were required to be equal for the both deformation paths, the final geometrical configuration must be chosen differently. Thus the conclusion generally deduced in Section 7 is now seen to hold actually for the special case of combined extension-torsion. It is worthy of notice that the difference (9.3) of the plastic strain due to the deformation path should correspond to the difference of the plastic properties such as anisotropy and the Bauschinger effect of the material, but not to the difference of the stress state according to the deformation path, which has its cause in the state equations.

Not to mention, the path-dependence of the plastic strain, i.e. (9.3) or (9.4), is of practical importance for the finite deformation, and the usual procedure to obtain the plastic strain from the difference of the geometrical configuration before and after deformation is seen to be meaningless. Also for the small deformation, it is expected to have practical importance, especially when the deformation is repreated and the remaining strain becomes finite after a number of cycles.

The above statement illustrated for the combined extension-torsion that the plastic strain for the same final geometrical configuration depends on the deformation path up to the state does not necessarily hold for any case. For example, *in case of the triaxial extension, the amount of slip, and consequently the micro-structural change, of the body is considered to be the same, no matter what the order of extension may be.* And in this case, the strain (8.47) has been found to be really independent of the order of integration with respect to l, m and n.

10. Transformation of Components of $\boldsymbol{DE}$ and $\boldsymbol{E}$

In the preceding section, we have illustrated for the case of pure shear that the transformation of the plastic strain components for the finite deformation, when the coordinate axes are rotated by 45°, is the same to that for the small one. In the present section, we will derive the transformation formulae of the components of $\boldsymbol{DE}$ and $\boldsymbol{E}$ for general coordinate transformations.

We consider two Lagrangian coordinate systems, and suppose that between the coordinates x^λ and $x^{\lambda'}$ of a material point referred to the respective systems there exist the single-valued and mutually independent relations

$$x^{\lambda'}=x^{\lambda'}(x^1, x^2, x^3), \qquad \lambda'=1', 2', 3', \tag{10.1}$$

which can be solved to give

$$x^{\lambda}=x^{\lambda}(x^{1'}, x^{2'}, x^{3'}), \qquad \lambda=1, 2, 3\,. \tag{10.1'}$$

Since the coordinates x^λ and $x^{\lambda'}$ are both Lagrangian, the relations (10.1) and (10.1′) hold for arbitrary t, when they hold for a particular t, say $t=0$.

As the reference frames for the coordinate systems x^λ and $x^{\lambda'}$ are given by

$$\boldsymbol{e}_\lambda=\partial_\lambda \boldsymbol{r} \quad \text{and} \quad \boldsymbol{e}_{\lambda'}=\partial_{\lambda'}\boldsymbol{r}\,, \tag{10.2}$$

respectively, where

$$\partial_{\lambda'}\equiv\frac{\partial}{\partial x^{\lambda'}}, \tag{10.3}$$

we have

$$\boldsymbol{e}_\lambda=A_\lambda^{\mu'}\boldsymbol{e}_{\mu'}\,, \qquad \boldsymbol{e}_{\lambda'}=A_{\lambda'}^{\mu}\boldsymbol{e}_\mu\,, \tag{10.4}$$

and

$$\boldsymbol{e}^\lambda=A_{\mu'}^{\lambda}\boldsymbol{e}^{\mu'}\,, \qquad \boldsymbol{e}^{\lambda'}=A_{\mu}^{\lambda'}\boldsymbol{e}^\mu\,, \tag{10.5}$$

putting

$$A_\lambda^{\mu'}\equiv\partial_\lambda x^{\mu'}\,, \qquad A_{\lambda'}^{\mu}\equiv\partial_{\lambda'}x^\mu\,. \tag{10.6}$$

It is evident from (10.6) that the relations

$$A_\lambda^{\rho'}A_{\rho'}^{\mu}=\delta_\lambda^\mu\,, \quad A_{\lambda'}^{\rho}A_\rho^{\mu'}=\delta_{\lambda'}^{\mu'} \tag{10.7}$$

hold between $A_\lambda^{\mu'}$ and $A_{\lambda'}^{\mu}$.

By virtue of (10.5), we have the transformation formulae between the strain components

$$\left.\begin{aligned}
\varepsilon_{\lambda'\mu'}&=A_{\lambda'}^{\lambda}A_{\mu'}^{\mu}\varepsilon_{\lambda\mu}\,, & \varepsilon_{\lambda\mu}&=A_\lambda^{\lambda'}A_\mu^{\mu'}\varepsilon_{\lambda'\mu'}\,,\\
\varepsilon^{\lambda'\mu'}&=A_\lambda^{\lambda'}A_\mu^{\mu'}\varepsilon^{\lambda\mu}\,, & \varepsilon^{\lambda\mu}&=A_{\lambda'}^{\lambda}A_{\mu'}^{\mu}\varepsilon^{\lambda'\mu'}\,,\\
\varepsilon_{\mu'}^{\lambda'}&=A_\lambda^{\lambda'}A_{\mu'}^{\mu}\varepsilon_\mu^\lambda\,, & \varepsilon_\mu^\lambda&=A_{\lambda'}^{\lambda}A_\mu^{\mu'}\varepsilon_{\mu'}^{\lambda'}\,,
\end{aligned}\right\} \tag{10.8}$$

and the same formulae between the strain increment components $(D\varepsilon)_{\lambda\mu}$, $(D\varepsilon)^{\lambda\mu},\cdots$ and $(D\varepsilon)_{\lambda'\mu'}$, $(D\varepsilon)^{\lambda'\mu'},\cdots$.

The above case is for the Lagrangian method. If we introduce local coordinate systems x^i and $x^{i'}$ which constitute, for $t=0$, certain curvilinear coordinate systems x^λ and $x^{\lambda'}$ respectively, which may be regarded as Lagrangian systems for $t=0$, then the reference frames for the local systems

$$\boldsymbol{e}_i=\partial_i\boldsymbol{r} \quad \text{and} \quad \boldsymbol{e}_{i'}=\partial_{i'}\boldsymbol{r}\,, \tag{10.9}$$

where

$$\partial_{i'}\equiv\frac{\partial}{\partial x^{i'}} \tag{10.10}$$

are equal to those $\mathring{\boldsymbol{e}}_\lambda$ and $\mathring{\boldsymbol{e}}_{\lambda'}$ for the Lagrangian systems at $t=0$. For the Lagrangian system, the relation between $\mathring{\boldsymbol{e}}_\lambda$ and $\mathring{\boldsymbol{e}}_{\lambda'}$ is the same to that between $\boldsymbol{e}_\lambda$ and $\boldsymbol{e}_{\lambda'}$, because the transformation equation (10.1) holds independently of t. Hence the transformation formulae between $\boldsymbol{e}_i$ and $\boldsymbol{e}_{i'}$, and $\boldsymbol{e}^i$ and $\boldsymbol{e}^{i'}$, being identical to (10.4) and (10.5) respectively, are given by

$$\boldsymbol{e}_i=A_i^{\kappa'}\boldsymbol{e}_{k'}\,, \qquad \boldsymbol{e}_{i'}=A_{\iota'}^{\kappa}\boldsymbol{e}_k \tag{10.11}$$

and

$$\boldsymbol{e}^i=A_{\kappa'}^{\iota}\boldsymbol{e}^{k'}\,, \qquad \boldsymbol{e}^{i'}=A_\kappa^{\iota'}\boldsymbol{e}^k\,. \tag{10.12}$$

Thus we obtain the same transformation formulae as (10.8) for ε_{ik}, $\varepsilon^{ik},\cdots$, viz.

$$\left.\begin{aligned} &\varepsilon_{i'k'}=A^{\iota}_{\iota'}A^{\kappa}_{\kappa'}\varepsilon_{ik}\,, && \varepsilon_{ik}=A^{\iota'}_{\iota}A^{\kappa'}_{\kappa}\varepsilon_{i'k'}\,,\\ &\varepsilon^{i'k'}=A^{\iota'}_{\iota}A^{\kappa'}_{\kappa}\varepsilon^{ik}\,, && \varepsilon^{ik}=A^{\iota}_{\iota'}A^{\kappa}_{\kappa'}\varepsilon^{i'k'}\,,\\ &\varepsilon^{i'}_{k'}=A^{\iota'}_{\iota}A^{\kappa}_{\kappa'}\varepsilon^{i}_{k}\,, && \varepsilon^{i}_{k}=A^{\iota}_{\iota'}A^{\kappa'}_{\kappa}\varepsilon^{i'}_{k'}\,, \end{aligned}\right\} \tag{10.13}$$

where ι, ι', κ and κ' have values corresponding to $\boldsymbol{i}, \boldsymbol{i}', \boldsymbol{k}$ and $\boldsymbol{k}'$ respectively. The formulae for $D\varepsilon_{ik}, \cdots$ are the same and therefore they need not be written down.

It is easily found that the coefficients $A^{\mu}_{\lambda'}$ and $A^{\mu'}_{\lambda}$ are also given by

$$\left.\begin{aligned} &A^{\iota'}_{\iota}=\boldsymbol{e}_{\iota}\cdot\boldsymbol{e}^{\iota'}=\boldsymbol{e}_{i}\cdot\boldsymbol{e}^{k'}\,,\\ &A^{\iota}_{\iota'}=\boldsymbol{e}_{\iota'}\cdot\boldsymbol{e}^{\iota}=\boldsymbol{e}_{i'}\cdot\boldsymbol{e}^{k}\,. \end{aligned}\right\} \tag{10.14}$$

When the local coordinate system, or the Lagrangian coordinate system for $t=0$, is rectangular Cartesian in particular, the transformation coefficients $A^{\mu}_{\lambda'}$ and $A^{\mu'}_{\lambda}$ represent the cosines of the angles between the axes of the coordinate systems dx^{i} and $dx^{i'}$. In this case the transformation formulae (10.8) and (10.13) are reduced to the usual ones obtained for small deformations in many text books.

