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Thermal Buckling of Rectangular Plates*

By

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Summary. Thermal buckling of the plate simply supported by the web is analysed when the system is subjected to an arbitrary symmetrical temperature distribution over the plate surface. The buckling criterion is established and, furthermore, a simple formula for buckling criterion is obtained with reasonable accuracy for engineering purposes. Special attention is paid in exactly satisfying the boundary conditions, and the solution thus obtained is compared with those of Hayashi, Klosner and Forray, who have treated it approximately. It is found that their solutions for buckling criterion are appreciably larger than the value in the present analysis and, either decrease in the stiffness of webs or increase in the non-uniformity of temperature can result in a much larger descrepancy with more accurate value.

INTRODUCTION

On thermal buckling of supersonic wing panels heated by the air flowing in the boundary layer many researches have already been made in both analytical and experimental procedures. One branch in this problem, that is, thermal buckling of a rectangular plate subjected to non-uniform temperature distributions has already been treated by N. J. Hoff [1], T. Hayashi [2], J. M. Klosner, and M. J. Forray [3] and others in the case where the edges of the plate are supported by shear webs.

Hoff analysed the problem by assuming one-dimensional temperature distributions. Hayashi solved the problem in the case where the temperature distribution is expressed as $\theta=\theta_{00}+\theta_{11}\sin\frac{\pi x}{2a}\sin\frac{\pi y}{2b}$. Klosner and Forray obtained a buckling criterion under more general temperature distributions, i.e., arbitrary symmetrical temperature distributions.

Before entering into the presentation of author's analysis, we had better inquire into several physical conditions imposed on thermal buckling of supersonic wing panels and then criticize the validity of analysises of earlier studies. If the thickness of the plate is small and if there are no large temperature gradients in the plane of the plate or through the thickness of the plate, the stress may be regarded as two-dimensional. Usually, these requirements may be fulfilled in the systems of

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the supersonic wing panels and, therefore, to solve the two-dimensional thermal-stress problem is to be presupposed for the analysis of thermal buckling. It is always possible to solve rigorously the two-dimensional thermal-stress problem of the rectangular plate whether the boundary is free from any external forces or, on the contrary, entirely rigid.

But the matter becomes quite difficult to solve, if the boundary of the plate is supported by the web having finite stiffness and thermal expansion and, accordingly, the web has both tangential and normal components of displacement to the edge of the plate. In this case, then, the matter becomes to the two-dimensional thermal-stress problem having changeable boundary conditions, the solution of which is difficult to obtain.

Moreover, one restriction should be added to the boundary conditions, since the system of a plate and webs now being treated is a part of the wing or the body of vehicles. This restriction is that the edges of the plate should deform in a manner in which the edges remain straight, because the edges of a system are connected by the other simmilar systems. Hayashi, Klosner and Forray solved the problem by using some approximate treatments and avoided the difficulty arising from the required complicated boundary conditions. Their approximate methods abandon the attempt to satisfy the boundary conditions for every part of the edges and rather use the integration of these boundary conditions for displacements, which are used in place of original conditions.

The analysis proposed by the author in this paper treats the rectangular plate subjected to the symmetrical temperature distributions expressed as:

$$\theta=\theta_{00}+\sum_{p:\text{odd}}^{\infty}\sum_{q:\text{odd}}^{\infty}\theta_{pq}\sin\frac{p\pi x}{2a}\sin\frac{q\pi y}{2b}$$

and it is easily applicable to more general symmetrical temperature distributions. The ideas and the principles of author's method are: first, by the use of Duhamel's analogy the thermal-stress problem can be converted into the iso-thermal problem subjected to an appropriate imaginary hydrostatic pressure and imaginary body forces; secondly, the two-dimensional plane stress problems subjected to body forces can be solved rigorously by the principle of virtual displacements provided that the displacements are given at the edges; and finally, these displacements expressing the boundary conditions can be so defined that the conservation of the straightness of the edges after deformation may be satisfied automatically.

In short, the Klosner and Forray's method is based on the principle of least work which affords a powerfull means in the case where the boundary conditions are given in terms of stresses and, on the contrary, the author's method is based on the principle of minimum potential energy which is suitable in the case where the boundary conditions are given in terms of displacement.

Thermal Stresses in Rectangular Plates

The temperature distribution θ acting on the plate is assumed to be symmetrical about the centerlines of the plate, and besides, the uniform temperature at the edges of the plate is assumed. (The latter restriction can easily be released if desired.) Then the temperature can be expressed by the Fourier's series as

$$\theta=\theta_{00}+\sum_{p:\mathrm{odd}}^{\infty}\sum_{q:\mathrm{odd}}^{\infty}\theta_{pq}\sin\frac{p\pi x}{2a}\sin\frac{q\pi y}{2b}\,. \tag{1}$$

Or, it is usually more convenient to write it in a dimensionless form

$$\frac{\theta}{\theta_{\mathrm{av}}}=T_{00}+\sum_{p:\mathrm{odd}}^{\infty}\sum_{q:\mathrm{odd}}^{\infty}T_{pq}\sin\frac{p\pi x}{2a}\sin\frac{q\pi y}{2b}, \tag{2}$$

where θ_{av} is the average temperature of the plate

$$\theta_{\mathrm{av}}=\frac{1}{4ab}\int_0^{2a}\int_0^{2b}\theta dydx \tag{3}$$

and, consequently, there exists the following relation between the uniform term, T_{00}, of the temperature and the non-uniform term, T_{pq}, both in dimensionless forms.

$$T_{00}+\frac{4}{\pi^2}\sum_{p:\mathrm{odd}}^{\infty}\sum_{q:\mathrm{odd}}^{\infty}\frac{1}{pq}T_{pq}=1\,. \tag{4}$$

It is usually accepted that the temperature gradient in the direction of the thickness of the plate is negligible provided that the plate is thin.

In solving the thermal-stress problem, Duhamel's analogy can be of great convenience since, by use of the relations comprising it, it is possible to establish a correspondence between a thermal-stress problem for a given body and a fictitious stress problem for the same body at a uniform temperature. In this plane thermal-stress problem, the thermal-stress is obtained by superposing fictitious 'hydrostatic pressures', σ_{xh}, σ_{yh}, and τ_{xyh}, on the stresses, σ_{xb}, σ_{yb}, and τ_{xyb}, produced by fictitious 'body forces', X and Y, provided that the edges of the plate are fixed. Thus,

$$\left.\begin{aligned}\sigma_x&=\sigma_{xh}+\sigma_{xb}\,,\\ \sigma_y&=\sigma_{yh}+\sigma_{yb}\,,\\ \tau_{xy}&=\tau_{xyh}+\tau_{xyb}\,,\end{aligned}\right\} \tag{5}$$

and

$$\left.\begin{aligned}&\sigma_{xh}=\sigma_{yh}=-\frac{\alpha E\theta}{1-\nu},\ \tau_{xyh}=0\,,\\ &X=-\frac{\alpha E}{1-\nu}\cdot\frac{\partial T}{\partial x},\\ &Y=-\frac{\alpha E}{1-\nu}\cdot\frac{\partial T}{\partial y}.\end{aligned}\right\} \tag{6}$$

The solution of this problem can be obtained rigorously by the standard use of the principle of virtual displacements. In this section, we shall treat the case where the edges are fixed and in the following section we shall treat another case the edges allow to displace in the plane of the plate, but not vertically to the plane.

When the fictitious body forces, X and Y, are applied to the plate, the equations of equilibrium are

$$\left.\begin{aligned}&\frac{\partial\sigma_{xb}}{\partial x}+\frac{\partial\tau_{xyb}}{\partial y}+X=0,\\ &\frac{\partial\sigma_{yb}}{\partial y}+\frac{\partial\tau_{xyb}}{\partial x}+Y=0.\end{aligned}\right\} \tag{7}$$

As the temperature distribution is taken to be symmetrical about the centerlines, x component of the displacement should be asymmetrical about x axis and be symmetrical about y axis. The same consideration is to be given to the y component of the displacements. Then the displacements, u and v can be represented by series as

$$\left.\begin{aligned} u(x,y) &= \sum_{m:\text{even}}^{\infty} \sum_{n:\text{odd}}^{\infty} A_{mn} \sin\frac{m\pi x}{2a} \sin\frac{n\pi y}{2b}, \\ v(x,y) &= \sum_{m:\text{odd}}^{\infty} \sum_{n:\text{even}}^{\infty} B_{mn} \sin\frac{m\pi x}{2a} \sin\frac{n\pi y}{2b}, \end{aligned}\right\} \tag{8}$$

where A_{mn} and B_{mn} are unknown coefficients. Obviously these displacements satisfy the prescribed boundary conditions in this case, i.e.,

$$\left.\begin{aligned} u=0, \ (x=0,\ 2a\ ;\ y=0,\ 2b), \\ v=0, \ (x=0,\ 2a\ ;\ y=0,\ 2b). \end{aligned}\right\} \tag{9}$$

For the calculation of the coefficients A_{mn} and B_{mn}, the principle of virtual displacements can be used. Taking virtual displacements in the form

$$\left.\begin{aligned} \delta u &= \delta A_{mn} \sin\frac{m\pi x}{2a} \sin\frac{n\pi y}{2b}, \\ \delta v &= \delta B_{mn} \sin\frac{m\pi x}{2a} \sin\frac{n\pi y}{2b}, \end{aligned}\right\} \tag{10}$$

we obtain the necessary equations for calculating the coefficients A_{mn} and B_{mn} in the following form:

$$\left.\begin{aligned} &\int_0^{2a}\int_0^{2b} \left(\frac{\partial \sigma_{xb}}{\partial x} + \frac{\partial \tau_{xyb}}{\partial y} + X\right) \sin\frac{m\pi x}{2a} \sin\frac{n\pi y}{2b}\, dy dx = 0, \\ &\qquad\qquad (m\text{: even},\ n\text{: odd}) \\ &\int_0^{2a}\int_0^{2b} \left(\frac{\partial \sigma_{yb}}{\partial y} + \frac{\partial \tau_{xyb}}{\partial x} + Y\right) \sin\frac{m\pi x}{2a} \sin\frac{n\pi y}{2b}\, dy dx = 0. \\ &\qquad\qquad (m\text{: odd},\ n\text{: even}) \end{aligned}\right\} \tag{11}$$

By using the stress-displacement relations, i.e.,

$$\left.\begin{aligned} \sigma_{xb} &= \frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y}\right), \\ \sigma_{yb} &= \frac{E}{1-\nu^2}\left(\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x}\right), \\ \tau_{xyb} &= \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \end{aligned}\right\} \tag{12}$$

Eq. (11) can be written in terms of the temperature:

$$\int_0^{2a}\int_0^{2b} \Big\{ \sum_{m:\text{even}}^{\infty} \sum_{n:\text{odd}}^{\infty} A_{mn} \left[-\frac{m^2\pi}{4a^2(1-\nu)} - \frac{n^2\pi}{8b^2(1+\nu)} \right] \sin\frac{m\pi x}{2a} \sin\frac{n\pi y}{2b}$$
$$+ \sum_{m:\text{odd}}^{\infty} \sum_{n:\text{even}}^{\infty} B_{mn} \left[\frac{mn\pi}{8ab(1-\nu)} \right] \cos\frac{m\pi x}{2a} \cos\frac{n\pi y}{2b}$$

$$
\left.
\begin{aligned}
&+\sum_{p:\text{odd}}^{\infty}\sum_{q:\text{odd}}^{\infty}\theta_{\text{av}}T_{pq}\left[\frac{-p\alpha}{2a(1-\nu)}\right]\cos\frac{p\pi x}{2a}\sin\frac{q\pi y}{2b}\Bigg\} \\
&\qquad\qquad\qquad\times\sin\frac{m\pi x}{2a}\sin\frac{n\pi y}{2b}\,dydx=0\,, \\
&\qquad\qquad\qquad(m\text{: even},\ n\text{: odd}) \\
&\int_0^{2a}\int_0^{2b}\Bigg\{\sum_{m:\text{odd}}^{\infty}\sum_{n:\text{even}}^{\infty}B_{mn}\left[-\frac{n^2\pi}{4b^2(1-\nu)}-\frac{m^2\pi}{8a^2(1+\nu)}\right]\sin\frac{m\pi x}{2a}\sin\frac{n\pi y}{2b} \\
&+\sum_{m:\text{even}}^{\infty}\sum_{n:\text{odd}}^{\infty}A_{mn}\left[\frac{mn\pi}{8ab(1-\nu)}\right]\cos\frac{m\pi x}{2a}\cos\frac{n\pi y}{2b} \\
&+\sum_{p:\text{odd}}^{\infty}\sum_{q:\text{odd}}^{\infty}\theta_{\text{av}}T_{pq}\left[-\frac{q\alpha}{2b(1-\nu)}\right]\sin\frac{p\pi x}{2a}\cos\frac{q\pi y}{2b}\Bigg\} \\
&\qquad\qquad\qquad\times\sin\frac{m\pi x}{2a}\sin\frac{n\pi y}{2b}\,dydx=0\,. \\
&\qquad\qquad\qquad(m\text{: odd},\ n\text{: even})
\end{aligned}
\right\}\qquad(13)
$$

