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New Estimation Methods for Attitude Determination Using GPS Carrier Phase Measurements; Part 1: Direct Attitude Matrix Element Solution by Least Squares

Yong LI, Masaaki MURATA

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NATIONAL AEROSPACE LABORATORY OF JAPAN

CHOFU, TOKYO, JAPAN

New Estimation Methods for Attitude Determination Using GPS Carrier Phase Measurements — Part 1: Direct Attitude Matrix Element Solution by Least Squares

Yong Li *1 Masaaki Murata *2

ABSTRACT

This paper presents a new approach to the problem of attitude determination using GPS. Unlike the traditional linearized, or quaternion-based methods, the attitude matrix element solution is calculated directly. This approach is computationally more economical than the linearized method, and does not need any initial attitude, as it is a non-iterative, forward procedure. As a generalization of the balanced condition, a symmetric condition is introduced which acts to simplify the derived solution, and makes the approach suitable for either coplanar or non-coplanar baseline configurations. It is also applicable to both symmetric and non-symmetric cases. Furthermore, it can act as a compass algorithm in the case of a two-antenna configuration which always fulfills the symmetric condition. The results of experiments demonstrate that algorithms derived from this new approach are highly efficient.

Keywords: attitude determination, GPS, carrier phase, least squares, Wahba's problem

概　　要

GPS搬送波位相データによる姿勢決定問題に対する新しいパラメータ推定アルゴリズムを提案する。新解法では、常套的な線形化法やクオタニオン法とは異なり、姿勢行列要素を直接求める。線形化法に比べて、本手法は計算負荷が少ないうえ、逐次解法ではないので初期姿勢情報を必要としない。また、GPSアンテナ配置に関する従来の平衡条件の一般化として、対称条件を導入する。対称条件が満たされるアンテナ配置の場合には、アンテナの同一平面上の有無、または対称・非対称配置に拘わらず、最適な姿勢行列要素解を極めて簡単なアルゴリズムによって計算できる。さらに2アンテナ構成の場合には、コンパスアルゴリズムとして適用できる。終わりにGPS姿勢決定用受信機による実験データの解析から、提案する新解法の有用性を実証する。

1. Introduction

During the past ten years, many authors have investigated the problem of attitude determination using the Global Positioning System (GPS). GPS has the potential to be the key system for spacecraft and aircraft attitude determination and navigation[1,2]. With the potential capability to provide angular velocity, a spacecraft on-board GPS receiver is really a full-capability sensor. Because of its advantages of full capability, long-term stable accuracy, low cost, and low power consumption, GPS attitude determination is very attractive for space applications.

The algorithms of attitude determination using GPS can

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＊1　STA fellow, Flight Division
＊2　Flight Division

be categorized into three kinds: those that employ the carrier phase measurements directly[4]; those utilizing the GPS vectorized observations[5-7]; and those based on state estimation theory, *i.e.* Kalman filtering, which yield the highest level of accuracy[7]. The use of GPS vectorized observations can be summarized as a problem of two-level optimal estimation, and certain conditions must be satisfied in order to guarantee that the solution is globally optimal.

Two traditional approaches to the first kind of algorithm are the non-linear least squares fit (NLLSFit) method and the Parallel to Wahba's Problem (PWP)[4, 8]. NLLSFit is the more computationally expensive due to its iterative nature, but although the PWP method is faster than NLLSFit, it has to satisfy the condition of a balanced baseline configuration in order to obtain the optimal solution[6, 8].

This paper presents a new approach, which is characterized by the fact that it calculates the attitude matrix element solution (AMES) directly. The key idea is to convert a non-linear problem of attitude angles into a linear weighted least squares problem of attitude matrix elements. The advantages of this method are that it is less computationally intensive and does not need any initial attitude information due to a non-iterative procedure, that it is suitable for both coplanar and non-coplanar baseline configurations as well as for a two-antenna configuration. The symmetric condition derived hereafter can be regarded as the generalization of the balanced condition, and is no longer the necessary condition to guarantee the AMES solution optimal and just acts as the role to simplify the derived solution. Due to this point, the approach can be applied to either symmetric or non-symmetric cases. It is worth noting that the two-antenna configuration always satisfies the symmetric condition as shown in the sequel. Results of experiments using a TANS Vector GPS receiver show that algorithms derived from the new approach are highly efficient.

2. Statement of Problem

The basic measurement for GPS-based attitude determination is the single difference carrier phase, which is the carrier phase difference between the GPS signals received by two antennae separated by a baseline. This kind of measurement also reflects the projection of the baseline vector onto the line-of-sight vector of a GPS satellite.

If there are m baselines and n visible satellites, one method of attitude determination is to find the attitude matrix A that minimizes the following cost function:

$$J(\mathbf{A})=\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^2(\Delta\phi_{ij}+n_{ij}\cdot\lambda-\beta_j-\mathbf{s}_i^T\mathbf{A}^T\mathbf{b}_j)^2 \tag{1}$$

where $\Delta\phi_{ij}$ is the single difference carrier phase measurement corresponding to ith satellite and jth baseline, n_{ij} is the integer ambiguity, λ is the wavelength of the GPS L_1 carrier signal, $\mathbf{s}_i$ is the unit LOS vector of ith satellite, $\mathbf{b}_j$ is jth baseline vector, β_j is the line-bias corresponding to jth baseline, and w_{ij}^2 is weighted factor.