The transformatian formulae (10.8) and (10.13) have beem derived for the deformation of arbitrary magnitudes. And it is not only noteworthy but fortunate that they have the same form as for the small deformation. Namely, as shown in Fig. 13, we

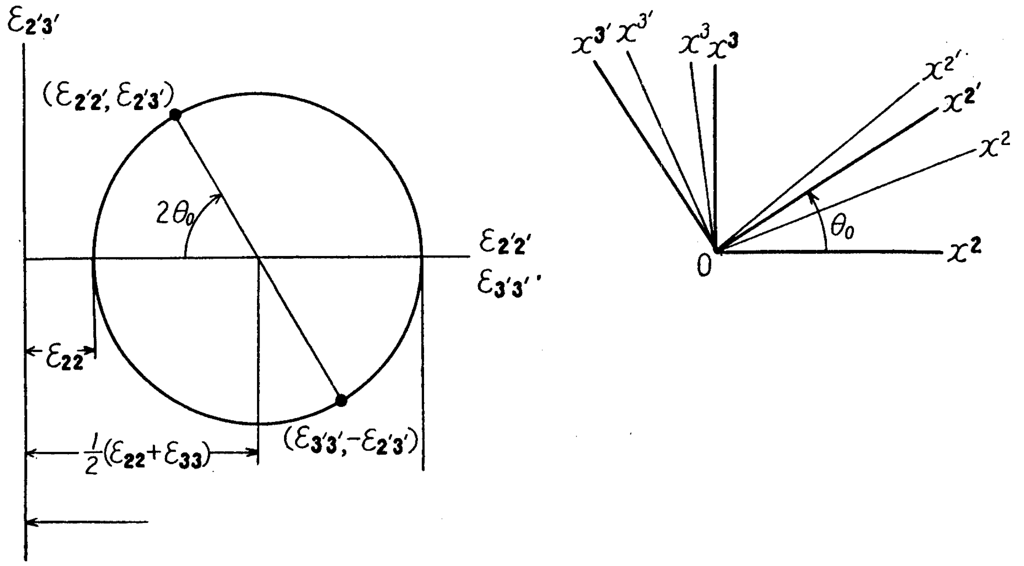

FIGURE 13.

will now consider two Lagrangian coordinate systems x^{λ} and $x^{\lambda'}$ which are rectangular Cartesian for $t=0$ and whose axes x^{1} and $x^{1'}$ are coincident with each other, and suppose a uniform plane strain in the x^{2}, x^{3} plane, or the $x^{2'}, x^{3'}$ plane, to take place. If the angle between the axes x^{2} and $x^{2'}$ is θ_0 for $t=0$, it becomes, for t, θ other than θ_0, according to the plane strain. On the other hand, when the local coordinate systems x^{i} and $x^{i'}$ (as we are dealing with uniform strain, we need not distinguish between dx^{i} and x^{i}, and $dx^{i'}$ and $x^{i'}$) which coincide at $t=0$ with the above Lagrangian coordinate systems x^{λ} and $x^{\lambda'}$ are considered, the angle between the axes x^{2} and $x^{2'}$ is maintained equal to θ_0 for all t. Accordingly $A^{\iota}_{\iota'}$ are given by Table 1.

TABLE 1. Value of $A^{\iota}_{\kappa'}=A^{\kappa'}_{\iota}$

κ $\boldsymbol{k}$ \ ι $\boldsymbol{i}$	1 $\boldsymbol{1}$	2 $\boldsymbol{2}$	3 $\boldsymbol{3}$
$1'$ $\boldsymbol{1'}$	1	0	0
$2'$ $\boldsymbol{2'}$	0	$\cos\theta_0$	$\sin\theta_0$
$3'$ $\boldsymbol{3'}$	0	$-\sin\theta_0$	$\cos\theta_0$

Hence (10.8) and (10.13) read

$$\left.\begin{aligned}
\varepsilon_{2'2'}&=\cos^2\theta_0\varepsilon_{22}+\sin^2\theta_0\varepsilon_{33}+\sin 2\theta_0\varepsilon_{23}\,,\\
\varepsilon_{3'3'}&=\sin^2\theta_0\varepsilon_{22}+\cos^2\theta_0\varepsilon_{33}-\sin 2\theta_0\varepsilon_{23}\,,\\
\varepsilon_{2'3'}&=-\frac{1}{2}\sin 2\theta_0(\varepsilon_{22}-\varepsilon_{33})-\cos 2\theta_0\varepsilon_{23}\,,
\end{aligned}\right\}\tag{10.15}$$

and

$$\left.\begin{aligned}
\varepsilon_{\boldsymbol{2'2'}}&=\cos^2\theta_0\varepsilon_{\boldsymbol{22}}+\sin^2\theta_0\varepsilon_{\boldsymbol{33}}+\sin 2\theta_0\varepsilon_{\boldsymbol{23}}\,,\\
\varepsilon_{\boldsymbol{3'3'}}&=\sin^2\theta_0\varepsilon_{\boldsymbol{22}}+\cos^2\theta_0\varepsilon_{\boldsymbol{33}}-\sin 2\theta_0\varepsilon_{\boldsymbol{23}}\,,\\
\varepsilon_{\boldsymbol{2'3'}}&=-\frac{1}{2}\sin 2\theta_0(\varepsilon_{\boldsymbol{22}}-\varepsilon_{\boldsymbol{33}})-\cos 2\theta_0\varepsilon_{\boldsymbol{23}}
\end{aligned}\right\}\tag{10.16}$$

respectively. These are seen to have the same form as the transformation formula for the case of small plane strain. *What must be remarked here is that θ_0 is the angle between the axes x^2 and $x^{2'}$ for the undeformed state $t=0$, i.e. between the axes $\boldsymbol{x}^{\boldsymbol{2}}$ and $\boldsymbol{x}^{\boldsymbol{2'}}$.* In particular, when the axes x^2 and $\boldsymbol{x}^{\boldsymbol{3}}$ are taken coincident with the principal directions of the strain, we can put $\varepsilon_{23}=0$ in (10.15) and $\varepsilon_{\boldsymbol{23}}=0$ in (10.16), and therefore have

$$\left.\begin{aligned}
\varepsilon_{2'2'}&=\frac{1}{2}(\varepsilon_{22}+\varepsilon_{33})+\frac{1}{2}(\varepsilon_{22}-\varepsilon_{33})\cos 2\theta_0\,,\\
\varepsilon_{3'3'}&=\frac{1}{2}(\varepsilon_{22}+\varepsilon_{33})-\frac{1}{2}(\varepsilon_{22}-\varepsilon_{33})\cos 2\theta_0\,,\\
\varepsilon_{2'3'}&=-\frac{1}{2}(\varepsilon_{22}-\varepsilon_{33})\sin 2\theta_0\,.
\end{aligned}\right\}\tag{10.17}$$

and

$$\left.\begin{aligned}
\varepsilon_{\boldsymbol{2'2'}}&=\frac{1}{2}(\varepsilon_{\boldsymbol{22}}+\varepsilon_{\boldsymbol{33}})+\frac{1}{2}(\varepsilon_{\boldsymbol{22}}-\varepsilon_{\boldsymbol{33}})\cos 2\theta_0\,,\\
\varepsilon_{\boldsymbol{3'3'}}&=\frac{1}{2}(\varepsilon_{\boldsymbol{22}}+\varepsilon_{\boldsymbol{33}})+\frac{1}{2}(\varepsilon_{\boldsymbol{22}}-\varepsilon_{\boldsymbol{33}})\cos 2\theta_0\,,\\
\varepsilon_{\boldsymbol{2'3'}}&=-\frac{1}{2}(\varepsilon_{\boldsymbol{22}}-\varepsilon_{\boldsymbol{33}})\sin 2\theta_0\,.
\end{aligned}\right\}\tag{10.18}$$

Thus the relations between ε_{22}, ε_{23} and $\varepsilon_{2'2'}$, $\varepsilon_{3'3'}$, $\varepsilon_{2'3'}$, and between $\varepsilon_{\boldsymbol{22}}$, $\varepsilon_{\boldsymbol{33}}$ and $\varepsilon_{\boldsymbol{2'2'}}$, $\varepsilon_{\boldsymbol{3'3'}}$, $\varepsilon_{\boldsymbol{2'3'}}$ are seen to be represented by the Mohr's circle of Fig. 13 just as in the case of small deformation.

When the plane strain is pure shear in particular, and its principal axes are taken to be the $\boldsymbol{x}^{\boldsymbol{2}}$ and $\boldsymbol{x}^{\boldsymbol{3}}$ axes, $\varepsilon_{\boldsymbol{ij}}$'s are given by (8.50), and accordingly (10.18) is written as

$$\left.\begin{aligned}\varepsilon_{2'2'}&=-\cos 2\theta_0 \log n\,,\\ \varepsilon_{3'3'}&=\cos 2\theta_0 \log n\,,\\ \varepsilon_{2'3'}&=\sin 2\theta_0 \log n\,.\end{aligned}\right\} \tag{10.19}$$

Further, when θ_0 is taken equal to $\pi/4$ as shown in Fig. 6, (10.19) is reduced to (8.60), i.e.

$$\varepsilon_{2'3'}=\log n\,,\qquad \varepsilon_{2'2'}=\varepsilon_{3'3'}=0\,. \tag{10.20}$$

This means that (8.60) is also obtained by the coordinate transformation from (8.50).