Integration of above equations results in the following expressions for the coefficients:

$$
\left.
\begin{aligned}
A_{mn}&=-\frac{8\alpha}{\pi^2a(1-\nu)}\cdot\frac{1}{\dfrac{m^2}{a^2(1-\nu^2)}+\dfrac{n^2}{2b^2(1+\nu)}}\sum_{p:\text{odd}}^{\infty}\theta_{\text{av}}T_{pq}\frac{mp}{m^2-p^2}\delta(q,n)\,, \\
B_{mn}&=-\frac{8\alpha}{\pi^2b(1-\nu)}\cdot\frac{1}{\dfrac{m^2}{2a^2(1+\nu)}+\dfrac{n^2}{b^2(1-\nu)}}\sum_{q:\text{odd}}^{\infty}\theta_{\text{av}}T_{pq}\frac{nq}{n^2-q^2}\delta(p,m)\,,
\end{aligned}
\right\}\qquad(14)
$$

where $\delta(q, n)$ and $\delta(p, m)$ are the Kronecker's deltas defined as follows:

$$
\delta(q,n)=\begin{cases}1\ ;\ q=n\,,\\ 0\ ;\ q\neq n\,.\end{cases}\qquad(15)
$$

Thus, the stresses σ_{xb}, σ_{yb}, and τ_{xyb}, caused by the fictitious body forces, are given by the following expressions:

$$
\left.
\begin{aligned}
\sigma_{xb}=&-\frac{4\alpha E}{\pi(1-\nu)}\cdot\sum_{m:\text{even}}^{\infty}\sum_{n:\text{odd}}^{\infty}\left[\frac{1}{m^2+\dfrac{n^2(1-\nu)a^2}{2b^2}}\sum_{p:\text{odd}}^{\infty}\theta_{\text{av}}T_{pq}\frac{m^2p}{m^2-p^2}\delta(q,n)\right] \\
&\qquad\qquad\qquad\times\cos\frac{m\pi x}{2a}\sin\frac{n\pi y}{2b} \\
&-\frac{4\alpha E}{\pi(1-\nu)}\nu\sum_{m:\text{odd}}^{\infty}\sum_{n:\text{even}}^{\infty}\left[\frac{1}{\dfrac{m^2(1-\nu)b^2}{2a^2}+n^2}\sum_{q:\text{odd}}^{\infty}\theta_{\text{av}}T_{pq}\frac{n^2q}{n^2-q^2}\delta(p,m)\right] \\
&\qquad\qquad\qquad\times\sin\frac{m\pi x}{2a}\cos\frac{n\pi y}{2b}\,, \\
\sigma_{yb}=&-\frac{4\alpha E}{\pi(1-\nu)}\sum_{m:\text{odd}}^{\infty}\sum_{n:\text{even}}^{\infty}\left[\frac{1}{\dfrac{m^2(1-\nu)b^2}{2a^2}+n^2}\sum_{q:\text{odd}}^{\infty}\theta_{\text{av}}T_{pq}\frac{n^2q}{n^2-q^2}\delta(p,m)\right] \\
&\qquad\qquad\qquad\times\sin\frac{m\pi x}{2a}\cos\frac{n\pi y}{2b} \\
&-\frac{4\alpha E}{\pi(1-\nu)}\sum_{m:\text{even}}^{\infty}\sum_{n:\text{odd}}^{\infty}\left[\frac{1}{m^2+\dfrac{n^2(1-\nu)a^2}{2b^2}}\sum_{p:\text{odd}}^{\infty}\theta_{\text{av}}T_{pq}\frac{m^2p}{m^2-p^2}\delta(q,n)\right] \\
&\qquad\qquad\qquad\times\cos\frac{m\pi x}{2a}\sin\frac{n\pi y}{2b}\,,
\end{aligned}
\right\}\qquad(16)
$$

$$\tau_{xyb}=-\frac{2\alpha E}{\pi}\sum_{m:\text{even}}^{\infty}\sum_{n:\text{odd}}^{\infty}\frac{1}{m^2\frac{b}{a}+\frac{n^2(1-\nu)}{2}\frac{a}{b}}\sum_{p:\text{odd}}^{\infty}\theta_{\text{av}}T_{pq}\frac{mn\pi}{m^2-p^2}\delta(q,n)\times\sin\frac{m\pi x}{2a}\cos\frac{n\pi y}{2b}$$
$$-\frac{2\alpha E}{\pi}\sum_{m:\text{odd}}^{\infty}\sum_{n:\text{even}}^{\infty}\frac{1}{\frac{m^2(1-\nu)}{2}\frac{b}{a}+n^2\frac{a}{b}}\sum_{q:\text{odd}}^{\infty}\theta_{\text{av}}T_{pq}\frac{mn\pi}{n^2-q^2}\delta(p,m)\times\cos\frac{m\pi x}{2a}\sin\frac{n\pi y}{2b}.$$

and these infinite series form the complete solutions of the stresses. Unfortunately these expressions present great complexity in their forms, and it is difficult to use these expressions in calculating the buckling load and, furthermore, these are in no way the forms which may suggest to us readily understandable relations between thermal-stresses and temperature.

After due consideration about these formulas, it will be convenient for succeeding analysis to calculate first the stress due to some particular term,

$$\theta_{\text{av}}T_{pq}\sin\frac{p\pi x}{2a}\sin\frac{q\pi y}{2b}$$

in the infinite series of the temperature expression and, superposing together, we can obtain planer expressions for stresses due to the arbitrary temperature. Namely, it means the rearrangement of the summations in Eq. (16). Thus, the thermal-stresses due to the particular term of the expression of the temperature are written, exclusive of the stresses σ_{xpqh} and σ_{ypqh}.

$$\left.\begin{aligned}
-\frac{1-\nu}{\alpha E}\sigma_{xpqb}&=\theta_{\text{av}}T_{pq}\sin\frac{q\pi y}{2b}\sum_{m:\text{even}}^{\infty}\frac{4}{\pi}\frac{1}{m^2+\frac{q^2(1-\nu)a^2}{2b^2}}\cdot\frac{m^2p}{m^2-p^2}\cos\frac{m\pi x}{2a}\\
&\quad+\nu\theta_{\text{av}}T_{pq}\sin\frac{p\pi x}{2a}\sum_{n:\text{even}}^{\infty}\frac{4}{\pi}\frac{1}{\frac{p^2(1-\nu)b^2}{2a^2}+n^2}\frac{n^2q}{n^2-q^2}\cos\frac{n\pi y}{2b},\\
-\frac{1-\nu}{\alpha E}\tau_{xypqb}&=\theta_{\text{av}}T_{pq}\cos\frac{q\pi y}{2b}\sum_{m:\text{even}}^{\infty}\frac{2}{\pi}\frac{1-\nu}{m^2\frac{b}{a}+\frac{q^2(1-\nu)}{2}\frac{a}{b}}\frac{mpq}{m^2-p^2}\sin\frac{m\pi x}{2a}\\
&\quad+\theta_{\text{av}}T_{pq}\cos\frac{p\pi x}{2a}\sum_{n:\text{even}}^{\infty}\frac{2}{\pi}\frac{1-\nu}{\frac{p^2(1-\nu)}{2}\frac{b}{a}+n^2\frac{a}{b}}\frac{npq}{n^2-q^2}\sin\frac{n\pi y}{2b}.
\end{aligned}\right\}\quad(17)$$

The close relations between stresses and the temperature are comprehensible. Superposing these stresses from $p=1$ to $p=\infty$ or $q=1$ to $q=\infty$, and adding to them the stresses σ_{xh} and σ_{yh} due to fictitious hydrostatic pressure, we finally obtain the thermal-stresses in the plate due to arbitrary symmetrical temperature distributions.

$$-\frac{1-\nu}{\alpha E}\sigma_x=\theta_{\text{av}}T_{00}+\theta_{\text{av}}\sum_{p:\text{odd}}^{\infty}\sum_{q:\text{odd}}^{\infty}T_{pq}\Big[\sin\frac{p\pi x}{2a}\sin\frac{q\pi y}{2b}$$
$$+\sin\frac{q\pi y}{2b}\sum_{m:\text{even}}^{\infty}C_{pqmo}\cos\frac{m\pi x}{2a}+\sin\frac{p\pi x}{2a}\nu\sum_{n:\text{even}}^{\infty}C_{pqon}\cos\frac{n\pi y}{2b}\Big],$$

$$
\left.
\begin{aligned}
-\frac{1-\nu}{\alpha E}\sigma_y&=\theta_{av}T_{00}+\theta_{av}\sum_{p:odd}^{\infty}\sum_{q:odd}^{\infty}T_{pq}\Big[\sin\frac{p\pi x}{2a}\sin\frac{q\pi y}{2b}\\
&\quad+\sin\frac{p\pi x}{2a}\sum_{n:even}^{\infty}C_{pqon}\cos\frac{n\pi y}{2b}+\sin\frac{q\pi y}{2b}\nu\sum_{m:even}^{\infty}C_{pqmo}\cos\frac{m\pi x}{2a}\Big],\\
-\frac{1-\nu}{\alpha E}\tau_{xy}&=\theta_{av}\sum_{p:odd}^{\infty}\sum_{q:odd}^{\infty}T_{pq}\Big[\cos\frac{q\pi y}{2b}\sum_{m:even}^{\infty}C^{s}_{pqmo}\sin\frac{m\pi x}{2a}\\
&\quad+\cos\frac{p\pi x}{2a}\sum_{n:even}^{\infty}C^{s}_{pqon}\sin\frac{n\pi y}{2b}\Big],
\end{aligned}
\right\}\quad(18)
$$

where coefficients, C_{pqmo}, C_{pqon}, C^{s}_{pqmo}, and C^{s}_{pqon}, are defined as follows:

$$
\left.
\begin{aligned}
C_{pqmo}&=\frac{4}{\pi}\frac{1}{m^2+\dfrac{q^2(1-\nu)a^2}{2b^2}}\frac{m^2p}{m^2-p^2},\\
C_{pqon}&=\frac{4}{\pi}\frac{1}{\dfrac{p^2(1-\nu)b^2}{2a^2}+n^2}\frac{n^2q}{n^2-q^2},\\
C^{s}_{pqmo}&=\frac{2}{\pi}\frac{1-\nu}{m^2\dfrac{b}{a}+\dfrac{q^2(1-\nu)}{2}\dfrac{a}{b}}\frac{mpq}{m^2-p^2},\\
C^{s}_{pqon}&=\frac{2}{\pi}\frac{1-\nu}{\dfrac{p^2(1-\nu)}{2}\dfrac{b}{a}+n^2\dfrac{a}{b}}\frac{npq}{n^2-q^2}.
\end{aligned}
\right\}\quad(19)
$$

Effects of the Temperature and the Stiffness of Web on Thermal-stresses in the Plate

In the preceding section we obtained the rigorous solution in the case where the midplane displacements at the edges of the plate are zero, i.e., the boundary is rigid. In practical cases, this condition usually can not be satisfied exactly and the midplane displacements at the edges are permissible to a certain extent. In consequence, the thermal-stresses in the plate are subjected to the influence of the edge displacements due to the temperature and the stiffness of the web. The edge displacement, however, can not be obtained independently on the stresses within the plate. Then the problem becomes the one having variable boundary conditions which is far more difficult to solve exactly than the former one. Subsequent analysis will show the principle of virtual displacements used in the preceding section can also afford a powerfull means for solving approximately the problem.