The typical method of attitude determination can be divided into two independent steps: resolving the integer ambiguities and determining attitude matrix A. Once the integer ambiguities are fixed, they no longer need to be resolved in the later procedure. Therefore, the following equivalent differential range can be defined if the integer ambiguities are known:

$$\Delta r_{ij}=\Delta\phi_{ij}+n_{ij}\cdot\lambda-\beta_j \tag{2}$$

Substituting Eq. (2) into Eq. (1), we obtain

$$J(\mathbf{A})=\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^2(\Delta r_{ij}-\mathbf{s}_i^T\mathbf{A}^T\mathbf{b}_j)^2 \tag{3}$$

One kind of candidates to resolve the integer ambiguities is the motion-based method, including platform motion-based methods[8, 9], GPS satellites motion-based methods[10]. Both of them collect carrier phase measurements over a few epochs till antennae platform moves or geometry of visible GPS satellites obviously changes. Although the motion-based methods have been proved to be efficient, another kind of method named instantaneous method is more suitable for real-time applications, i.e. spacecraft attitude control. An effective instantaneous ambiguity resolution is Knight method which has been applied on the Trimble TANS Vector attitude receiver[11]. Knight method has the capability to determine the ambiguities only utilizing the measurements at an epoch. Mathematically, it is based on the search principle, and employs Kalman filter to do the search procedure in order to find the most possible solution from a number of candidates.

Although the integer ambiguities solution is very crucial to the problem of attitude determination using GPS, the discussion below will focus on the method to resolve the attitude matrix, and suppose the integer ambiguities have been resolved by a method. In fact, we employ Knight method to resolve the integer ambiguities in the experiments.

The antenna configuration is assumed rigid in this paper, and flexibility of the configuration is out of the scope of the present paper.

3. Traditional Approach

Two traditional approaches to the problem of Eq. (3) are the non-linear least squares fit and the Parallel to Wahba's Problem. These are summarized in the following paragraphs.

a) Non-Linear Least Squares Fit method

As is well known, the attitude matrix A can be expressed by a first-order linearization about a nominal attitude $\mathbf{A}_0$ as follows:

$$\mathbf{A}=[\mathbf{I}+\boldsymbol{\Omega}(\delta\theta)]\,\mathbf{A}_0 \quad (4)$$

$$\boldsymbol{\Omega}(\delta\theta)=\begin{bmatrix} 0 & \delta\theta_3 & -\delta\theta_2 \\ -\delta\theta_3 & 0 & \delta\theta_1 \\ \delta\theta_2 & -\delta\theta_1 & 0 \end{bmatrix} \quad (5)$$

where $\delta\theta_i(i=1,2,3)$ are the three components of $\delta\theta$.

To define $\boldsymbol{\Omega}(\mathbf{b}_j)$ similarly as Eq .(5), only using $\mathbf{b}_j$ instead of $\delta\theta$, then Eq. (3) can be converted to the following form:

$$J(\mathbf{A})=\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}{}^2(\Delta r_{ij}-\mathbf{s}_i^T\mathbf{A}_0^T\mathbf{b}_j-\mathbf{s}_i^T\mathbf{A}_0^T\boldsymbol{\Omega}(\mathbf{b}_j)\delta\theta)^2 \quad (6)$$

Introducing a variables z_{ij} and a (1×3) matrix $\mathbf{h}_{ij}$ as

$$z_{ij}=\Delta r_{ij}-\mathbf{s}_i^T\mathbf{A}_0^T\mathbf{b}_j \quad (7)$$

$$\mathbf{h}_{ij}=\mathbf{s}_i^T\mathbf{A}_0^T\boldsymbol{\Omega}(\mathbf{b}_j) \quad (8)$$

then the linear weighted least squares solution of Eq. (6) is given by:

$$\delta\theta=[\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}{}^2\mathbf{h}_{ij}^T\mathbf{h}_{ij}]^{-1}\,[\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}{}^2\mathbf{h}_{ij}^T z_{ij}] \quad (9)$$

From Eq. (4), the nominal $\mathbf{A}_0$ can be corrected by $\delta\theta$:

$$\mathbf{A}=[\mathbf{I}+\boldsymbol{\Omega}(\delta\theta)]\,\mathbf{A}_0$$

$\mathbf{A}_0$ is used instead of $\mathbf{A}$ in order to calculate a new correction parameter $\delta\theta$ to correct the new $\mathbf{A}_0$. This iterative procedure continues until $\|\delta\theta\|$ is smaller than a certain specified accuracy.