We will next consider simple shear, as an example of the plane strain. If the x^i axes are taken as in Fig. 6, the strain components ε_{ij} are given by (8.23), and hence (10.18) becomes

$$\left.\begin{aligned}\varepsilon_{2'2'}&=\frac{1}{2}\gamma\sin 2\theta_0\,,\\ \varepsilon_{3'3'}&=-\frac{1}{2}\gamma\sin 2\theta_0\,,\\ \varepsilon_{2'3'}&=-\frac{1}{2}\gamma\cos 2\theta_0\,,\end{aligned}\right\} \tag{10.21}$$

For $\theta_0=\pi/4$ particularly, (10.21) is written

$$\varepsilon_{2'2'}=\frac{1}{2}\gamma\,,\quad \varepsilon_{3'3'}=-\frac{1}{2}\gamma\,,\quad \varepsilon_{2'3'}=0\,. \tag{10.22}$$

This means that *one of the principal directions of simple shear lies in the direction inclined by 45° with the slip direction (x^2 direction), no matter how large the deformation may be.* This is a matter of much importance. As stated in the introduction, the recognition commonly accepted that the principal directions of simple shear are parallel to the principal axes of the ellipse resulting from a circle before deformation by the simple shear, or to the corresponding axes of the circle, is wrong for the plastic deformation at least. It can easily be found that the principal direction of DE for the simple shear also makes 45° with the slip direction (x^2 direction). That is, *in the plastic simple shear, by the strain increment which has a principal direction of 45° with the slip direction, the strain with the same principal direction is produced. And without this idea of strain indeed we can not explain the very fact for an isotropic thin tube that the plastic pure torsion is caused by a pure torque, without any axial elongation or contraction of the tube.* This is the reason which enforced us to our present theory of plasticity.

To avoid confusion, it should be noticed here that the elastic simple shear, contrary to the plastic one, is specified by the ellipse which was a circle before the deformation and its principal axes are coincident with those of the ellipse [3].

11. Definitions and Expressions of Stress and Stress Deviator

In the preceding sections, we have introduced the plastic strain increment $\boldsymbol{DE}$ and the plastic strain $\boldsymbol{E}$ reasonable for describing plastic deformation viewed from its physical picture. Our next problem is to define stress, and give its expression, which, together with $\boldsymbol{DE}$ and $\boldsymbol{E}$, can describe plastic deformation with logical and physical legitimacy.

The first condition for the stress to satisfy for this purpose is that it is a tensor, as was dilivered in Section 2. This condition is not sufficient to define stress uniquely, for there can exist not a few tensors which correspond in some manner to the state of forces at a point. Mathematically, a stress tensor is a quantity which transforms the vector representing a sectional area in the material to the force vector acting on the area. As the sectional area one may choose that before deformation $\mathring{\boldsymbol{n}}d\mathring{S}$ or that after deformation $\boldsymbol{n}dS$ ($\mathring{\boldsymbol{n}}$ and $\boldsymbol{n}$ are unit normals on $d\mathring{S}$ and dS respectively), and as the force vector $\boldsymbol{f}dS$ or $\boldsymbol{f}d\mathring{S}$. The stress tensor depends on the way of combining these vectors.

Thus we need further condition for the stress to satisfy, in order to reach its unique definition. And as such condition serves the virtual work principle (2.1). What is suggestive in selecting the stress tensor, in the light of the condition (2.1), out of the various transformation tensors mentioned above, is the conception that the stress tensor should also be referred to the deformed state in the plastic deformation, as it was for the plastic strain tensor. Thus we are tentatively led to the definition of the stress tensor

$$\boldsymbol{f}dS = (\boldsymbol{n}dS)\cdot\boldsymbol{T}$$

or

$$\boldsymbol{f} = \boldsymbol{n}\cdot\boldsymbol{T} \tag{11.1}$$

where, as touched above, dS represents the elementary area at a point considered in the material, $\boldsymbol{n}$ the unit normal on dS and $\boldsymbol{f}dS$ the force vector which the part on the side of dS to which $\boldsymbol{n}$ is directed exerts on the other side of dS.

Leaving the verification to a later occasion that the tensor $\boldsymbol{T}$ defined by (11.1) satisfies the condition (2.1), we will now continue our subject on the assumption that (11.1) is reasonable. If in (11.1) three independent normals $\boldsymbol{n}_A$ (A=I, II, III) and the corresponding forces $\boldsymbol{f}_A$ (A=I, II, III) are given*, the stress tensor $\boldsymbol{T}$ is completely determined in the form

$$\boldsymbol{T} = \boldsymbol{n}^A \boldsymbol{f}_A\,. \tag{11.2}$$

This is the definition of stress in the absolute form valid independently of the coordinate system.

Having recourse to the Lagrangian method, we can write

$$\boldsymbol{n}^A = n^{A\lambda}\boldsymbol{e}_\lambda = n^A_\lambda \boldsymbol{e}^\lambda\,, \tag{11.3}$$

$$\boldsymbol{f}_A = f^\mu_A \boldsymbol{e}_\mu = f_{A\mu}\boldsymbol{e}^\mu\,. \tag{11.4}$$

---

* Here the capitalized index, say A, is used to denote the directions I, II, III independent of those referred to any of the coordinate systems ever introduced.

Substituting (11.3) and (11.4) into (11.2), and putting

$$\sigma^{\lambda\mu}=n^{A\lambda}f^{\mu}_{A}, \quad \sigma_{\lambda\mu}=n^{A}_{\lambda}f_{A\mu}, \quad \sigma^{\lambda}_{\mu}=n^{A\lambda}f_{A\mu}, \tag{11.5}$$

we obtain the expression of $\boldsymbol{T}$

$$\boldsymbol{T}=\sigma^{\lambda\mu}\boldsymbol{e}_{\lambda}\boldsymbol{e}_{\mu}=\sigma_{\lambda\mu}\boldsymbol{e}^{\lambda}\boldsymbol{e}^{\mu}=\sigma^{\lambda}_{\mu}\boldsymbol{e}_{\lambda}\boldsymbol{e}^{\mu}, \tag{11.6}$$

where $\sigma^{\lambda\mu}$, $\sigma_{\lambda\mu}$ and σ^{λ}_{μ} are transformed to one another by the metric tensor $g_{\lambda\mu}$ and $g^{\lambda\mu}$, i.e. for example

$$\sigma_{\lambda\mu}=g_{\lambda\epsilon}g_{\mu\rho}\sigma^{\epsilon\rho}, \quad \sigma^{\lambda}_{\mu}=g_{\mu\epsilon}\sigma^{\lambda\epsilon}. \tag{11.7}$$

When the local coordinate system is used, similarly we have the following expressions:

$$\boldsymbol{n}^{A}=n^{Ai}\boldsymbol{e}_{i}=n^{A}_{i}\boldsymbol{e}^{i}, \tag{11.8}$$

$$\boldsymbol{f}_{A}=f^{j}_{A}\boldsymbol{e}_{j}=f_{Aj}\boldsymbol{e}^{j}, \tag{11.9}$$

$$\sigma^{ij}=n^{Ai}f^{j}_{A}, \quad \sigma_{ij}=n^{A}_{i}f_{Aj}, \quad \sigma^{i}_{j}=n^{Ai}f_{Aj}, \tag{11.10}$$

$$\boldsymbol{T}=\sigma^{ij}\boldsymbol{e}_{i}\boldsymbol{e}_{j}=\sigma_{ij}\boldsymbol{e}^{i}\boldsymbol{e}^{j}=\sigma^{i}_{j}\boldsymbol{e}_{i}\boldsymbol{e}^{j}, \tag{11.11}$$

$$\sigma_{ij}=g_{ik}g_{jl}\sigma^{kl}, \quad \sigma^{i}_{j}=g_{jk}\sigma^{ik}, \tag{11.12}$$

When the local coordinate system being rectangular Cartesian in particular, the reference frame $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ constitute a system of mutually perpendicular unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, the distinction among the contra-, co-variant and mixed components of stress disappears, and they can all be substituted by $\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy}$ as considered in many text books.

We will next introduce deviatoric stress. In the first place the hydrostatic tension is given by one third of the first scalar invariant of the stress tensor $\boldsymbol{T}$, i.e. by

$$\bar{\sigma}=\frac{1}{3}\sigma^{\lambda\mu}g_{\lambda\mu}=\frac{1}{3}\sigma_{\lambda\mu}g^{\lambda\mu}=\frac{1}{3}\sigma^{\lambda}_{\mu}\delta^{\mu}_{\lambda} \tag{11.13}$$

for the Lagrangian coordinate system, and by

$$\bar{\sigma}=\frac{1}{3}\sigma^{ij}g \;=\frac{1}{3}\sigma_{ij}g^{ij}=\frac{1}{3}\sigma^{i}_{j}\delta^{j}_{i} \tag{11.14}$$

for the local coordinate system respectively.

The stress deviatoric tensor $\boldsymbol{T}$ is therefore defined by

$$\boldsymbol{T}'=\boldsymbol{T}-\sigma\boldsymbol{I} \tag{11.15}$$

where the unit tensor $\boldsymbol{I}$ is expressed by (3.25) for the Lagrangian system and by

$$\boldsymbol{I}=\boldsymbol{e}_{i}\boldsymbol{e}^{i}=\boldsymbol{e}^{i}\boldsymbol{e}_{i}=g_{ij}\boldsymbol{e}^{i}\boldsymbol{e}^{j}=g^{ij}\boldsymbol{e}_{i}\boldsymbol{e}_{j} \tag{11.16}$$

for the local system. Representing the components of the deviatoric stress tensor $\boldsymbol{T}'$ by $\sigma'^{\lambda\mu},\cdots,\sigma'^{ij},\cdots$ for the Lagrangian and local system respectively, we can write

$$\begin{aligned}\boldsymbol{T}'&=\sigma'^{\lambda\mu}\boldsymbol{e}_{\lambda}\boldsymbol{e}_{\mu}=\cdots,\\ &=\sigma'^{ij}\boldsymbol{e}_{i}\boldsymbol{e}_{j}=\cdots,\end{aligned} \tag{11.17}$$

hence from (11.15) obtain

$$\left.\begin{array}{lll}\sigma'^{\lambda\mu}=\sigma^{\lambda\mu}-\bar{\sigma}g^{\lambda\mu}, & \sigma'_{\lambda\mu}=\sigma_{\lambda\mu}-\bar{\sigma}g_{\lambda\mu}, & \sigma'^{\lambda}_{\mu}=\sigma^{\lambda}_{\mu}-\bar{\sigma}\delta^{\lambda}_{\mu},\\ \sigma'^{ij}=\sigma^{ij}-\bar{\sigma}g^{ij}, & \sigma'_{ij}=\sigma_{ij}-\bar{\sigma}g_{ij}, & \sigma'_{j}=\sigma^{i}_{j}-\bar{\sigma}\delta^{i}_{j}.\end{array}\right\} \tag{11.18}$$

It is easily found from (11.18) that the first scalar invariant of $\boldsymbol{T}'$ vanishes, viz.

$$\sigma'^{\lambda\mu}g_{\lambda\mu}=0\,,\qquad \sigma'^{ij}g_{ij}=0\,. \tag{11.19}$$

When the local coordinate system is rectangular Cartesian in particular, (11.18) gives the usual relations

$$\sigma'_x=\sigma_x-\bar{\sigma},\cdots,\qquad \tau'_{yz}=\tau_{yz},\cdots \tag{11.20}$$

no matter what the components may be, contra- or co-variant or mixed.