Let us start from the step of Eq. (8). In the case now under consideration, the component of the midplane displacement normal to the edge must be uniform along the edge so as to conform the requirement about the conservation of straightness of the web. The displacements then sould be expressed as

$$
\left.
\begin{aligned}
u(x,y)&=u_1(x)+\sum_{m:even}^{\infty}\sum_{n:odd}^{\infty}A_{mn}\sin\frac{m\pi x}{2a}\sin\frac{n\pi y}{2b},\\
v(x,y)&=v_1(x)+\sum_{m:odd}^{\infty}\sum_{n:even}^{\infty}B_{mn}\sin\frac{m\pi x}{2a}\sin\frac{n\pi y}{2b},
\end{aligned}
\right\}\quad(20)
$$

where $u_1(x)$ and $v_1(y)$ are the unknown functions of x only or y only, respectively, and are the displacement functions of the boundary. From this it follows that the requirement about the conservation of straightness of the web after deformation can be satisfied in all cases, since $u_1(x)$ and $v_1(y)$ are constant at $x=0$ and $x=2a$, $y=0$ and $y=2b$, respectively, and the second terms of the right hand of Eq. (20) vanishes at the boundary.

To decide the correct form of the boundary displacement functions, $u_1(x)$ and $v_1(y)$, satisfying the condition aforementioned is relating to the solution of the problem itself which we shall now intend to obtain. This inconsistency arises from the approximate manner of treating the problem, i.e., the single system consist of a plate and webs is divided into separate but yet relating two systems of a plate and webs and then, the latter system is regarded as the boundary condition as if it were given previously. Thus, some ingenious procedure is required to avoid this difficulty.

It is reasonable for first order approximation to assume that the edge displacement parallel to the edge is linear along it. This corresponds to the condition of uniform temperature of both plate and web and it seems to assure a good approximation unless the temperature differs considerably from the uniform one. Then the following functions are taken for the boundary displacement functions.

$$\left.\begin{aligned} u_1(x)&=Ax\,,\\ v_1(y)&=By\,,\end{aligned}\right\} \tag{21}$$

where A and B are the unknown coefficients which are to be determined by equating resultant displacements of both the plate and the web.

The principle of virtual displacements affords a means of making the approximation more correctly. For example, we can take the boundary displacement functions as follows:

$$\left.\begin{aligned} u_1(x)&=A_2u_2(x)\,,\\ v_1(y)&=B_2v_2(y)\,,\end{aligned}\right\} \tag{22}$$

where unknown coefficients A_2 and B_2 are to be determined by the same method as the preceding one, while the functions $u_2(x)$ and $v_2(y)$, which are constant at $x=0$ and $x=2a$, $y=0$ and $y=2b$, respectively, are to be determined by the equilibrium conditions within the narrow strip of the plate and the web being cut by the two lines of x and $x+\Delta x$ (or y and $y+\Delta y$). In other words, the total amount of the boundary displacement is decided by the method mentioned before, while the local distribution of the displacement is determined by the forms of function $u_2(x)$ and $v_2(y)$.

This local distribution, however, seems to have merely a secondary effect on the stresses in the plate, and so it would not be advisable for us to use the latter complicated form for the purpose of comprising this second order effect, because it can not exclude the possibility of missing a wide view of the whole matter. We, therefore, will use Eq. (21) for the boundary displacement functions in the following analysis.

Using Eq. (20) in place of Eq. (8) and after the same manipulation as before, we have the following equations in place of Eq. (18):

$$\left.\begin{aligned}
-\frac{1-\nu}{\alpha E}\sigma_x &= -\frac{A+\nu B}{\alpha(1+\nu)}+\theta_{av}T_{00}+\theta_{av}\sum_{p:odd}^{\infty}\sum_{q:odd}^{\infty}T_{pq}\left[\sin\frac{p\pi x}{2a}\cdots\right],\\
-\frac{1-\nu}{\alpha E}\sigma_y &= -\frac{B+\nu A}{\alpha(1+\nu)}+\theta_{av}T_{00}+\theta_{av}\sum_{p:odd}^{\infty}\sum_{q:odd}^{\infty}T_{pq}\left[\sin\frac{p\pi x}{2a}\cdots\right],\\
-\frac{1-\nu}{\alpha E}\tau_{xy} &= \theta_{av}\sum_{p:odd}^{\infty}\sum_{q:odd}^{\infty}T_{pq}\left[\cos\frac{q\pi y}{2b}\cdots\right].
\end{aligned}\right\} \tag{23}$$

The first terms of the right hand of first two equations are newly added to the solution for the rigid boundary problem and apparently indicate the effect of boundary displacement.

In the calculation of the coefficients A and B the equilibrium conditions of applied loads and stresses are to be used. They are

$$\left.\begin{aligned}
\int_0^{2b} h\sigma_x dy+2P_x=F_x\,,\\
\int_0^{2a} h\sigma_y dx+2P_y=F_y\,,
\end{aligned}\right\} \tag{24}$$

where F_x and F_y are the external forces in the x and y directions, respectively, and P_x and P_y are the axial forces in the webs in the x and y directions, respectively.

The boundary condition about the edge displacement parallel to the edge should be satisfied approximately by equating the elongation of the web due to the temperature and the axial force to the elongation of the plate. Denoting A_x, A_y, E_x, and E_y the sectional areas and the Young's moduli of webs, respectively, then the total elongation of the web in the x direction by the force P_x is

$$u_{1,2a}=\frac{2abh\alpha E}{A_xE_x(1-\nu)}\left[-\frac{A+\nu B}{\alpha(1+\nu)}+\theta_{av}\left(T_0+\sum_{p:odd}^{\infty}\sum_{q:odd}^{\infty}\frac{4}{\pi^2}\frac{1}{pq}T_{pq}\right)\right]+\frac{aF_x}{A_xE_x}. \tag{25}$$

Substituting Eq. (4) into (25) we obtain the following expression for $u_{1,2a}$ and $v_{1,2b}$:

$$\left.\begin{aligned}
u_{1,2a}&=\frac{2abh\alpha E}{A_xE_x(1-\nu)}\left[-\frac{A+\nu B}{\alpha(1+\nu)}+\theta_{av}\right]+\frac{aF_x}{A_xE_x}\,,\\
v_{1,2b}&=\frac{2abh\alpha E}{A_yE_y(1-\nu)}\left[-\frac{B+\nu A}{\alpha(1+\nu)}+\theta_{av}\right]+\frac{bF_y}{A_yE_y}.
\end{aligned}\right\} \tag{26}$$

And if we express the temperature rises and the coefficients of thermal expansion of the web in the directions of x and y by the term θ_x, θ_y, α_x, and α_y, respectively, there exist the following relations:

$$\left.\begin{aligned}
A(2a)&=u_{1,2a}+\alpha_x\theta_x(2\mathrm{a})\,,\\
B(2a)&=v_{1,2b}+\alpha_y\theta_y(2b)\,.
\end{aligned}\right\} \tag{27}$$

Substituting Eq. (26) into Eq. (27), we obtain

$$\begin{aligned}
A=\Bigg[&\left\{(1+\nu)\alpha\theta_{av}+(1-\nu^2)\frac{A_xE_x}{bhE}\alpha_x\theta_x+(1-\nu^2)\frac{F_x}{2bhE}\right\}\left\{(1-\nu^2)\frac{A_yE_y}{ahE}+1\right\}\\
&-\left\{(1+\nu)\alpha\theta_{av}+(1-\nu^2)\frac{A_yE_y}{ahE}\alpha_y\theta_y+(1-\nu^2)\frac{F_y}{2ahE}\right\}\nu\Bigg]\\
&\div\left[\left\{(1-\nu^2)\frac{A_xE_x}{bhE}+1\right\}\left\{(1-\nu^2)\frac{A_yE_y}{ahE}+1\right\}-\nu^2\right],
\end{aligned} \tag{28}$$

$$B=\left[\left\{(1+\nu)\alpha\theta_{\mathrm{av}}+(1-\nu^2)\frac{A_yE_y}{ahE}\alpha_y\theta_y+(1-\nu^2)\frac{F_y}{2ahE}\right\}\left\{(1-\nu^2)\frac{A_xE_x}{bhE}+1\right\}\right.$$
$$\left.-\left\{(1+\nu)\alpha\theta_{\mathrm{av}}+(1-\nu^2)\frac{A_xE_x}{bhE}\alpha_x\theta_x+(1-\nu^2)\frac{F_x}{2bhE}\right\}\nu\right]$$
$$\div\left[\left\{(1-\nu^2)\frac{A_yE_y}{ahE}+1\right\}\left\{(1-\nu^2)\frac{A_xE_x}{bhE}+1\right\}-\nu^2\right].$$

Here, for both convenience and better understanding, we will define 'equivalent uniform temperatures' T_{0x} and T_{0y} as follows:

$$\left.\begin{aligned}\theta_{\mathrm{av}}T_{0x}&=\theta_{\mathrm{av}}T_{00}\left\{1-\frac{A+\nu B}{\alpha(1+\nu)\theta_{\mathrm{av}}T_0}\right\},\\ \theta_{\mathrm{av}}T_{0y}&=\theta_{\mathrm{av}}T_{00}\left\{1-\frac{B+\nu A}{\alpha(1+\nu)\theta_{\mathrm{av}}T_0}\right\}.\end{aligned}\right\} \tag{29}$$

When A equals to B, T_{0x} equals to T_{0y}, too. Further we will define the stiffness ratios β_x and β_y of the webs to the plate as follows:

$$\left.\begin{aligned}\beta_x&=\frac{A_xE_x}{bhE},\\ \beta_y&=\frac{A_yE_y}{ahE}.\end{aligned}\right\} \tag{30}$$

Substituting Eqs. (28) and (30) into Eq. (29) we obtain the following expressions for the equivalent uniform temperatures:

$$\left.\begin{aligned}T_{0x}=T_{00}-&\left[\left\{1+(1-\nu)\beta_x\frac{\alpha_x\theta_x}{\alpha\theta_{\mathrm{av}}}+(1-\nu)\frac{F_x}{2bhE\alpha\theta_{\mathrm{av}}}\right\}(\beta_y+1)(1-\nu^2)\right.\\ &\left.+\left\{1+(1-\nu)\beta_y\frac{\alpha_y\theta_y}{\alpha\theta_{\mathrm{av}}}+(1-\nu)\frac{F_y}{2ahE\alpha\theta_{\mathrm{av}}}\right\}\beta_x\nu(1-\nu^2)\right]\\ &\div[\{(1-\nu^2)\beta_x+1\}\{(1-\nu^2)\beta_y+1\}-\nu^2],\\ T_{0y}=T_{00}-&\left[\left\{1+(1-\nu)\beta_y\frac{\alpha_y\theta_y}{\alpha\theta_{\mathrm{av}}}+(1-\nu)\frac{F_y}{2ahE\alpha\theta_{\mathrm{av}}}\right\}(\beta_x+1)(1-\nu^2)\right.\\ &\left.+\left\{1+(1-\nu)\beta_x\frac{\alpha_x\theta_x}{\alpha\theta_{\mathrm{av}}}+(1-\nu)\frac{F_x}{2bhE\alpha\theta_{\mathrm{av}}}\right\}\beta_y\nu(1-\nu^2)\right]\\ &\div[\{(1-\nu^2)\beta_y+1\}\{(1-\nu^2)\beta_x+1\}-\nu^2].\end{aligned}\right\} \tag{31}$$

Thus, in cases where the midplane displacements at the boundary are allowed, we can use entirely the same equation (18) as in cases of the rigid boundary so long as we use the equivalent uniform temperature T_{0x} and T_{0y} in place of the uniform temperature T_{00} in the formulas of σ_x and σ_y, respectively.