b) Parallel to Wahba's Problem

Using baseline vectors and LOS vectors, we define the matrices $\mathbf{B}(3\times m)$ and $\mathbf{S}(3\times n)$:

$$\mathbf{B}=[\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_m] \quad (10)$$

$$\mathbf{S}=[\mathbf{s}_1\ \mathbf{s}_2\ \cdots\ \mathbf{s}_n] \quad (11)$$

We also construct $\Delta\mathbf{R}(m\times n)$ of single difference range measurements:

$$\Delta\mathbf{R}=\begin{bmatrix} \Delta r_{11} & \Delta r_{12} & \cdots & \Delta r_{1n} \\ \Delta r_{21} & \Delta r_{22} & \cdots & \Delta r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta r_{m1} & \Delta r_{m2} & \cdots & \Delta r_{mn} \end{bmatrix} \quad (12)$$

If at the same time we introduce two positive numbers w_{sii}^2 which is the $(i,i)^{th}$ element of a diagonal positive definite matrix $\mathbf{W}_s(n\times n)$, and w_{bjj}^2 which is the $(j,j)^{th}$ element of a diagonal positive definite matrix $\mathbf{W}_B(m\times m)$, such that $w_{ij}^2=w_{sii}^2\cdot w_{bjj}^2$ which replace the weighted factor w_{ij}^2, then the problem of Eq. (3) can be converted into the following equivalent form[8]:

$$J(\mathbf{A})=\|\mathbf{W}_B^{1/2}(\Delta\mathbf{R}-\mathbf{B}^T\mathbf{A}\mathbf{S})\mathbf{W}_B^{1/2}\|_F^2 \quad (13)$$

where $\|\bullet\|_F^2=tr([\bullet]^T[\bullet])$ is the Frobenius norm of a matrix, $\mathbf{W}_B^{1/2}$ is the square root of of $\mathbf{W}_B$, and $\mathbf{W}_S^{1/2}$ is the square root of $\mathbf{W}_S$. If $\mathbf{W}_B$ is chosen so as to satisfy the condition

$$\mathbf{B}\mathbf{W}_B\mathbf{B}^T=\mathbf{I} \quad (14)$$

then the form of Eq. (13) can be made identical to that of Wahba's problem, which maximizes the new cost function:

$$J'(\mathbf{A})=tr(\mathbf{A}\mathbf{S}\mathbf{W}_S\Delta\mathbf{R}^T\mathbf{W}_B\mathbf{B}^T)=tr(\mathbf{A}\mathbf{G}^T) \quad (15)$$

where $\mathbf{G}=\mathbf{B}\mathbf{W}_B\Delta\mathbf{R}\mathbf{W}_S\mathbf{S}^T$. Eq.(14) is referred to as Cohen's balanced condition. There are several methods to solve Eq. (15), such as QUEST, SVD, the Euler-q method *et al.*[12], *i.e.,* QUEST employs quaternion $\tilde{\mathbf{q}}$ to represent the attitude matrix A. After doing so, the problem of Eq‾. (15) is converted to one of finding the optimal solution $\tilde{\mathbf{q}}_{opt}$, which is the eigenvector of a (4×4) matrix K corresponding to the maximum eigenvalue of K, λ_{max}. So

$$\mathbf{K}\tilde{\mathbf{q}}_{opt}=\lambda_{max}\tilde{\mathbf{q}}_{opt} \quad (16)$$

Here,

$$\mathbf{K}=\left[\begin{array}{c|c}\mathbf{G}+\mathbf{G}^T-tr(G)\cdot\mathbf{I}_{3\times3} & \mathbf{v}\\ \hline \mathbf{v}^T & tr(G)\end{array}\right]$$

and $\mathbf{v}$ is a (3×1) vector defined as

$$\mathrm{v}_1=\mathbf{G}_{23}-\mathbf{G}_{32},\quad \mathrm{v}_2=\mathbf{G}_{31}-\mathbf{G}_{13},\quad \mathrm{v}_3=\mathbf{G}_{12}-\mathbf{G}_{21}$$

where $\mathbf{v}_j$ is the jth element of $\mathbf{v}$, and $\mathbf{G}_{ij}$ is the (i, j)th element of G.

Up to this point, the optimal solution $\tilde{\mathbf{q}}_{opt}$ has been obtained and the attitude matrix A may also be calculated by $\tilde{\mathbf{q}}_{opt}$. However, the condition of Eq. (14) is referred to as a "balanced" baseline configuration. Obviously, only non-coplanar baseline configurations may satisfy Eq. (14). For the case of coplanar baseline configuration, this method can obtain the sub-optimal solution[4, 6].

