No. 72.

(Published June, 1931.)

On the Yield Point of Mild Steel

By

Fujio NAKANISHI, *Kôgakuhakusi*,

Member of the Institute.

Abstract.

Various theories concerning the elastic limit have been advanced, but none of them can account for the yielding of mild steel. The object of this paper is to propose a new theory. The author considers that the yielding of material is a problem of stability, analogous to the critical point of viscous flow through a pipe. The material will yield when the state of stress becomes unstable; all the stress distribution in the body must therefore have effect on the yield point.

When a specimen yields under tension, there exist, during the yielding of the specimen, two portions, yielded and not yet yielded; the strain in the yielded portion is constant for the material and is very large compared with that in the portion not yet yielded. When some part in the body yields, the strain of this part shifts suddenly from the elastic strain to the yielded strain. The average strain of the specimen increases as such yielded parts increase.

In like manner, when a cylinder yields under torsion, a thin part between some two cross-sections yields at first, and the yielding spreads all over the cross-section. Then such parts come out in succession; thus the twisting moment remains constant during the yielding of the whole length. This fact confirms the author's idea. Moreover, it has been shown by torsion tests of hollow cylinders that the stresses at yield points are not constant. Such a way of yielding cannot be accounted for by any of the theories hitherto advanced.

Corrigenda for the Rep. Aeron. Res. Inst., No. 72.

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128	5	τ_y	Circular	15·5	12·5

Before the yielded portion is strained further, the adjacent portion not yet yielded will begin to yield; hence the stress will be constant where two portions, yielded and not yet yielded, coexist. At the end of the horizontal part of the twisting-moment-twisted-angle diagram the two portions exist in the inner part of the specimen, so in that state the stress may be considered to be uniform. From this fact we have,

$$M_y = \tau_y K,$$

where M_y is the twisting moment at the yield point of a solid or hollow cylinder, τ_y the shearing stress definite for the material, and K the moment of area of the cross-section about the axis. The same relation can be derived from the fact that, at the yield point of a cylinder, the yielding has the tendency to spread throughout the cross-section when a very small part of it begins to yield. The yield point of prisms under torsion can be obtained in the same manner if we take the value of K properly.

The yield points of beams of symmetrical cross-section under uniform bending can also be obtained similarly, viz.,

$$M_y = \sigma_y K,$$

where M_y is the bending moment at the yield point, σ_y the tensile stress definite for the material, and K the moment of area of the cross-section about the neutral axis. By the author's theory, $\sigma_y = 2\tau_y$.

All these relations were confirmed by experiments.

I. Introduction.

1. *Yield Point.*

It is one of most important problems for engineers to know the strength of materials under various kinds of stresses. Nowadays the strength of materials under uniform tensile stress is comparatively well known. But though the strength of a material under a certain kind of stress is known, it has been almost impossible to estimate the stress which this material can bear without failure when it undergoes some other kind of stress. Of course there have been made many researches concerning the relation between the stresses which the material can bear under various kinds of stresses. And as to the yield point of mild steel many theories have already been advanced. But none of them are satisfactory. They coincide with experiments in some cases, but they cannot be said to coincide in every case. All of them assume that a part in the material yields when the state of stress or strain exceeds a certain limit. But I have a different idea. I believe that the stress under which a part in the material yields is not only affected by the kinds of stresses but also by the stress distribution in the neighbouring parts. The object of this paper is to study the effect of the kind of stress and its distribution on the yield point of mild steel, and to advance a new theory concerning the yield point.

The strength under uniform tensile stress is easily determined by measuring the stress and strain in a test piece of uniform cross section under a tensile load. Such mechanical tests of mild steel have been carried out for many years by many experimenters. The stress-strain diagram of previous experiments show that the line which expresses the relation between stress and strain is at first straight but it bends considerably before it arrives at the yield point. The fact that it bends considerably is, in my opinion, due to the lack of uniformity of the tested materials. Such phenomena may occurs if the material is fatigued even though it is uniform, but with virgin specimens of moderately uniform material the relation between stress and strain will be almost linear up to the yield point. Of course the limit of perfect elasticity in the physically strict sense is much lower than the yield point. But if we use the words "elastic limit" in the sense ordinarily used by engineers, or in the sense that the extension is nearly proportional to the load up to this point, we may consider that the elastic limit and yield point are the same for mild steel.

When a tensile load is applied very slowly to a test piece cut out from a moderately uniform material, the stress-strain diagram generally becomes as shown in Fig. 1. At first, strain is proportional to stress, the relation between them being expressed by a straight line OA in the figure. If we increase the load gradually, the stress will at last arrive at a point like A', then it decreases suddenly to a point like B. From B to C it remains practically constant if the strain is made to increase slowly. From C the stress increases again as the strain increases.

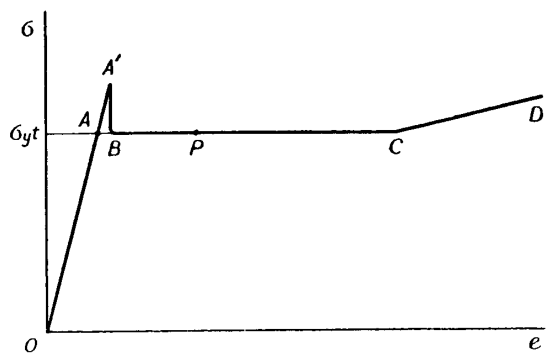

Fig. 1.

An extensometer is generally used to measure the strain. The extension between the gauge marks of a test piece is measured by this means, and the strain is calculated from this measured extension assuming that the strain is uniform all over the gauge length. The strain calculated in such a way is the average strain between the gauge marks. And yet we generally draw the stress-strain diagram by taking this average strain in abscissa as if it were the real strain. The stress-strain diagram drawn in such a way is in Fig. 1. When the strain is uniform all over the gauge length this average strain and the real strain will be the same. But the strain is not always uniform. If we study the strain minutely we will notice that, between B and C in Fig. 1, the strain is not uniform. When the material is on the way of yielding, for instance when the average strain of the test piece is expressed by some point like P in Fig. 1, there exist two portions where the strains are quite different. Of these two portions, one is already yielded and the other is not yet yielded. The strain of the yielded portion is to be denoted by the point C, and that of the portion not yet yielded is to be denoted by the point A. Thus the state of the parts of the test piece is denoted either by A or C, and there exists no intermediate state when the test piece is in equilibrium. During the yielding the parts yield in succession under a constant load, and the state of the yielding part shifts suddenly from A to C. The average strain between the gauge marks increases along the horizontal line BC as such yielded parts increase, and the point C is reached when all the parts between the gauge marks has yielded.

From C the stress increases again as the strain increases, as shown by the curve CD in the figure. In this case the strain is practically uniform until the so called local deformation begins to occur.

If the stress is made to decrease before it arrives at A, it decreases practically along the straight line AO, and the permanent set caused by this loading is of the order 0.001%. So in this region the material may be thought to be practically elastic, and the mathematical theory of elasticity can be applied without an appreciable error. Beyond the point A' the permanent set becomes very large compared with the former case, and for ordinary mild steel the permanent set in the yielded portion amounts to 2～3%.

There are many people who consider that the point A' in Fig. 1—so called *oberestreckgrenze*—is the yield point, and it must have some physical meaning. But from what is explained above it is now clear that we must take the point A, B or C—so called *unterestreckgrenze*—as the yield point. For σ_{yt}, the stress at A, is the stress under which the parts of the test piece yields in succession. The stress, in some cases, goes up at first to a point like A', but this seems to be entirely accidental. With the same material the magnitude of the stress σ_{yt} is always definite, but the stress at A' is not at all definite. Such a phenomenon always occurs in the problem of stability, and, I think, the yielding of material is also to be considered as a problem of stability. The régime of strain expressed by a straight line OA and that expressed by a curve CD are quite different; OA is the strain almost elastic and CD plastic. It seems that at A the régime of elastic strain becomes unstable and the state of the material changes suddenly from A to C.

And it is quite natural from the point of view of stability that the stress at first goes up to a point like A' higher than A, when the material is moderately uniform, and the stress increases very slowly. It is rather strange that no one, so far as I know, has treated the yielding of material as a problem of stability. The idea to consider the yielding as a problem of stability is very simple, and yet it seems to account for many facts concerning the yield point of mild steel.

2. *Analogies.*

Similar phenomena as the yielding of material always occur, as explained above, in the problems of stability. For example, the buckling of a long column under a compressive load may be considered to be analogous to the yielding of mild steel. As long as the load is small the column is stable and the contraction is proportional to the compressive stress. But if we increase the load gradually it will at last arrive at a critical point. Above this critical point the column is unstable to lateral vibration and buckling of the column will take place.

When a long column with round ends buckles under a compressive load, it is well known that the buckling load, P, is given by

$$P = \frac{\pi^2 EI}{l^2},$$

where E = modulus of elasticity of the material of the column,
I = least moment of inertia of the cross-section,
l = length of the column.

As far as the load is small the contraction of the column is proportional to the compressive stress. We see, however, that the buckling load is not given by the compressive stress. But it is given by the above formula. In other words the compressive stresses at the critical points of columns are not definite; they depend on the forms of cross-section. From this analogy we may consider that, though the strain is proportional to stress below the yield point, the stress at the yield point will not be definite; it will depend on the form of specimen and the stress distribution. All the theories hitherto advanced assume that the material yields when the state of stress or strain exceeds a certain limit, and that the forms of specimens and stress distributions have no effect on the yield points. I think these assumptions are nonsense just as to assume that the long columns of any cross-sectional forms would buckle when the compressive stress exceeds a certain limit.

I will mention another example. When a fluid flows through a pipe the flow is laminar as far as the velocity is low, and the surface resistance is proportional to the mean velocity of the fluid. But as we increase the velocity gradually it reaches a critical value. The laminar flow is practically unstable above this critical point, and if there exist some disturbance the state of flow suddenly changes and the flow becomes turbulent. The relation between the surface resistance and the mean velocity also changes suddenly. Fig. 2 shows the relation for the flow of fluid through a pipe of circular section. In the figure $R\frac{D^2}{\rho\nu^2}$ is taken in abscissa and $V\frac{D}{\nu}$ in ordinate, where

V = mean velocity of the fluid,
R = surface resistance per unit area,
ν = kinematic viscosity of the fluid,
ρ = density of the fluid.
D = diameter of the pipe.

We will notice some resemblance between the general form of this curve and the stress-strain diagram of mild steel. Moreover, if the disturbance in the flow is made very small, the laminar flow continues up to a velocity much higher than the critical point; this phenomenon is quite natural from the point of view of stability, and it corresponds to the fact that, referring to the stress-strain diagram in Fig. 1, the stress sometimes goes up to a point like A', higher than A, when the material is moderately uniform and the load is increased very slowly.

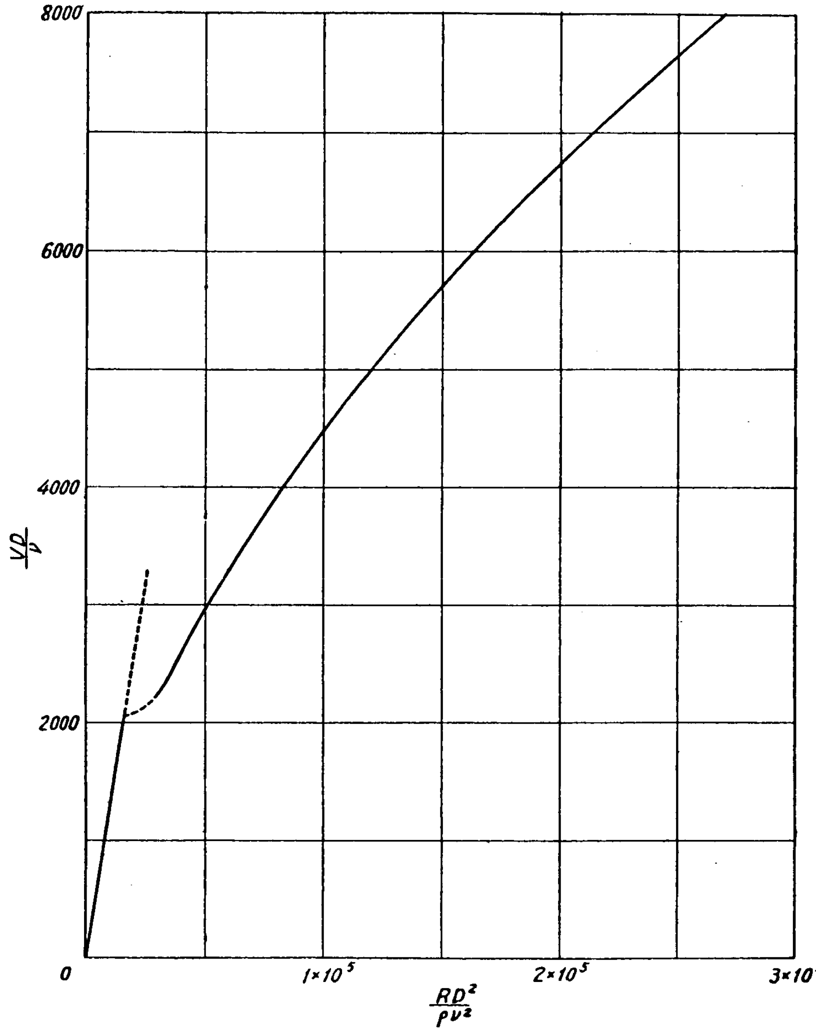

Fig. 2.

It is well known that the critical point of viscous flow throgh a pipe is given by

$$\frac{VD}{\nu} = \text{const.}$$

It means that, when the flows are similar, the transition from the laminar flow to the turbulent flow takes place for a definite value of $\frac{VD}{\nu}$. For instance, this valve is always 2030 for the flow of any fluid through a pipe of circular section of any diameter. But as it is a law of similarity, it holds only when the flows are similar, and $\frac{VD}{\nu}$ must take other values for pipes of other sections.

In this analogy the stress at yield point corresponds to the value of $\frac{VD}{\nu}$, and it is suggested by this analogy that the stresses at yield points will be definite only when the stress distributions are similar.

3. *Law of Similarity.*

We have seen in the previous article that a law of similarity holds for the critical point of viscous flow. Similarly we get the following law of similarity for the same material at the same temperature and for similar stress distributions:—

$$\text{Max. stress at yield point} = \text{const.} \quad \text{..................} \quad (1)$$

Here the max. stress was taken, but as this relation holds only when the stress distributions are similar, the max. strain or any quantity like it may be taken instead of max. stress. If the stress distributions are not similar the max. stresses at yield points will generally differ.

4. *Theories of Yield Point.*

There have been advanced many theories or hypotheses concerning the elastic limit or the yield point of ductile materials like mild steel. The principal ones among them are:—

(a) The Max. Principal Stress Theory,
(b) The Max. Principal Strain Theory,
(c) The Max. Shearing Stress Theory,
(d) The Constant Proof Resilience Theory.

