

On Torsional Rigidity and Torsional Vibration of
Aerodynamically Heated Wings Having
a Small Amount of Pretwist*

By

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Summary. This paper presents an approximate analysis of torsional rigidity and of torsional vibration of aerodynamically heated wings having a small amount of pretwist. The combined effects of pretwist and thermal stress on the torsional rigidity are made clear and are shown to be considerably large even if the amount of pretwist is small. As for the free vibration of pretwisted wing, the author shows that, regardless of whether the thermal stress is present or not, the vibration can be characterized by the following differential equation

$$\ddot{x} + c_1\dot{x} + c_2x^2 + c_3x^3 = 0,$$

whose solution can be expressed by the Weierstrass's elliptic function. The third term of the left hand side of the equation shows the asymmetry of vibration for such pretwisted wings.

INTRODUCTION

When the thin wing is subjected to the aerodynamic heating, the reduction of torsional rigidity may occur because of the thermal-stresses arising in the plane of the wing. This effect of thermal-stresses is individually pointed out by N. J. Hoff [1], B. Budiansky and M. J. Mayers [2], and John Duberg [3]. This reduction can be expressed as :

$$\frac{GJ_{\text{eff}}}{GJ} = 1 + \left[\int_A \sigma_{\text{th}} r^2 dA / GJ \right],$$

where GJ is the torsional rigidity of the wing, GJ_{eff} is the effective torsional rigidity, σ_{th} is the thermal stress in the direction of the span, r is the distance from the torsional axis and A is the sectional area of the wing.

In general, as the integration in above equation becomes negative when σ_{th} is the compressive stress in the region of both leading edge and trailing edge of the wing, the increase in thermal-stress results in a reduction in torsional rigidity. The

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distribution of thermal-stress which makes GJ_{eff} zero gives the criterion for torsional thermal buckling of the wing, and such a distribution of the temperature may take place in the vehicle usually considered.

The temperature distribution in the wing is, in general, a function of time and so is the effective torsional rigidity. Thus, Budiansky and Mayers calculated the reduction of torsional rigidity of the wing having a diamond section for various types of flight missions. Hoff and Singer [4] further developed the theory by the finite deflection theory of the plate and showed that the torsional rigidity does not become equal to zero even if the thermal-stress reaches at such value that thermal buckling should take place as predicted by a small deflection plate theory, and also analysed the torsional vibration of the wing subjected to thermal-stresses by the Rayleigh-Ritz's procedure [5]. It was shown that if the torsional rigidity reduces, the vibration frequency reduces as a matter of course, but if the effect of finite deflection is taken into account, the rigidity increases as the increase of amplitude and thus the reduction of frequency is moderated, and also that this effect is notable even if the amplitude is in a moderate value.

Above-mentioned discussions, however, are based on the assumption that the wing does not possess a pretwist and, therefore, the wing should have an additional twist, if the wing possesses some amount of pretwist. Budiansky and Mayers comment on this effect that the torsional rigidity of wing having a moderate value of pretwist will be the same as that of the straight wing, unless the thermal-stress comes up to such a large value as to cause buckling, where the wing suffers excessive deformation.

In contrast with this weakening influence of a pretwist, however, it was observed that a pretwisted bar, even a slightly pretwisted one, is considerably stiffer against torsion than the same straight bar. This was explained by Chen Chu [6] to be caused by the appearance of secondary longitudinal stresses. Physically the reason is as follows: a longitudinal fiber, following the same element dA of the cross section, is not a straight line but a spiral about the center fiber of a symmetrical section. When the bar is twisted (keeping the length of the center fiber unchanged), the spiral becomes longer, if the elastic twist is in the same direction as the pretwist, and shorter in the opposite case. Elongation of the spiral causes tension in it, following the spiral direction. This tension is mostly longitudinal (parallel to the center fiber), but it has a small component in the plane of the cross section and directed tangentially. All these horizontal components over a complete section form a torque, which must be added to the Saint Venant torque caused by the shear-force distribution, and this secondary torque can become substantially larger than the Saint Venant torque itself. This fact was afterwards discussed by E. Reissner and K. Washizu [7], too.

Chen's theory is comprehensible to us that it extends the Saint Venant's theory held for the straight bar to the theory including the effect of the pretwist, and that, in conclusion, it gives consideration to the contribution of the normal stress in the plane of the bar besides the shear stress calculated by the Saint Venant's theory. Thus, the thermal-stress in the plane of the wing which may become

considerably large value, must have naturally notable influences upon the torsional rigidity of a pretwisted wing. Then it is required to establish a unified theory treating this problem having these overlapping effects and it is necessary that the theory is based on the finite deflection theory for purpose of including the effect of amplitude.