4. New Approach

Although the problem of Eq. (3) is a non-linear function of the attitude angles, it is a linear function of the attitude matrix A. The key idea is to convert the problem of Eq. (3) to a weighted least squares problem of elements of A. Introducing a (9×1) state vector $\mathbf{a}$ to express A as follows:

$$\mathbf{a}=[\mathbf{a}_1^T \mid \mathbf{a}_2^T \mid \mathbf{a}_3^T]^T \tag{17}$$

where $\mathbf{a}_i^T(i=1,2,3)$ is the ith row of A. Then

$$\mathbf{s}_i^T\mathbf{A}^T\mathbf{b}_j=[b_{jx}\mathbf{s}_i^T \mid b_{jy}\mathbf{s}_i^T \mid b_{jz}\mathbf{s}_i^T]\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix}\tilde{\mathbf{h}}_{ij}\mathbf{a} \tag{18}$$

where b_{jx}, b_{jy}, b_{jz} are the three components of vector $\mathbf{b}_j$, $\tilde{\mathbf{h}}_{ij}=[b_{jx}\mathbf{s}_i^T \mid b_{jy}\mathbf{s}_i^T \mid b_{jz}\mathbf{s}_i^T]$ is a (1×9) matrix.

Substituting Eq. (18) into Eq. (3), we obtain

$$J(\mathbf{A})=\sum_{i=1}^{n}\sum_{j=1}^{m}w_{ij}^2(\Delta r_{ij}-\tilde{\mathbf{h}}_{ij}\mathbf{a})^2 \tag{19}$$

The weighted least squares solution of Eq. (19) is that

$$\mathbf{a}=[\sum_{i=1}^{n}\sum_{j=1}^{m}w_{ij}^2\tilde{\mathbf{h}}_{ij}^T\tilde{\mathbf{h}}_{ij}]^{-1}[\sum_{i=1}^{n}\sum_{j=1}^{m}w_{ij}^2\tilde{\mathbf{h}}_{ij}^T\Delta r_{ij}] \tag{20}$$

Taking into account Eq. (18), we obtain

$$\tilde{\mathbf{h}}_{ij}^T\tilde{\mathbf{h}}_{ij}=\left[\begin{array}{c|c|c}b_{jx}^2(\mathbf{s}_i\mathbf{s}_i^T) & b_{jx}b_{jy}(\mathbf{s}_i\mathbf{s}_i^T) & b_{jx}b_{jz}(\mathbf{s}_i\mathbf{s}_i^T)\\ \hline b_{jy}b_{jx}(\mathbf{s}_i\mathbf{s}_i^T) & b_{jy}^2(\mathbf{s}_i\mathbf{s}_i^T) & b_{jy}b_{jz}(\mathbf{s}_i\mathbf{s}_i^T)\\ \hline b_{jz}b_{jx}(\mathbf{s}_i\mathbf{s}_i^T) & b_{jz}b_{jy}(\mathbf{s}_i\mathbf{s}_i^T) & b_{jz}^2(\mathbf{s}_i\mathbf{s}_i^T)\end{array}\right] \tag{21}$$

By selecting the baseline configuration to satisfy the condition, which hereafter will be named the symmetric condition,

$$\sum_{j=1}^{m}w_{ij}^2b_{jx}b_{jy}=0,\quad \sum_{j=1}^{m}w_{ij}^2b_{jx}b_{jz}=0,\quad \sum_{j=1}^{m}w_{ij}^2b_{jy}b_{jz}=0 \tag{22}$$

Eq. (20) can be decomposed into three independent parts:

$$\mathbf{a}_1=[\sum_{i=1}^{n}\sum_{j=1}^{m}w_{ij}^2b_{jx}^2(\mathbf{s}_i\mathbf{s}_i^T)]^{-1}[\sum_{i=1}^{n}\sum_{j=1}^{m}(w_{ij}^2b_{jx}\cdot\Delta r_{ij})\mathbf{s}_i] \tag{23a}$$

$$\mathbf{a}_2=[\sum_{i=1}^{n}\sum_{j=1}^{m}w_{ij}^2b_{jy}^2(\mathbf{s}_i\mathbf{s}_i^T)]^{-1}[\sum_{i=1}^{n}\sum_{j=1}^{m}(w_{ij}^2b_{jy}\cdot\Delta r_{ij})\mathbf{s}_i] \tag{23b}$$

$$\mathbf{a}_3=[\sum_{i=1}^{n}\sum_{j=1}^{m}w_{ij}^2b_{jz}^2(\mathbf{s}_i\mathbf{s}_i^T)]^{-1}[\sum_{i=1}^{n}\sum_{j=1}^{m}(w_{ij}^2b_{jz}\cdot\Delta r_{ij})\mathbf{s}_i] \tag{23c}$$

Thus far, the attitude matrix A can be calculated using Eq. (20). Moreover, if the conditions of Eq. (22) are satisfied, A can be simplified so as to be computed from Eqs. (23a)~(23c).