The material yields when, the max. principal stress (a), the max. principal strain (b), the max. shearing stress (c) or the strain energy per unit volume (d) exceeds a certain limit. By those theories the stress distribution is not taken into account, and it is assumed that a part in the material yields when its state of stress or strain exceeds a certain limit. In this respect, they are in discord with my idea. Their assumptions seem to be only permissible when the stress distributions are similar, or they are only the laws of similarity. Yet by those theories it is attempted to apply those laws of similarity beyond their applicability, and assumed that the material yields when its state exceeds a certain limit notwithstanding the variation of stress distribution.

It will easily be seen that those theories do not accord with the results of torsion tests of solid cylinders. When a solid circular cylinder is twisted, it is well known that the relation between the twisting moment, M, and the twisted angle per unit length, θ, becomes as shown in Fig. 6. The line OA is practically straight and AC is horizontal. Such relations, however, are not obtained by those theories. If we suppose those theories to be true, the part of the greatest stress or the outermost cylindrical layer would begin to yield when it reaches a

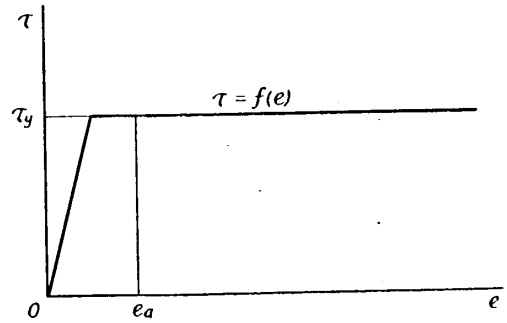

Fig. 3.

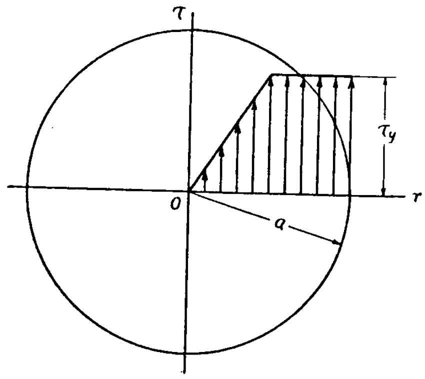

Fig. 4.

certain limit, and the yielding would gradually spread inwards. The ordinary method of calculation of the relation between M and θ with those theories is as follows; — Let the shearing stress, τ, be a function of shearing strain, e, or

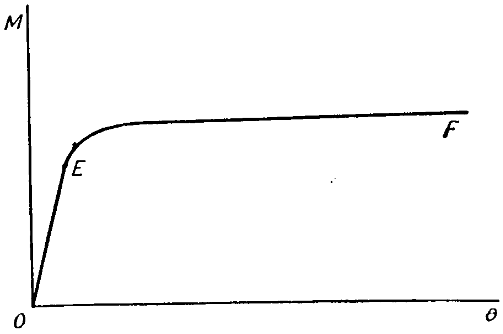

Fig. 5.

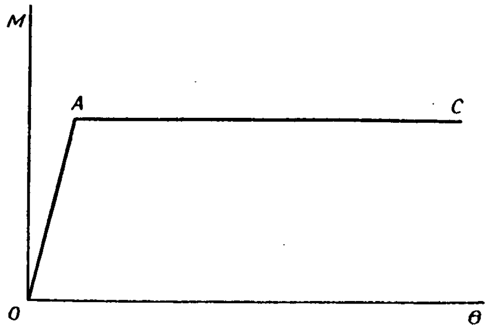

Fig. 6.

$$\tau = f(e)\,.$$

This relation is shown in Fig. 3. When the strain of the outermost layer is e_a, the stress distribution in the cylinder may be as shown in Fig. 4, and the twisting moment, M, is given by

$$M = 2\pi \int_0^a \tau r^2 dr$$

$$= 2\pi \frac{a^3}{e_a^3} \int_0^{e_a} f(e) e^2 de\,.$$

This relation is shown in Fig. 5. E is the point where the outermost cylindrical layer would begin to yield, and the twisting moment would still increase as the yielding spreads to the inner part. But in reality this is not the case. Experiments show that the relation between M and θ is as shown in Fig. 6. Thus there is a distinct difference between the theories and experiments.

The same things can be said with the relation between bending moment and deflection of a beam of symmetrical cross-section under uniform bending. Experiments show that the bending moment remains constant after the beginning of yielding. But with the theories hitherto advanced such a relation cannot be obtained. Moreover it was already remarked by Kennedy[1] that the stresses at yield points of various beams under transverse tests are not definite. On the other hand there are yet some who insist that the accordance of one of those theories with transverse tests is fairly good. But it is perhaps due to rough experiments with materials of not so uniform quality, for with such materials the yield points is not sharp and it appears as if the former theories were correct.

Guest[2] made many experiments on the yield points of thin tubes under various systems of combined stress, and he developed the Max. Shearing Stress Theory with the results of his experiments. His experiments show that the material is made to yield by the shearing stress, and in this respect the Max. Shearing Stress Theory is not so far from the truth. But this theory has also a defect as it does not take into account the form of specimen or stress distribution. This theory accords well with Guest's experiments, but this is accounted for by the fact that the tubes used in his experiments were thin, and the stress in them can be considered, without much error, to be uniform. Thus the stress distributions may be considered to be the same whether the tubes are under torsion or under internal pressure, or their combination. So it is natural that the shearing stress at yield points were nearly constant and the theory accorded well with his experiments. But this accordance is only when the stress distributions are similar, and his experiments do not show that a specimen of any form always yields under a definite shearing stress. I believe that the yielding is caused by instability and the shearing stresses at yield points are different when the stress distributions are not similar. This idea is not at all contradictory to Guest's experiments. It is, I think, not only consistent with his experiments but also the only one able to account for the phenomena of yielding of mild steel.

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(1) Engineering, 15 June, 1923.
(2) Phil. Mag., July 1900.

II. Strain Due to Yielding.

5. *Yielding of a Thin Tube under Torsion.*

When a bar of uniform cross-section yields under tension, the yielding takes place in a slightly curved surface inclined nearly 45 degrees to the axis. This fact suggests that the material is made to yield by the shearing stress, and the strain due to yielding is simple shear. So the simplest case of yielding will be the one where the specimen is subjected to uniform shearing stress only: this is the case of a very thin tube under torsion. The simplest case next to this will be the one where the specimen is under an uniformly varying shearing stress: this is the case of a solid cylinder under torsion. We will now study the strain due to yielding in these cases.

When a material is under uniform shearing stress, for instance a very thin tube under torsion, the relation between the shearing stress, τ, and the strain, e, becomes as shown in Fig. 7. So far as the stress is not so great, the strain is nearly proportional to the stress, the relation between them being expressed by a straight line OA. The slipping of the crystals scarcely takes place in this region. Single-crystal iron slips under a comparatively small stress, but the crystals of ordinary mild steel do not slip so easily, as the orientations of crystals are varied and their slipping in their own definite directions is prevented by the mutual interference of the crystals. But if the shearing stress becomes sufficiently great, the interference between crystals will be overcome and another mode of strain will come into existence. Of course this mode of strain will differ with the material.

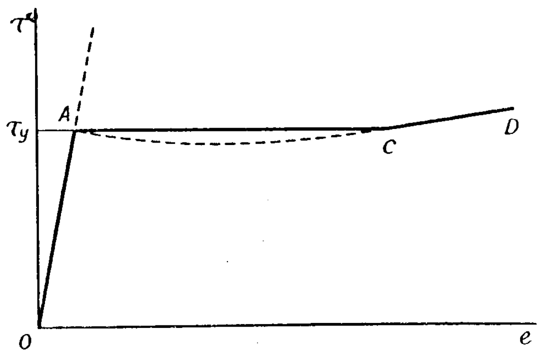


Fig. 7.

If we assume that, in the case of mild steel, the relation between stress and strain in this mode is as shown by the broken line AC in Fig. 7, all the phenomena of yielding seem to be accounted for. The elastic strain is unstable above the point A, and a part which begins to yield shifts suddenly from a state denoted by A, to a state C. At C the stress is equal to that at A or τ_y. To attain a state above C the stress must become greater than τ_y, so the change of the state of the yielding part will cease when it arrives at C, and before this part is strained further the adjacent part not yet yielded will begin to yield. As the yielded parts increase the average strain increases from A to C along the horizontal line AC. Thus when the average strain is between A and C, there exist two portions; one has yielded already, its state being denoted by C, and the other is yet almost elastic, its state being denoted by A.

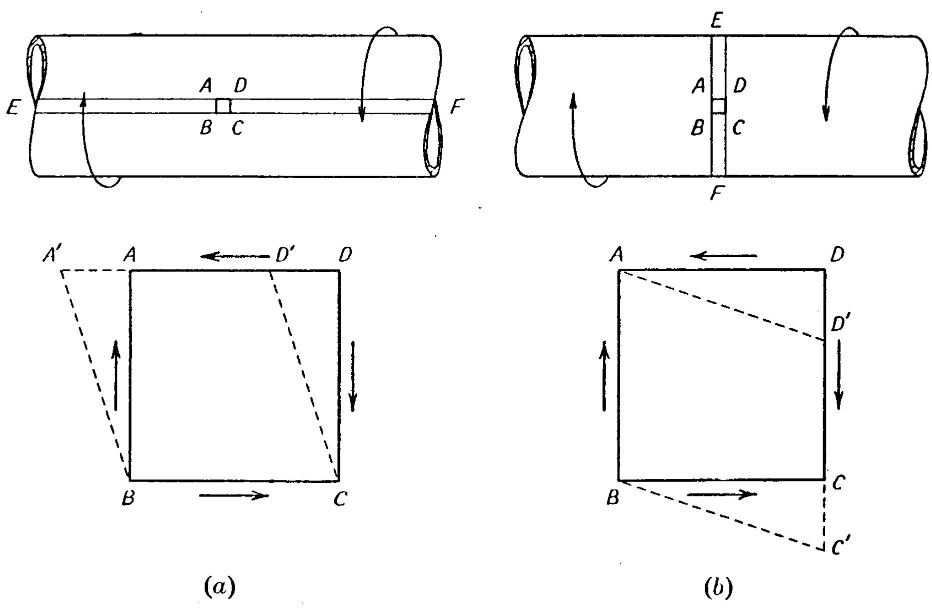


Fig. 8.

When an elemental part, $ABCD$, of a thin tube is in equilibrium under the shearing stress, referring to Fig. 8, the magnitude of shearing stress across the plane DA is equal to that across DC. As the strain due to yielding is simple shear, the yielding may in some cases take place in the plane parallel to DA, i.e., in the plane through the axis of the tube as shown in Fig. 8–a, and in other cases in the plane parallel to DC, i.e., in the cross-sectional plane of the tube as shown in Fig. 8–b. In the former case the elemental part $ABCD$ becomes $A'BCD'$. But $ABCD$ cannot deform by itself; the left-hand and right-hand part of $ABCD$ must deform simultaneously, or it means that all the parts in the thin rod EF in Fig. 8–a yield simultaneously. Similarly, in the latter case, all the parts in the thin ring EF in Fig. 8–b yield simultaneously. Of course the quality of strain in the yielded part is the same in both cases, and when all the parts of the tube have yielded there will be no difference between the two kinds of yielding. But in process of yielding there will be two different kinds of strian due to yielding as shown in Fig. 8.

6. *Yielding of A Solid Cylinder under Torsion.*

Fig. 9 shows the relation between the twisting moment, M, and the twisted angle per unit length, θ, of a solid cylinder. Experiments show that the strain is nearly proportional to the stress until A, and from A to C the twisting moment is constant. As explained in the previous article, the state of the yielding part shifts suddenly from A to C in Fig. 7 under the uniform shearing stress. Comparing Fig. 7 with Fig. 9, it is reasonable to consider that, when a solid cylinder yields under torsion, the state of the yielding part changes also suddenly from some state to some other state. When a thin tube yields under torsion there will be two kinds of strain as

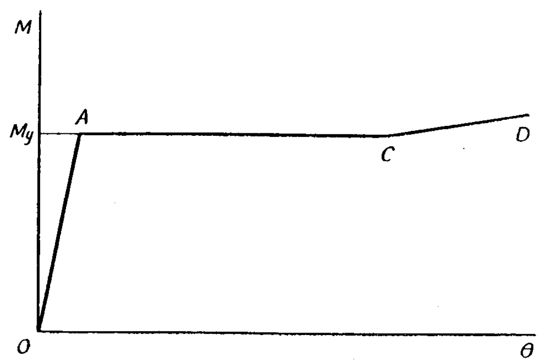


Fig. 9.

shown in Fig. 8. There will be also two kinds of strain with the solid cylinder under torsion; one is the strain when the yielding takes place in the cross-sectional plane and the other in the plane through the axis. I explained already that, when the yielding took place in the cross-sectional plane of a thin tube, all the part in the thin ring like EF in Fig. 8–*b* would yield simultaneously. Similarly it is reasonable to consider for a solid cylinder that, when the yielding takes place in the cross-sectional plane, at first a thin part between some two cross-sections yields suddenly, and then the yielding will gradually spread lengthwise. The twisted angle per unit length or the rate of twist in the yielded part will be denoted by C in Fig. 9, and the rate of twist of the other part which still remains almost elastic will be denoted by A. The parts of the cylinder yield in succession under the constant twisting moment M_y, and the state of the yielding part will change suddenly from A to C.

As already explained the greatest stress at yield point will be affected by the stress distribution. And the stress distribution in a thin tube under torsion differs from that in a solid cylinder under torsion; in the former case the stress may be considered to be uniform, and in the latter case it varies uniformly. So the greatest sheartng stress τ_s at A in Fig. 9 will not be equal to the stress τ_y in Fig. 7; τ_s will be greater than τ_y.

For a solid cylinder under torsion the stress distribution on the plane through the axis is equal to that on the cross-sectional plane. So in some cases the yielding will take place in the plane through the axis, and in such cases a thin wedge-shaped part between some two planes through the axis will yield at first, and then such wedge-shaped yielded parts will successively come out into existence. The strain of the yielded part will, also in these cases, be equal to the strain at C in Fig. 9. But as the thin wedge-shaped yielded part will, similarly to the case shown in Fig. 8-a, spread to the whole length of the cylinder, the rate of twist will be uniform all over the length. And as such yielded parts increase the rate of twist of the cylinder increases from A to C.

7. *Experiments I.*

By the theories hitterto advanced it is assumed that, when a solid cylinder yields under torsion, the part of the greatest stress or the outermost cylindrical layer yields at first and then the yielding spreads gradually to the inner parts. My idea concerning the yielding was explained in the previous article; it is not in accord with the theories. Simple experiments will show whether my idea is correct or not.