The author conducted the analysis which includes the effect of finite deformation in the same degree as Hoff's analysis and he analysed these overlapping effects in the pretwisted, thermally stressed wing by using the theoretical method analogous to the one of Chen. It should be noted that the effects of finite deformation can not be limited to the one mentioned above and there is such effect as the change of thermal stress due to this finite deformation. In applying the method described in the paper, therefore, it is necessary to pay attention to that influence, though it is not yet solved.

TORSIONAL RIGIDITY

Now let us consider the thin wing having a small amount of pretwist, whose thickness is given by a function of y as shown in Fig. 1. The span of the wing is assumed to be so long that the local effects of the tip may be neglected and that both the temperature and the thermal-stress may be expressed by a function of y alone.

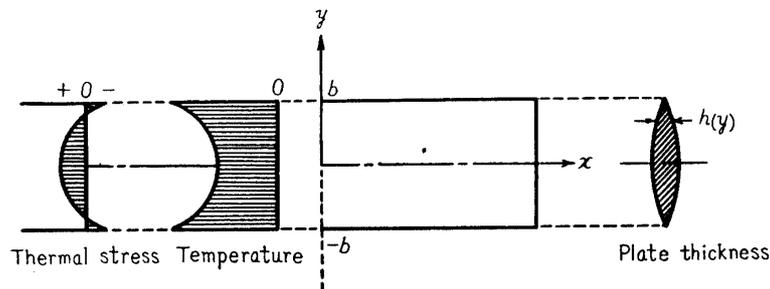


FIGURE 1. Configuration of wing considered.

It is assumed that the deflection of the midplane of the wing can be expressed by the sum of w_0 and w_1 , where w_0 is due to the pretwist and w_1 is due to the elastic torsion by the torque applied at the tip of the wing. Then these deflection functions are assumed as follows :

$$w_0 = k_0 xy, \quad (1)$$

$$w_1 = k_1 xy, \quad (2)$$

where k_0 (rad/m) and k_1 (rad/m) are the angles of twist per unit length of the wing due to the pretwist and the applied torque, respectively. J. Singer discussed about the validity of above expression as an deflection of the thin wing. Singer reported that there is 5% discrepancy in torsional vibration frequency between the value obtained by Eq. (2) and the one obtained by using the following expression when thermal-stress is so large as to be sufficient to cause thermal buckling (the pretwist is not considered).

$$w_1 = k_{11}xy + k_{31}x^3y + k_{13}xy^3. \quad (3)$$

This result is obtained in case of constant thick wing, but it was also verified in case of parabolic distribution of the thickness. It may, therefore, be allowed to use Eqs. (1) and (2) as deflection functions of the present problem.

The strain in the direction of x caused by the deflection w_1 is apparently

$$\frac{1}{2} \left[\frac{\partial(w_0 + w_1)}{\partial x} \right]^2 - \frac{1}{2} \left[\frac{\partial w_0}{\partial x} \right]^2. \quad (4)$$

On account of the condition that no external normal force is applied at the tip in the plane of the wing, the integration of the stress over a complete section of the wing must be zero and, therefore, the aforementioned strain is accompanied by the shortening of the wing, ϵ_0 , in the direction of x uniformly distributed over the section. This can be expressed by the following relation:

$$\int_{-b}^b \left\{ \frac{1}{2} \left[\frac{\partial(w_0 + w_1)}{\partial x} \right]^2 - \frac{1}{2} \left[\frac{\partial w_0}{\partial x} \right]^2 \right\} h dy - \epsilon_0 \int_{-b}^b h dy = 0. \quad (5)$$

Then,

$$\epsilon_0 = \frac{1}{\int_{-b}^b h dy} \int_{-b}^b \left\{ \frac{1}{2} \left[\frac{\partial(w_0 + w_1)}{\partial x} \right]^2 - \frac{1}{2} \left[\frac{\partial w_0}{\partial x} \right]^2 \right\} h dy. \quad (6)$$

In consequence, the strain in the direction of x is the sum of Eqs. (4) and (6).

$$\epsilon = \left(\frac{1}{2} k_1^2 + k_1 k_0 \right) \left[y^2 - \frac{1}{\int_{-b}^b h dy} \int_{-b}^b h y^2 dy \right]. \quad (7)$$

If the thermal-stress in the direction of x is denoted by σ_{th} , then the total stress can be expressed as follows:

$$\sigma_x = \sigma_{th} + E \left(\frac{1}{2} k_1^2 + k_1 k_0 \right) \left[y^2 - \frac{1}{\int_{-b}^b h dy} \int_{-b}^b h y^2 dy \right]. \quad (8)$$