Only one step now remains to reach the LS solution of the problem. Suppose all w_{ij}^2 to be equal to unity, and introduce the following weighted differential ranges,

$$\Delta\tilde{r}_{ix}=\sum_{j=1}^{m}(b_{jx}\cdot\Delta r_{ij})\Big/\sum_{j=1}^{m}b_{jx}^2 \tag{24a}$$

$$\Delta\tilde{r}_{iy}=\sum_{j=1}^{m}(b_{jy}\cdot\Delta r_{ij})\Big/\sum_{j=1}^{m}b_{jy}^2 \tag{24b}$$

$$\Delta\tilde{r}_{iz}=\sum_{j=1}^{m}(b_{jz}\cdot\Delta r_{ij})\Big/\sum_{j=1}^{m}b_{jz}^2 \tag{24c}$$

Eqs. (23a)~(23c) can then be converted to the following LS solution:

$$\mathbf{a}_1=[\sum_{i=1}^{n}(\mathbf{s}_i\mathbf{s}_i^T)]^{-1}[\sum_{i=1}^{n}(\mathbf{s}_i\Delta\tilde{r}_{ix})] \tag{25a}$$

$$\mathbf{a}_2=[\sum_{i=1}^{n}(\mathbf{s}_i\mathbf{s}_i^T)]^{-1}[\sum_{i=1}^{n}(\mathbf{s}_i\Delta\tilde{r}_{iy})] \tag{25b}$$

$$\mathbf{a}_3=[\sum_{i=1}^{n}(\mathbf{s}_i\mathbf{s}_i^T)]^{-1}[\sum_{i=1}^{n}(\mathbf{s}_i\Delta\tilde{r}_{iz})] \tag{25c}$$

Note that these solutions are sub-optimal as in the derivation we have not taken into account the orthogonal constraint

$$\sum_{k=1}^{3}(\mathbf{a}_k\mathbf{a}_k^T)=\mathbf{I}$$

As shown in the sequel, orthogonalization procedures exist to improve the accuracy of solution.

5. Symmetric Condition

The configuration which satisfies the symmetric condition of Eq.(22) is here termed the "symmetric" baseline configuration. We can prove the following condition by noting that $w_{ij}^2 = w_{sii}^2 \cdot w_{bjj}^2$.

Condition 1 *If there exists a positive (m × m) diagonal marix* $\mathbf{W}_B$ *such that*

$$\mathbf{B}\mathbf{W}_B\mathbf{B}^T = \Lambda \tag{26}$$

where Λ *is an arbitrary (3 × 3) non-negative diagonal matrix, m is the number of baselines, and* $\mathbf{B}$ *is the baseline matrix as shown in Eq.(10), then the configuration is a symmetric baseline configuration.*

Specifically, if $\mathbf{W}_B$ can be taken as a unit matrix, the configuration will be called the innately symmetric configuration. Then the innately symmetric condition can be expressed as

$$\mathbf{B}\mathbf{B}^T = \Lambda \tag{27}$$

Now the symmetric condition is expressed as Eq.(26) instead of Eq.(22). The results derived in the previous section can be summarized by the following conditions.

Condition 2 *If the baseline configuration is not symmetric, the rows of attitude matrix can be estimated by Eq.(20).*

Condition 3 *If the baseline configuration is symmetric, the rows of attitude matrix can be estimated by a combination of Eqs.(23a) ~ (23c).*

Condition 4 *If the baseline configuration is innately symmetric, the rows of attitude matrix can be estimated by a combination of Eqs.(25a) ~ (25c).*

Figure 1 shows some examples of innately symmetric baseline configuration, where "M" represents the master antenna, and "i" represents the ith slave antenna.

If the matrix Λ happens to be a (3 × 3) unit matrix, Eq.(26) will be completely the same as Eq.(14). Therefore, the symmetric condition is the generalization of the balanced condition.

To show that the symmetric condition differs from the balanced condition, the left hand side of Eq. (14) is expanded as

$$\mathbf{B}\mathbf{W}_B\mathbf{B}^T = \sum_{j=1}^{m} w_{bjj}^2 \mathbf{b}_j\mathbf{b}_j^T = \sum_{j=1}^{m} w_{bjj}^2 \begin{bmatrix} b_{jx}^2 & b_{jx}b_{jy} & b_{jx}b_{jz} \\ b_{jy}b_{jx} & b_{jy}^2 & b_{jy}b_{jz} \\ b_{jz}b_{jx} & b_{jz}b_{jy} & b_{jz}^2 \end{bmatrix} \tag{28}$$

Substituting Eq. (28) into Eq. (14), we derive the following equations:

$$\sum_{j=1}^{m} w_{bjj}^2 b_{jx}^2 = 1, \quad \sum_{j=1}^{m} w_{bjj}^2 b_{jy}^2 = 1, \quad \sum_{j=1}^{m} w_{bjj}^2 b_{jz}^2 = 1 \tag{29a}$$

$$\sum_{j=1}^{m} w_{bjj}^2 b_{jx}b_{jy} = 0, \quad \sum_{j=1}^{m} w_{bjj}^2 b_{jx}b_{jz} = 0, \quad \sum_{j=1}^{m} w_{bjj}^2 b_{jy}b_{jz} = 0 \tag{29b}$$

This shows that Cohen's balanced condition requires Eqs. (29a) and (29b) to be satisfied; however, the symmetric condition only requires Eq. (29b) to be satisfied. Due to