When a bar of rectangular cross-section yields under tension, the states of the material denoted by A and C in Fig. 1 can be clearly distinguished by the eye. But when a cylinder yields under torsion the distinction cannot be seen. Yet we can make its yielded portion visible in the following way. The strain in the state A in Fig. 9 is very small, while that in the state C is rather large. Now suppose that some brittle material is coated on the surface of a test piece of cylindrical form. If this brittle material can bear the strain of the state A but not the strain of the state C, then, supposing my idea concerning the yielding to be correct, the coated material will be broken on the surface of the yielded portion. I used enamel as this coating material and satisfactory results were obtained. Figs. 10 and 11 show examples of test pieces coated with enamel and made to yield. We see clearly the parts where the enamel was broken off the surface. Fig. 10 shows a test piece in the

Fig. 10.

state just after the beginning of yielding, and Fig. 11 shows another test piece, of which about a third has yielded. In both cases the yielding took place in the cross-sectional plane. The fact that the yielding takes place in such a way confirms my idea.

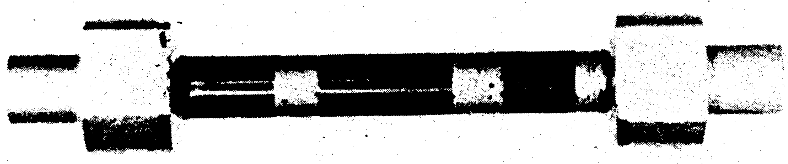


Fig. 11.

8. *Experiments II.*

My idea concerning the strain due to yielding was confirmed qualitatively by previous experiments. I next made quantitative measurements of the amount of strain in the yielded portions. Many lines—lines parallel to the axis and lines perpendicular to them—were drawn on the surface of a test piece of circular section as shown in Fig. 16. The test piece was then twisted by means of a torsion testing mashine. The experimental data are in Table 1. Fig. 12 shows the relation between the twisting moment, M, and the twisted angle, α. It was at first twisted up to the point D in the figure, and then it was released; the average permanent set of the test piece in this state is denoted by the point D'. In this state some portion of the test piece was in the yielded state, while the other portion remained almost elastic. In the yielded portion the rectangles described on the surface were distorted and became parallelograms, and in the portion not yet yielded they remained as they were before. Fig. 16-D' is the photograph of the test piece in this state. We see clearly that in some portion the lines which were parallel to the axis became inclined; this is the yielded portion. The inclinations of the lines were measured by means of a comparator. The results of measurements are in Table 2-D'. Fig. 13 shows these results diagramatically. In this figure the position of the part of the test piece was taken in abscissa and the angle of inclination, φ, in ordinate.

The same test piece was again twisted up to the point E in Fig. 12 and was released; the average permanent set in this state is denoted by the point E'. As before the inclinations of the lines on the surface were measured. The results are in Table 2-E', and shown diagramatically in Fig. 14. Once more the test piece was twisted, and the inclinations were measured in the state denoted by F' in Fig. 12. The results of measurements are in Table 2-F' and shown diagramatically in Fig. 15. Figs. 16-E' and F' are respectively the photographs of the test piece in the states E' and F' in Fig. 12.

These experiments show that the inclinations of lines, which were at first parallel to the axis, is almost constant in the yielded portion of the test piece; the angles of inclination were about $2^\circ 40'$ with the material tested. In the part not yet yielded the lines remained as they were before, showing that the material is yet rather elastic. The ratio of AD to AC in Fig. 12 is equal to the ratio of the length of the yielded portion, shown in Fig. 13, to the whole length of the test piece.

As the test piece was twisted further, the yielding spread lengthwise and the point C in Fig. 12 was reached when the whole length had yielded, and after that the strain increased almost uniformly.

These experiments confirm my idea concerning the yield point; none of the theories hitherto advanced allows such a way of yielding as observed in these experiments.

TABLE I.

TORSION TEST. Test Piece of Circular Section.
Material: Mild Steel.
Diameter: 11·7 mm.
Length: 80 mm.

Twisted Angle, α, in degrees	Moment, M, in cm-kg.	Twisted Angle, α, in degrees	Moment, M, in cm-kg.	Twisted Angle, α, in degrees	Moment, M, in cm-kg.
0	0	12·7	0	23·9	0
0·2	20	13·3	160	24·5	180
0·4	45	13·9	315	25·1	320
0·6	80	14·5	475	25·7	445
0·8	130	15·0	530	26·0	545
1·0	170	16	530	27	545
1·2	210	17	535	28	530
1·4	260	18	525	29	530
1·6	300	19	520	30	520
1·8	360	20	530	31	530
2·0	395	21	530	32	530
2·2	425	22	535	33	525
2·4	490	23	530	34	520
2·6	535	24	525	31·9	0
2·8	565	25	535		
4	515	26	530		
5	520	23·9	0		
6	530				
7	530				
8	520				
9	535				
10	525				
11	515				
12	530				
13	525				
14	525				
15	525				
12·7	0				

TABLE 2.

STRAIN DUE TO YIELDING. Test Piece of Circular Section.
Material: Mild Steel.
Diameter: 11·7 mm.
Length: 80 mm.

Distance from one end, in mm.	Angle of Inclination, φ, in degrees and minutes.					
	D'		E'		F'	
3	0°	0′	2°	18′	2°	26′
6	−0	5	2	31	2	28
9	2	15	2	24	2	24
12	2	27	2	33	2	33
15	2	28	2	33	2	37
18	2	38	2	38	2	37
21	2	40	2	38	2	36
24	2	39	2	40	2	37
27	2	40	2	40	2	39
30	2	38	2	41	2	39
33	2	34	2	38	2	39
34·5	2	32				
36	1	49	2	40	2	39
39	0	0	2	39	2	39
42	0	0	2	39	2	39
45	0	0	2	39	2	39
48	0	0	2	36	2	39
51	0	0	2	32	2	39
54	0	0	2	29	2	39
54·5			1	47	2	38
57	0	0	0	0	2	41
60	0	0	0	0	2	41
63	0	0	0	0	2	36
66	0	0	0	0	2	34
69	0	0	0	0	2	27
72	0	0	0	0	1	47
75	0	0	0	0	0	0

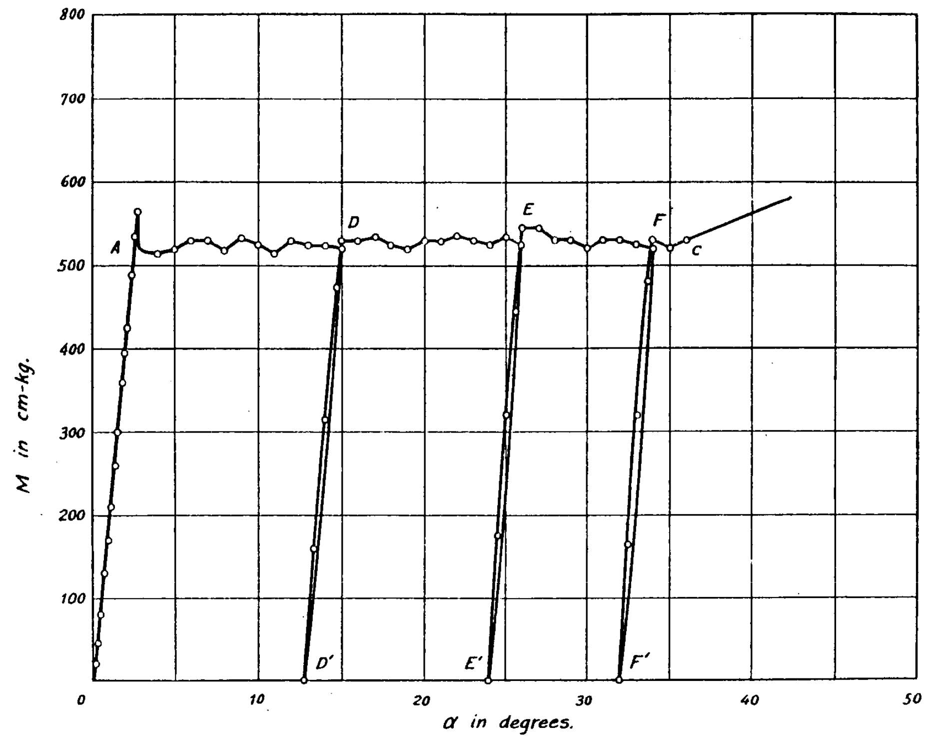

800
700
600
500
400
300
200
100
0
10
20
30
40
50
M in cm-kg.
α in degrees.
A
D
E
F
C
D'
E'
F'

Fig. 12.

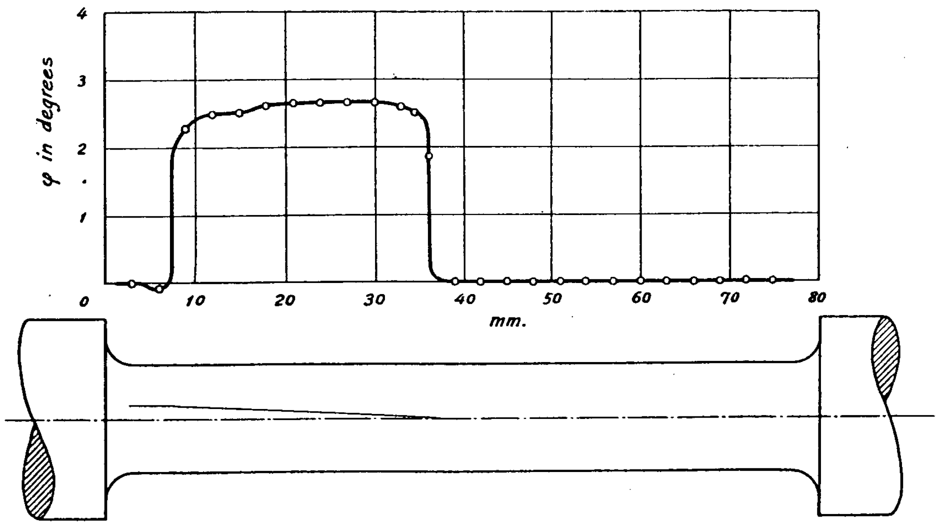
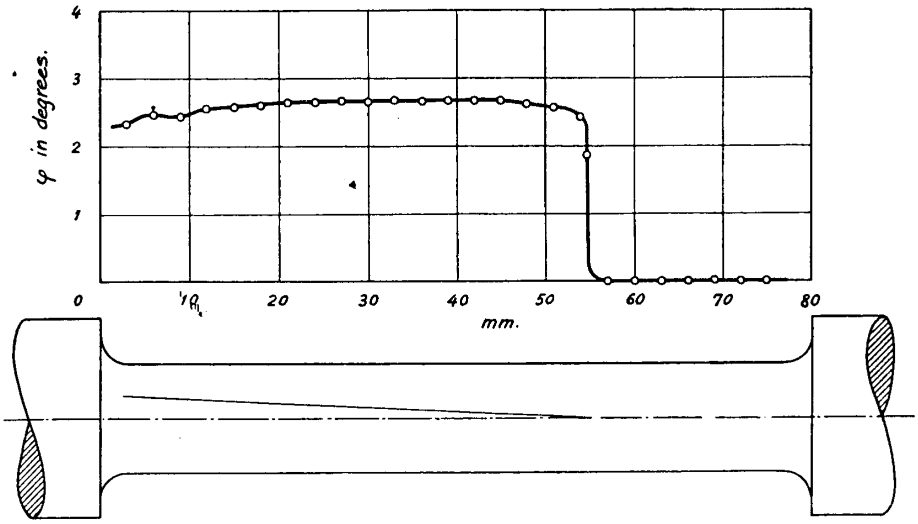
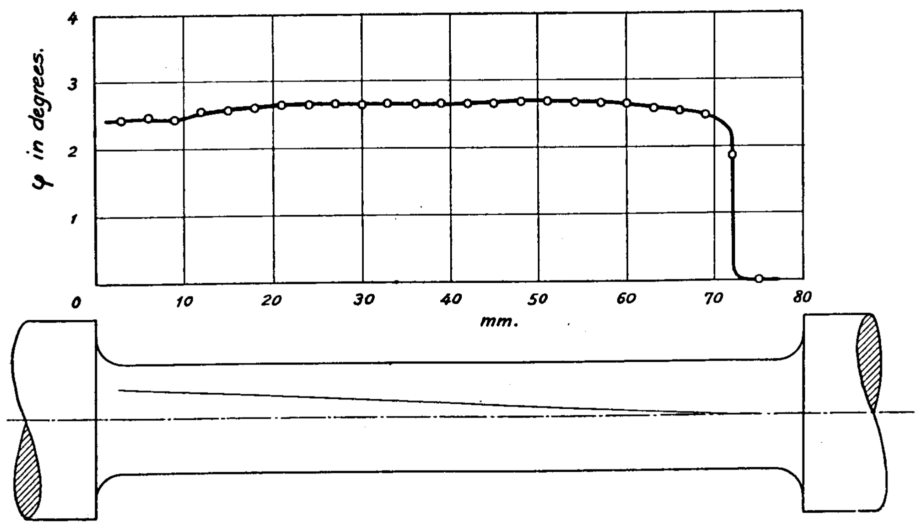

4
3
2
1
0
10
20
30
40
50
60
70
80
φ in degrees
mm.

Fig. 13.



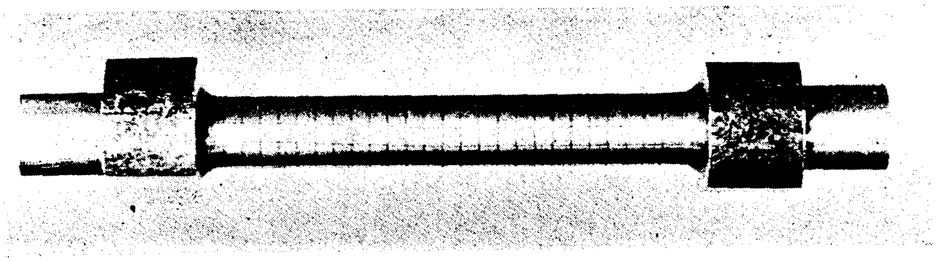
4
3
2
1
0
φ in degrees.
10
20
30
40
50
60
70
80
mm.

Fig. 14.

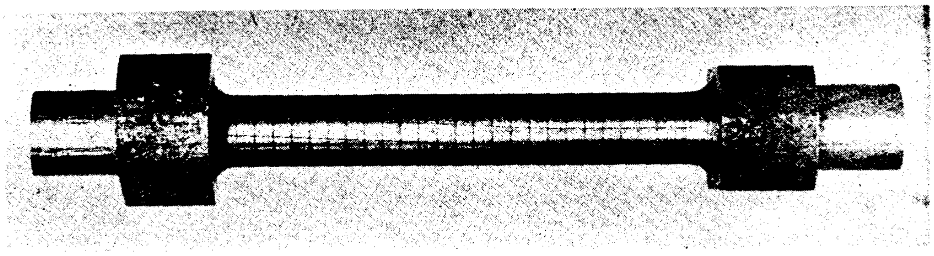


4
3
2
1
0
φ in degrees.
10
20
30
40
50
60
70
80
mm.