12. Some Examples of $\boldsymbol{T}'$

We will now consider some cases of simple stress states, for the purpose of both deepening our understanding for the stress tensors $\boldsymbol{T}$ and $\boldsymbol{T}'$ and clarifying their correspondence with the strains $\boldsymbol{DE}$ and $\boldsymbol{E}$ previously introduced.

(1) *Simple Tension*

We consider, as shown in Fig. 14, a rectangular Cartesian local coordinate system x^i and suppose that an uniform simple tension of the amount σ per unit

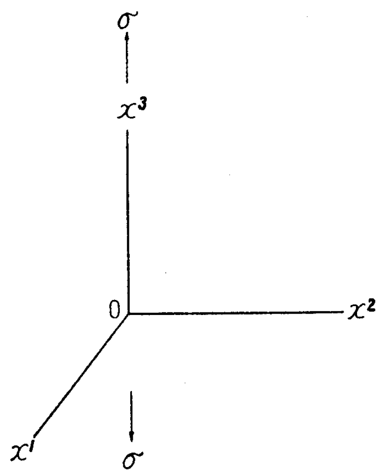

FIGURE 14.

cross section is exerted in the x^3 direction. Then the stress tensor $\boldsymbol{T}$ is expressed in the form

$$\boldsymbol{T}=\sigma\boldsymbol{e}_3\boldsymbol{e}_3=\sigma\boldsymbol{e}^3\boldsymbol{e}^3=\cdots\,. \tag{12.1}$$

The σ in this case is the so-called true stress. If the plastic deformation due to this simple tension is a simple extension by n times in the x^3 direction as described in Article (1.1), Section 8, then (12.1) is transformed into the form

$$\boldsymbol{T}=\frac{\sigma}{n^2}\boldsymbol{e}_3\boldsymbol{e}_3=n^2\sigma\boldsymbol{e}^3\boldsymbol{e}^3=\sigma\boldsymbol{e}^3\boldsymbol{e}_3 \tag{12.2}$$

referred to the Lagrangian system coincident for $t=0$ with the local one, because $\boldsymbol{e}_3=n\boldsymbol{e}_3$, $n\boldsymbol{e}^3=\boldsymbol{e}^3$. Thus it is seen that the Lagrangian component of the tensile stress of the amount σ is σ/n^2 or $n^2\sigma$ or σ, according as it is contravariant or co-variant or mixed.

The hydrostatic tension being

$$\bar{\sigma}=\frac{1}{3}\sigma\,, \tag{12.3}$$

the components of $\boldsymbol{T}'$ are obtained from (11.20) as

$$\left.\begin{array}{l}\sigma'^{11}=\sigma'_{11}=\sigma'^{1}_{1}=-\dfrac{1}{3}\sigma\,,\quad \sigma'^{22}=\sigma'_{22}=\sigma'^{2}_{2}=-\dfrac{1}{2}\sigma\,,\\ \sigma'^{33}=\sigma'_{33}=\sigma'^{3}_{3}=\dfrac{2}{3}\sigma\,,\\ \qquad\qquad \text{the other}\quad \sigma'^{ij}=\sigma'^{j}_{i}=\sigma'^{j}_{i}=0\,.\end{array}\right\} \tag{12.4}$$

The Lagrangian components of $\boldsymbol{T}'$ are obtained from (12.4), by means of (8.1′) and (8.2′), in the form

$$\left.\begin{array}{c}\sigma'^{11}=-\dfrac{1}{3}n\sigma\,,\quad \sigma'^{22}=-\dfrac{1}{3}n\sigma\,,\quad \sigma'^{33}=\dfrac{2}{3}\dfrac{\sigma}{n^2}\,,\\ \text{the other}\quad \sigma'^{\lambda\mu}=0\,,\end{array}\right\} \tag{12.5}$$

$$\left.\begin{array}{c}\sigma'_{11}=-\dfrac{1}{3}\dfrac{\sigma}{n}\,,\quad \sigma'_{22}=-\dfrac{1}{3}\dfrac{\sigma}{n}\,,\quad \sigma'_{33}=\dfrac{2}{3}n^2\sigma\,,\\ \text{the other}\quad \sigma'_{\lambda\mu}=0\,,\end{array}\right\} \tag{12.5′}$$

$$\left.\begin{array}{c}\sigma'^{1}_{1}=-\dfrac{1}{3}\sigma\,,\quad \sigma'^{2}_{2}=-\dfrac{1}{3}\sigma\,,\quad \sigma'^{3}_{3}=\dfrac{2}{3}\sigma\,,\\ \text{the other}\quad \sigma'^{\lambda}_{\mu}=0\,,\end{array}\right\} \tag{12.5″}$$

according to the contra- or co-variant or mixed components. Although the mixed components have the same values as those by the rectangular Cartesian coordinate system, their mechanical meaning is not the same.

It is at our disposal to adopt what components of the stresses for what coordinate system, but the components of $D\boldsymbol{E}$ and $\boldsymbol{E}$ must be chosen correponding to those of the stresses. That is, in considering the stress-strain relation, they must be $(D\varepsilon)^{\lambda\mu}$ and $\varepsilon^{\lambda\mu}$ for $\sigma'^{\lambda\mu}$, and $(D\varepsilon)_{ij}$ and ε_{ij} for σ'_{ij}, for example, but in considering work, $(D\varepsilon)^{\lambda\mu}$ for $\sigma_{\lambda\mu}$, and so on. And no matter what kind of components may be considered, the $\boldsymbol{T}'\sim\boldsymbol{E}$, $D\boldsymbol{E}$ relation is seen to be reduced to

$$\frac{2}{3}\sigma\sim\log n\,,\qquad \frac{Dn}{n} \tag{12.6}$$

and the work $\boldsymbol{T}\cdot\cdot D\boldsymbol{E}$ to

$$\sigma\frac{Dn}{n}\,. \tag{12.7}$$

(2) *Torsion*

We consider a shearing stress state as shown in Fig. 15. Then it is expressed by

$$\boldsymbol{T}=\boldsymbol{T}'=\tau(\boldsymbol{e}_2\boldsymbol{e}_3+\boldsymbol{e}_3\boldsymbol{e}_2) \tag{12.8}$$

or

$$\left.\begin{array}{c}\sigma^{23}=\sigma_{23}=\sigma^{2}_{3}=\sigma^{3}_{2}=\sigma'^{23}=\sigma'_{23}=\sigma'^{2}_{3}=\sigma'^{3}_{2}=\tau\,,\\ \text{the other components}=0\,.\end{array}\right\} \tag{12.9}$$

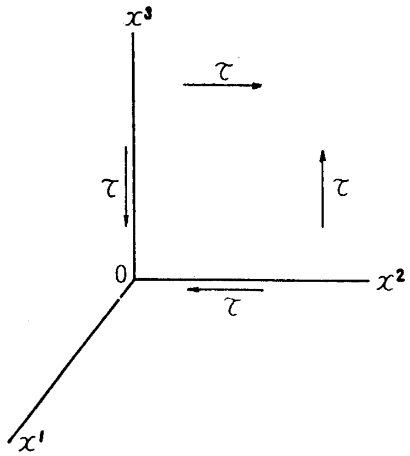

FIGURE 15.

Their Lagrangian form is

$$\sigma^{22}=\sigma'^{22}=-2\gamma\tau\,,\quad \sigma^{23}=\sigma'^{23}=\tau\,, \tag{12.10}$$

$$\sigma_{23}=\sigma'_{23}=\tau\,,\qquad \sigma_{33}=\sigma'_{33}=2\gamma\tau\,, \tag{12.10'}$$

$$\left.\begin{array}{ll}\sigma_2^2=\sigma_2'^2=-\gamma\tau\,, & \sigma_3^2=\sigma_3'^2=(1-\gamma^2)\tau\,,\\ \sigma_2^3=\sigma_2'^3=\tau\,, & \sigma_3^3=\sigma_3'^3=\gamma\tau\,,\end{array}\right\} \tag{12.10''}$$

the components not written in the above being zero. When the deformation due to this shear stress is the simple shear shown in Section 8, the relation between $\boldsymbol{T}$ or $\boldsymbol{T}'$ and $\boldsymbol{E}$, $D\boldsymbol{E}$ is, likewise to the above case, reduced to

$$\tau\sim\frac{1}{2}\gamma\,,\quad \frac{1}{2}D\gamma \tag{12.11}$$

and the work $\boldsymbol{T}\cdot\cdot D\boldsymbol{E}$ to

$$\frac{1}{2}\tau D\gamma \tag{12.12}$$

regardlesss of the kind of components and of coordinate system selected. When the deformation due to the stress is a pure shear, it is sufficient for the $\frac{1}{2}\gamma$ and $\frac{1}{2}D\gamma$ in these relations to be replaced by $\log n$ and Dn/n respectively.