THERMAL BUCKLING OF PLATES

When the temperature of the plate rises to some value, the plate will becomes to buckle. The deflection function of the plate can be expressed by the infinite series as follows:

$$w=\sum_{s;\mathrm{odd}}^{\infty}\sum_{t;\mathrm{odd}}^{\infty}w_{st}\sin\frac{s i\pi x}{2a}\sin\frac{tj\pi y}{2b}. \tag{32}$$

The deflection mode in the x direction, for example, will be symmetrical or a-symmetrical depending upon whether i is odd or even, respectively.

The total change of energy of the system at buckling is due to (1) the energy of bending of the plate U_1, and (2) the total work done by the forces acting on the middle plane of the plate during buckling U_2. The assumption is made that the stresses acting in the middle plane of the plate remain constant during the deflection, and thus the stresses in the webs must also remain constant. Thus the total change of energy is the sum of U_1 and U_2, then,

$$\begin{aligned}U&=U_1+U_2\\&=\frac{1}{2}D\int_0^{2a}\int_0^{2b}\left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2-2(1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right\}\right]dydx\\&\quad+\frac{1}{2}h\int_0^{2a}\int_0^{2b}\left[\sigma_x\left(\frac{\partial w}{\partial x}\right)^2+\sigma_y\left(\frac{\partial w}{\partial y}\right)^2+2\tau_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]dydx\,.\end{aligned}\tag{33}$$

Since the total change of energy at buckling must be stationary, then,

$$\frac{\partial U}{\partial w_{\sigma\tau}}=0\quad,\quad\begin{pmatrix}\sigma=1,\,3,\,5\cdots\\\tau=1,\,3,\,5\cdots\end{pmatrix}\tag{34}$$

Substituting Eq. (32) into Eq. (33) we obtain the following expressions for U_1 and U_2:

$$\left.\begin{aligned}U_1&=\frac{1}{2}D\sum_{s:\mathrm{odd}}^{\infty}\sum_{t:\mathrm{odd}}^{\infty}\left\{\left(\frac{si\pi}{2a}\right)^2+\left(\frac{tj\pi}{2b}\right)^2\right\}^2 w_{st}{}^2ab\,,\\U_2&=\frac{1}{2}h\sum_{s:\mathrm{odd}}^{\infty}\sum_{\sigma:\mathrm{odd}}^{\infty}\sum_{t:\mathrm{odd}}^{\infty}\sum_{\tau:\mathrm{odd}}^{\infty}w_{st}w_{\sigma\tau}\left\{\left(\frac{si\pi}{2a}\right)\left(\frac{\sigma i\pi}{2b}\right)I_{xx}(s,\sigma,t,\tau)\right.\\&\quad\left.+\left(\frac{tj\pi}{2b}\right)\left(\frac{\tau j\pi}{2b}\right)I_{yy}(s,\sigma,t,\tau)+2\left(\frac{si\pi}{2a}\right)\left(\frac{tj\pi}{2b}\right)I_{xy}(s,\sigma,t,\tau)\right\}.\end{aligned}\right\}\tag{35}$$

In these formulas, $I_{xx}(s, \sigma, t, \tau)$, $I_{yy}(s, \sigma, t, \tau)$, and $I_{xy}(s, \sigma, t, \tau)$ are

$$\left.\begin{aligned}I_{xx}(s,\sigma,t,\tau)&=\int_0^{2a}\int_0^{2b}\sigma_x\cos\frac{si\pi x}{2a}\cos\frac{\sigma i\pi x}{2a}\sin\frac{tj\pi y}{2b}\sin\frac{\tau j\pi y}{2b}\,dydx\,,\\I_{yy}(s,\sigma,t,\tau)&=\int_0^{2a}\int_0^{2b}\sigma_y\sin\frac{si\pi x}{2a}\sin\frac{\sigma i\pi x}{2a}\cos\frac{tj\pi y}{2b}\cos\frac{\tau j\pi y}{2b}\,dydx\,,\\I_{xy}(s,\sigma,t,\tau)&=\int_0^{2a}\int_0^{2b}\tau_{xy}\cos\frac{si\pi x}{2a}\sin\frac{\sigma i\pi x}{2a}\sin\frac{tj\pi y}{2b}\cos\frac{\tau j\pi y}{2b}\,dydx\,,\end{aligned}\right\}\tag{36}$$

and if I is connected to $\boldsymbol{I}$ by the following relation,

$$\boldsymbol{I}=-\frac{1}{ab}\frac{1-\nu}{\alpha E\theta_{\mathrm{av}}}I\,,\tag{37}$$

then, $\boldsymbol{I}$ are expressed by the following equations:

$$\begin{aligned}\boldsymbol{I}_{xx}(s,\sigma,t,\tau)&=T_{0x}\delta(s,\sigma)\cdot\delta(t,\tau)\\&\quad+\sum_p^{\infty}\sum_q^{\infty}{}'T_{pq}\frac{4}{\pi^2}\left\{\frac{p}{p^2-(s-\sigma)^2i^2}+\frac{p}{p^2-(s+\sigma)^2i}\right\}\\&\qquad\times\left\{\frac{q}{q^2-(t-\tau)^2j^2}-\frac{q}{q^2-(t+\tau)^2j^2}\right\}\end{aligned}$$

$$
\left.
\begin{aligned}
&+\sum_p^\infty\sum_q^\infty T_{pq}\sum_m^\infty C_{pqmo}[\delta\{m, |s-\sigma|i\}+\delta\{m, (s+\sigma)i\}]\\
&\qquad\qquad\times\frac{1}{\pi}\left\{\frac{q}{q^2-(t-\tau)^2j^2}-\frac{q}{q^2-(t+\tau)^2j^2}\right\}\\
&+\sum_p^\infty\sum_q^\infty T_{pq}\nu\sum_n^\infty C_{pqon}[\delta\{m, |t-\tau|j\}-\delta\{n, (t+\tau)j\}]\\
&\qquad\qquad\times\frac{1}{\pi}\left\{\frac{p}{p^2-(s-\sigma)^2i^2}+\frac{p}{p^2-(s+\sigma)^2i^2}\right\},\\
&\boldsymbol{I}_{yy}(s, \sigma, t, \tau)=T_{0y}\delta(s, \sigma)\cdot\delta(t, \tau)\\
&+\sum_p^\infty\sum_q^\infty T_{pq}\frac{4}{\pi^2}\left\{\frac{p}{p^2-(s-\sigma)^2i^2}-\frac{p}{p^2-(s+\sigma)^2i^2}\right\}\\
&\qquad\qquad\times\left\{\frac{q}{q^2-(t-\tau)^2j^2}+\frac{q}{q^2-(t+\tau)^2j^2}\right\}\\
&+\sum_p^\infty\sum_q^\infty T_{pq}\sum_n^\infty C_{pqon}[\delta\{n, |t-\tau|j\}+\delta\{n, (t+\tau)j\}]\\
&\qquad\qquad\times\frac{1}{\pi}\left\{\frac{p}{p^2-(s-\sigma)^2i^2}-\frac{p}{p^2-(s+\sigma)^2i^2}\right\}\\
&+\sum_p^\infty\sum_q^\infty T_{pq}\nu\sum_m^\infty C_{pqmo}[\delta\{m, |s-\sigma|i\}-\delta\{m, (s+\sigma)i\}]\\
&\qquad\qquad\times\frac{1}{\pi}\left\{\frac{q}{q^2-(t-\tau)^2j^2}+\frac{q}{q^2-(t+\tau)^2j^2}\right\},\\
&\boldsymbol{I}_{xy}(s, \sigma, t, \tau)\\
&=\sum_p^\infty\sum_q^\infty T_{pq}\sum_m^\infty C^s_{pqmo}\big[\delta\{m, (s+\sigma)i\}+\delta\{m, |s-\sigma|i\}[1-2\delta\{m, (s-\sigma)i\}]\big]\\
&\qquad\qquad\times\frac{1}{\pi}\left\{\frac{(t+\tau)j}{(t+\tau)^2j^2-q^2}+\frac{(t-\tau)j}{(t-\tau)^2j^2-q^2}\right\}\\
&+\sum_p^\infty\sum_q^\infty T_{pq}\sum_n^\infty C^s_{pqon}\big[\delta\{n, (t+\tau)j\}+\delta\{n, |t-\tau|j\}[1-2\delta\{n, (t-\tau)j\}]\big]\\
&\qquad\qquad\times\frac{1}{\pi}\left\{\frac{(s+\sigma)i}{(s+\sigma)^2i^2-p^2}-\frac{(s-\sigma)i}{(s-\sigma)^2i^2-p^2}\right\},
\end{aligned}
\right\}
\tag{38}
$$

where δ is the Klonecker's delta defined by the following formula

$$
\delta\{m, (s+\sigma)i\}=\begin{cases}1\ ; & m=(s+\sigma)i\,,\\ 0\ ; & m\neq(s+\sigma)i\,.\end{cases}
\tag{39}
$$

Substituting Eqs. (37) and (38) into Eqs. (33) and (34) we obtain an infinite set of linear homogeneous simultaneous algebraic equations as follows:

$$
\begin{aligned}
&Dw_{\sigma\tau}\left\{\left(\frac{\sigma i\pi}{2a}\right)^2+\left(\frac{\tau j\pi}{2b}\right)^2\right\}^2ab\\
&+h\sum_s^\infty\sum_t^\infty w_{st}\left\{\left(\frac{si\pi}{2a}\right)\left(\frac{\sigma i\pi}{2b}\right)\boldsymbol{I}_{xx}(s, \sigma, t, \tau)+\left(\frac{tj\pi}{2b}\right)\left(\frac{\tau j\pi}{2b}\right)\boldsymbol{I}_{yy}(s, \sigma, t, \tau)\right.\\
&\qquad\left.+\left(\frac{si\pi}{2a}\right)\left(\frac{\tau j\pi}{2b}\right)\boldsymbol{I}_{xy}(s, \sigma, t, \tau)+\left(\frac{\sigma i\pi}{2a}\right)\left(\frac{tj\pi}{2b}\right)\boldsymbol{I}_{xy}(\sigma, s, \tau, t)\right\}=0
\end{aligned}
\tag{40}
$$

$$
\begin{pmatrix}\sigma=1, 3, 5\cdots\\ \tau=1, 3, 5\cdots\end{pmatrix}
$$

It will be convenient to define the buckling parameter K_T instead of the temperature itself in the following analysis, i.e.,

$$
K_T=\frac{48}{\pi^2}(1+\nu)\alpha\theta_{\mathrm{av}}\left(\frac{b}{h}\right)^2.
\tag{41}
$$

Using this buckling parameter, Eq. (40) can be written as

$$-\frac{1}{K_T}\left(\frac{a}{b}\right)\left\{\sigma^2\frac{b^2}{a^2}i^2+\tau^2j^2\right\}^2 w_{\sigma\tau}$$
$$+\sum_s^\infty\sum_t^\infty w_{st}\left\{s\sigma i^2\frac{b}{a}\boldsymbol{I}_{xx}(s,\sigma,t,\tau)+t\tau j^2\frac{a}{b}\boldsymbol{I}_{yy}(s,\sigma,t,\tau)\right.$$
$$\left.+s\tau ij\boldsymbol{I}_{xy}(s,\sigma,t,\tau)+\sigma tij\boldsymbol{I}_{xy}(\sigma,s,\tau,t)\right\}=0 \tag{42}$$
$$\begin{pmatrix}\sigma=1,3,5\cdots\\ \tau=1,3,5\cdots\end{pmatrix}$$

This set of simultaneous equations constitute a characteristic-value problem, the solution of which gives sets of relative values of the coefficients w_{st} and associated values of the buckling parameter. One method of solving Eq. (42) is a matrix iteration process which is described in reference [5].