As the direction of this stress is inclined with respect to the plane by $\partial(w_1 + w_0)/\partial x = (k_1 + k_0)y$ on account of the twisting, it gives rise to the force as much as $\sigma_x(k_1 + k_0)y h dy$ per infinitesimal sectional element, $h dy$, normal to the x - y plane. The torque increment ΔT due to this force can be obtained by integrating the product of this force and arm, y , over a complete section. Thus

$$\begin{aligned} \Delta T = & \int_{-b}^b \sigma_x (k_1 + k_0) y^2 h dy = (k_1 + k_0) \int_{-b}^b \sigma_{th} y^2 h dy \\ & + E k_1 \left(\frac{1}{2} k_1^2 + \frac{3}{2} k_1 k_0 + k_0^2 \right) \left[\int_{-b}^b h y^4 dy - \frac{1}{\int_{-b}^b h dy} \left(\int_{-b}^b h y^2 dy \right)^2 \right]. \quad (9) \end{aligned}$$

The total torque T is assumed to be the algebraic sum of the torque due to the Saint Venant's theory, provided that the angle of twist and the pretwist remain reasonably small value. It has been verified experimentally by Chen that the limitation in which this assumption is valid is that the angle which the leading or the trailing edge makes to x axis is about 0.15 radian. This limitation which is thought not to be exceeded in almost any practical case of the problem considered here is followed by us in this analysis.

Then we have

$$T = GJk_1 + \Delta T. \quad (10)$$

Substituting Eq. (9) in Eq. (10) we obtain the following expression for torque.

$$T = GJk_1 + (k_1 + k_0) \int_{-b}^b \sigma_{th} y^2 h dy + Ek_1 \left(\frac{1}{2} k_1^2 + \frac{3}{2} k_1 k_0 + k_0^2 \right) \left[\int_{-b}^b h y^4 dy - \frac{1}{\int_{-b}^b h dy} \left(\int_{-b}^b h y^2 dy \right)^2 \right]. \quad (11)$$

If we define the effective torsional rigidity by the following formula,

$$T = GJ_{eff} k_1 \quad (12)$$

then, GJ_{eff} can be expressed as

$$\frac{GJ_{eff}}{GJ} = 1 + \frac{1}{GJ} \left(1 + \frac{k_0}{k_1} \right) \int_{-b}^b \sigma_{th} y^2 h dy + \frac{1+\nu}{J} (k_1^2 + 3k_1 k_0 + 2k_0^2) \left[\int_{-b}^b h y^4 dy - \frac{1}{\int_{-b}^b h dy} \left(\int_{-b}^b h y^2 dy \right)^2 \right]. \quad (13)$$

Eq. (11) contains Chen's solution when there is no thermal-stress and Hoff's solution when there is no pretwist and, as expected, the mutually related term exists when both the thermal-stress and the pretwist are present. The concept of the effective torsional rigidity defined by Eq. (12) is due to the direct extension of the concept in case of linear problem to non-linear problem in which the angle of twist is no longer the linear function of the applied torque. In other words, this definition is similar to that of secant modulus. Naturally, the new concept of effective torsional rigidity corresponds to so-called tangent modulus may be defined and let it called the effective torsional rigidity GJ'_{eff} , which is the tangent of the torque-to-angle-of-twist curve.

$$GJ'_{eff} = \frac{\partial T}{\partial k_1} = GJ + \int_{-b}^b \sigma_{th} h y^2 dy + E \left(\frac{3}{2} k_1^2 + 3k_1 k_0 + k_0^2 \right) \left[\int_{-b}^b h y^4 dy - \frac{1}{\int_{-b}^b h dy} \left(\int_{-b}^b h y^2 dy \right)^2 \right]. \quad (14)$$

$$\frac{GJ'_{eff}}{GJ} = 1 + \frac{1}{GJ} \int_{-b}^b \sigma_{th} h y^2 dy + \frac{1+\nu}{J} (3k_1^2 + 6k_1 k_0 + 2k_0^2) \left[\int_{-b}^b h y^4 dy - \frac{1}{\int_{-b}^b h dy} \left(\int_{-b}^b h y^2 dy \right)^2 \right]. \quad (15)$$

Let us consider the magnitude of the following terms.

$$\left. \begin{aligned} I &= \int_{-b}^b \sigma_{th} h y^2 dy, \\ K &= \int_{-b}^b h y^4 dy - \frac{1}{\int_{-b}^b h dy} \left(\int_{-b}^b h y^2 dy \right)^2. \end{aligned} \right\} \quad (16)$$

I , the only terms including the thermal-stress, is the quantity calculated from the temperature distribution and the shape of the wing section and it takes usually negative value, when the vehicle is in a accelerated stage.