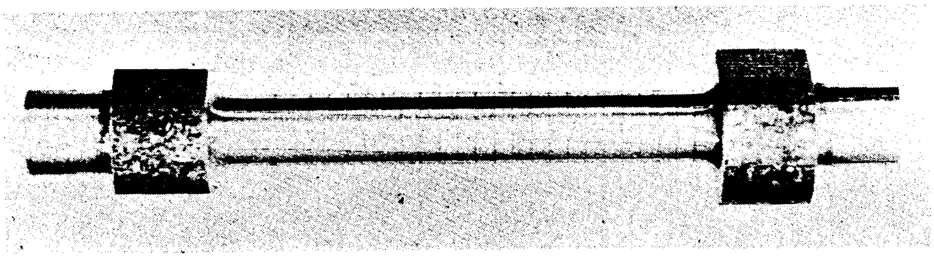
Fig. 15.



D′



E′



F′

Fig. 16.

9. *Strain Figure.*

Recently a method to determine the yielded portion was developed by Fry.(1) Mild steel, in process of yielding, is either yielded or elastic; there is no intermediate state. There is a remarkable difference between these two states as shown by A and C in Fig. 7. It is, I think, due to this remarkable difference that a clear strain figure is obtained by annealing and etching. It is already known that strain figures cannot be obtained with materials which have no horizontal part in the stress-strain diagram. Thus it is also an evidence of my idea concerning the yield point that a clear strain figure is obtained with mild steel.

Figs. 17 and 18 are the strain figures obtained by Fry's method. They show respectively the yielded portions in the interior of the test pieces shown in Figs. 10 and 11. In both cases the yielding took place in the cross-sectional plane. We see that the yielding has spread throughout the cross-section in the yielded portions.

In some cases, as already described, the yielding takes place in a plane through the axis. Most of the strain figures so far published are of this type. For example, Figs. 19-*a* and *b* are reproductions of strain figures obtained by Bader and Nádai.(2) We see that the yielded parts

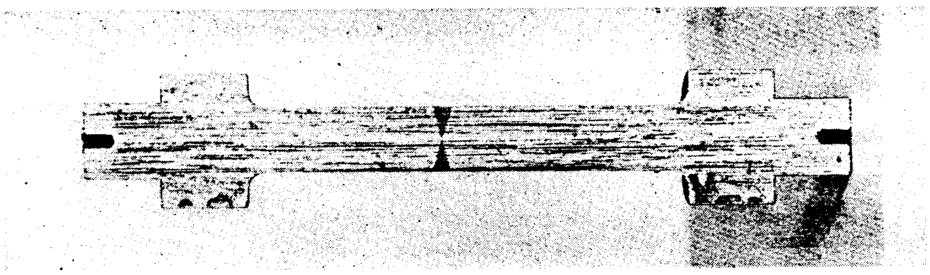


Fig. 17.

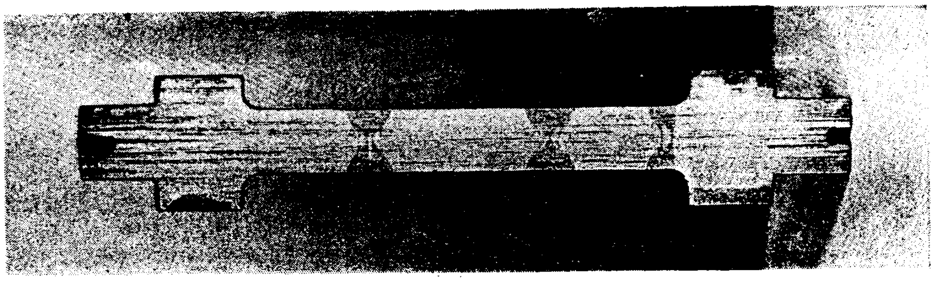


Fig. 18.

(1) Fry, Stahl und Eisen, 11 August, 1921, and Takenaka, J. Soc. Mech. Eng., July 1928.
(2) VDI, 5 März, 1927. They obtained many beautiful strain figures, yet they attempt to account for these figures from the stand point of the Max. Shearing Stress Theory.

are like wedges and they spread from the surface of the cylinder to the neighbourhood of the axis.

(*a*) (*b*)

Fig. 19.

By former theories, the strain would be continuous, so clear strain figures are not to be obtained. Moreover, admitting that strain figures could be obtained, they would be of a form shown in Fig. 20, as the yielded parts would at first appear along the outermost layer of the cylinder. In reality, however, this is not the case, proving that former theories are erroneous in this respect. According to my theory concerning the yield point, the phenomena appearing in the strain figures in Figs. 17 and 18 and Figs. 19-*a* and *b* can be well interpreted.

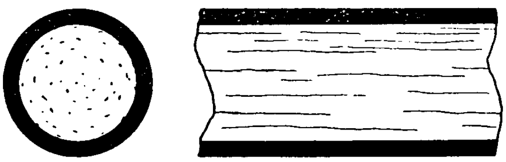


Fig. 20.

When a test piece of circular section is twisted, the strain must be proportional to the distance from the axis. But by my theory, when the test piece is in the act of yielding, some parts of it are in the state C in Fig. 7 and the other parts are in the state of being almost elastic, there being no intermediate state. This idea necessitates the coexistence of yielded and unyielded portions in the inner layer, so that the average strain is proportional to the distance from the axis. The strain figures in Fig. 19 confirm this idea. We see that the inner layers contain a greater portion of material in the elastic state and a less portion in the yielded state.

III. Yield Points of Cylinders Under Torsion.

10. *Preliminary Experiments.*

It has been already described that the greatest shearing stress at yield point will not be definite, but a function of stress distribution. Hence the greatest stresses at yield points of hollow cylinders of various thickness under torsion will not be equal, because the stress distributions in them are different. To find out these stresses experimentally, it is necessary that the material of each test piece must be the same throughout. Preliminary tests were made to study how far this condition can be fulfilled. Tension test pieces of various diameters were cnt out from a bar of mild steel, and were tested. The Amsler 20-tons Universal Testing Machine was used for these tests. Yield points were observed by allowing the strain to increase very slowly. The results of these tests are in Table 3. They are also shown diagrammatically in Fig. 21, the cross-sectional area, S, being taken in abscissa and the load at yield point, L_y in ordinate. All the points plotted by the test data are, as shown in the figure, on a straight line passing through the origin. This fact shows that the material is the same within the experimental error.

Next, preliminary tests under torsion were made with test pieces of solid circular section. Several torsion test pieces were cut out from a bar of mild steel, and were twisted very slowly. The Inokuty-Yamanaka Torsion Testing Machine was used for these tests. In tension tests local deformation takes place before the fracture. But such local deformation does not takes place in torsion tests. Hence, if the material of each test piece is the same, the following relation must hold until fracture:—

$$\frac{M}{d^3} = f(\theta d),$$

where $d =$ diameter of the test piece,

$\theta =$ angle of twist per unit length,

$M =$ twisting moment.

This is an expression of the law of similarity. The results of tests are shown in Fig. 22, θd being taken in abscissa and M/d^3 in ordinate. It will be noticed that all the points plotted by the experimental data are on one curve, showing that the law of similarity holds well. This fact shows that the test pieces made in this way may be considered to be of the same material within the experimental error.

TABLE 3.

TENSION TEST. Material: Mild Steel.

Diameter, D, in mm.	Sectional Area, S, in sq. mm.	Yield Point, L_y, in tons.
20·02	314·6	8·70
20·02	314·6	8·85
14·00	153·9	4·20
14·01	154·1	4·25
10·02	78·8	2·20
10·01	78·7	2·25
7·01	38·6	1·10
7·02	38·7	1·15

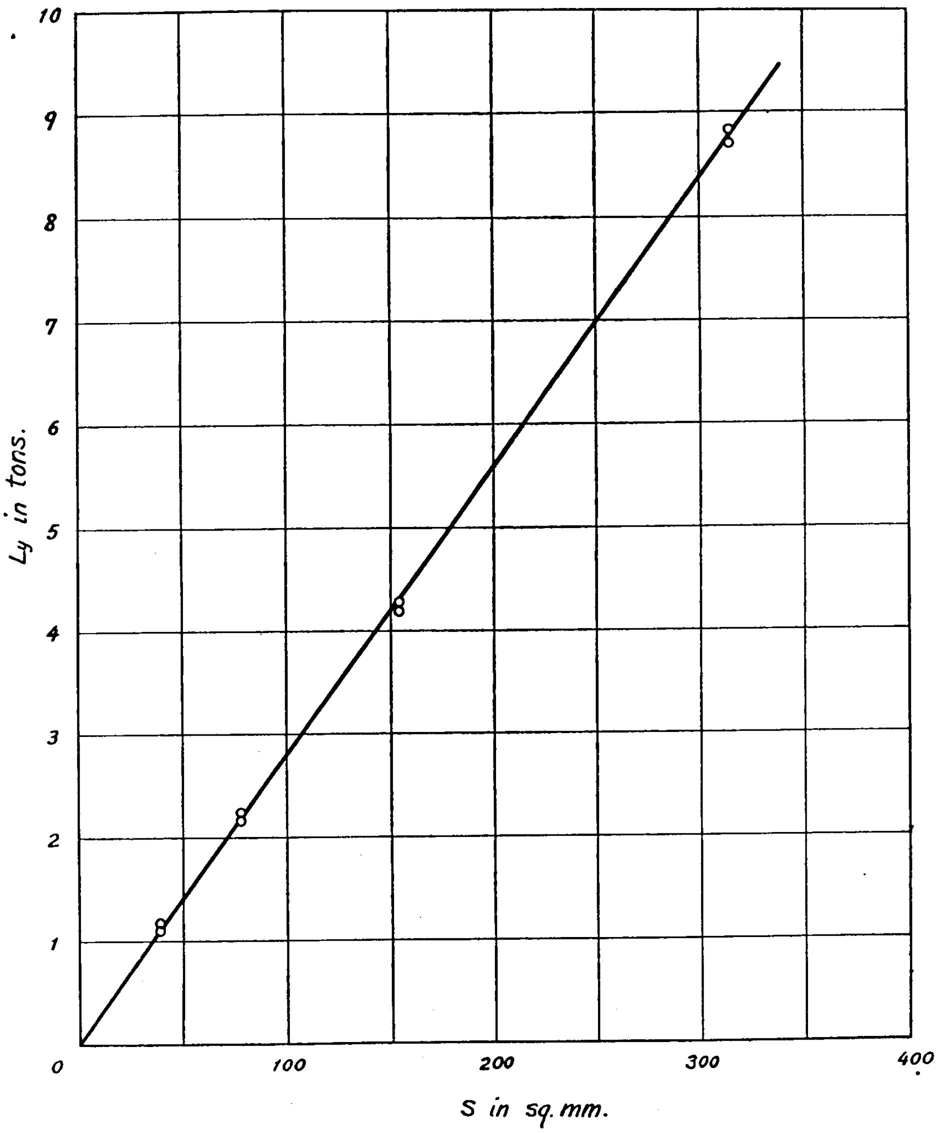


Fig. 21.

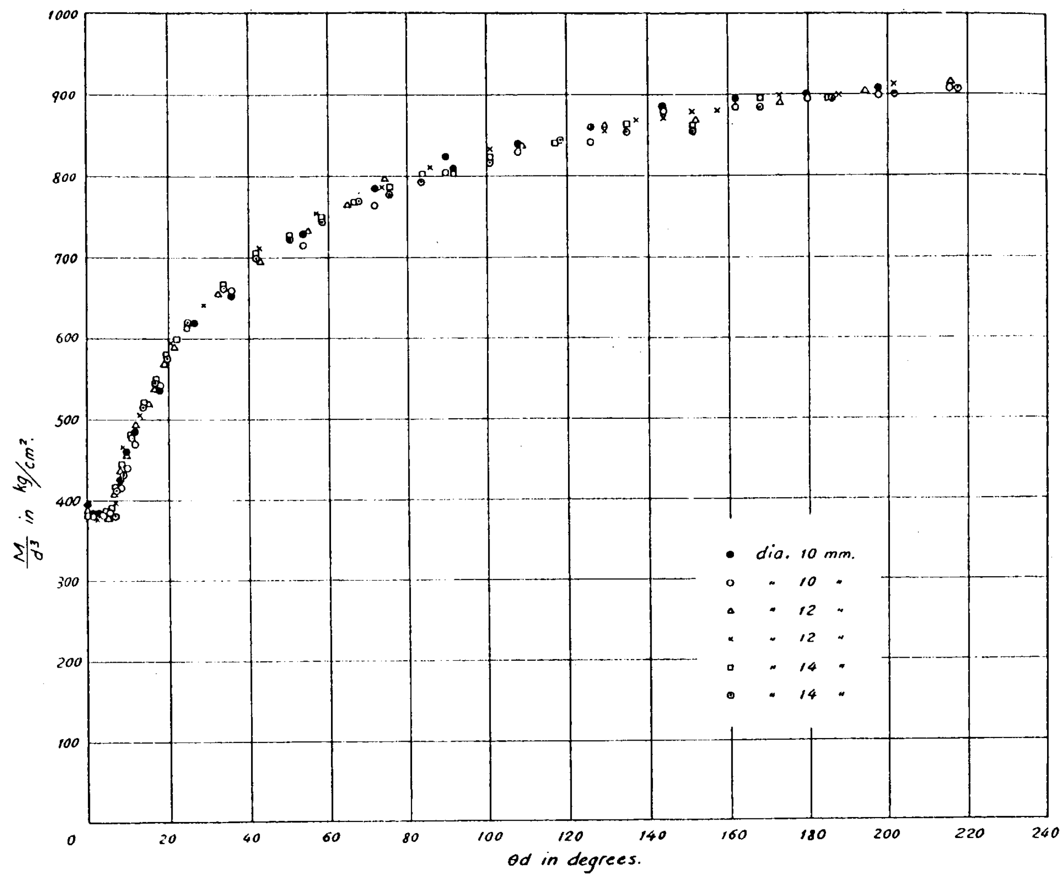


Fig. 22.

11. *Experiments with Hollow Cylinders.*

We have seen in the above experiments that the dimension of the test piece has no effect on the stress at yield point. Hence, for hollow cylinders under torsion the greatest shearing stress at yield point must only be a function of the ratio of internal diameter, d_1, to external diameter, d_0. So test pieces of various ratios of d_1 to d_0 were cut from bars of mild steel, and were tested. The testing machine used was, as before, the Inokuty-Yamanaka Torsion Testing Machine. As the twisting moment is practically constant during the yielding, the yield point can be determined accurately. The results of experiments are in Table 4. An example of the form of test piece and the diagram showing the relation between the twisted angle, α, and the twisting moment, M, of this test piece are shown in Fig. 23.

The material is almost elastic until it begins to yield, and the stress may be considered to be proportional to the distance from the axis. Hence, the greatest stress, τ_s, in the elastic part when the test piece is under the yielding moment can be calculated from Table 4. Let this value for solid cylinder be τ_{ss}. The relation between the ratios of τ_s to τ_{ss} and d_1 to d_0 is shown in Fig. 24. As I expected, the ratio of τ_s to τ_{ss} is not a constant. This fact shows that the greatest stresses appearing just before yielding are not definite even with the same material, but depend upon the sectional form of the test piece, or are a function of stress distribution. This shows directly the erroneousness of all the theories hitherto advanced, as they assume that any part in the material yields when the state of that part exceeds a certain limit.