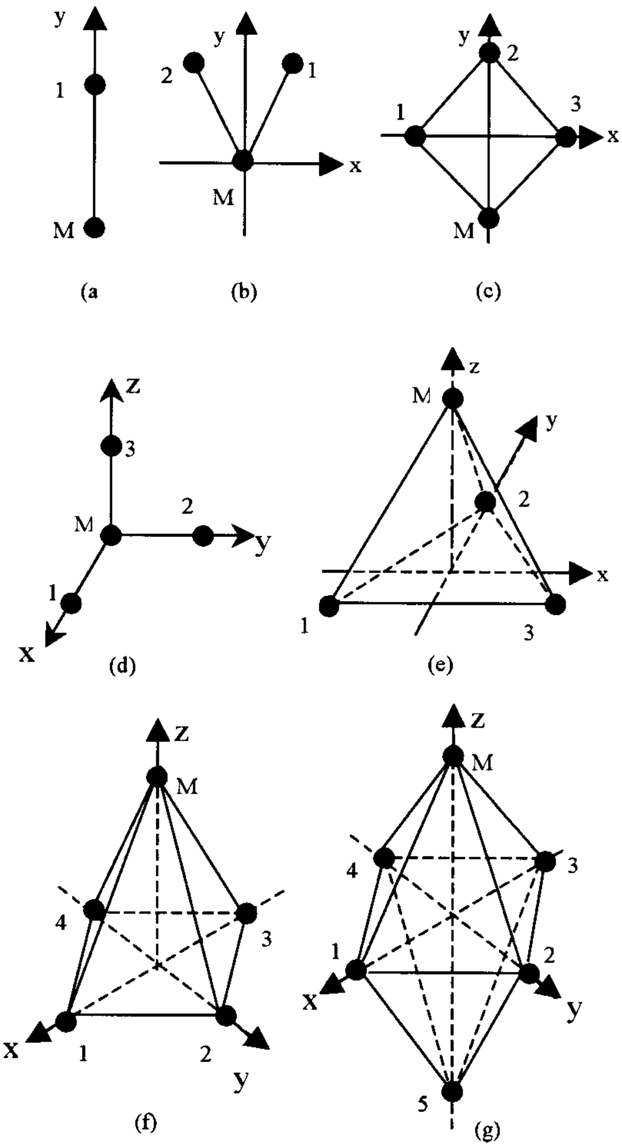

Fig. 1 Examples of innately symmetric baseline configuration

this, the concept of symmetric condition is suitable for both coplanar and non-coplanar configurations, and for two-antenna configuration. As shown above, the symmetric configuration is no longer the necessary condition to guarantee the AMES solution optimal and just acts as the role to simplify the derived solution.

6. New Algorithm for Coplanar Baseline Configurations

The symmetric condition is more easily satisfied by coplanar baseline configurations than by non-coplanar baseline configurations. Because the z-components of all baseline vectors $\mathbf{b}_j$ are zero, the last two equations of Eq. (22) are always satisfied. For the same reason, $\mathbf{a}_3$ cannot be estimated by Eq. (23c) — only $\mathbf{a}_1$ and $\mathbf{a}_2$ can be estimated by Eqs. (23a) and (23b). Fortunately, since A is an orthogonal matrix, $\mathbf{a}_3$ can be derived as follows, once $\mathbf{a}_1$ and $\mathbf{a}_2$ have been estimated:

$$\hat{\mathbf{a}}_3 = \hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_2 \tag{30}$$

a) Three-Antenna Configuration

The typical symmetric baseline configuration of three antennae is shown in Fig. 1(b). If the lengths of two baselines are the same, their positions in the antenna coordinate system are given as

$$\mathbf{b}_1 = [b_x \;\; b_y \;\; 0]^T, \;\; \mathbf{b}_2 = [-b_x \;\; b_y \;\; 0]^T$$

It is easy to show that, according to *Condition 4*, this configuration is innately symmetric. The LS solution is then

$$\hat{\mathbf{a}}_1 = \frac{1}{2b_x} [\sum_{i=1}^{n} (\hat{\mathbf{s}}_i \hat{\mathbf{s}}_i^T)]^{-1} [\sum_{i=1}^{n} \hat{\mathbf{s}}_i (\Delta r_{i1} - \Delta r_{i2})] \tag{31a}$$

$$\hat{\mathbf{a}}_2 = \frac{1}{2b_y} [\sum_{i=1}^{n} (\hat{\mathbf{s}}_i \hat{\mathbf{s}}_i^T)]^{-1} [\sum_{i=1}^{n} \hat{\mathbf{s}}_i (\Delta r_{i1} + \Delta r_{i2})] \tag{31b}$$

$$\hat{\mathbf{a}}_3 = \hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_2$$

b) Four-Antenna Configuration

The typical symmetric baseline configuration of four antennae arranged in a square is shown in Fig. 1(c). The positions of three baselines in the antenna coordinate system are

$$\mathbf{b}_1 = [-b \;\; b \;\; 0]^T, \;\; \mathbf{b}_2 = [0 \;\; 2b \;\; 0]^T, \;\; \mathbf{b}_3 = [b \;\; b \;\; 0]^T$$