K is the quantity calculable from the shape of the wing section and is positive. If the Eqs. (11) and (15) are rewritten using these quantity, the following equations

are obtained.

$$T = (GJ + I + EKk_0^2)k_1 + Ik_0 + EK\left(\frac{1}{2}k_1^3 + \frac{3}{2}k_1^2k_0\right), \quad (17)$$

$$GJ'_{\text{eff}} = (GJ + I + EKk_0^2) + EK\left(\frac{3}{2}k_1^2 + 3k_1k_0\right). \quad (18)$$

T is expressed by the third order polynomials of k_1 , and if $GJ + I + EKk_0^2 > 0$, the equation, that the right hands of Eq. (17) is taken to be zero, has one positive root and two either negative or imaginary roots (Fig. 2). It is necessary to note

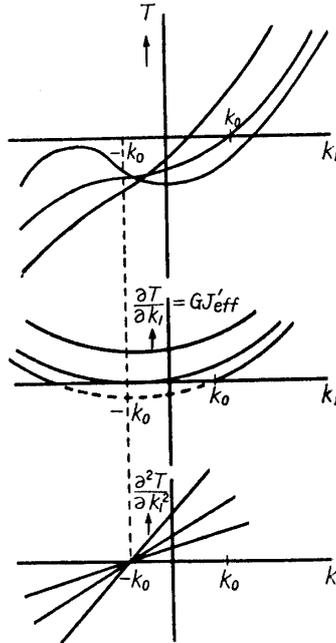


FIGURE 2. Torque and 'effective' torsional rigidity versus angle of twist for pre-twisted wing.

that $T-k_1$ curves do not pass through the origin since the torque $-Ik_0$ is induced by the interaction of the thermal-stress and the pretwist even if the torque is not applied. That is

$$-Ik_0 = (GJ + I + EKk_0^2)k_1 + EK\left(\frac{1}{2}k_1^3 + \frac{3}{2}k_1^2k_0\right). \quad (19)$$

The effective torsional rigidity GJ'_{eff} becomes minimum when $k_1 = -k_0$. GJ'_{eff} , when the angle of twist k_1 equals to zero, are $GJ + I + EKk_0^2$, whose second term is the reduction due to the thermal-stress predicted by Hoff and others and the third term is the strengthening effect of the pretwist predicted by Chen and others. Then, if the deflection is infinitesimally small, the total effect of the pretwist and the thermal-stress on the torsional rigidity of the wing is merely the sum of each independent effect. But if the deflection is finite, the aforementioned formula, $GJ + I + EKk_0^2$, does not correspond to the rigidity for zero applied torque, because the wing is still subjected to the induced torque $-Ik_0$ due to the thermal-stress and the pretwist, and the wing is twisted as much as an angle calculable from

Eq. (19) and, therefore, the effective torsional rigidity GJ'_{eff} usually increases from the value mentioned above.

Next, the condition, that the GJ'_{eff} must be neither zero nor negative, can be obtained by $(\partial T/\partial k_1)_{k_1=k_0} > 0$.

$$GJ + I - \frac{1}{2}EKk_0^2 > 0. \tag{20}$$

In a case where this condition is not fulfilled, there exists either an inflection point or a maximum and a minimum, which corresponds to the unstable condition. This case is shown to exist practically in case of a propeller which has considerably large k_0 by Niedenfuhr [8], though it is an isothermal problem. But in every case of the thin wing of high speed vehicles considered in this paper Eq. (20) is satisfied, and it is no use to take into account, unless the thermal-stress reaches closely to the buckling criterion.

It is to be noted that the lowering of torsional rigidity itself by the thermal-stress results in a more rapid increase in the influence of the pretwist upon the torsional rigidity. For better understanding of the aspect of the problem it is