If only the terms w_{11}, w_{13}, w_{31}, and w_{33} are retained in the deflection function (Eq. (32)), Eq. (42) may be written in a matrix form as

$$\begin{bmatrix}K_{11} & K_{13} & K_{31} & K_{33}\\ L_{11} & L_{13} & L_{31} & L_{33}\\ M_{11} & M_{13} & M_{31} & M_{33}\\ N_{11} & N_{13} & N_{31} & N_{33}\end{bmatrix}\begin{Bmatrix}w_{11}\\ w_{13}\\ w_{31}\\ w_{33}\end{Bmatrix}=\frac{1}{K_T}\begin{Bmatrix}w_{11}\\ w_{13}\\ w_{31}\\ w_{33}\end{Bmatrix}. \tag{43}$$

The elements K_{st}, L_{st}, M_{st}, and N_{st} are defined as follows:

$$\frac{1}{\left(\frac{a}{b}\right)\left\{\sigma^2\frac{b^2}{a^2}i^2+\tau^2j^2\right\}^2}\left\{s\sigma i^2\frac{b}{a}\boldsymbol{I}_{xx}(s,\sigma,t,\tau)+t\tau j^2\frac{a}{b}\boldsymbol{I}_{yy}(s,\sigma,t,\tau)\right.$$
$$\left.+s\tau ij\boldsymbol{I}_{xy}(s,\sigma,t,\tau)+\sigma tij\boldsymbol{I}_{xy}(\sigma,s,\tau,t)\right\}$$
$$=K_{st},\ L_{st},\ M_{st},\ N_{st} \tag{44}$$

where K_{st}, for example, is taken in the right hand of the equation, if $(\sigma,\tau)=(1,1)$ and so on. That is,

$$\begin{aligned}K_{st}\ &;\ (\sigma,\tau)=(1,1)\\ L_{st}\ &;\ (\sigma,\tau)=(1,3)\\ M_{st}\ &;\ (\sigma,\tau)=(3,1)\\ N_{st}\ &;\ (\sigma,\tau)=(3,3)\,.\end{aligned}$$

The solution for the largest value of $\frac{1}{K_T}$, and hence for the smallest value of K_T, is obtained from the matrix iteration of Eq. (43).

Examination of the Solution

By the method developed in the foregoing sections it is always possible to calculate the non-dimensional thermal buckling parameter K_T for various types of temperature distribution. In this section K_T will be calculated in the case where the temperature distribution in the plate is expressed as

$$\theta=\theta_{\mathrm{av}}\left(T_{00}+T_{11}\sin\frac{\pi x}{2a}\sin\frac{\pi y}{2b}\right) \tag{45}$$

which is the most representative of non-uniform temperature distributions and is the most easily applicable form. It is assumed that β_x and β_y are infinitely large and θ_x, θ_y, F_x, and F_y are all zero. These assumptions are the very condition that every component of the midplane boundary displacements is zero and, therefore, the solution thus obtained does not contain any error due to stresses.

K_T contains only two independent parameters, a/b and T_{11}/T_{00}, (Poisson's ratio ν may be another parameter, but its effect is negligibly small and $\nu=0.3$ is used in the calculation.) which are plotted in various forms as shown in Fig. 1.

The accuracy of the solution will be examined here. In Fig. 1, $K_T^{(1)}$ is the solution obtained by approximating the deflection by the first term only of the deflection function (chain lines) and in like manner $K_T^{(4)}$ by the first four terms (solid lines).

Physical meaning of approximation in the former case, i.e., the use of the first term in the double series for w, is to use the exact solution for the buckling pattern of the plate subjected to the uniform temperature as a substitute for the case of non-uniform temperature. Therefore, if the non-uniformity of the temperature T_{11}/T_{00} is small, this should be a good approximation. Let us now compare the relative magnitude of first four terms of the deflection function with each other for various values of non-uniformity T_{11}/T_{00} when $a/b=1$. (Table 2)

It is seen in this table that the first term always predominates over the other

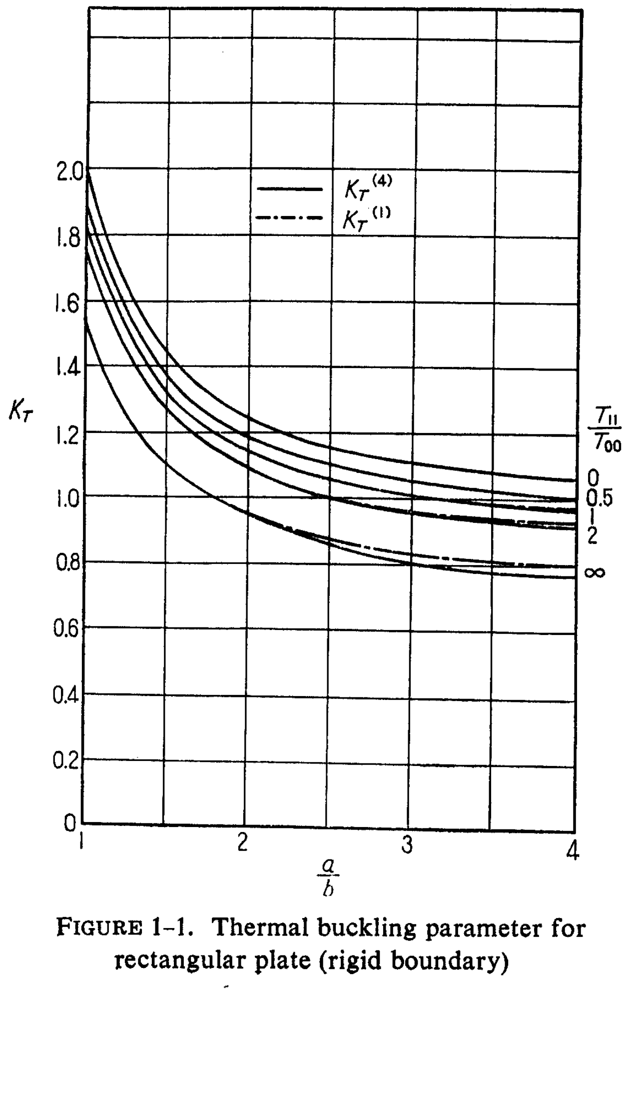

FIGURE 1-1. Thermal buckling parameter for rectangular plate (rigid boundary)

Temperature distribution in the plate:

$$\theta=\theta_{av}\left(T_{00}+T_{11}\sin\frac{\pi x}{2a}\sin\frac{\pi y}{2b}\right)$$

Boundary coditions:
(deflection)—simply supported
(displacement in the plane of the plate)
—rigid boundary

Thermal buckling parameter:

$$K_T=\frac{48}{\pi^2}(1+\nu)\alpha\theta_{av}\left(\frac{b}{h}\right)^2$$

Symbols:

θ_{av}=average temperature rise of the plate

α=linear thermal expansion of the plate

$2a$, $2b$=length and width of the plate

h=plate thickness

$K_T^{(1)}$=solution obtained by approximating the deflection by only the first term of deflection function (dotted lines in Fig. 1-1)

$$K_T^{(1)}=\left\{\frac{b}{a}\left(\frac{b}{a}+\frac{a}{b}\right)^2\right\}\div\Bigg[\left(\frac{b}{a}+\frac{a}{b}\right)T_{00}+\frac{16}{9\pi^2}\Bigg\{2\left(\frac{b}{a}+\frac{a}{b}\right)+\frac{4(b/a)^2-3\nu+1}{4\left(\frac{b}{a}\right)+\frac{1-\nu}{2}\left(\frac{a}{b}\right)}+\frac{4(a/b)^2-3\nu+1}{4\left(\frac{a}{b}\right)+\frac{1-\nu}{2}\left(\frac{b}{a}\right)}\Bigg\}T_{11}\Bigg]$$

(see Eq. (55))

$K_T^{(4)}$=solution obtained by approximating the deflection by the first four terms of the deflection function (solid lines in Fig. 1-1)

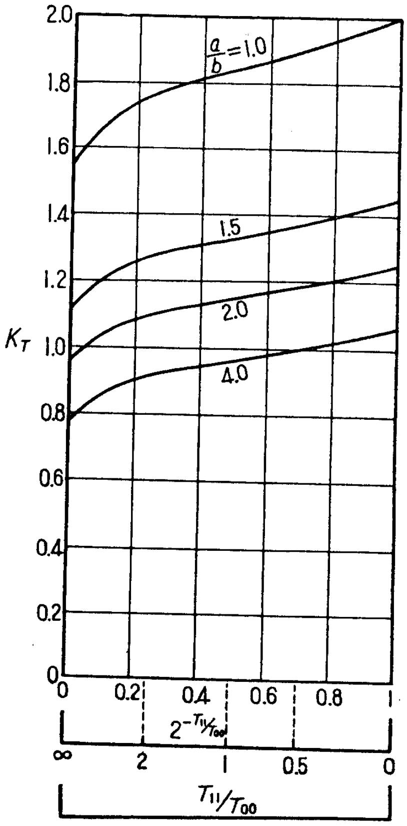

FIGURE 1-2. Thermal buckling parameter for rectangular plate (rigid boundary).

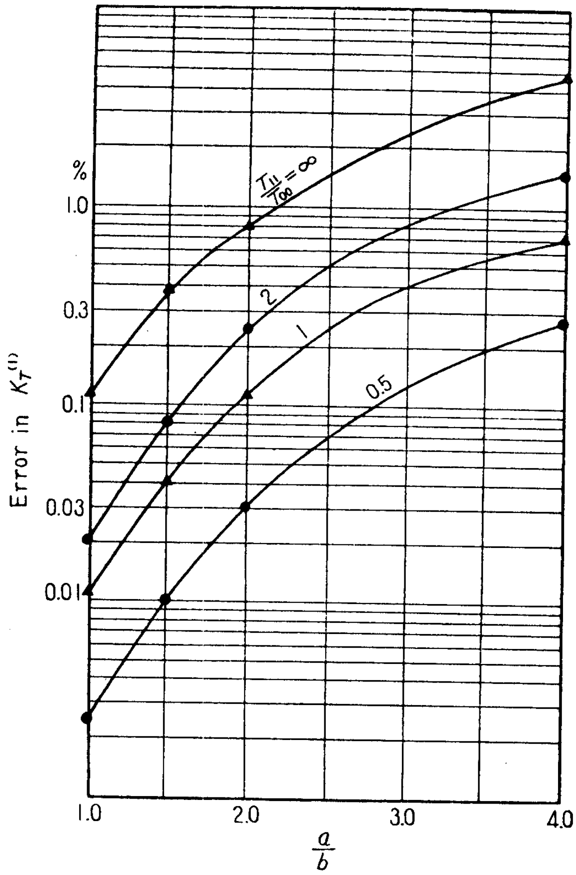

FIGURE 1-3. Error in $K_T^{(1)}$

TABLE 1. Numerical values of K_T

K_T		$\frac{T_{11}}{T_{00}}$	∞ ($T_{00}=0$)	2	1	0.5	0 ($T_{11}=0$)
		$2^{-\frac{T_{11}}{T_{00}}}$	0	0.25	0.5	0.707	1
$\frac{a}{b}$	1.0	$K_T^{(1)}$	1.5396	1.7637	1.8410	1.9040	2.0000
		$K_T^{(4)}$	1.5378	1.7632	1.8410	1.9040	2.0000
	1.5	$K_T^{(1)}$	1.1114	1.2734	1.3295	1.3750	1.4444
		$K_T^{(4)}$	1.1081	1.2723	1.3289	1.3749	1.4444
	2.0	$K_T^{(1)}$	0.9590	1.1004	1.1493	1.1890	1.2500
		$K_T^{(4)}$	0.9516	1.0977	1.1480	1.1886	1.2500
	4.0	$K_T^{(1)}$	0.8076	0.9308	0.9738	1.0089	1.0625
		$K_T^{(4)}$	0.7709	0.9171	0.9669	1.0062	1.0625

TABLE 2. Relative magnitude of first four terms of the deflection function ($a/b=1$)

T_{11}/T_{00}	0	0.5	1	2	∞
w_{11}/w_{11}	1.0000	1.0000	1.0000	1.0000	1.0000
w_{13}/w_{11}	0.0000	−0.0009	−0.0016	−0.0024	−0.0045
w_{31}/w_{11}	0.0000	−0.0009	−0.0016	−0.0024	−0.0045
w_{33}/w_{11}	0.0000	−0.0005	−0.0008	−0.0013	−0.0022

terms, provided the temperature distribution is expressed as Eq. (45). Hence, the deflection of the plate submitted to the action of such a non-uniformly distributed temperature will be closely akin to that of the uniform temperature, even if the non-uniformity T_{11}/T_{00} is considerably large. This very important fact found here will readily suggest us that the use of the first term of the series may be a good approximation and also that it affords us some possible way of obtaining the compact formulas for thermal buckling, which we shall discuss in the following section.