(3) *Combined Tension-Torsion*

We consider the stress state to be given by Fig. 16. Then we have

$$\boldsymbol{T}=\sigma\boldsymbol{e}_3\boldsymbol{e}_3+\tau(\boldsymbol{e}_2\boldsymbol{e}_3+\boldsymbol{e}_3\boldsymbol{e}_2)\,, \tag{12.13}$$

$$\left.\begin{array}{l}\sigma'^{11}=\sigma'_{11}=\cdots=\sigma'^{22}=\sigma'_{22}=\cdots=-\dfrac{1}{3}\sigma\,,\\ \sigma'^{33}=\sigma'_{33}=\cdots=\dfrac{2}{3}\sigma\,,\\ \sigma'^{23}=\sigma'_{23}=\cdots=\sigma'^{32}=\cdots=\tau\,,\\ \qquad\text{the other components}=0.\end{array}\right\} \tag{12.14}$$

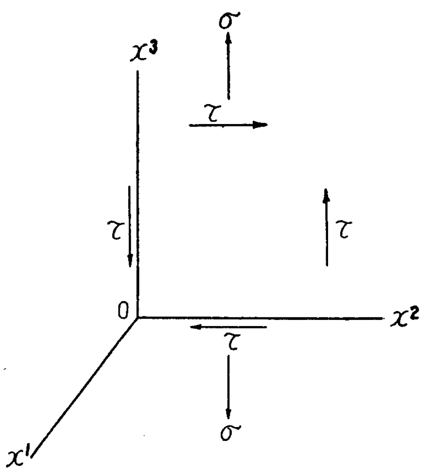

FIGURE 16.

If the deformation due to this stress is the combined extension-torsion specified by n, γ as shown in Fig. 5b, the Lagrangian expression of (12.14) is given by

$$\left.\begin{aligned}
&\sigma'_{11}=\sigma'_{22}=-\frac{1}{3n}\sigma\,,\\
&\sigma'_{33}=2n\gamma\tau-\frac{1}{3}\gamma^2\sigma+\frac{2}{3}n^2\sigma\,,\\
&\sigma'_{23}=\sqrt{n}\,\tau-\frac{1}{3}\frac{\gamma}{\sqrt{n}}\sigma\,,\\
&\sigma'_{31}=\sigma'_{12}=0\,.
\end{aligned}\right\} \tag{12.15}$$

It seems rather strange that σ'_{33} and σ'_{23} of (12.15) become negative for the positive σ and τ when they both, or only σ, become large beyond some extent. The like has already been shown for the strain components given in (8.26). This same question encountered in both (8.26) and (12.15) is now seen to be solved by considering them in correspondence with each other. Namely, their comparison is found to lead to the correspondence

$$\frac{2}{3}\sigma\sim\log n\,, \qquad \tau\sim\frac{\gamma}{2n}$$

which is very natural and is also obtainable by the comparison of the other kind of components as well.

From the above examples, we can see that the interdependence of $\boldsymbol{T}'$ and $\boldsymbol{E}$, $\boldsymbol{DE}$ and the work $\boldsymbol{T}\cdot\cdot\boldsymbol{DE}$ are quite the same for any kind of the coordinate system and the components. This result, though rather natural in view of their being tensor, is considered to show the self-consistency of our present theory.

13. Theoretical Justification of the True Stress and the Logarithmic Strain

In the preceding sections, it was made clear that the tensors $\boldsymbol{DE}$, $\boldsymbol{E}$, $\boldsymbol{T}$ and $\boldsymbol{T}'$ themselves, and therefore their interdependence, are the same (invariant), though their analytical expressions depend on the kind of the coordinate system and of the components used, and especially that in the case of simple extension due to simple tension the $\boldsymbol{T}'\sim\boldsymbol{E}$ correspondence is reduced to the true stress-logarithmic strain correspondence

$$\sigma=\frac{P}{A}\sim\varepsilon=\log n \tag{13.1}$$

which was first introduced by P. Ludwik [1] to describe extensional plastic deformation. (Here the coefficient 2/3 in (12.6) is omitted, for the sake of simplicity.) That is, it is one of the conclusion of our theory that *any stress and strain other than the true stress and the logarithmic strain must not be used to describe extensional plastic deformation.* We will now show that this conclusion, and hence the basis of our present theory, is valid in view of the logical and practical considerations on the description of the extensional deformation.

For this purpose we will compare the characteristics, which our true stress and logarithmic strain exhibit for the description of deformation, when the origin of the strain is displaced, with those shown by other ones commonly used. And further for this, we choose an arbitrary state, work-hardened or annealed, as the state $t=0$, and suppose a test piece of unit of both length and cross section in this state to have the lengths and cross sections shown in Table 2 in the respective state $t=t_0$ and t. And then we represent by the notations in Table 3 the logarithmic

TABLE 2.

state	$t=0$	$t=t_0$	t
length	1	n_0	n
cross section	1	A_0	A

TABLE 3.

origin of strain	$t=0$	$t=t_0$
logarithmic strain	ε	ε'
usual strain	e	e'
true stress	σ	σ'
nominal stress	s	s'

and usual strains and the true and nominal stresses when the origin of strain, i.e. the undeformed state, is taken at the states $t=0$ and t_0 respectively. Further we indicate the logarithmic and usual strains of the state $t=t_0$, when the state $t=0$ is regarded as the origin of strain, by ε_0 and e_0 respectively. Then we have from the definitions of usual and logarithmic strains,

$$1+e=n, \quad 1+e'=\frac{n}{n_0}, \quad 1+e_0=n_0, \tag{13.2}$$

$$\varepsilon=\log n, \quad \varepsilon'=\log n-\log n_0, \quad \varepsilon_0=\log n_0. \tag{13.3}$$

From (13.2) we obtain

$$1+e'=\frac{1+e}{1+e_0},$$

and hence

$$e'=\frac{e-e_0}{1+e_0}, \tag{13.4}$$

and similarly from (13.3)

$$\varepsilon' = \varepsilon - \varepsilon_0 \,. \tag{13.5}$$

It can be seen from (13.4) and (13.5) that by shifting the origin of strain from the state $t=0$ to the state $t=t_0$, the scale of measuring the usual strain is increased by $(1+e_0)$ times, while that of the logarithmic strain is maintained unchanged.

Next, indicating the load acting at the state t by P, we have from the definitions,

$$s = P, \quad s' = \frac{P}{A_0}. \tag{13.6}$$

By virtue of (13.2), (13.6) and the incompressibility condition

$$n_0 A_0 = 1 \,, \tag{13.7}$$

we obtain the relation

$$s' = s(1+e_0) \tag{13.8}$$

between s and s'. As for the true stress, on the other hand, the relation

$$\sigma = \sigma' = \frac{P}{A} \tag{13.9}$$

is obvious to hold. From (13.8) and (13.9), we can see the value of the nominal stress to be increased by $(1+e_0)$ times, but that of the true stress kept unchanged on the transposition of the origin of strain from the state $t=0$ to the state $t=t_0$.

It is found from the above consideration that the $\sigma' \sim \varepsilon'$ relation when the state $t=t_0$ is selected as the origin of strain, is represented by the curve O′BC′ in Fig. 17a which is coincident with the curve OABC representing the $\sigma \sim \varepsilon$ relation when

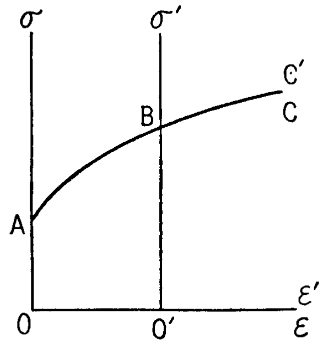

FIGURE 17a.

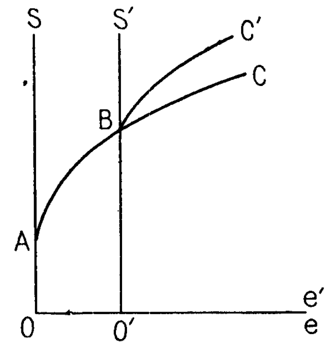

FIGURE 17b.

the state $t=0$ is taken as the origin of strain, namely that *the stress strain curve represented by means of σ and ε does not change its form by the transposition of the origin of strain.* This result being consistent with the essential character of plastic deformation that the current deformed state can be regarded at the same time as an undeformed state, is considered to be a matter of good fortune also from the practical viewpoint.

As against the above case, when the stress and strain, or either one of them, other than σ and ε are employed in representing the stress-strain curve, it is found that the above relation does not hold. We will now consider the case, for example, when the nominal stress s and the usual strain e are used. Let the $s \sim e$ curve when the state $t=0$ is chosen as the origin of strain be represented by the curve

OABC of Fig. 17b, then it is found from (13.4) and (13.8) that the $s' \sim e'$ curve whose origin of strain is taken at the state $t=t_0$ is given by the curve O'BC' different from the original curve OABC. *This result that the same mechanical process of deformation is represented by different curves according to the state where the origin of strain is situated, seems inconsistent with the conception that the origin of strain can be chosen at any state in the plastic deformation. And this is considered to show the $s \sim e$ combination not to be reasonable for the description of plastic deformation.* The same circumstances are seen to hold for the $\sigma \sim e$ and $s \sim \varepsilon$ combinations. What must be remarked here is that the stress and strain reasonable for describing elastic extensional deformation where the origin of strain can not be admitted to choose arbitrarily, being uniquely determined, are those other than the true stress and the logarithmic strain.

The above fact is considered to give a further affirmation on the foundation of our present theory of plasticity.

14. Equilibrium Equations

The stress tensor introduced in Section 11 for the plastic deformation is such tensor as to give the force exerted actually on unit of sectional area in the material in the deformed state t. And accordingly it will be natural to consider also the equilibrium condition referring to the deformed state.