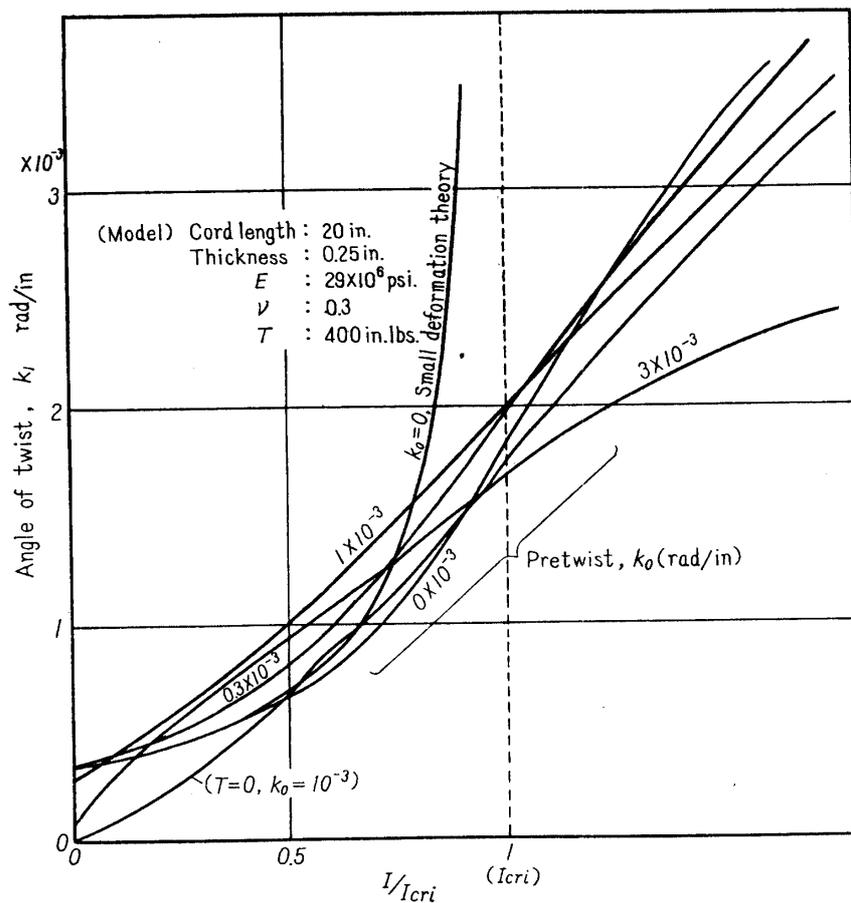


FIGURE 3. Angle of twist versus thermal-stress for pretwisted wings.

usefull to evaluate the examples of the wing with various values of pretwist (Fig. 3). Taking a wide view of the matter in these examples, the following summary may be concluded.

- 1) If $I=0$, i.e., there is no thermal-stress, the angle of twist k_1 reduces as the pretwist k_0 increases.
- 2) In the region of $0 < I < I_{\text{cri}}$, the increase in k_0 results in the increase in k_1 .
- 3) In the region of $I_{\text{cri}} < I$, the increase in k_0 results in the decrease in k_1 .
- 4) Even if there is no torque applied, almost the same order of twist occurs as in the case with torque.

Budiansky and Mayers comment that the increase of the influence of the pretwist will be appreciable, only if the thermal-stress closely reaches to the buckling criterion. This view of the matter, however, is not true in respect of the examples considered here, where the effect of pretwist is rather small, if the thermal-stress in wing is so large as to cause buckling. In general, the effect of pretwist must be predicted by the quantitative evaluation of Eq. (17) and the conclusion derived from the results may be considerably different in every individual case.

SOLUTION FOR TUBULAR SECTION

The method of analysis developed in the preceding section can easily be applied to the case of thin tubular section. In this case, the integration must be done along the closed line S (Fig. 4).

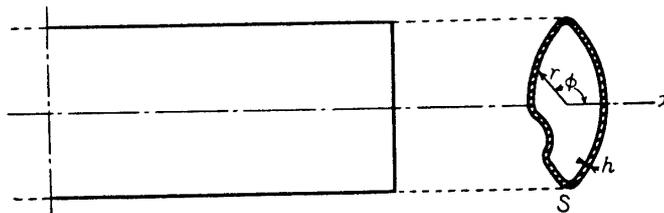


FIGURE 4. Configuration of thin tubular member.

The stress can be expressed as follows :

$$\sigma_x = \sigma_{\text{th}} + E \left(\frac{1}{2} k_1^2 + k_1 k_0 \right) \left[r(\phi)^2 - \frac{1}{\oint r(\phi) ds} \oint r(\phi)^2 ds \right]. \quad (21)$$

The relation between the torque and the angle of twist is

$$T = GJ k_1 + (k_1 + k_0) \oint \sigma_{\text{th}} r(\phi)^2 h ds + E k_1 \left(\frac{1}{2} k_1^2 + \frac{3}{2} k_1 k_0 + k_0^2 \right) h \left[\oint r(\phi)^2 ds - \frac{1}{\oint r(\phi) ds} \left(\oint r(\phi)^2 ds \right)^2 \right]. \quad (22)$$

TORSIONAL VIBRATION

In this section the torsional vibration of the thin wing with a small amount of pretwist subjected to the aerodynamic heating is analysed. It seems that the non-linearity as well as the asymmetry will appear in the torsional vibration, as these characteristics are observed in the case of statical twist of the pretwisted wing. It should be noted that the torsional vibration of the wing in the flow is

influenced by the aerodynamic moment. These effect, however, may be neglected in the case of the wing with symmetrical section in the supersonic air flow, because the center of the aerodynamic forces acting on the wing coincides with its elastic axis.