In Fig. 1 it is seen that the buckling parameter $K_T^{(1)}$ closely coincides with the

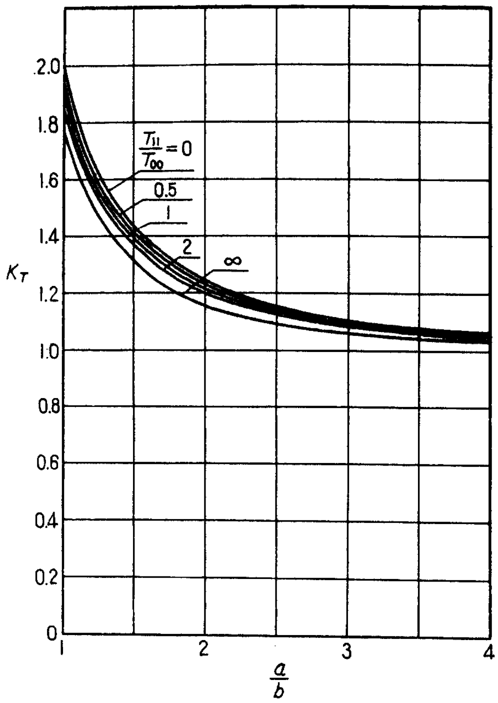

FIGURE 2. Solution by T. Hayashi (rigid boundary).

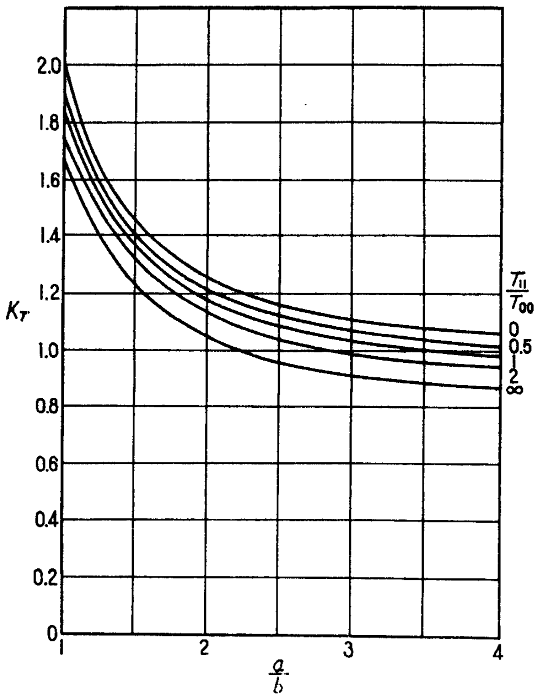

FIGURE 3. Solution by Klosner & Forray (rigid boundary).
Temperature distribution:

$$\theta=\theta_{av}\left[T_{00}+T_{11}\left\{1-\left(\frac{x-a}{a}\right)^2\right\}\left\{1-\left(\frac{y-b}{b}\right)^2\right\}\right]$$

$K_T^{(4)}$. This may be a logical conclusion deduced from the consideration about the relative magnitude of each term of the deflection function, and then it is reasonable, if we expect that the use of the first four terms will be sufficient for our present purpose.

The uppermost curve in Fig. 1 corresponds to K_T of the uniform temperature and is given by the formula $1+b^2/a^2$. The larger the non-uniformity, the lower the K_T. In cases where the temperature is high in the vicinity of the center of the plate compared with that of the edges of the plate, that may occur quite likely in supersonic flight structures, the true K_T should be lower than K_T obtained by assuming as if this temperature were uniformly distributed over the plate, provided the total heat contained are the same in both cases.

The comparison with the earlier theories Figs. 2 and 3 (It is not a exact comparison with Klosner and Forray's result as their example is evaluated in case of parabolic temperature distribution.) shows us that these estimated buckling temperatures are higher than those given by the present theory, or it may be said that the earlier theories rather underestimate the effect of non-uniformity of temperature on thermal buckling. In fact, the effect of uniform temperature on buckling is directly weakened by the reduction of the stiffness of the web, and on the other hand, the effect of non-uniformity seems to increase its relative importance. Take, for instance, a plate without any web where the uniform temperature never participates in buckling and suppose buckling solely depends upon the non-uniformity of the temperature. In short, the K_T for $T_{11}/T_{00}=\infty$ takes far lower value than the K_T for $T_{11}/T_{00}=0$ as the stiffness of the webs becomes weak and it should be noted that in these cases the differences between the present theory and the earlier theories, too, become remarkably large. We will examine this web effect in detail in the subsequent section.

FORMULATION OF THE BUCKLING CRITERION

The use of the first term alone of the series of deflection has been proved to be a good approximation and suggested as a possible way of obtaining an approximate formula for the buckling criterion. That conclusion may be largely based on the type of temperature distribution used, but if the temperature distribution is roughly akin to the one used in the preceding analysis, the same conclusion may be formed on a reasonable basis. A compact general formula for the criterion of thermal buckling will now be presented in the following.

The temperature distribution is assumed to be given by Eq. (1), which is repeated here for convenience, i.e.

$$\theta=\theta_{00}+\sum_{p:\mathrm{odd}}^{\infty}\sum_{q:\mathrm{odd}}^{\infty}\theta_{pq}\sin\frac{p\pi x}{2a}\sin\frac{q\pi y}{2b}. \tag{46}$$

The deflection function is assumed to be expressed as

$$w=w_{st}\sin\frac{si\pi x}{2a}\sin\frac{tj\pi y}{2b} \tag{47}$$

$$(s:\ \text{odd},\ t:\ \text{odd}).$$

Substituting Eq. (47) in Eq. (44) we obtain the following simple expression for the buckling conditions:

$$\left(K_{st}-\frac{1}{K_T}\right)w_{st}=0 \tag{48}$$

or

$$K_T=\frac{\dfrac{b}{a}\left(s^2i^2\dfrac{b}{a}+t^2j^2\dfrac{a}{b}\right)^2}{s^2i^2\dfrac{b}{a}\boldsymbol{I}_{xx}(s,s,t,t)+2stij\,\boldsymbol{I}_{xy}(s,s,t,t)+t^2j^2\dfrac{a}{b}\boldsymbol{I}_{yy}(s,s,t,t)}, \tag{49}$$

where $\boldsymbol{I}_{xx}$, $\boldsymbol{I}_{yy}$, and $\boldsymbol{I}_{xy}$ are given in Eq. (38).

In the following, several typical examples are presented.

1) $\theta=\theta_{00}$ (uniform temperature)

$\boldsymbol{I}_{xx}$, $\boldsymbol{I}_{yy}$, and $\boldsymbol{I}_{xy}$ are

$$\left.\begin{aligned} I_{xx}(s,s,t,t)&=1\,,\\ I_{yy}(s,s,t,t)&=1\,,\\ I_{xy}(s,s,t,t)&=0\,,\end{aligned}\right\} \tag{50}$$

then, K_T can be written as

$$K_T=s^2i^2\frac{b^2}{a^2}+t^2j^2 \qquad \begin{pmatrix} s\text{: odd, } t\text{: odd}\\ i,j=1 \text{ or } 2\end{pmatrix} \tag{51}$$

where i and j are the number of half-waves in the directions of x and y, respectively. These are obviously the exact solution of the problem.
If $s=t=1$, then

$$K_T=i^2\frac{b^2}{a^2}+j^2\,. \tag{52}$$

2) $$\theta=\theta_{\mathrm{av}}\left(T_{00}+T_{11}\sin\frac{\pi x}{2a}\sin\frac{\pi y}{2b}\right) \tag{53}$$

When $s=t=1$ and $i=j=1$, then

$$K_T=\frac{\dfrac{b}{a}\left(\dfrac{b}{a}+\dfrac{a}{b}\right)^2}{\dfrac{b}{a}T_{0x}+\dfrac{a}{b}T_{0y}+\dfrac{16}{9\pi^2}\left[2\left(\dfrac{b}{a}+\dfrac{a}{b}\right)+\dfrac{4\left(\dfrac{b}{a}\right)^2-3\nu+1}{4\left(\dfrac{b}{a}\right)+\dfrac{1-\nu}{2}\left(\dfrac{a}{b}\right)}+\dfrac{4\left(\dfrac{a}{b}\right)^2-3\nu+1}{4\left(\dfrac{a}{b}\right)+\dfrac{1-\nu}{2}\left(\dfrac{b}{a}\right)}\right]T_{11}}\,. \tag{54}$$

When the boundary is rigid ($\beta_x=\beta_y=\infty$, $\theta_x=\theta_y=0$), thus $T_{0x}=T_{0y}=T_{00}$ and Eq. (54) reduces to

$$K_T=\frac{\dfrac{b}{a}\left(\dfrac{b}{a}+\dfrac{a}{b}\right)^2}{\left(\dfrac{b}{a}+\dfrac{a}{b}\right)T_{00}+\dfrac{16}{9\pi^2}\left[2\left(\dfrac{b}{a}+\dfrac{a}{b}\right)+\dfrac{4\left(\dfrac{b}{a}\right)^2-3\nu+1}{4\left(\dfrac{b}{a}\right)+\dfrac{1-\nu}{2}\left(\dfrac{a}{b}\right)}+\dfrac{4\left(\dfrac{a}{b}\right)^2-3\nu+1}{4\left(\dfrac{a}{b}\right)+\dfrac{1-\nu}{2}\left(\dfrac{b}{a}\right)}\right]T_{11}}\,. \tag{55}$$

This is the formula for $K_T^{(1)}$ shown by the chain lines in Fig. 1–1.

3) $$\begin{aligned}\theta&=\theta_{\mathrm{av}}\left[T_{00}+T_{\mathrm{tent}}\left(1-\left|\frac{x-a}{a}\right|\right)\left(1-\left|\frac{y-b}{b}\right|\right)\right]\\ &=\theta_{\mathrm{av}}\left[T_{00}+T_{\mathrm{tent}}\frac{64}{\pi^4}\sum_{p:\mathrm{odd}}^{\infty}\sum_{q:\mathrm{odd}}^{\infty}\frac{1}{p^2q^2}(-1)^{\frac{p-1}{2}}(-1)^{\frac{q-1}{2}}\sin\frac{p\pi x}{2a}\sin\frac{q\pi y}{2b}\right]\end{aligned} \tag{56}$$

(tent-like temperature distribution).
T_{tent} is connected with the T_{00} by the following equation:

$$T_{00}+\frac{1}{4}T_{\mathrm{tent}}=1\,. \tag{57}$$

The result is shown in Fig. 4.

It is interesting to note that the curves in Fig. 4 not only bear a resemblance to the curves in Fig. 1, but give almost the same value for K_T provided the non-uniformity parameters T_{tent}/T_{00} and T_{11}/T_{00} are the same, notwithstanding that the temperature distributions are fairly different between these two cases.

From all these considerations, it seems that the most important temperature

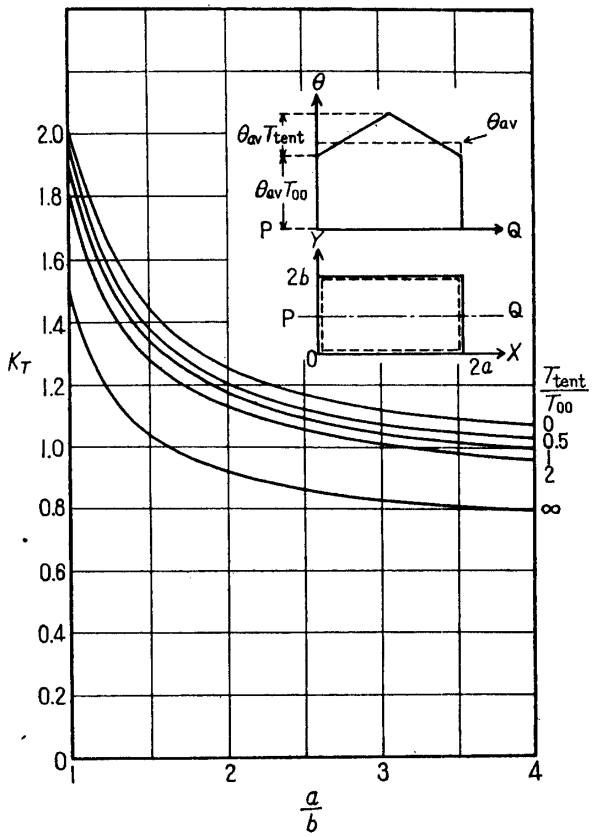

FIGURE 4. Thermal buckling parameter for tent-like temperature distribution (rigid boundary).
Temperature distribution:

$$\theta=\theta_{av}\left[T_{00}+T_{tent}\left(1-\left|\frac{x-a}{a}\right|\right)\left(1-\left|\frac{y-b}{b}\right|\right)\right].$$

parameters affecting the buckling criterion of plate subjected to such types of temperature distributions are the average temperature rise θ_{av} and the non-uniformity parameter representing the overall non-uniformity of the temperature and not the local distribution of the temperature. Thus, the buckling parameters for most representative temperature distributions must find there wide and convenient applications in practical cases.