According to *Condition 4*, this configuration is also innately symmetric. The LS solution is then

$$\hat{\mathbf{a}}_1 = \frac{1}{2b} [\sum_{i=1}^{n} (\hat{\mathbf{s}}_i \hat{\mathbf{s}}_i^T)]^{-1} [\sum_{i=1}^{n} \hat{\mathbf{s}}_i (\Delta r_{i3} - \Delta r_{i1})] \tag{32a}$$

$$\hat{\mathbf{a}}_2 = \frac{1}{6b} [\sum_{i=1}^{n} (\hat{\mathbf{s}}_i \hat{\mathbf{s}}_i^T)]^{-1} [\sum_{i=1}^{n} \hat{\mathbf{s}}_i (\Delta r_{i1} + 2\Delta r_{i2} + \Delta r_{i3})] \tag{32b}$$

$$\hat{\mathbf{a}}_3 = \hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_2$$

7. Compass Algorithm

The two-antenna configuration always satisfies the innately symmetric condition, as shown in Fig. 1(a). Although only a row of A can be estimated, the baseline vector separated by two antennae acts like a compass.

Suppose A is the 3(ψ)-1(θ)-2(γ) sequence as follows:

$$\mathbf{A} = \begin{bmatrix} c\gamma \cdot c\psi - s\gamma \cdot s\theta \cdot s\psi & c\gamma \cdot s\psi + s\gamma \cdot s\theta \cdot c\psi & -s\gamma \cdot c\theta \\ -c\theta \cdot s\psi & c\theta \cdot c\psi & s\theta \\ s\gamma \cdot c\psi + c\gamma \cdot s\theta \cdot s\psi & s\gamma \cdot s\psi - c\gamma \cdot s\theta \cdot c\psi & c\gamma \cdot c\theta \end{bmatrix} \tag{33}$$

where ψ is azimuth, θ is pitch, γ is roll, and c denotes the *cosine* function and s denotes the *sine* function.

According to *Condition 4*, the second row of A can be estimated to be

$$\hat{\mathbf{a}}_2 = \frac{1}{b} [\sum_{i=1}^{n} (\hat{\mathbf{s}}_i \hat{\mathbf{s}}_i^T)]^{-1} [\sum_{i=1}^{n} \hat{\mathbf{s}}_i \Delta r_i] \tag{34}$$

so that the azimuth and pitch are given by

$$\psi = -\tan^{-1}(a_{21}/a_{22}), \;\; \theta = \tan^{-1}(a_{23}/\sqrt{a_{21}^2 + a_{22}^2}) \tag{35}$$

8. Experiments

a) Description of Experiments

This section presents some results of experiments to test the new algorithm. The raw single difference carrier phase measurements and LOS vector were received from a TANS Vector GPS receiver, which is a solid-state attitude-determination and position location system with a four-antenna array[13]. Fig. 2 shows four antennae arranged in a *41cm* × *41cm* square. The definition of antenna coordinate system is also shown as in Fig. 1(c). The positions of three baselines in the antenna coordinate system are

$$\mathbf{b}_1 = [-b \;\; b \;\; 0]^T, \;\; \mathbf{b}_2 = [0 \;\; 2b \;\; 0]^T, \;\; \mathbf{b}_3 = [b \;\; b \;\; 0]^T$$

where $b = 29cm$.

One of experiments was carried out on December 23, 1998. The experiment was conducted continuously for *1*

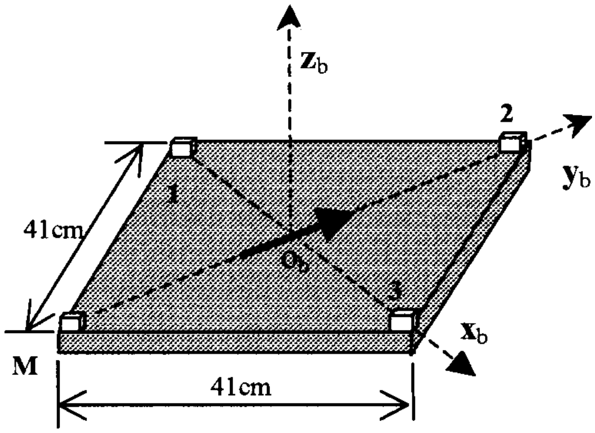

Fig. 2 TANS Vector's 4-antenna square configuration

hour, from GPS time *259180.75s* to *262780.75s*. The LOS (line-of-sight) and single difference carrier phase measurements were recorded in order to utilize them in post-processing. A packet EC containing the LOS vectors was updated by the Vector GPS receiver periodically at intervals of approximately *30s*, while a packet E3 containing the single difference carrier phase measurements was output by the Vector receiver at about *2.0s* intervals. This means that the LOS vector persists equally during about *15* epochs. As mentioned above, the integer ambiguities were resolved by Knight method.

b) Comparison of Accuracy

Fig. 3 shows the results of the AMES algorithm for the four-antenna configuration proposed in this paper, especially shown as Eqs. (32a), (32b) and (30). The azimuth accuracy will benefit from orthogonalization of A such that $\mathbf{A}^{(c)} = (\mathbf{A} + \mathbf{A}^{-T})/2$, where $\mathbf{A}$ and $\mathbf{A}^{(c)}$ represent attitude matrices (see Eq. (17)) before and after orthogonalization, respectively. The procedures shown in the above equations originally need iterations[14]. However, as $\mathbf{A}$ itself is close to be orthogonal, one orthogonalization cycle is sufficient.