The pretwist is assumed to be given by the following formula :

$$w_0 = k_0 xy. \quad (23)$$

It is assumed that the same deflection function as in the case of statical twist expresses the deflection of the wing in vibration, too. The displacement at any time, then, can be expressed as follows :

$$w_1 = k_1 xy F(t), \quad (24)$$

where $F(t)$ is the time function accompanied to the deflection w_1 . The total displacement from the plane, then, is the sum of w_0 and w_1 . The potential energy measured from the position of w_0 is

$$\begin{aligned} U = & \frac{1}{2} GJ k_1^2 F^2 \times 2a \\ & + \int_{-a}^a \int_{-b}^b h \left\{ \sigma_{th} + \frac{1}{2} E \left(\frac{1}{2} k_1^2 F^2 + k_1 k_0 F \right) \left[y^2 - \frac{1}{\int_{-b}^b h dy} \int_{-b}^b h y^2 dy \right] \right\} \\ & \times \left(\frac{1}{2} k_1^2 F^2 + k_1 k_0 F \right) \left[y^2 - \frac{1}{\int_{-b}^b h dy} \int_{-b}^b h y^2 dy \right] dy dx. \quad (25) \end{aligned}$$

It is to be noted that the coefficient 1/2 of the second term in the braces { } in the right hand of the equation is for the following reason; the first term σ_{th} in the braces is constant with any deformation but the second term varies from zero where $w_1=0$ to the following value for $w_1=w_1$,

$$E \left(\frac{1}{2} k_1^2 F^2 + k_1 k_0 F \right) \left[y^2 - \frac{1}{\int_{-b}^b h dy} \int_{-b}^b h y^2 dy \right],$$

and the strain changes from zero to

$$\left(\frac{1}{2} k_1^2 F^2 + k_1 k_0 F \right) \left[y^2 - \frac{1}{\int_{-b}^b h dy} \int_{-b}^b h y^2 dy \right],$$

thus, the former stress term has a linear relation to the latter strain term. The same results can also be obtained by integrating Eq. (11).

The kinetic energy is

$$T = \frac{\gamma}{2g} \int_{-a}^a \int_{-b}^b h w_1^2 dy dx.$$

After integration, Eqs. (25) and (26) can be written respectively as follows :

$$U = 2I a k_1 k_0 F + (GJ + I + EK k_0^2) a k_1^2 F^2 + EK k_0 a k_1^3 F^3 + \frac{1}{4} EK a k_1^4 F^4, \quad (27)$$

$$T = \frac{\gamma a^3 k_1^2 H}{3g} \dot{F}^2, \quad (28)$$

where

$$\left. \begin{aligned} H &= \int_{-b}^b hy^2 dy, \\ \dot{F} &= \frac{\partial F}{\partial t}. \end{aligned} \right\} \quad (29)$$

The Lagrange's equation is

$$\frac{d}{dt} \left(\frac{\partial \mathbf{T}}{\partial \dot{F}} \right) + \frac{\partial U}{\partial F} = 0. \quad (30)$$

Substituting Eqs. (27) and (28) in Eq. (30) we obtain the following ordinary differential equation for $F(t)$.

$$\ddot{F} + \lambda F + \mu F^2 + \nu F^3 = \kappa, \quad (31)$$

where λ , μ , ν , and κ are

$$\left. \begin{aligned} \lambda &= \frac{GJ + I + EK k_0^2}{\frac{\gamma a^3 H}{3g}}, \\ \mu &= \frac{\frac{3}{2} EK k_1 k_0}{\frac{\gamma a^2 H}{3g}}, \\ \nu &= \frac{\frac{1}{2} EK k_1^2}{\frac{\gamma a^2 H}{3g}}, \\ \kappa &= - \frac{I \frac{k_0}{k_1}}{\frac{\gamma a^2 H}{3g}}. \end{aligned} \right\} \quad (32)$$

The initial conditions are

$$\left. \begin{aligned} F(0) &= 1, \\ \dot{F}(0) &= 0. \end{aligned} \right\} \quad (33)$$

If the coordinate of $F(t)$ is shifted as much as $-\xi$, $F(t)$ can be written as follows:

$$F(t) = f(t) + \xi, \quad (34)$$

where ξ is to be determined by the following relation.