EFFECTS OF THE TEMPERATURE AND THE STIFFNESS OF WEB ON THERMAL BUCKLING OF THE PLATE

The temperature and the stiffness of web exert predominating influences upon thermal buckling. Because, as it will easily be seen, the forces that produces thermal buckling originate mainly from the differences of the temperature and the stiffness between the plate and the web.

Let us consider first the case where the stiffness ratio β are the same in both x and y directions, i.e.,

$$\beta_x=\beta_y=\beta\,. \tag{58}$$

For simplicity, $F_x=F_y=0$ and $\theta_x=\theta_y=0$ are assumed. Then, the equivalent

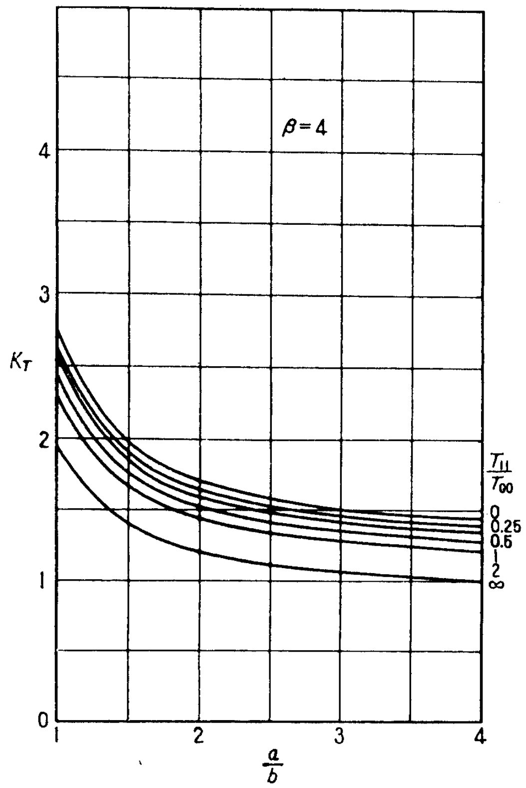

FIGURE 5-1. Thermal buckling parameter of the plate (when the stiffness of web is finite), $\beta_x = \beta_y$.

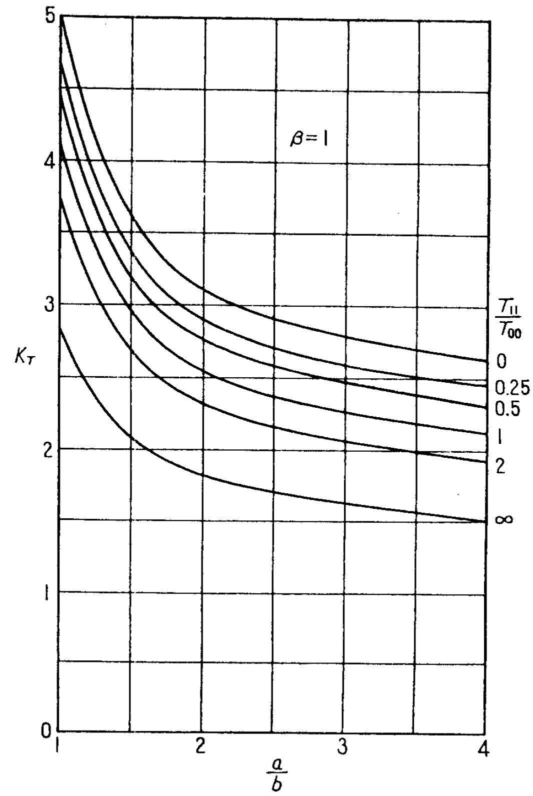

FIGURE 5-2. Thermal buckling parameter of the plate (when the stiffness of web is finite), $\beta_x = \beta_y$.

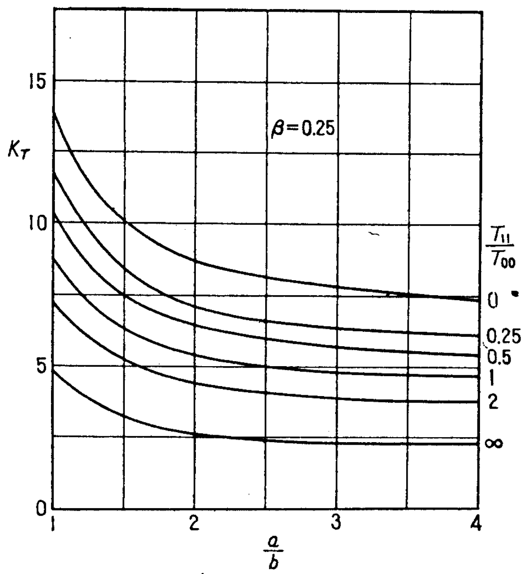

FIGURE 5-3. Thermal buckling parameter of the plate (when the stiffness of web is finite), $\beta_x = \beta_y$.

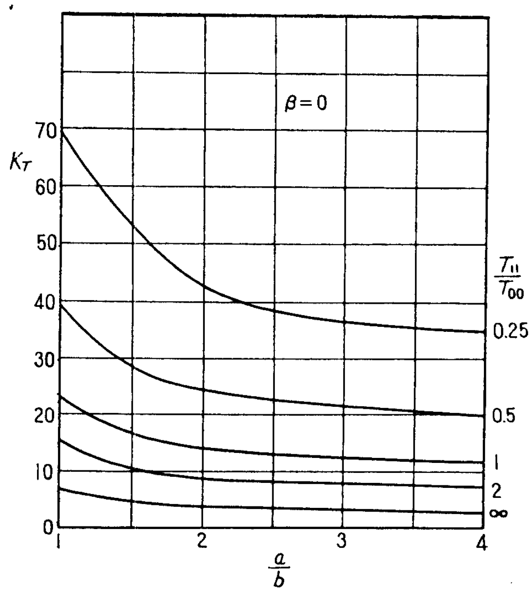

FIGURE 5-4. Thermal buckling parmeter of the plate (when the stiffness of web is finite), $\beta_x = \beta_y$.

uniform temperature in the x and y directions are the same and can be expressed as follows:

$$T_{0x}=T_{0y}=T_{00}-\frac{1}{(1-\nu)\beta+1}. \tag{59}$$

If the temperature is expressed by Eq. (53) the following result is obtained by substituting Eq. (59) in Eq. (54).

$$K_T=\left[\frac{b}{a}\left(\frac{b}{a}+\frac{a}{b}\right)^2\right]\div\left[\left(\frac{b}{a}+\frac{a}{b}\right)\left(T_0-\frac{1}{(1-\nu)\beta+1}\right)+\frac{16}{9\pi^2}\left\{2\left(\frac{b}{a}+\frac{a}{b}\right)\right.\right.$$
$$\left.\left.+\frac{4\left(\frac{b}{a}\right)^2-3\nu+1}{4\left(\frac{b}{a}\right)+\frac{1-\nu}{2}\left(\frac{a}{b}\right)}+\frac{4\left(\frac{a}{b}\right)^2-3\nu+1}{4\left(\frac{a}{b}\right)+\frac{1-\nu}{2}\left(\frac{b}{a}\right)}\right\}T_{11}\right]. \tag{60}$$

In the case where $\beta=\infty$, it follows that

$$T_{0x}=T_{0y}=T_{00} \tag{61}$$

thus, T_{0x} and T_{0y} are the largest and so K_T is the smallest. In the case where $\beta=0$, it follows that

$$T_{0x}=T_{0y}=T_{00}-1\,, \tag{62}$$

T_{0x} and T_{0y} have negative values ($T_{00}\leqq 1$). This 'negative' temperature produce tensile stress over the whole plate, and buckling is solely due to the non-uniformity of the temperature. The K_T, thus, is the largest as shown in Fig. 5.

For finite and non-zero values of β, T_{0x} and T_{0y} decrease and, therefore, the effect of non-uniformity increases, while the effect of uniform temperature decreases.

These effects can be plainly observed in Fig. 6 where K_T is plotted for various values of T_{11}/T_{00} and β ($a/b=1$). When $\beta=\infty$, the effect of T_{11}/T_{00} is 20% at the most. Decreasing β results in a rapid increase in the difference of K_T for different values of T_{11}/T_{00}.

Next, let us consider the case where the stiffness ratios are different in x and y directions. In such case, primary characteristics will be found in the theory of buckling of the rectangular plate uniformly compressed in two parpendicular directions, where the deformation other than the aforementioned one ($i=1$, $j=1$) is possible for some combination of applied forces. But it may be said that these deformations may not likely happen provided the buckling is due mainly to the temperature difference between the plate and the web and not to the external forces.

The equivalent uniform temperatures T_{0x} and T_{0y} can be expressed by the following equations when $F_x=F_y=0$, and $\theta_x=\theta_y=0$:

$$\left.\begin{aligned}T_{0x}&=T_{00}-\frac{(1-\nu^2)[\beta_y+1+\beta_x\nu]}{[(1-\nu^2)\beta_x+1][(1-\nu^2)\beta_y+1]-\nu^2},\\ T_{0y}&=T_{00}-\frac{(1-\nu^2)[\beta_x+1+\beta_y\nu]}{[(1-\nu^2)\beta_y+1][(1-\nu^2)\beta_x+1]-\nu^2}.\end{aligned}\right\} \tag{63}$$

Taking the temperature distribution Eq. (53) and assuming $s=t=1$, we can evaluate K_T by the following formulas.

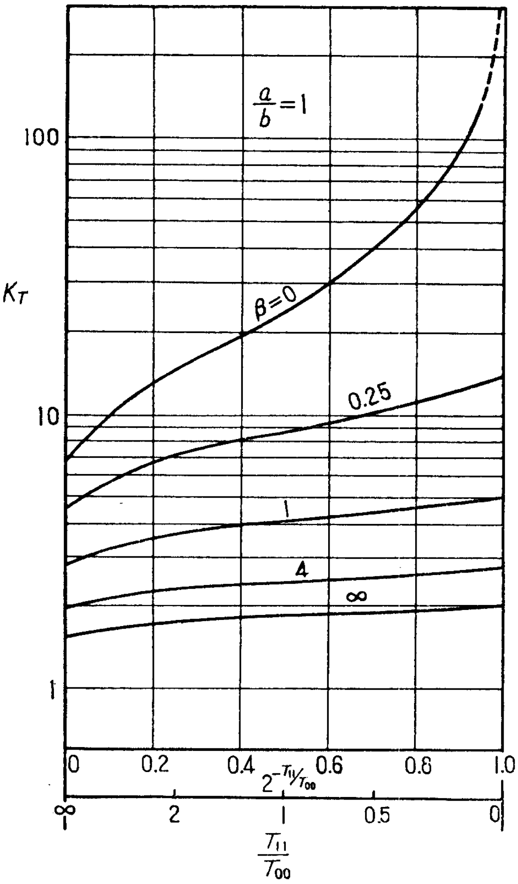

FIGURE 6. Effects of the web stiffness and the non-uniformity of plate temperature on thermal buckling.