TABLE 4.

TORSION TEST. Material: Mild Steel.

Test Piece	External Diameter, d_0, in mm.	Internal Diameter, d_1, in mm.	Yield Point, M_y, in cm-kg.
A—1	10·00	0	375
A—2	12·01	0	640
A—3	14·00	0	1020
A—4	14·00	12·00	380
A—5	13·50	12·00	260
A - 6	13·50	12·00	265
B—1	12·04	0	535
B - 2	12·04	0	525
B—3	12·02	5·00	495
B—4	12·03	5·00	485
B—5	14·01	10·07	545
B—6	14·01	10·06	550
B—7	13·00	10·09	360
B—8	15.00	12.05	495

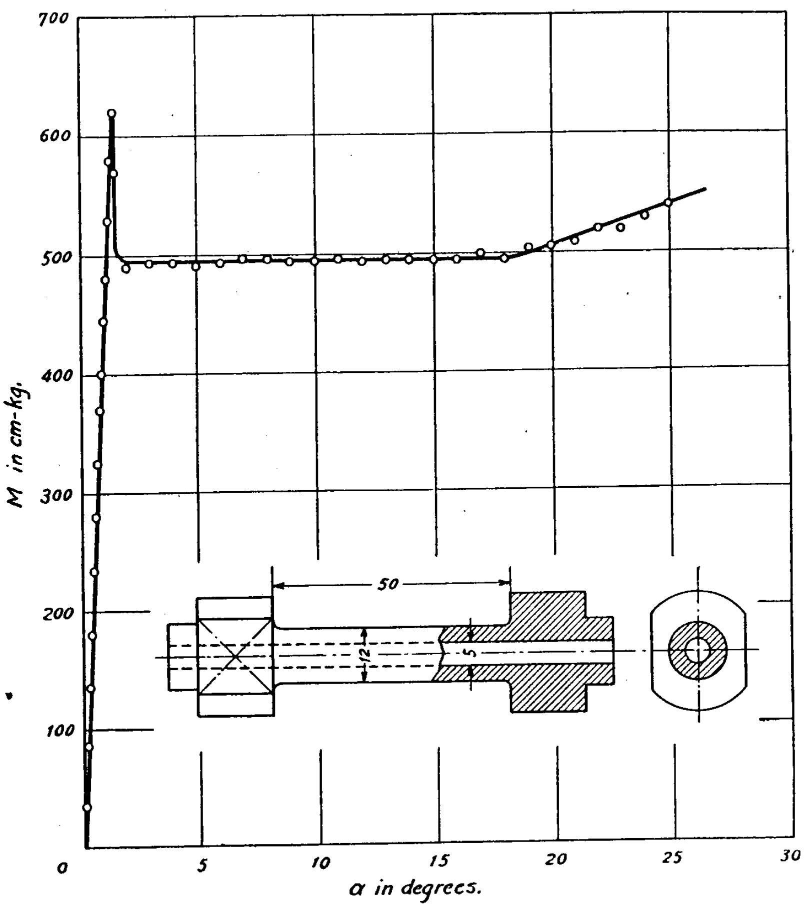


Fig. 23.

12. *Stress at Yield Point, I.*

When the whole length of the test piece has yielded, the state of the test piece is denoted by the point C in Fig. 9; in this state the outermost layer of the test piece has entirely yielded, and in the inner layer two portions, one yielded and the other not yet yielded, coexist. As already described, before the yielded part is strained further the adjacent part not yet yielded will begin to yield. Hence, so far as the two parts coexist, it may be considered that the magnitude of the stress in that part is a constant. This fact holds good for any layer except in the vicinity of the axis, so for a hollow cylinder the shearing stress must be uniform when its whole length has just yielded. Let this stress be τ_y, and the moment of area of the cross-section about the axis be K; then the twisting moment M_y in this state is expressed in the form

$$M_y = \tau_y K \quad \text{(2)}$$

This is the twisting moment when the outermost layer of the cylinder has just entirely yielded, but as the twisting moment is constant during the yielding this is also the moment at which the yielding begins. What is explained above may not be applied in the vicinity of the axis of a solid cylinder but so far as the moment is concerned we may consider without an appreciable error that the equation (2) holds also for a solid cylinder.

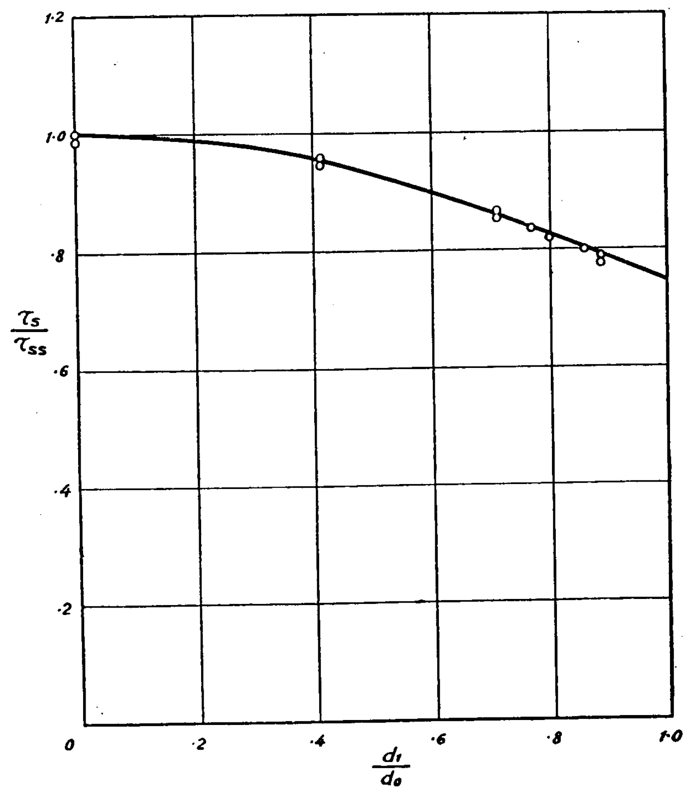


Fig. 24.

The correctness of the equation (2) can be easily shown by comparing this equation with the test results in Table 4. Fig. 25 shows the relation between K and M_y. As shown in the figure, the relation between K and M_y for the test pieces of the same material is denoted by a straight line passing through the origin. This fact shows that the equation (2) holds well for any ratio of d_1 to d_0.

Now the greatest stress τ_s at yield point can be expressed in terms of τ_y. By (2)

$$M_y = \tau_y K = \tau_s \frac{2I}{d_0},$$

where, I = moment of inertia of the cross-section.

Or
$$\tau_s = \frac{4d_0(d_0^3 - d_1^3)\tau_y}{3(d_0^4 - d_1^4)}$$

$$= \frac{4(1-\gamma^3)\tau_y}{3(1-\gamma^4)}, \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (3)$$

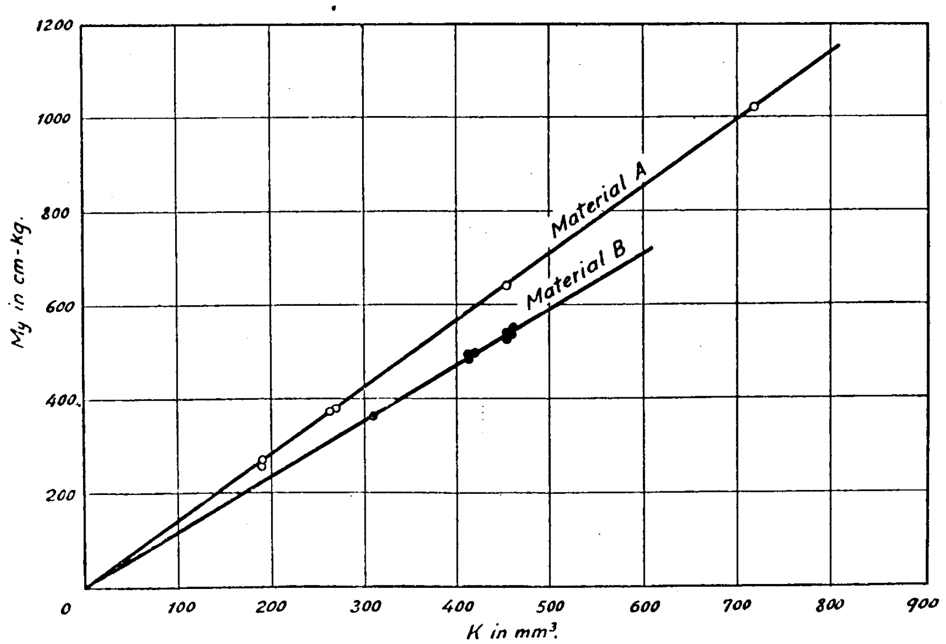


Fig. 25.

in which, γ stands for d_1/d_0.
For a solid cylinder in which $\gamma = 0$, we have

$$\tau_{ss} = \frac{4}{3}\tau_y \qquad (3)'$$

The ratio of τ_s to τ_{ss} is

$$\frac{\tau_s}{\tau_{ss}} = \frac{1-\gamma^3}{1-\gamma^4} \qquad (4)$$

The curve in Fig. 24 was drawn by this formula, all the points plotted by the experimental data lying well on the curve.

13. *Stress at Yield Point, II.*

The equations in the previous article were deduced from the fact that the twisting moment is constant during the yielding, and that the stress in the test piece is uniform when the outermost layer has just entirely yielded. We shall now consider the stress at yield point from another point of view. The manner of yielding explained in the previous chapter shows that, when a cylinder is under the yielding moment, the yielding has the tendency to spread throughout the cross-section of the test piece if an infinitesimal part of it has begun to yield. We seek to determine the state of stress under which such a tendency exists.

Now we shall begin with the simplest case, or the yielding of a solid cylinder under torsion. In the elastic state the stress is proportional to the distance from the axis, r. Let the radius of the cylinder be r_0, and the stress at $r = r_0$ be τ_s, then the twisting moment, M, is expressed in the form

$$M = \frac{1}{2}\pi r_0^3 \tau_s \qquad (5)$$

The yielding will perhaps start from the part where the shearing stress is greatest, so let us assume that the outermost annular part between

r_0 and $r_0-\delta r$ has begun to yield. This part formerly supported the stress τ_s, but as it has begun to yield the stress which this part supports may be considered to have decreased to τ_y. As the twisting moment remains constant, the inner cylindrical part of radius $r_0-\delta r$ must become to support greater moment than before by increasing its strain. Thus the stress distribution will become as shown by the broken line in Fig. 26. The moment, M', which the cylindrical part of radius $r_0-\delta r$ must support is

$$M' = M-2\pi r_0^2 \tau_y \delta r \qquad (5)'$$

Let the stress at $r = r_0-\delta r$ be τ_s', then

$$M' = \frac{1}{2}\pi(r_0-\delta r)^3\tau_s' \qquad (5)''$$

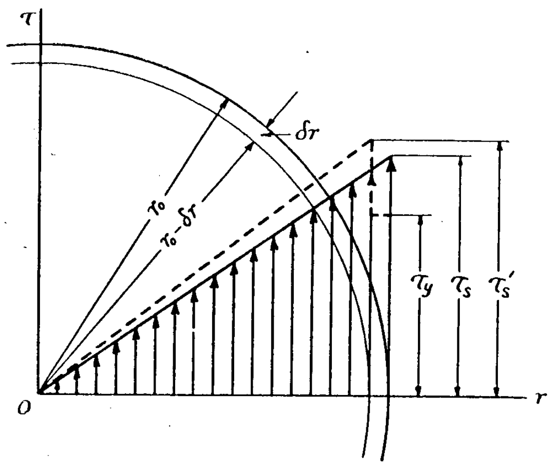


Fig. 26.

The fact that the yielding has the tendency to spread inwards means that this state of stress is unstable and the layer at $r = r_0-\delta r$ will yield. As the annular part between $r_0-\delta r$ and r_0 was assumed to have already begun to yield, the strain of this part can be increased without any change of stress; hence this part can be put out of consideration, and we may consider that the state of stress is similar to the case when the twisting moment M' is applied to a solid cylinder of radius $r_0-\delta r$, So it will be also similar to the case when the moment M is applied to a cylinder of radius r_0. Supposing τ_s to be the critical stress, the state of stress will be unstable for the value of τ_s' equal to or greater than τ_s. Or at the yield point the following equation must hold:—

$$\tau_s' = \tau_s \quad \cdots\cdots\cdots\cdots \quad (6)$$

This is the condition of yield point of a solid cylinder under torsion.

From equations (5), (5)', (5)'' and (6), neglecting the small quantities of higher order, we get

$$\tau_s = \frac{4}{3}\tau_y ,$$

which is the same as equation (3)'.

Next we shall consider the yielding of a hollow cylinder under torsion. Let the external radius be r_0 and the internal radius r_1. As before, assume that the part between r_0 and $r_0-\delta r$ has begun to yield. Then

$$M = \frac{\pi(r_0^4-r_1^4)\tau_s}{2r_0} \quad \cdots\cdots\cdots\cdots \quad (7)$$

$$M' = M - 2\pi r_0^2 \tau_y \delta r \quad \cdots\cdots\cdots\cdots \quad (7)'$$

$$M' = \frac{\pi\left\{(r_0-\delta r)^4 - r_1^4\right\}\tau_s'}{2(r_0-\delta r)} \quad \cdots\cdots\cdots\cdots \quad (7)''$$

The only difference from the case of a solid cylinder is that the law of similarity does not hold between the two states of stress where the external radius is r_0 and where it was thought to be $r_0-\delta r$. But here the greatest stress at yield point is a function of the radius ratio,

γ, of r_1 to r_0 only. If the variation of γ is $\delta\gamma$ when the external radius is diminished by δr, we have

$$\tau_s' = \tau_s + \frac{d\tau_s}{d\gamma}\delta\gamma \quad \cdots\cdots\cdots\cdots \quad (8)$$

Inserting the relation

$$\delta\gamma = \frac{r_1\delta r}{r_0^2},$$

we have

$$\frac{\gamma d\tau_s}{d\gamma} - \frac{3+\gamma^4}{1-\gamma^4}\tau_s + \frac{4\tau_y}{1-\gamma^4} = 0 \quad \cdots\cdots\cdots\cdots \quad (9)$$

Or

$$\tau_s = \frac{4(1+Cr^3)\tau_y}{3(1-\gamma^4)},$$

C being the integration constant. When $\gamma = 1$, τ_s must be equal to τ_y, hence $C = -1$, and

$$\tau_s = \frac{4(1-r^3)\tau_y}{3(1-r^4)}.$$

Or

$$M_y = \tau_y K.$$

These two equations were obtained previously as equations (3) and (2). When the twisting moment arrives at the value expressed by the equation (2), the state of stress becomes unstable and the material begins to yield. And as it is unstable the yielding takes place locally and spreads as explained in the previous chapter.