We suppose the material part of the volume V_0 enclosed by the closed surface S_0 at the state $t=0$ to occupy at the state t the space of the volume V enclosed by the surface S as shown in Fig. 18. Then, representing, for the state t, the density

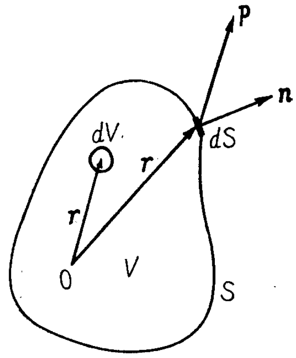

Figure 18.

by ρ, the body force per unit of mass by $\boldsymbol{F}$ and the unit outer normal on the surface S by $\boldsymbol{n}$, we have the equilibrium conditions of forces and moments

$$\int \rho \boldsymbol{F} dV + \int \boldsymbol{n} \cdot \boldsymbol{T} dS = 0 \tag{14.1}$$

and

$$\int \boldsymbol{r} \times \rho \boldsymbol{F} dV + \int \boldsymbol{r} \times (\boldsymbol{n} \cdot \boldsymbol{T}) S = 0 \tag{14.2}$$

for the deformation of any magnitude. Applying the Gauss' theorem

$$\int \boldsymbol{n}\cdot\boldsymbol{T}dS=\int \nabla\cdot\boldsymbol{T}dV \tag{14.3}$$

to (14.1), we obtain

$$\int (\rho\boldsymbol{F}+\nabla\cdot\boldsymbol{T})dV=0\,, \tag{14.4}$$

so that

$$\rho\boldsymbol{F}+\nabla\cdot\boldsymbol{T}=0\,. \tag{14.5}$$

This is the equilibrium condition of the form, which holds for arbitrary coordinate system, whether Eulerian or Lagrangian or local, over the whole range of the small and finite deformations.

The equilibrium condition (14.5), however, is not of the form invariant for coordinate transformations. Namely, for the Lagrangian coordinate transformation

$$x^{\lambda'}=x^{\lambda'}(x^1, x^2, x^3)\,, \tag{14.6}$$

for example, as given in (10.1), the integral (14.4) is transformed as

$$\int (\nabla\cdot\boldsymbol{T}+\rho\boldsymbol{F})\sqrt{g}\,dx^1dx^2dx^3=\int (\nabla\cdot\boldsymbol{T}+\rho\boldsymbol{F})\sqrt{g}\,|\partial_{\mu'}x^\lambda|\,dx^{1'}dx^{2'}dx^{3'} \tag{14.7}$$

where $|\partial_{\mu'}x^\lambda|$ indicates the determinant composed of the elements $\partial_{\mu'}x^\lambda$. Since, g defined by (3.17) being a scalar of weight 2, $\sqrt{g}$ is a scalar density which obeys the transformation law

$$\sqrt{g'}=|\partial_{\mu'}x^\lambda|\sqrt{g}\,, \tag{14.8}$$

(14.7) can be written

$$\int (\nabla\cdot\boldsymbol{T}+\rho\boldsymbol{F})\sqrt{g}\,dx^1dx^2dx^3=\int (\nabla\cdot\boldsymbol{T}+\rho\boldsymbol{F})\sqrt{g'}\,dx^{1'}dx^{2'}dx^{3'}\,. \tag{14.9}$$

Consequently we obtain the equilibrium equation of the form

$$\sqrt{g}\,\nabla\cdot\boldsymbol{T}+\sqrt{g}\,\rho\boldsymbol{F}=0 \tag{14.10}$$

invariant for coordinate transformations. Introducing the relative tensor

$$\boldsymbol{X}=\sqrt{g}\,\boldsymbol{T}\,, \tag{14.11}$$

its divergence is given by

$$\mathrm{div}\,\boldsymbol{X}=\sqrt{g}\,\nabla\cdot\boldsymbol{T}=\sqrt{g}\,\boldsymbol{e}^\lambda\cdot\partial_\lambda[\sqrt{g}^{-1}\boldsymbol{T}]=\partial_\lambda(\boldsymbol{e}^\lambda\cdot\boldsymbol{X})\,. \tag{14.12}$$

Hence (14.10) is also written in the form

$$\mathrm{div}\,\boldsymbol{X}+\sqrt{g}\,\rho\boldsymbol{F}=0\,. \tag{14.13}$$

That the stress tensors $\boldsymbol{T}$ and $\boldsymbol{X}$ are both symmetric is seen to result from the equilibrium condition (14.2) of moments. That is, applying to (14.2) the Gauss' theorem whose $\boldsymbol{T}$ is replaced by $\boldsymbol{T}\times\boldsymbol{r}$ and taking account of (14.5), we obtain

$$\boldsymbol{e}^\lambda\cdot\boldsymbol{T}\times\boldsymbol{e}_\lambda=0\,, \tag{14.14}$$

so that

$$\boldsymbol{e}^\lambda\cdot\boldsymbol{X}\times\boldsymbol{e}_\lambda=0\,. \tag{14.15}$$

This result obtained in particular for the Lagrangian coordinate system show the coordinate independent property of the tensors $\boldsymbol{T}$ and $\boldsymbol{X}$ being symmetric.

For practical purpose, it is needed to give the equilibrium equations (14.5) and

(14.13) an analytical expression for some coordinate system appropriate for the problem under consideration. We will now begin with the case of the Lagrangian coordinate system. In this case we may put

$$\boldsymbol{F} = F^{\lambda}\boldsymbol{e}_{\lambda} \tag{14.16}$$

and substitute the expressions (11.6) and (3.21) of $\boldsymbol{T}$ and ∇, (14.16) and

$$\partial_{\lambda}\boldsymbol{e}_{\mu} = \Gamma^{\kappa}_{\lambda\mu}\boldsymbol{e}_{\kappa} \tag{14.17}$$

into (14.5) to give

$$(\nabla_{\mu}\sigma^{\lambda\mu} + \rho F^{\lambda})\boldsymbol{e}_{\lambda} = 0\,, \tag{14.18}$$

where $\Gamma^{\nu}_{\lambda\mu}$ is given by (4.4), and the covariant derivative $\nabla_{\mu}\sigma^{\lambda\mu}$ of $\sigma^{\lambda\mu}$ by

$$\nabla_{\mu}\sigma^{\lambda\mu} = \partial_{\mu}\sigma^{\lambda\mu} + \sigma^{\kappa\mu}\Gamma^{\lambda}_{\mu\kappa} + \sigma^{\lambda\kappa}\Gamma^{\mu}_{\mu\kappa}\,. \tag{14.19}$$

We have from (14.18) the equilibrium equations

$$\nabla_{\mu}\sigma^{\lambda\mu} + \rho F^{\lambda} = 0 \tag{14.20}$$

in the direction $\boldsymbol{e}_{\lambda}$.

For the stress tensor density

$$\boldsymbol{X} = X^{\lambda\mu}\boldsymbol{e}_{\lambda}\boldsymbol{e}_{\mu} = \sqrt{g}\,\sigma^{\lambda\mu}\boldsymbol{e}_{\lambda}\boldsymbol{e}_{\mu} \tag{14.21}$$

we obtain the component equations

$$\nabla_{\mu}X^{\lambda\mu} + \sqrt{g}\,\rho F^{\lambda} = 0 \tag{14.22}$$

in the $\boldsymbol{e}_{\lambda}$ direction, making use of the definition (14.12) and the relations

$$\partial_{\lambda}\sqrt{g} = \Gamma^{\kappa}_{\lambda\kappa}\sqrt{g}\,, \qquad \partial_{\lambda}\sqrt{g}^{-1} = -\Gamma^{\kappa}_{\lambda\kappa}\sqrt{g}^{-1}\,. \tag{14.23}$$

Where the covariant derivative $\nabla_{\mu}X^{\lambda\mu}$ is given by

$$\nabla_{\mu}X^{\lambda\mu} = \partial_{\mu}X^{\lambda\mu} + X^{\kappa\mu}\Gamma^{\lambda}_{\mu\kappa}\,. \tag{14.24}$$

When the coordinate system used is Eulerian or local, the equilibrium equations are also obtained in the similar form

$$\nabla_{j}\sigma^{ij} + \rho F^{i} = 0\,, \tag{14.25}$$

$$\nabla_{j}X^{ij} + \sqrt{g}\,\rho F^{i} = 0\,, \tag{14.26}$$

or

$$\nabla_{j}\sigma \ \ + \rho F \ = 0\,, \tag{14.27}$$

$$\nabla_{j}X \ \ + \sqrt{\boldsymbol{g}}\,\rho F \ = 0 \tag{14.28}$$

where

$$g = |g_{ij}|\,, \qquad \boldsymbol{g} = |g_{ij}| \tag{14.29}$$

and the covariant derivatives $\nabla_{j}\sigma^{ij}$, $\nabla_{j}\sigma$,$\cdots$ are of the form quite similar to those for the Lagrangian coordinate system, and therefore we may save the trouble to write them down. Since both the Eulerian and the local coordinate systems are usually orthogonal, the covariant derivative are simplified, many of the coefficients Γ^{k}_{ij} and Γ^{k}_{ij} vanishing. Of course, all what has been said in this section applies not only to the small deformation but to the finite one.

15. Principle of Virtual Work

It has been stated in Section 11 that for the definition (11.2) of the stress, together with those of strain and strain increment (7.1) and (3.27), to be justifiable for describing plastic deformation, they must satisfy the virtual work principle of the form (2.1) as well as the condition of their being tensors. And now we are in the stage to examine this problem.

Likewise to the virtual work principle

$$\delta W = \boldsymbol{K}\cdot\delta\boldsymbol{u} = X\delta u + Y\delta v + Z\delta w = 0$$

of a particle exerted by the force $\boldsymbol{K}(X, Y, Z)$ in equilibrium, that for the material of volume V composed of the volume element $dV=\sqrt{g}\,dx^1dx^2dx^3$, in the deformed state t, exerted by the force

$$(\nabla\cdot\boldsymbol{T}+\rho\boldsymbol{F})dV = (\mathrm{div}\,\boldsymbol{X}+\sqrt{g}\,\rho\boldsymbol{F})dx^1dx^2dx^3\,, \tag{15.1}$$

in equilibrium is given by

$$\int(\nabla\cdot\boldsymbol{T}+\rho\boldsymbol{F})\cdot\delta\boldsymbol{u}\,dV = \int(\nabla_\mu\sigma^{\lambda\mu}+\rho F^\lambda)(\delta u)_\lambda dV = 0\,, \tag{15.2}$$

where $\delta\boldsymbol{u}$ indicates the virtual displacement of a material point.