$$\lambda \xi + \mu \xi^2 + \nu \xi^3 = \kappa. \quad (35)$$

By this transformation of the coordinates Eq. (31) can be transferred to the following equation,

$$\dot{f} + (\lambda + 2\mu\xi + 3\nu\xi^2)f + (\mu + 3\nu\xi)f^2 + \nu f^3 = 0, \quad (36)$$

with the initial conditions

$$\left. \begin{aligned} f(0) &= 1 - \xi, \\ \dot{f}(0) &= 0. \end{aligned} \right\} \quad (37)$$

Now, let us examine into the roots of the following equation of ξ .

$$X(\xi) = \nu \xi^3 + \mu \xi^2 + \lambda \xi - \kappa = 0. \quad (38)$$

The initial conditions are transformed to the following equations.

$$\left. \begin{aligned} F(0) &= 1, \\ \dot{F}(0) &= 0, \end{aligned} \right\} \quad (43)$$

λ , μ , and ν are

$$\left. \begin{aligned} \lambda &= \lambda + 2\mu\xi + 3\nu\xi^2, \\ \mu &= (1-\xi)(\kappa + 3\nu\xi), \\ \nu &= (1-\xi^2)\nu. \end{aligned} \right\} \quad (44)$$

The relation between $F(t)$ and $F'(t)$ is defined by the Eqs. (31) and (41) and is written as

$$F'(t) = (1-\xi)F(t) + \xi. \quad (45)$$

Thus, the initial conditions are the same in both functions. Conclusively, the vibration frequency of the function $F'(t)$ never changes by the upper transformation and is the same as the one of the function $F(t)$ and, further, the amplitude of the $F'(t)$ is $1/(1-\xi)$ times as much as the one of the function $F(t)$. Every characteristic of the time function $F(t)$, therefore, is included in the function $F'(t)$, which we will discuss in the following.

Integrating Eq. (42) we obtain the following equation.

$$t = \int_1^{F'} \frac{dF'}{\sqrt{\lambda(1-F'^2) + \frac{2}{3}\mu(1-F'^3) + \frac{1}{2}\nu(1-F'^4)}}. \quad (46)$$

The solution of this can be written as

$$F'(t) = 1 - \frac{6(\lambda + \mu + \nu)}{\wp(t) + \lambda + 2\mu + 3\nu}, \quad (47)$$

where $s = \wp(t)$ is the Weierstrass's elliptic function. It is defined by the following formula.

$$t = \int_s^\infty \frac{ds}{\sqrt{4s^2 - g_2s - g_3}} = \int_s^\infty \frac{ds}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}}, \quad (48)$$

where the zeros e_1 , e_2 , and e_3 are related to the invariants g_2 and g_3 in the following manner.

$$\left. \begin{aligned} e_1 + e_2 + e_3 &= 0, \\ e_2e_3 + e_3e_1 + e_1e_2 &= -\frac{1}{4}g_2 = -\frac{1}{48}\{-\nu(3\nu + 4\mu + 6\lambda) + \lambda^2\}, \\ e_1e_2e_3 &= \frac{1}{4}g_3 = \frac{1}{144}\{3\nu\lambda(3\nu + 4\mu + 6\lambda) + \lambda^3 - \mu^2(3\nu + 4\mu + 6\lambda)\}. \end{aligned} \right\} \quad (49)$$

For practical evaluation of the results, it is more convenient to rely on the Jacobian elliptic function sn [9]. First, if the time is transferred to the dimensionless form by the following relation,

$$\tau = \sqrt{\lambda} t, \quad (50)$$

and the following parameters are defined,

$$\left. \begin{aligned} \bar{\mu} &= \frac{\mu}{\lambda}, \\ \bar{\nu} &= \frac{\nu}{\lambda}, \end{aligned} \right\} \quad (51)$$

finally, Eq. (42) can be transformed to the following differential equation where the independent variable is the dimensionless time τ .

$$\left. \begin{aligned} \frac{d^2 F}{d\tau^2} + F + \bar{\mu} F^2 + \bar{\nu} F^3 &= 0, \\ F(0) &= 1, \\ \frac{dF(0)}{d\tau} &= 0. \end{aligned} \right\} \quad (52)$$

The solution of this is

$$\tau = \int_F^1 \frac{dZ}{\sqrt{-\frac{1}{2}\bar{\nu}Z^4 - \frac{2}{3}\bar{\mu}Z^3 - Z^2 + \frac{1}{2}\bar{\nu} + \frac{2}{3}\bar{\mu} + 1}}. \quad (53)$$

In this problem $\bar{\nu}$ is positive. The zeros of the inner terms of the denominator of Eq. (53) are denoted by Z_1, Z_2, Z_3 , and Z_4 in the decreasing order and $Z_{ki}, m', m'', m, n', n''$, and k are defined by the following equations.