14. *Comparison of New and Former Theories.*

The shearing stress, τ, is a function of shearing strain, e. But there are two sorts of strain, viz., elastic and plastic strain. In Fig. 27, OA' is elastic strain and CD is plastic strain.

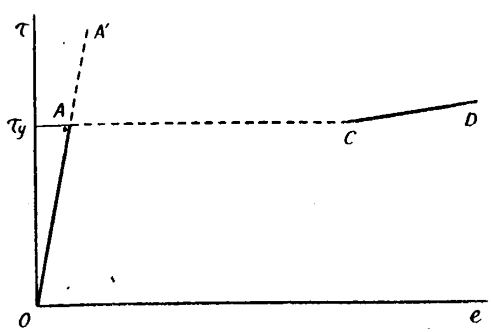


Fig. 27.

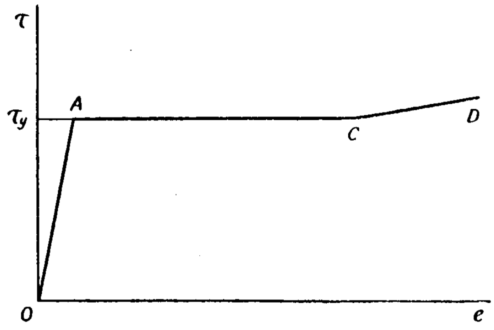
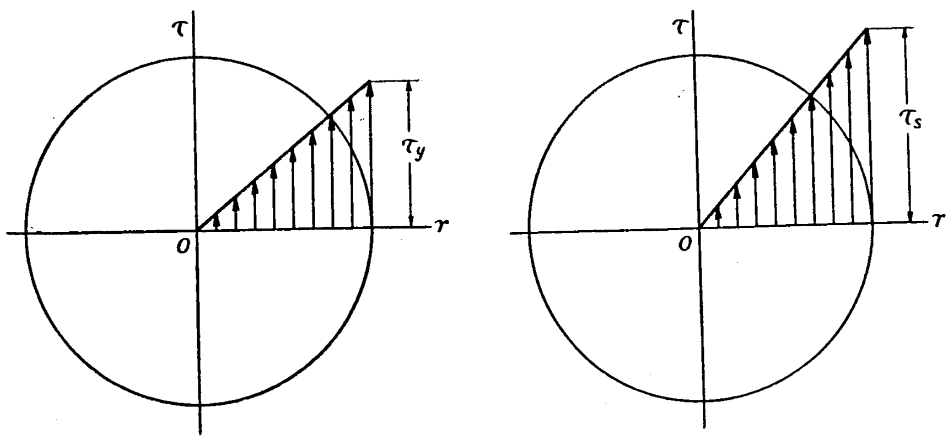


Fig. 28.

For a very thin tube under torsion, both my theory and former theories are the same as to the shearing stress under which the material yields, viz., referring to Fig. 28, the material begins to yield when the shearing stress exceeds a certain limit, τ_y, which is definite for the material.

When a solid cylinder is twisted, the strain is proportional to radius, And as the relation between stress and strain is denoted by the diagram shown in Fig. 27, the stress distribution in the twisted cylinder will be similar to the diagram in Fig. 27. In this respect all the theories accord. But my theory and former theories differ as to the stress under which



(*a*) (*b*)

Fig. 29.

the state of material changes from elastic to plastic. By my theory the material begins to yield under the stress τ_s, which is larger than τ_y; while by former theories the material will begin to yield under the stress τ_y.

The stress distribution in the twisted cylinder by my theory and by former theories are as follows:—The stress distribution when the material begins to yield is shown in Fig. 29. Fig. 29–*a* is the stress distribution by former theories. The material begins to yield when the greatest shearing stress exceeds τ_y. Fig. 29–*b* is the stress distribution by my theory. The material begins to yield when the greatest shearing stress is τ_s, which is equal to $\frac{4}{3}\tau_y$.

Fig. 30 is the stress distribution by fomer theories when the cylinder is a little further twisted above the yield point. The yielding has spread inwards, and the stress in the yielded part is τ_y. By my theory there is no such intermediate state. When the yielding takes place in the cross-sectional plane, the stress distribution, which was as in Fig. 29–*b*, suddenly becomes as shown in Fig. 31–*b*, the twisting moment remaining unaltered. In some cases the yielding takes place in the plane through the axis; in such cases, the same things can be said with the stress distribution on the plane through the axis.

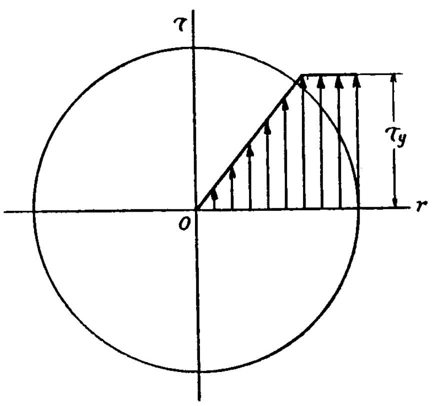
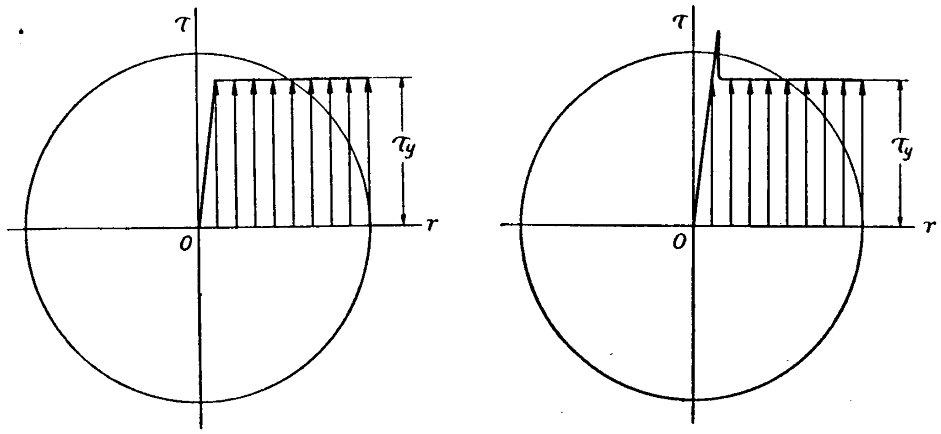


Fig. 30.

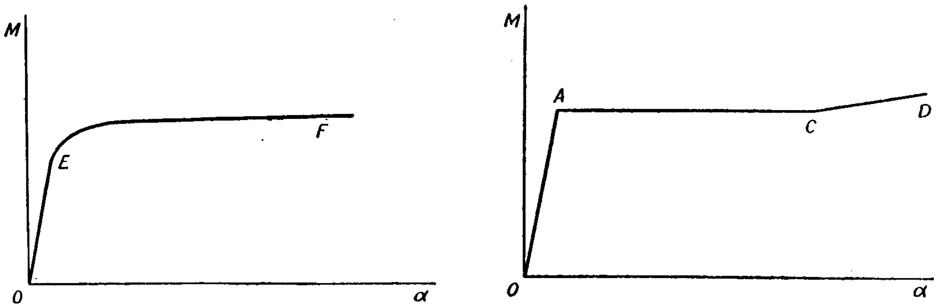
Stress distribution when the outermost layer of the cylinder has entirely yielded is shown in Fig. 31. Fig. 31–*a* is the stress distribution by former theories. The stress is uniform except in the vicinity of the axis where the material is yet in the elastic state. Fig. 31–*b* is the stress distribution by my theory. It is almost equal to that by former theories.



(*a*) (*b*)

Fig. 31.

Fig. 32 shows the relation between the twisted angle and the twisting moment. Fig. 32–*a* is the relation by former theories. The outermost layer of the cylinder begins to yield at some point like E where



(*a*) (*b*)

Fig. 32.

the stress in that layer becomes τ_y. When the cylinder is twisted further the yielding spreads inwards and the moment increases along the curve EF. Fig. 32–*b* is the relation by my theory. The material begins to yield at A where the greatest stress is τ_s or $\frac{4}{3}\tau_y$, and the twisting moment remains constant during the yielding.

IV. Yield Points of Prisms under Torsion.

15. *Theoretical Consideration.*

The yield points of prisms under torsion can be obtained in the same way as in cases of cylinders under torsion, if the forms of their cross-sections are symmetrical about the axes. Fig. 33 shows the relation between the twisting moment, M, and the twisted angle, α, of a prism of symmetrical cross-section; the prism begins to yield at A and the twisting moment remains constant during the yielding. At first, in some cases, the twisting moment may go up higher than the constant moment M_y, but M_y must be taken as the yield point because the yielding spreads from part to part under this moment.

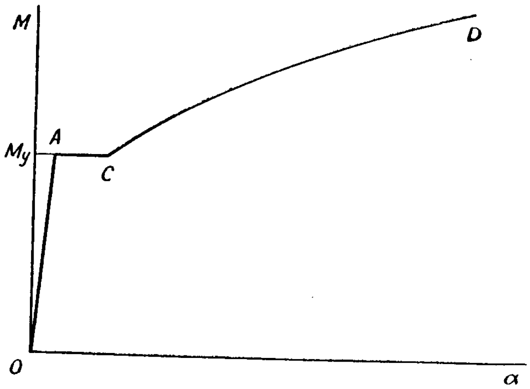


Fig. 33.

We have seen already that the stress is uniform in a circular cylinder, when it is twisted until the end of the horizontal part of the M-α (twisting moment-twisted angle) diagram, and the magnitude of this shearing stress is a constant for this material. In like manner we may consider that the magnitude of the shearing stress is also constant when a prism is twisted until the end of the M-α diagram, because in this state the two portions, one yielded and the other not yet yielded, coexist in the inner part of the prism. This coexistence of two portions can be seen clearly in strain figures. Various investigators have already obtained many such figures. Those shown in Fig. 34 are

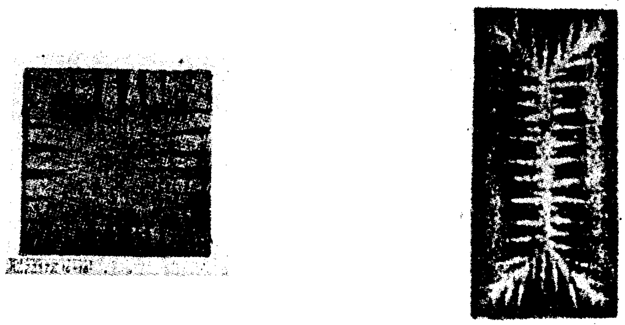


Fig. 34.

reproductions of strain figures of square and rectangular prisms obtained by Bader and Nádai.[1] In those figures we see that the cross-sections are clearly devided into four regions, their boundaries being the lines which bisect the angles. The forms of yielded parts are like wedges, and they are perpendicular to the surfaces of prisms. This fact shows that the yielding has taken place in the planes

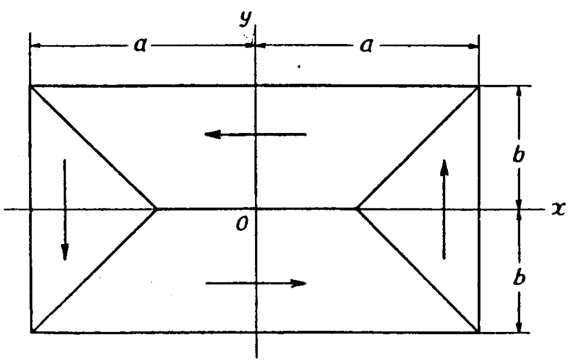


Fig. 35.

(1) loc. cit.

perpendicular to the surfaces of prisms and parallel to their axes. Hence, when the prism is in the state C in Fig. 33, the direction of the shearing stress on the cross-sectional plane will be parallel to the sides, and the magnitude of this stress is a constant.

From above facts the twisting moment under which rectangular prisms yield can easily be calculated. Let the length of the longer side be 2a and shorter 2b, then, referring to Fig. 35, the yield point will be expressed in the form

$$M_y = \tau_y K, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

where,
$$K = 4\int_{a-b}^{a} \{x-(a-b)\} x\,dx + 4\int_{0}^{b} \{y+(a-b)\}\, y\,dy$$

$$= 4b^2\left(a-\frac{1}{3}b\right).$$

16. *Experiments.*

Experiments were made to confirm the equation (10). A circular cylinder, a square prism and a rectangular prism of the same material were tested. The dimensions of test pieces and the results of experiments are in Table 5. The testing machine used was the Inokuty-Yamanaka Torsion Testing Machine. The relations between twisting moment and twisted angle are shown in Figs. 36, 37 and 38. The values of τ_y calculated by the equation (10) are in the last column of Table 5. They are equal within the experimental error. This fact shows that the equation (10) is correct.

The material is almost elastic below the yield point. And so far as the material is elastic, the stress in a prism can be calculated by the theory of elasticity. The formulae given by Saint-Venant(1) are as follows:—

(1) Love, The Mathematical Theory of Elasticity, 4th ed.

$$\tau_s = G\theta \cdot 2b\left[1-\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\operatorname{sech}\frac{(2n+1)\pi a}{2b}\right], \quad \dots\dots\dots\dots (11)$$

$$M = G\theta \cdot ab^3\left[\frac{16}{3}-\frac{b}{a}\left(\frac{4}{\pi}\right)^5\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\tanh\frac{(2n+1)\pi a}{2b}\right], \quad \dots (12)$$

where τ_s = the greatest stress in a rectangular prism,

G = modulus of rigidity,

θ = angle of twist per unit length.

For a rectangular prism of a : b = 2.44,

$$\tau_s = 0.9649 \times 2\mathrm{b}G\theta\,,$$

$$M = 3.9561 \times \mathrm{ab}^3 G\theta\,.$$

For a square prism,

$$\tau_s = 0.6753 \times 2\mathrm{a}G\theta\,,$$

$$M = 2.2492 \times \mathrm{a}^4 G\theta\,.$$

When M is equal to M_y shown in Table 5, we obtain

τ_s for rectangular prism = 21.2 kg / mm.^{2} ,

τ_s for square prism = 19.4 ,, ,

and τ_s for circular cylinder = 16.7 ,, .

As shown above, the greatest stresses at yield points differ from one another considerably.

The examples of calculation of yield points shown here were only for prisms of square and rectangular sections, but the yield points of other prisms, so far as the forms of cross-sections are symmetrical about the axes, can be calculated by a similar process.

TABLE 5.

TORSION TEST. Material: Mild Steel.