1) $i=j=1$ (1 half-wave in both x and y directions)

$$K_T=\left[\frac{b}{a}\left(\frac{b}{a}+\frac{a}{b}\right)^2\right]\div\left[\frac{b}{a}T_{0x}+\frac{a}{b}T_{0y}+\frac{16}{9\pi^2}\left\{2\left(\frac{b}{a}+\frac{a}{b}\right)+\frac{4\left(\frac{b}{a}\right)^2-3\nu+1}{4\left(\frac{b}{a}\right)+\frac{1-\nu}{2}\left(\frac{a}{b}\right)}+\frac{4\left(\frac{a}{b}\right)^2-3\nu+1}{4\left(\frac{a}{b}\right)+\frac{1-\nu}{2}\left(\frac{b}{a}\right)}\right\}T_{11}\right]. \tag{64}$$

2) $i=2, j=1$ (2 half-waves in x direction and 1 in y direction)

$$K_T=\left[\frac{b}{a}\left(4\frac{b}{a}+\frac{a}{b}\right)^2\right]\div\left[4\frac{b}{a}T_{0x}+\frac{a}{b}T_{0y}+\frac{64}{45\pi^2}\left\{14\left(\frac{b}{a}\right)+2\left(\frac{a}{b}\right)+\frac{16\left(\frac{b}{a}\right)^2-3\nu+1}{16\left(\frac{b}{a}\right)+\frac{1-\nu}{2}\left(\frac{a}{b}\right)}+\frac{4\left(\frac{a}{b}\right)^2-15\nu+1}{4\left(\frac{a}{b}\right)+\frac{1-\nu}{2}\left(\frac{b}{a}\right)}\right\}T_{11}\right]. \tag{65}$$

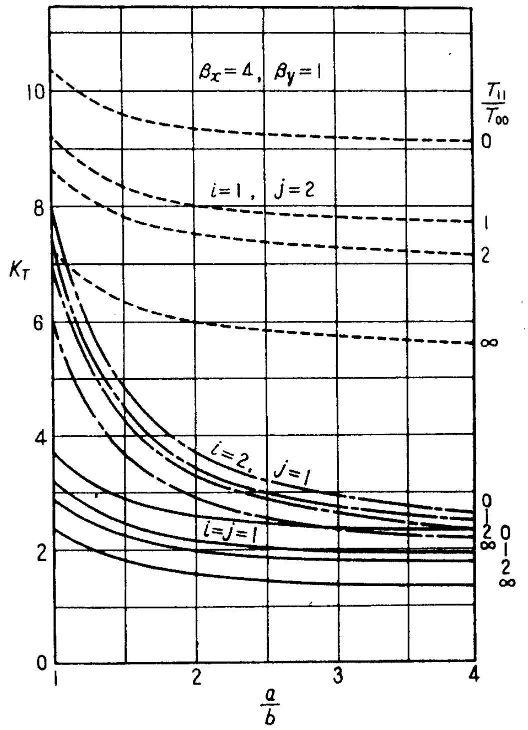

FIGURE 7–1. Thermal buckling parameter of the plate (when the stiffness of web is finite), $\beta_x \neq \beta_y$.

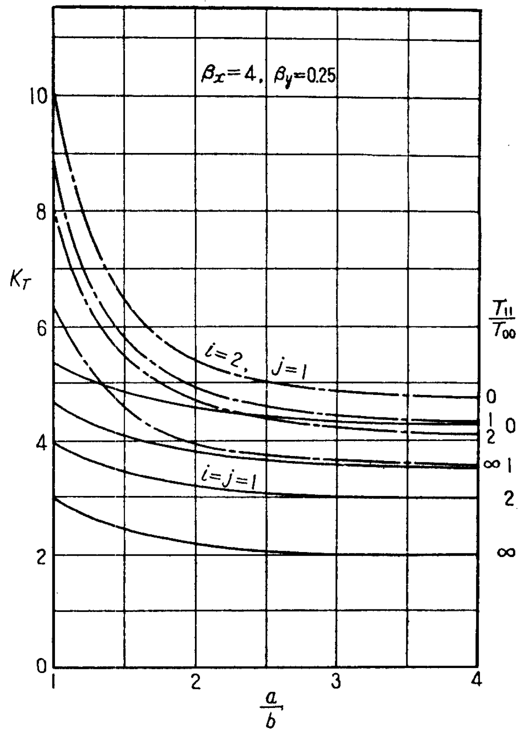

FIGURE 7–2. Thermal buckling parameter of the plate (when the stiffness of web is finite), $\beta_x \neq \beta_y$.

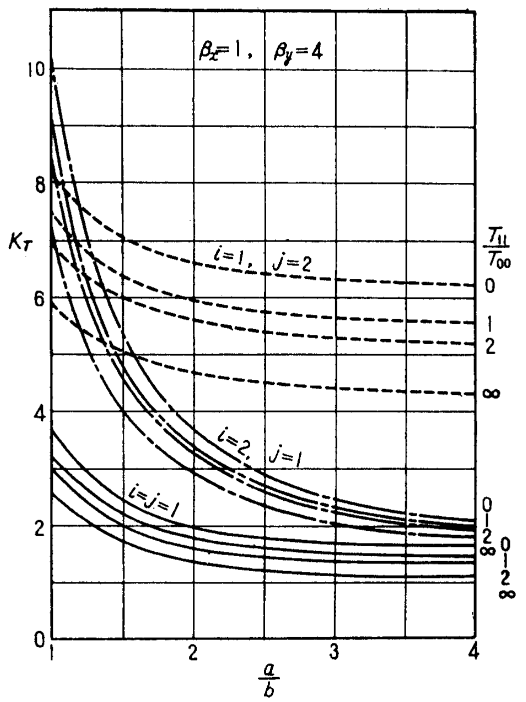

FIGURE 7–3. Thermal buckling parameter of the plate (when the stiffness of web is finite), $\beta_x \neq \beta_y$.

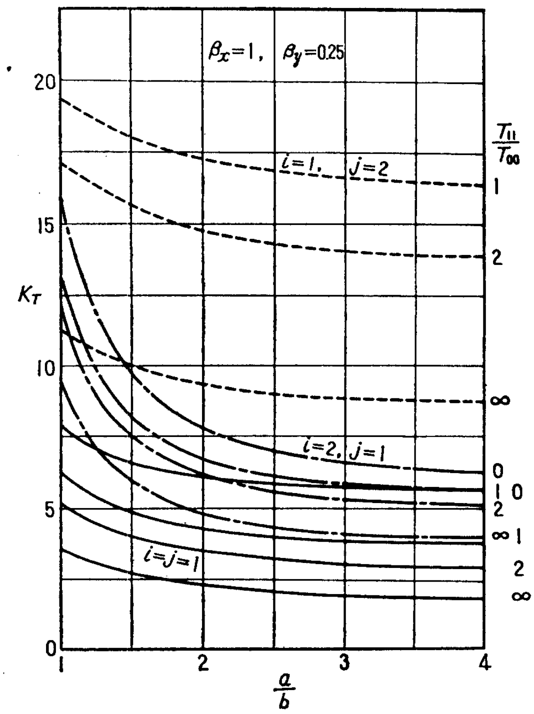

FIGURE 7–4. Thermal buckling parameter of the plate (when the stiffness of web is finite), $\beta_x \neq \beta_y$.

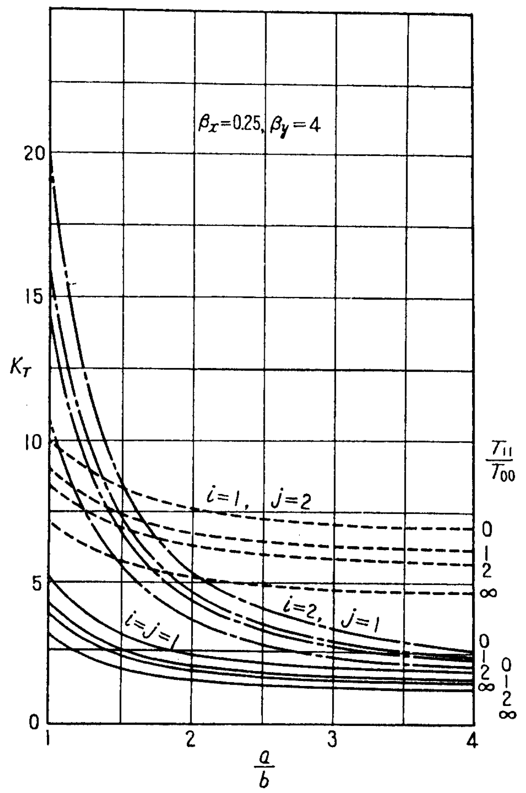

FIGURE 7-5. Thermal buckling parameter of the plate (when the stiffness of web is finite), $\beta_x \neq \beta_y$.

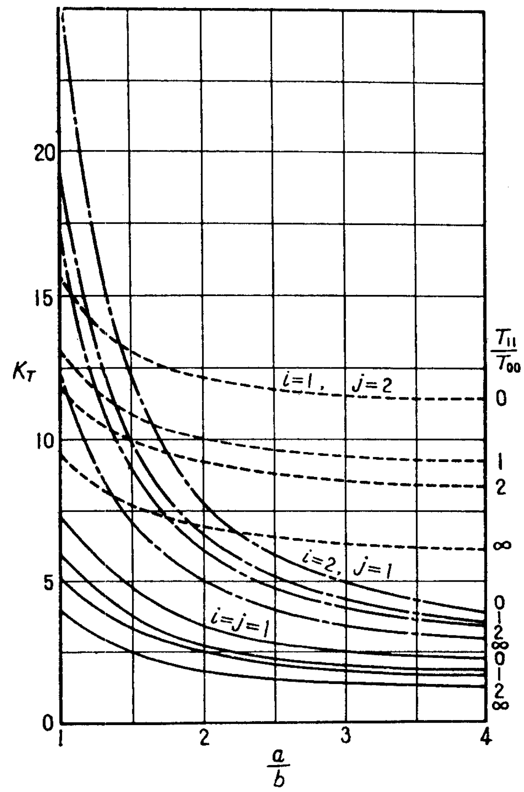

FIGURE 7-6. Thermal buckling paremeter of the plate (when the stiffness of web is finite), $\beta_x \neq \beta_y$.

3) $i=1$, $j=2$ (1 half-wave in x direction and 2 in y direction)

$$K_T=\left[\frac{b}{a}\left(\frac{b}{a}+4\frac{a}{b}\right)^2\right]\div\left[\frac{b}{a}T_{0x}+4\frac{a}{b}T_{0y}+\frac{64}{45\pi^2}\left\{2\left(\frac{b}{a}\right)+14\left(\frac{a}{b}\right)\right.\right.$$
$$\left.\left.+\frac{4\left(\frac{b}{a}\right)^2-15\nu+1}{4\left(\frac{b}{a}\right)+\frac{1-\nu}{2}\left(\frac{a}{b}\right)}+\frac{16\left(\frac{a}{b}\right)^2-3\nu+1}{16\left(\frac{a}{b}\right)+\frac{1-\nu}{2}\left(\frac{b}{a}\right)}\right\}T_{11}\right]. \quad (66)$$

The results are shown in Fig. 7.

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SYMBOLS

θ = temperature rise of plate
T = dimensionless temperature
p, q = integral numbers
$2a, 2b$ = length and width of plate
x, y = coordinates
$\sigma_{xh}, \sigma_{yh}, \tau_{xyh}$ = stresses due to fictitious hydrostatic pressure
$\sigma_{xb}, \sigma_{yb}, \tau_{xyb}$ = stresses due to fictitious body forces
X, Y = body forces
u, v = displacements in the directions of x and y, respectively
m, n = integral numbers
A_{mn}, B_{mn} = coefficients
$\delta(q, n)$ = Kronecker's delta
A, B, A_2, B_2 = coefficients
C_{pqmo} etc. = coefficients
P_x, P_y = axial forces in webs
F_x, F_y = external forces
θ_x, θ_y = temperature rises of webs
α_x, α_y = coefficients of linear expansion of webs
β_x, β_y = stiffness ratios of webs to plate
T_{0x}, T_{0y} = equivalent uniform temperatures
s, t, σ, τ = integral numbers
U = energy
h = thickness of plate