The first term of the integral (15.2) can be transformed as

$$\begin{aligned}\int\nabla\cdot\boldsymbol{T}\cdot\delta\boldsymbol{u}\,d\nabla &= \int\nabla_\mu\sigma^{\lambda\mu}(\delta u)_\lambda dV\\ &= \int\{\nabla_\mu[\sigma^{\lambda\mu}(\delta u)_\lambda] - \sigma^{\lambda\mu}\nabla_\mu(\delta u)_\lambda\}dV\\ &= \int[\nabla\cdot(\boldsymbol{T}\cdot\delta\boldsymbol{u}) - \boldsymbol{T}\cdot\cdot\nabla\delta\boldsymbol{u}]dV\,,\end{aligned} \tag{15.3}$$

and the first integral of the right-hand side of this equation is further transformed by the Gauss' theorem to a surface integral as

$$\int\nabla\cdot(\boldsymbol{T}\cdot\delta\boldsymbol{u})dV = \int\boldsymbol{n}\cdot\boldsymbol{T}\cdot\delta\boldsymbol{u}\,dS\,. \tag{15.4}$$

Since the surface traction on S is given by

$$\boldsymbol{p} = \boldsymbol{n}\cdot\boldsymbol{T}\,, \tag{15.5}$$

(15.4) is written in the form

$$\int\boldsymbol{p}\cdot\delta\boldsymbol{u}\,dS\,. \tag{15.6}$$

The second integral of the right-hand side of (15.3) is seen to be represented as

$$\int\boldsymbol{T}\cdot\cdot\nabla\delta u\,dV = \int\boldsymbol{T}\cdot\cdot\delta\boldsymbol{E}dV \tag{15.7}$$

taking account of the symmetric character of $\boldsymbol{T}$ and the definition (3.27) of $\delta\boldsymbol{E}$.

In consequence, the virtual work principle (15.2) is written in the form

$$\int\boldsymbol{p}\cdot\delta\boldsymbol{u}dS + \int\rho\boldsymbol{F}\cdot\delta\boldsymbol{u}dV = \int\boldsymbol{T}\cdot\cdot\delta\boldsymbol{E}dV \tag{15.8}$$

or, represented in terms of the components referred, for example, to the Lagrangian coordinate system, in the form

$$\int p^{\lambda}(\delta u)_{\lambda} dS + \int \rho F^{\lambda}(\delta u)_{\lambda} dV = \int \sigma^{\lambda\mu}(\delta\varepsilon)_{\lambda\mu} dV. \tag{15.8'}$$

The expressions for other coordinate systems are given in the same form. The left hand side of (15.8) representing the work δW_1 done by the external forces, i.e. the surface tractions $\boldsymbol{p}$ and the body forces $\boldsymbol{F}$, during the virtual displacement $\delta\boldsymbol{u}$, we can put

$$\delta W_1 = \int \boldsymbol{p}\cdot\delta\boldsymbol{u} dS + \int \rho\boldsymbol{F}\cdot\delta\boldsymbol{u} dV, \tag{15.9}$$

hence we obtain the result

$$\delta W_1 = \int \boldsymbol{T}\cdot\cdot\delta\boldsymbol{E} dV = \int \sigma^{\lambda\mu}(\delta\varepsilon)_{\lambda\mu} dV. \tag{15.10}$$

This relation, obtained as the result of the virtual work principle (15.2), representing that the work by the external forces is equal to that done by the stress during the strain, is usually called by the principle of the same name.

The relation (15.10) is seen to be of the same form as that given in (2.1), which was regarded as the condition for the stress $\boldsymbol{T}$, together with the strain increment $\delta\boldsymbol{E}$, to satisfy, in order that its definition may be legitimate for describing plastic deformation. Thus the definition (11.2) of the stress tensor $\boldsymbol{T}$ is concluded to be justifiable.

For finite deformation the relation (15.10) does not necessarily hold, according to the definition of $\boldsymbol{T}$ and $\delta\boldsymbol{E}$. And *the fact it holds for the small and finite deformations has a great significance, because on account of this there exists the plastic potential and the theory of plasticity for finite deformation is possible to be organized in the same way as that for the small deformation,* as will be shown later.

Of course, (15.10) is possible to be expressed in any coordinate system; for example, for the Eulerian and the local coordinate systems, it reads

$$\delta W_1 = \int \sigma^{ij}(\delta\varepsilon)_{ij} dV \tag{15.10'}$$

and

$$\delta W_1 = \int \sigma \ (\delta\varepsilon) \ dV \tag{15.10''}$$

respectively.

16. Plastic Potential

By taking $\delta\boldsymbol{u}$ in (15.2), and hence $\delta\boldsymbol{E}$ in (15.10) to be an actual displacement $D\boldsymbol{u}$ and an actual strain increment $D\boldsymbol{E}$ from t to $t+dt$, we have

$$DW = \boldsymbol{T}\cdot\cdot D\boldsymbol{E} = \sigma^{\lambda\mu}(D\varepsilon)_{\lambda\mu} \tag{16.1}$$

for unit of volume of the material in the deformed state, or for a constant mass, because the plastic deformation is incompressible.*

* Rigorously, we should consider the work for a constant mass

$$DW' = \boldsymbol{X}\cdot\cdot D\boldsymbol{E} = X^{\lambda\mu}(D\varepsilon)_{\lambda\mu},$$

but for the reason of the incompressibility of plastic deformation, this is equivalent to consider (16.1).

Now if we represent the internal energy per unit volume (a constant mass) stored in the material due to the whole plastic deformation up to the state t, by U, and the heat which flows out during dt, by DQ, then the first law of thermodynamics is written in the form

$$DU = DW - DQ\,. \tag{16.2}$$

Denoting the entropy per unit volume (constant mass) by S, and the temperature, by T, the irreversibility of the plastic deformation is generally represented by

$$DS > -\frac{DQ}{T}\,. \tag{16.3}$$

This is a matter of course in view of the character of plastic deformation in which $-DQ<0$, and $DS>0$ on account of the increase of the extent of disorder.

From (16.2) and (16.3), we obtain

$$DW = T\cdot\cdot\boldsymbol{DE} > DU - \boldsymbol{T}DS\,. \tag{16.4}$$

When the process is especially isothermal, the irreversibility condition is written in the form

$$DW = \boldsymbol{T}\cdot\cdot\boldsymbol{DE} > DF\,, \tag{16.5)*}$$

where F means the free energy

$$F = U - TS\,. \tag{16.6}$$

In metals subjected to plastic deformation, DS is negligibly small in general compared with DU, accordingly DF is regarded as being approximately equal to DU. We will herewith consider what restriction is put on the relation between $\boldsymbol{T}$ and $\boldsymbol{DE}$ by the irreversibility condition (16.5). For this purpose, we suppose $\boldsymbol{T}$ and $\boldsymbol{DE}$ to be vectors in six dimensional space, as usual, which have the components $\sigma^{\lambda\mu}$ and $(D\varepsilon)_{\lambda\mu}$ respectively. Then $DW=\sigma^{\lambda\mu}(D\varepsilon)_{\lambda\mu}$ is regarded as the scaler product of the both vectors. Therefore, *if the stress vector $\sigma^{\lambda\mu}$ is as shown in Fig. 19a, the irreversibility condition (16.5) prohibits the vector $(D\varepsilon)_{\lambda\mu}$ to exist in the hatched region of the figure. In the case where the material under consideration is perfectly plastic*

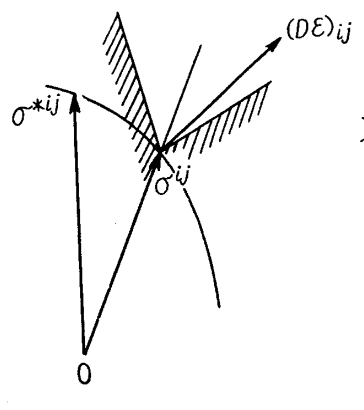

FIGURE 19a.

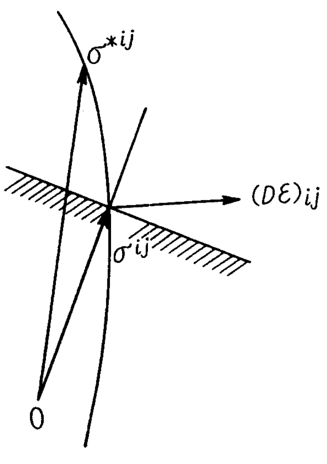

FIGURE 19b.

* It is not correct that the inequality $DW=\sigma_x d\varepsilon_x+\cdots>0$ is used in many literatures as the trreversibility condition of plastic deformation.

in particular, the relation $DF=0$ is considered to hold, so that the prohibited region of $(D\varepsilon)_{\lambda\mu}$ is of the shape shown in Fig. 19b.

The other condition, which put some restraint upon the relation between $\boldsymbol{T}$ and $\boldsymbol{DE}$, is the principle of maximum plastic work. It should be remarked here that this principle is properly derivable from that of minimum slip, although it seems to have been considered by many people to hold without any premise. At any rate, the maximum work principle is stated as follows. The work $\boldsymbol{T}\cdot\cdot\boldsymbol{DE}=\sigma^{\lambda\mu}(D\varepsilon)_{\lambda\mu}$ done by the actual stress $\boldsymbol{T}$ in case of the strain $\boldsymbol{DE}$ is greater than the work $\boldsymbol{T}^*\cdot\cdot\boldsymbol{DE}=\sigma^{*\lambda\mu}(D\varepsilon)_{\lambda\mu}$ done by the virtual stress on or inside the yield surface

$$f(\boldsymbol{T}, \boldsymbol{E})=c\,, \tag{16.7}$$

namely,

$$(\boldsymbol{T}-\boldsymbol{T}^*)\cdot\cdot\boldsymbol{DE}=(\sigma^{\lambda\mu}-\sigma^{*\lambda\mu})(D\varepsilon)_{\lambda\mu}\geqq 0\,. \tag{16.8}$$

The yield function f is considered to involve in general the strain history tensor $\boldsymbol{E}$. What is important here is that f is a function of the strain history $\boldsymbol{E}$ introduced in our present paper, but not of the strain used in general, which is specified by the geometrical configuration, independent of the deformation path. This is because the mechanical state of a plastically deformed body is determined by the path dependent strain history tensor $\boldsymbol{E}$. As for the form of the yield function f involving $\boldsymbol{E}$ we will make a full investigation in a subsequent paper.