Form of Cross-section of Test Piece	Dimension in mm.	K in mm.^{3}	M_y in cm-kg.	τ_y in kg/mm.^{2}
Circular	$d=11{\cdot}7$	419	525	15·5
Square	$2a=2b=11{\cdot}6$	520	630	12·1
Rectangular	$2a=17{\cdot}8$, $2b=7{\cdot}3$	409	515	12·6

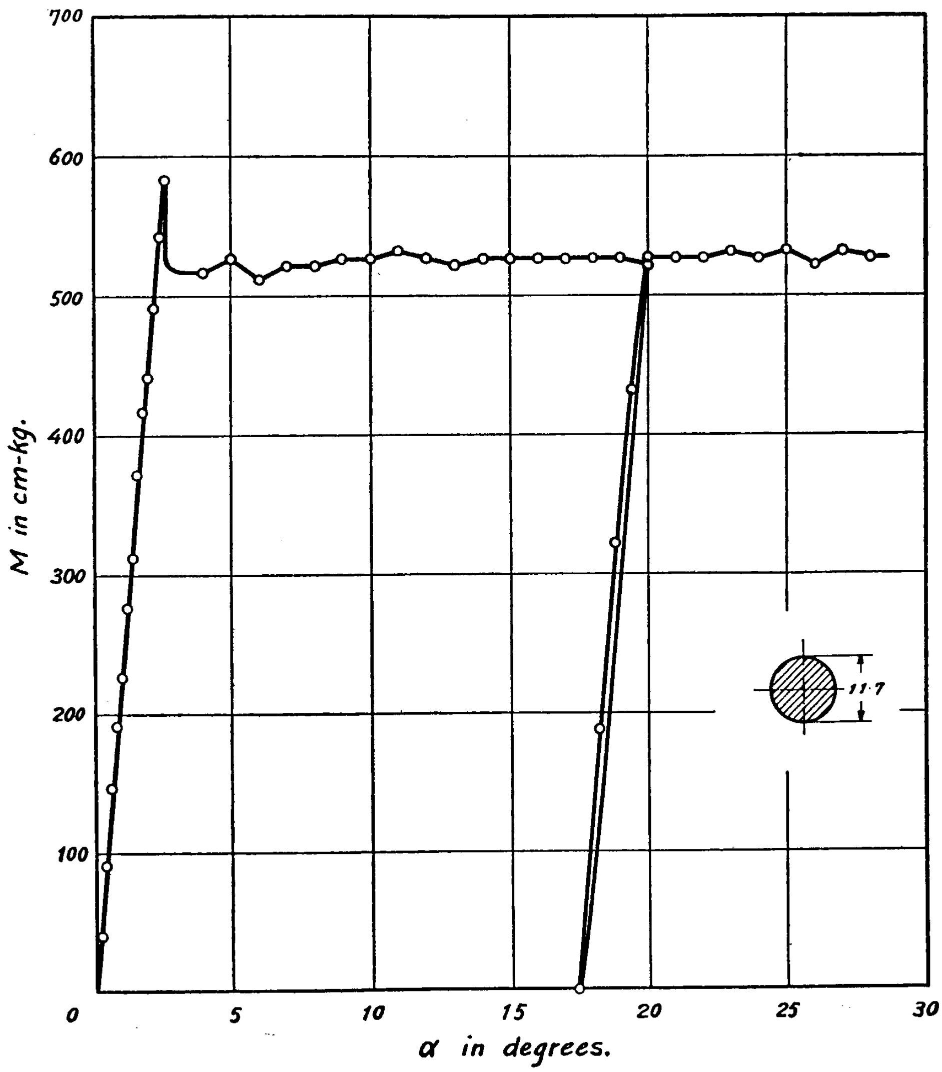


Fig. 36.

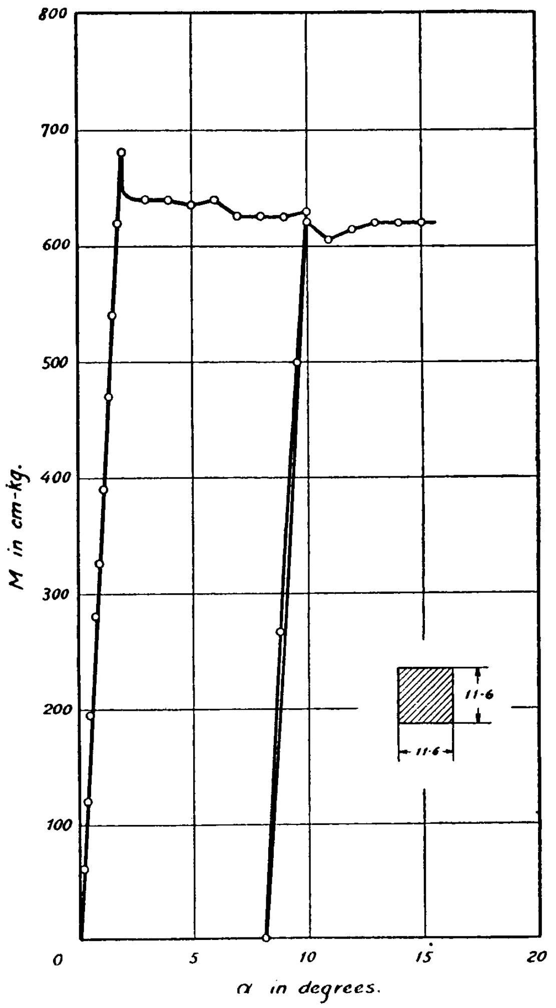


Fig. 37.

V. Yield Points of Beams Under Uniform Bending.

17. *Theoretical Consideration.*

The yield points of beams under uniform bending can be calculated in the same way, if the forms of their cross-sections are symmetrical about the neutral axes. Similarly to the case of a cylinder under torsion, a beam of mild steel is almost elastic below the yield point, and once it begins to yield, the bending moment remains constant during the yielding of the whole length of the beam.

The strain figure shown in Fig. 39 was obtained by etchidg the longitudinal section of a square beam after it was partly made to yield. Referring to the figure, we see that the strain figures in the compression side are nearly similar to those in the tension side. This fact shows

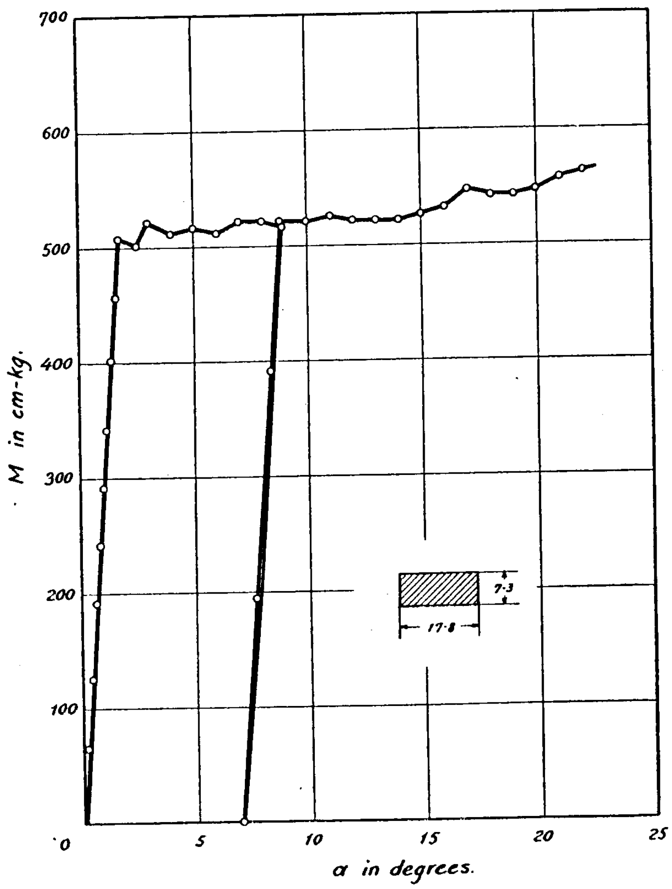


Fig. 38.

that the magnitude of the compressive stress under which the material yields is equal to that of tensile stress. The form of yielded parts are like wedges and their inclination is about 45 degrees to the axis; this fact shows that the material was made to yield by the shearing stress.

The yielded parts have undergone plastic deformation; hence the beam has some curvature in the middle part of the figure. There appears no strain figures in the right-hand part of the figure; this fact shows that this part is yet almost elastic, and, in fact, this part shows no trace of permanent set. Under a constant bending moment, the elastic part, like the right-hand part of the figure, yields in succession and comes to the state like the middle part of the figure. Thus the M-δ (bending moment-deflection) diagram becomes horizontal during the yielding of the whole length.

In the part where two portions, yielded already and not yet yielded, coexist, we may consider that the shearing stress is uniform, and its

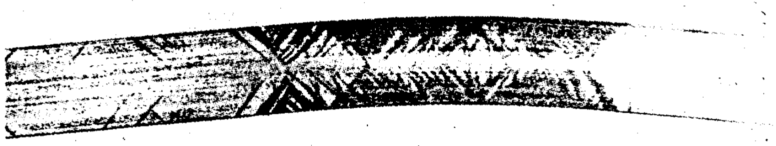


Fig. 39.

magnitude is τ_y. Let the tensile stress across the cross-section be σ_y, then, as the plane on which the stress τ_y acts is inclined 45 degrees to the axis, the following relation must hold:—

$$\sigma_y = 2\tau_y \qquad (13)$$

And the bending moment in such a state is

$$M_y = \sigma_y K, \qquad (14)$$

where K is the moment of area of the cross-section about the neutral axis, or

$$K = \int\int |y| \, dS,$$

y being the distance of elemental area dS from the neutral axis, and the integration being taken all over the cross-section.

The equation (14) can also be obtained by a similar process as equations (3) and (2) were deduced from the fact that the yielding has the tendency to spread throughout the cross-section. For example, the stress at yield point of a rectangular beam under uniform bending can be obtained as follows:— So far as the material is elastic, the relation between the bending moment, M, and the greatest normal stress, σ_s, is expressed in the form

$$M = \frac{4}{3} b h^2 \sigma_s, \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (15)$$

where $2h$ = height of the beam,

$2b$ = breadth of the beam.

Assume that the outermost layer of thickness δ_y has yielded. This part formerly supported the stress σ_s, but as it has yielded already, the stress which this part supports may be considered to have decreased to

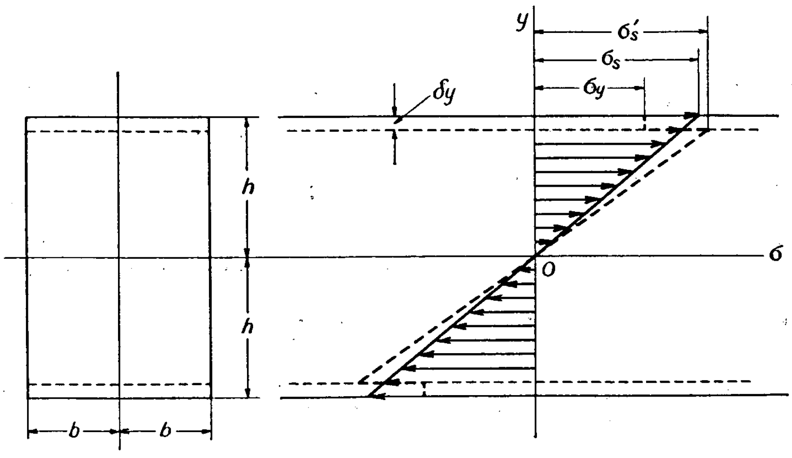


Fig. 40.

σ_y. As the bending moment remains constant, the inner part of thickness $2(h\text{-}\delta y)$ must support the greater moment than before by increasing its strain. Thus the stress distribution will become as shown by the broken line in Fig. 40. The bending moment, M', which the inner part of thickness $2(h\text{-}\delta y)$ must support is

$$M' = M - 4bh\,\sigma_y\,\delta y \quad \text{(15)}'$$

Let the stress at $y = h\text{-}\delta y$ be σ_s', then

$$M' = \frac{4}{3}b(h-\delta y)^2\sigma_s' \quad \text{(15)}''$$

By neglecting the small quantities of higher order, we have

$$\sigma_s' = \sigma_s + 2\frac{\delta y}{h}\sigma_s - 3\frac{\delta y}{h}\sigma_y \quad \text{(16)}$$

As the strain of the layer, assumed to have begun yielding, can be made to increase without any change of stress, this part can be put out of consideration. Then, both σ_s and σ_s' are the stresses under which rectangular beams yield, so

$$\sigma_s' = \sigma_s \quad \text{(17)}$$

This is the condition of yield point of a rectangular beam under uniform bending. From (16) and (17) we obtain

$$\sigma_s = \frac{3}{2}\sigma_y \quad \text{(18)}$$

This is a special case of equation (14).

The example shown here was the yielding of a rectangular beam. But the yield points of beams of any cross-sectional form, so far as the forms are symmetrical about the neutral axes, can be obtained by a similar process.

18. *Experiments.*

Experiments were made to confirm the equation (14). Test pieces of four different sections shown in Fig. 41 were cut out from a bar of mild steel and were made to yield under uniform bending. Yield points were measured by increasing the deflection very slowly. Some examples of the relations between bending moment and deflection are shown in Fig. 42. Dimensions of test pieces and results of tests are in Table 6.

The values of σ_y were calculated from the results of tests by equation (14). They are shown in the table. They may be considered to be a constant. Fig. 43 shows the relation between M_y and K. As shown in the figure, this relation is a straight line passing through the origin; this fact shows directly that the equation (14) is correct. The greatest stresses σ_s before yielding can be calculated easily from the results of tests. They are in the last column of the table. By former theories they would be constant, but as shown in the table they are not at all constant. But the stresses for square and rectangular beams are equal within the experimental error, as the stress distributions in them are similar.

TABLE 6.

BENDING TEST. Material: Mild Steel.

Test Piece	Dimension in mm.				Yield Point, M_y, in cm-kg.	σ_y in kg./mm.^{2}	σ_s in kg./mm.^{2}
	$2b$	$2h$	$2l'$	$2h'$			
No. 12	13·03	13·03	9·99	10·09	770	25·8	32·5
13	13·01	13·01	9·98	10·09	760	25·7	32·3
14	13·02	13·02	9·99	10·09	775	26·1	32·8
No. 22	10·03	10·16			650	25·3	37·7
23	10·02	10·03			640	25·4	38·1
24	9·97	10·02			640	25·6	38·4
No. 32	16·02	6·02			355	24·5	36·7
33	16·03	6·01			370	25·6	38·4
No. 42	d = 10·00				430	25·8	43·8
43	d = 10·02				420	25·1	42·5

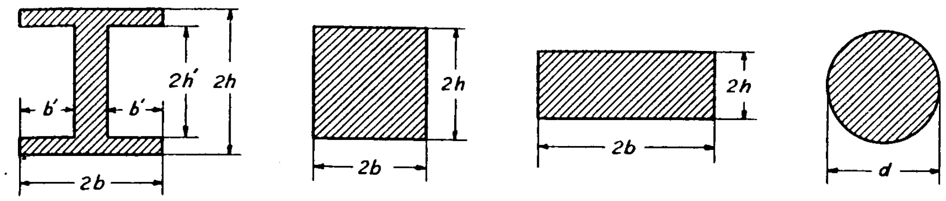

2h'
2h
b'
b'
2b
2h
2b
2h
2b
d

Fig. 41.