$$\left. \begin{aligned} Z_{ki} &= Z_k - Z_i, \\ m' &= \sqrt{Z_{13}Z_{24}}, \quad m'' = \sqrt{Z_{14}Z_{23}}, \quad m = \frac{m' + m''}{2}, \\ n' &= \sqrt{Z_{13}Z_{23}}, \quad n'' = \sqrt{Z_{14}Z_{24}}, \quad k = \frac{m' - m''}{m' + m''}. \end{aligned} \right\} \quad (54)$$

Further, the new variable φ is introduced.

$$\left. \begin{aligned} \frac{1 - \sin \varphi}{1 + \sin \varphi} &= \frac{n'Z_4 - Z}{n''Z - Z_{13}}, \\ \sin \varphi &= \frac{n'Z_4 + n''Z_3 - (n' + n'')Z}{-n'Z_4 + n''Z_3 + (n' - n'')Z}. \end{aligned} \right\} \quad (55)$$

Finally we obtain the following expression as the solution of $F(\tau)$.

$$F(\tau) = \frac{n' + n''Z_3 + (n' - n'')\text{sn}\left[\mathbf{K} - \sqrt{\frac{\bar{\nu}}{2}}\tau; k^2\right]}{n' + n'' - (n'' - n')\text{sn}\left[\mathbf{K} - \sqrt{\frac{\bar{\nu}}{2}}\tau; k^2\right]}. \quad (56)$$

The period of the vibration is

$$\left. \begin{aligned} T' &= \frac{4\mathbf{K}}{\sqrt{\lambda} m \sqrt{\frac{\bar{\nu}}{2}}}, \\ \mathbf{K} &= \int_0^1 \frac{dZ}{\sqrt{(1-Z^2)(1-k^2Z^2)}}. \end{aligned} \right\} \quad (57)$$

It is also useful to evaluate the approximate value of the function $F(t)$ by the successive approximation procedure. It is assumed that the function $F(t)$ can be

approximated by the following cosine series.

$$F(t) = A_0 + A_1 \cos \omega t + A_2 \cos 2\omega t + A_3 \cos 3\omega t. \quad (58)$$

Substituting Eq. (58) into Eq. (42) and performing the term by term comparison we obtain the following approximate expression for the vibration frequency.

$$\omega^2 = \lambda - \frac{5}{6} \frac{\mu^2}{\lambda} + \frac{3}{4} \nu - \frac{\nu \mu^2}{\lambda}. \quad (59)$$

The reasonable accuracy of this method was assured by the solution obtained by an electronic analogue computer provided that the vibration is not excessively asymmetric.

In conclusion, we can briefly state influences of the pretwist upon the torsional vibration of thermally stressed thin wings as follows :

1) In the case where there exists the thermal-stress ($I < 0$) and not the pretwist, the vibration frequency reduces as the thermal-stress increases and this decrement in the frequency decreases as the increase in amplitude. This special case is the same as the one treated by Singer [5].

2) In the case where there exists some amount of the pretwist and not thermal-stress, the vibration frequency increases as the pretwist increases unless the amplitude is large. This special case is the same one treated by Reissner and Washizu. The matter, however, will be fairly changed, if the amplitude increases. In this case the reduction of the frequency due to the second and the fourth term of Eq. (59) may become predominant.

3) Finally, the most general case treated in this paper is the one where there exist both the pretwist and the thermal-stress. The aspect of vibration in this category is featured by many complicated phenomena and it can not be stated in a unified manner. The overall aspect can, however, be predicted by the quantitative evaluation of Eq. (59). The more strict discussion on the matter must be done by the use of Eqs. (47) and (56).

In short, the variations in such parameters as the pretwist, the thermal-stress distribution, the sectional form of the wing, may results in various conclusions about the aspect of the subject. And, such lack of the knowledge about the combined effects of the pretwist and the thermal-stress stated in this paper may lead to an erroneous conclusion.

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SYMBOLS

GJ	=torsional rigidity
GJ_{eff}	=effective torsional rigidity
σ_{th}	=thermal stress in the direction of span
r	=distance from the center of twist
A	=sectional area of wing
x, y	=coordinates
$h=h(y)$	=thickness of wing
k_0, w_0	=pretwist and its deflection
k_1, w_1	=angle of twist and its deflection
k_{11}, k_{13}, k_{31}	=coefficients
T	=applied torque
ϵ_0, ϵ	=strains
σ_x	=stress
r, ϕ, X	=cylindrical coordinates
$r(\phi)$	=section of tubular member
$F(t)$	=time function
U	=potential energy
γ	=specific weight
$s=\wp(t)$	=Weierstress's elliptic function
g_2, g_3	=invariants
τ	=dimensionless time
sn	=Jacobi's elliptic function
ω	=vibration frequency