The maximum work principle (16.8) indicates that the yield surface (16.7) is concave toward the origin and the vector $(D\varepsilon)_{\lambda\mu}$ is parallel to the outer normal to the surface at the point $\sigma^{\lambda\mu}$. *What follows from both the irreversibility condition and the maximum work principle is that the direction of the yield surface at each point $\sigma^{\lambda\mu}$ is restricted in a certain range.*

The result that the vector $(D\varepsilon)_{\lambda\mu}$ is the normal to the yield surface is represented by the analytical expression

$$(D\varepsilon)_{\lambda\mu}=\frac{\partial f(\sigma^{\lambda\mu}, \varepsilon_{\lambda\mu})}{\partial\,\sigma^{\lambda\mu}}D\lambda\,, \tag{16.9}$$

where $D\lambda$ is obtained from the law of work-hardening as will be shown in the next section. As for the other coordinate systems, we have the similar relations

$$(D\varepsilon)_{ij}=\frac{\partial f(\sigma^{ij}, \varepsilon_{ij})}{\partial\,\sigma^{ij}}D\lambda\,, \tag{16.9$'$}$$

$$(D\varepsilon)_{ij}=\frac{\partial f\,\sigma^{ij}, \varepsilon_{ij}}{\partial\,\sigma^{ij}}D\lambda\,. \tag{16.9$''$}$$

17. Law of Work-Hardening

Work-hardening is as well one of the strain history phenomena as anisotropy and the Bauschinger effect. But they are distinguished from each other in respect that the former is due to the density change of dislocations in the metal crystals, while the latter, to the change in the mode of arrangement of dislocations. Otherwise expressed, the former is regarded as the expansion of the yield surface due to a scalar effect of the strain history, and the latter as the distorsion and the transposition of the yield surface due to its some tensor effects.

The theory of work-hardening was first introduced by G. I. Taylor and H. Quinney [4], and R. Schmidt [5], and it makes the assertion that the value of the yield function is a function of the plastic work done so far. But this conception, though generally accepted hitherto, seems desirous to be modified somewhat. That is, the amount of work-hardening should be regarded as a function of the internal energy U, not the plastic work W, because the state variable in the equation of energy conservation

$$DU=\sigma^{\lambda\mu}(D\varepsilon)_{\lambda\mu}-DQ \tag{17.1}$$

is only the internal energy U, and not the work or the heat quantity Q. This notion is seen to be justifiable also from the fact that DU obtained by the measurement of $\sigma^{\lambda\mu}(D\varepsilon)_{\lambda\mu}$ and DQ by Taylor and Quinney [6] is approximately equal to DU due to dislocations, obtained by the dislocation theory of Taylor [7].

For the above reason, we should put

$$f(\sigma^{\lambda\mu},\ \varepsilon_{\lambda\mu})=H(U)\,, \tag{17.2}$$

but not

$$f(\sigma^{\lambda\mu},\ \varepsilon_{\lambda\mu})=F\Big(\int\sigma^{\lambda\mu}(D\varepsilon)_{\lambda\mu}\Big). \tag{17.3}$$

In case, in particular, when the proportion of U to $\int\sigma^{\lambda\mu}(D\varepsilon)_{\lambda\mu}$ varies in the same way for all the deformation paths, (17.3) can be used instead of (17.2).

Provided the law of work-hardening to have the form (17.2), $D\lambda$ in (16.9) is expressed as follows. From (16.9), we have

$$\sigma^{\lambda\mu}(D\varepsilon)_{\lambda\mu}=\sigma^{\lambda\mu}\frac{\partial f(\sigma^{\lambda\mu},\ \varepsilon_{\lambda\mu})}{\partial\sigma^{\lambda\mu}}D\lambda\,, \tag{17.4}$$

so that from (17.1)

$$DU+DQ=\sigma^{\lambda\mu}\frac{\partial f(\sigma^{\lambda\mu},\ \varepsilon_{\lambda\mu})}{\partial\sigma^{\lambda\mu}}D\lambda\,. \tag{17.5}$$

On the other hand, from (17.2) it follows that

$$Df(\sigma^{\lambda\mu},\ \varepsilon_{\lambda\mu})=H'(U)DU\,, \tag{17.6}$$

where $H'(U)$ represents DH/DU. Substituting (17.6) into (17.5), we obtain

$$\sigma^{\lambda\mu}\frac{\partial f(\sigma^{\lambda\mu},\ \varepsilon_{\lambda\mu})}{\partial\sigma^{\lambda\mu}}D\lambda=\frac{Df(\sigma^{\lambda\mu},\ \varepsilon_{\lambda\mu})}{H'}+DQ\,,$$

and therefore

$$D\lambda=\frac{Df(\sigma^{\lambda\mu},\ \varepsilon_{\lambda\mu})+H'(U)DQ}{H'(U)\sigma^{\lambda\mu}\dfrac{\partial f(\sigma^{\lambda\mu},\ \varepsilon_{\lambda\mu})}{\partial\sigma^{\lambda\mu}}}\,. \tag{17.7}$$

Similarly, according to the other coordinate system,

$$D\lambda=\frac{Df(\sigma^{ij},\ \varepsilon_{ij})+H'(U)DQ}{H'(U)\sigma^{ij}\dfrac{\partial f(\sigma^{ij},\ \varepsilon_{ij})}{\partial\sigma^{ij}}}\,, \tag{17.7'}$$

$$D\lambda=\frac{Df(\sigma^{ij},\varepsilon_{ij})-H'(U)DQ}{H'(U)\sigma^{ij}\dfrac{\partial f(\sigma^{ij},\varepsilon_{ij})}{\partial\sigma^{ij}}}. \tag{17.7''}$$

If the yield function f and the work-hardening function H are known in any way, the state equation for plastic deformation is seen to be completely determined by (16.9) and (17.7).

18. CONCLUTION

On account of it being impossible to bring the theory of plasticity, more precisely the incremental strain theory, into a logical system, especially for finite deformation, when based on the existing concept of strain specified by the change in the geometrical configuration, and its increment, we introduced a new definition of strain increment, and accordingly the strain, denominated otherwise as strain history tensor, which depends on deformation path, but not on the change in the geometrical shape directly. This strain is reduced to the so-called logarithmic strain in the special case of simple extension. The stress tensor is defined in such a way that it gives for unit of area in the deformed state the actual force exerted through it and is reduced to the so-called true stress in case of simple tension.

According to this concept of strain and its increment, the theory of plasticity become free from the essential inconsistencies ever lurked in it, and is brought into a perfectly logical system. More than that, in consequence of this, it is extended quite naturally to the theory for the small and finite deformations, which holds in just the same form as in the case of small deformation.

Finally it must be noticed that this concept of strain plays an important roll indispensable for the theory of dependence of anisotropy and the Bauschinger effect on the deformation history, which will be introduced in the following papers in succession.

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July 27, 1959

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概　要

歪履歴塑性理論

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この研究の目的は，従来塑性力学において慣用されている，歪増分の概念に本質的な修正を行なうことによって，塑性力学の理論に内在する矛盾を除去し，該理論を微小，有限変形の全領域にわたって完全に矛盾のない論理的体系に拡張かつ改良することである．

まず，一般に弾性論において用いられている，物体要素の幾何学的形状の変化によって規定される，歪およびその増分は塑性変形を記述する目的のためには不合理であることが例証される．現在の塑性力学，更に正確にいえば歪増分理論，は特殊の変形を除いて，このような歪および歪増分を用いている点で本質的な誤りを侵しており，そのための矛盾は変形の増大と共に顕著になる．

塑性変形の記述のために合法的な歪および歪増分の概念を導入するために，著者は塑性変形の本質的性格についての明確な検討を行ない，その結果，応力と共に，歪，歪増分の補足すべき基本条件を誘導した．かかる必然的推理に基づいて，ある変形状態における歪増分はその変形状態が同時に無変形の状態であるように定義される．歪はこのような歪増分を与えられた変形経路に沿って積分することによって得られ，それはその経路に依存し，変形後の幾何学的形状には直接には依らないことが示される．この歪は対象とする物質の微視的構造変化に対応するものと考えられ，塑性変形を記述するための歪テンソルとしてのみならず，たとえば異方性のような変形履歴に依存する状態を規定するところの歪履歴テンソルとしても役立つ．更にこの歪は，単純伸張に対していわゆる対数歪を与えることが示される．したがってそれは履歴依存性一般自然歪と名付けることのできるものである．かくして塑性変形は二重の意味において，すなわち第1に歪それ自身において，第2に応力・歪関係において，履歴現象であることが明らかとなる．

応力は物質中の単位面積に対して，それに作用する現実の力を与えるようなテンソルとし定義される．この応力は，特に単純引張りに対しては，いわゆる真応力を与える．

歪，歪増分および応力をこのように定義することによって初めて，仮想仕事の原理が，微小，有限変影の全領域にわたって，微小変形の場合と全く同じ形式で表現されることが示される．この結果，このような一般の変形に対する平衡方程式，状態方程式等のすべての関係がまた，微小変形の場合と同様の形で成立する．このようにして塑性力学，すなわち歪増分理論，はその根本から組換えられ，極めて自然に微小および有限の一般の変形の場合に拡張される．