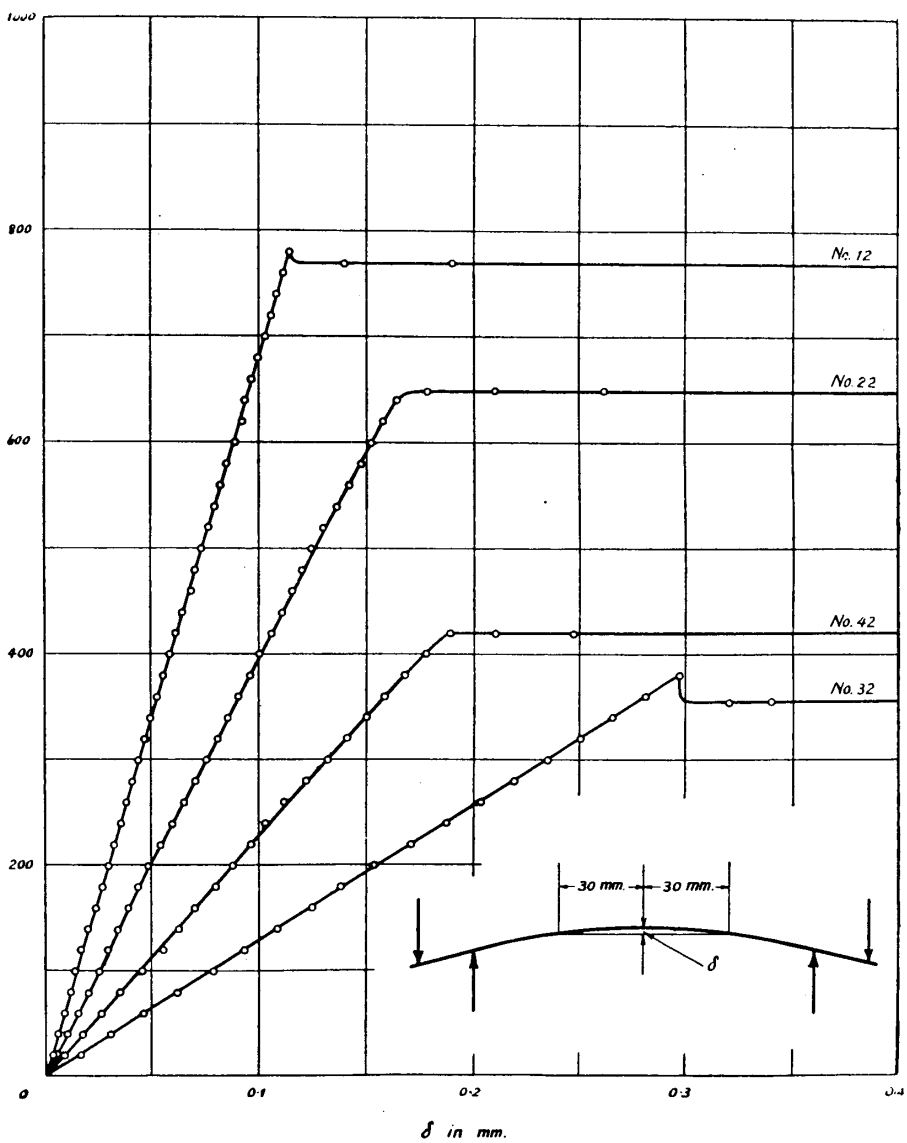

1000
800
600
400
200
0
M in cm-kg.
No. 12
No. 22
No. 42
No. 32
30 mm.
30 mm.
δ
0·1
0·2
0·3
0·4
δ in mm.

Fig. 42.

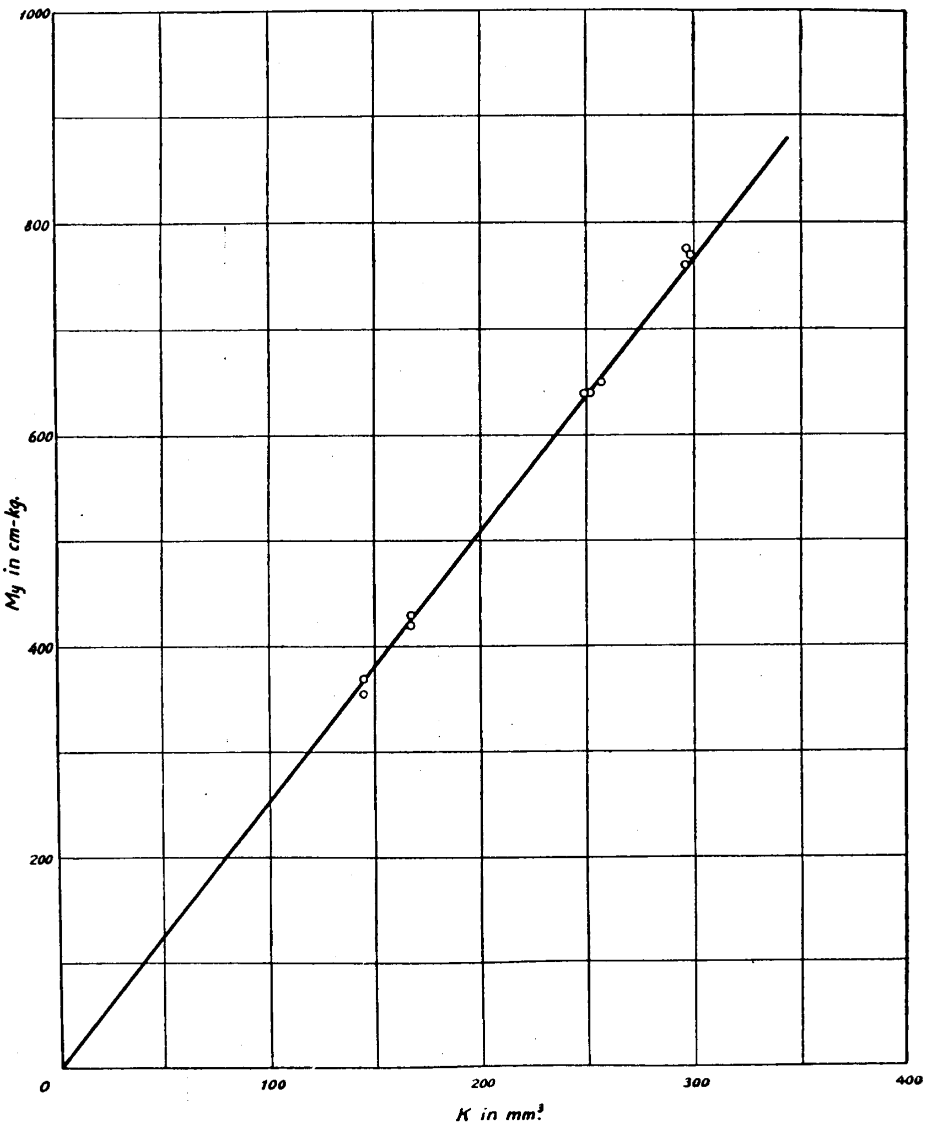


Fig. 43.

VI. Comparison of Yield Points under Torsion, Uniform Bending and Tension.

19. *Experiments on Yield Points under Torsion and Uniform Bending.*

By my theory, the yield point under torsion is given by the equation (2), and the yield point under uniform bending is given by the equation (14), and between them the relation (13) must hold.[1] The equations (2) and (14) were already confirmed by experiments. Now we seek to confirm the relation (13). The relation between the yield point and the cross-sectional form is already known, hence test pieces of any sectional form may be taken for this purpose. Here, test pieces of circular cross-section were taken as they can be made very easily and accurately. All of them were cut out from the same bar of mild steel. The torsion tests were made by means of the Inokuty-Yamanaka Torsion Testing Machine, and the bending tests by the Amsler 5-tons Universal Testing Machine. Dimensions of test pieces and results of experiments are in Table 7.

The greatest stresses at yield points, τ_s and σ_s were calculated; they are in the last column of the table. For test pieces of circular cross-section the ratio of σ_s to τ_s is about 2.6. This fact cannot be accounted for by any of the theories hitherto advanced. The ratio of σ_y to τ_y is 2.06; thus the relation (13) accords fairly well with experiments.

---

(1) The relation between the stress at the yield point under torsion and that under tension was often used to confirm which theory accords best with experiments. There are the following relations between σ_s and τ_s:—

The Max. Principal Stress Theory,	$\sigma_s = \tau_s$.
The Max. Principal Strain Theory,	$\sigma_s = 1.30\,\tau_s$.
The Constant Proof Resilience Theory,	$\sigma_s = 1.61\,\tau_s$.
The Max. Shearing Stress Theory,	$\sigma_s = 2\,\tau_s$.

TABLE 7.

(A) TORSION TEST. Material: Mild Steel.

Test Piece	Diameter in mm.	Yield Point, M_y, in cm-kg.	τ_y in kg/mm.^{2}	τ_s in kg/mm.^{2}
No. 131	12·00	615	13·6	18·2
132	11·99	620	13·7	18·3
133	12·00	640	14·2	18·9
134	12·00	600	13·3	17·7
135	12·00	620	13·7	18·3
136	11·99	620	13·7	18·3
137	11·99	620	13·7	18·3
138	12·00	610	13·5	18·0
			Mean Value = 13·7	Mean Value = 18·3

(B) BENDING TEST. Material: Mild Steel.

Test Piece	Diameter in mm.	Yield Point, M_y, in cm-kg.	σ_y in kg/mm.^{2}	σ_s in kg/mm.^{2}
No. 151	18·00	2720	28·1	47·6
152	17·99	2780	28·7	48·6
153	18·00	2700	27·9	47·2
154	18·00	2720	28·1	47·6
			Mean Value = 28·2	Mean Value = 47·8

20. *Experiments on the Yield Point under Tension.*

Experiments on the yield point under tension were made with test pieces of the same material as used in torsion and bending tests. The testing machine used was the Amsler 5-tons Universal Testing Machine, the same one as used for bending tests. Dimensions of test pieces and the results of experiments are in Table 8. We see that the mean stress at yield point under tension is a few percent larger than σ_y shown in Table 7.

If we assume that, when the yielding spreads under tension, the tensile stress were uniform over the cross-section, and the shear due to yielding would take place in a plane inclined 45 degrees to the axis, then the stress under which the material yields would be equal to σ_y. But experiments show that the yielding takes place in a somewhat complicated curved surface. As an example, photographs of a test piece

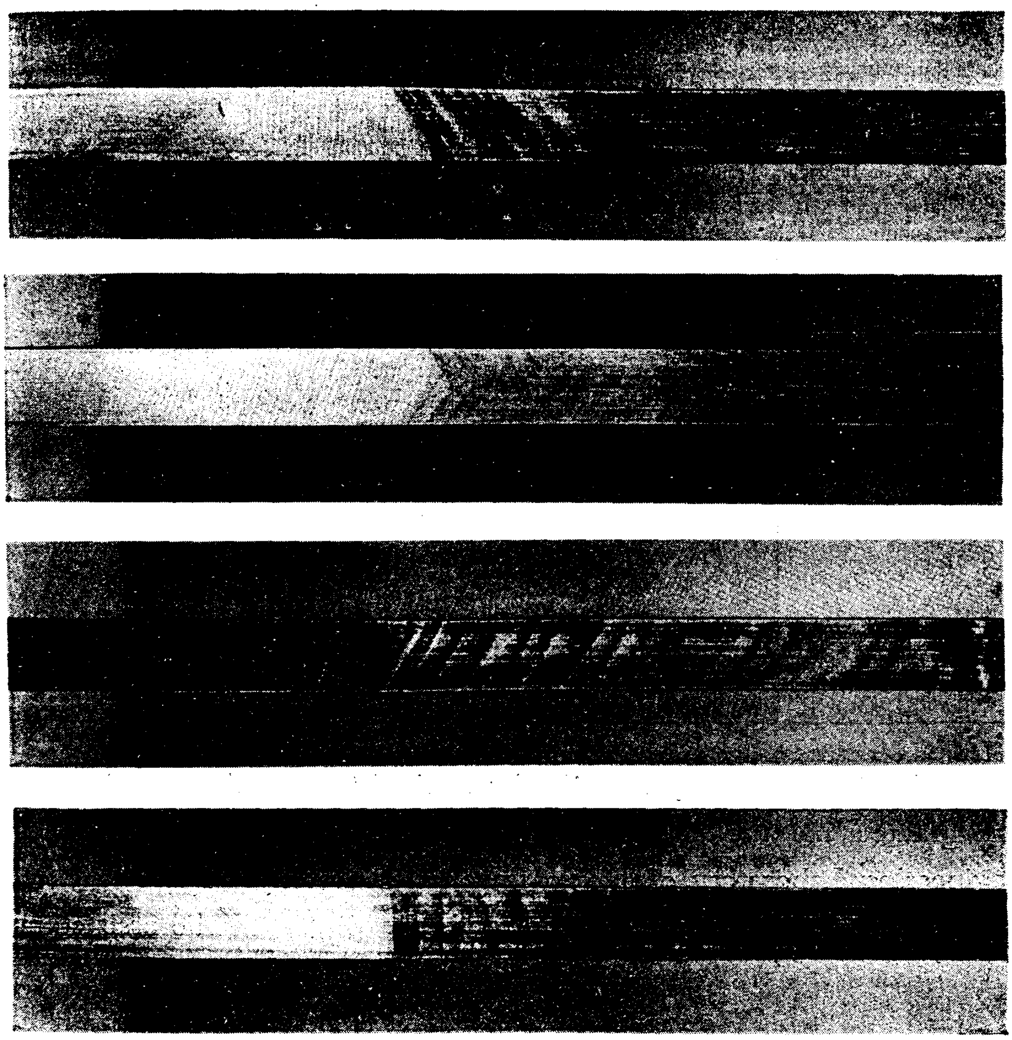


Fig. 44.

of square section, which was made to yield partly, are shown in Fig. 44. They show the four surfaces of the test piece. We see in the figure that the yielding has taken place in a somewhat complicated curved surface. Moreover, the axis in the elastic portion and that in the yielded portion do not coincide. Hence, the bending necessarily takes place in the test piece during the yielding, and so the normal stress across the cross-section cannot be considered to be uniform. Perhaps the minimum normal stress will be equal to σ_y, and the mean stress, σ_{yt}, will necessarily be greater than σ_y. By experiments the mean stress, σ_{yt}, is about 7% greater than σ_y.

TABLE 8.

TENSION TEST. Material: Mild Steel.

Test Piece	Diameter in mm.	Yeild Point in kg.	σ_{yt} in kg/mm.^{2}
No. 141	8·00	1525	30·3
142	8·00	1525	30·3
143	8·00	1470	29·3
144	8·00	1500	29·9
145	8·01	1525	30·3
146	8·00	1515	30·2
147	8·00	1510	30·1
148	8.00	1510	30·1
			Mean Value = 30·1

I wish to express my best thanks to Assistant Professor Takenaka, who granted me the use of the testing machines in the Mechanical Engineering Laboratory of Tokyo Imperial University. I also wish to express my best thanks to Mr. Tani and Mr. Kitamura who assisted me in testing materials.