The standard deviation (STD) of azimuth will be improved from *0.502* °to *0.424* °by the orthogonalizing process. Note that the results shown in Fig. 3 are those before applying orthogonalization.

Table 1 compares three kinds of algorithms. One of these is the TRIAD algorithm[15], which determines A by employing two vectors. Here, we utilize one of the following two-baseline combinations:

$$(\mathbf{b}_1, \mathbf{b}_2),\ (\mathbf{b}_2, \mathbf{b}_3) \text{ or } (\mathbf{b}_1, \mathbf{b}_3)$$

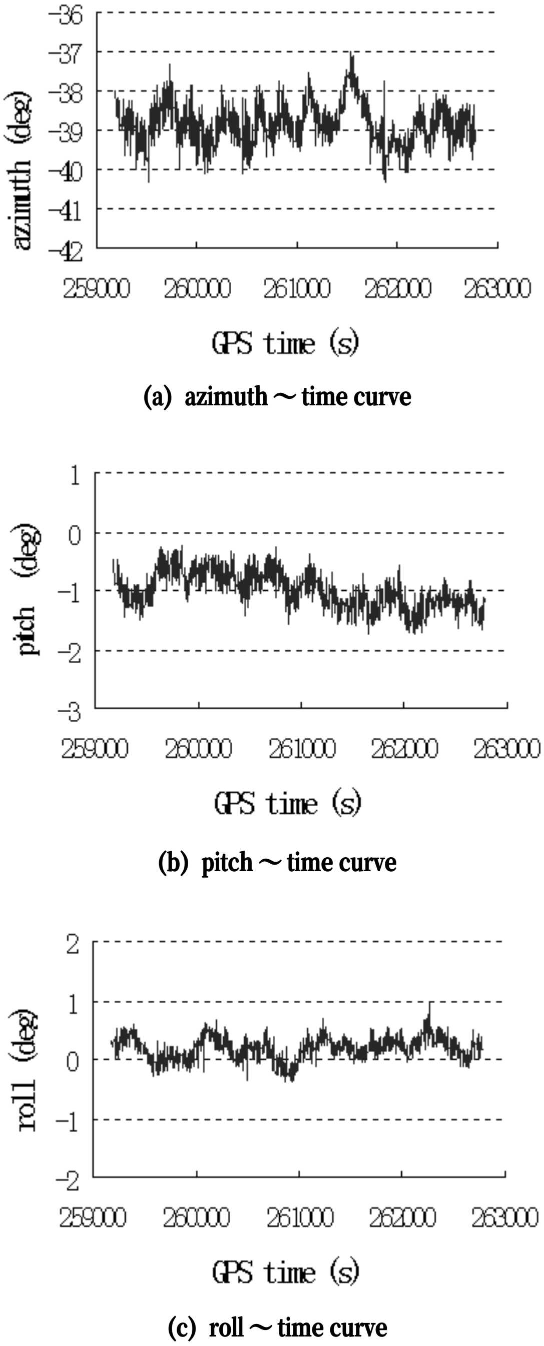

Fig. 3 Results of AMES for conplanar configuration

Table 1 Comparison of accuracy for coplanar configuration

Algorithm		STD(deg) Azimuth	Pitch	Roll
AMES(b1, b2, b3)		0.424	0.302	0.184
NLLSFit		0.417	0.288	0.170
TRIAD	b_1, b_2	0.459	0.254	0.337
	b_2, b_3	0.561	0.254	0.332
	b_1, b_3	0.680	0.443	0.184
AMES(b_1, b_3)		0.442	0.388	0.195
AMES(b_2)		0.499	0.254	—

The NLLSFit algorithm requires an initial nominal solution of A, and the (b_1, b_2) solution of TRIAD is employed as this initial value of NLLSFit. Table 1 also lists the results of AMES which involve two baselines (b_1, b_3), and one baseline (b_2), respectively, compared with the results of TRIAD also using b_1 and b_3. In this case, the result of AMES without orthogonalization has almost the same accuracy as the TRIAD.

The results of compass algorithm, which employs only the 2nd baseline b_2 are shown in Table 1, too. The STDs of azimuth and pitch are *0.499* °and *0.254* °respectively. It is very interesting to note that the accuracy of compass algorithm is higher than that of TRIAD employing two baselines (b_1, b_3). However, it is almost as same as the average of other two TRIAD solutions which employ (b_1, b_2) and (b_2, b_3) separately, referring to Table 1.

Multipath effects are obvious in Figs. 3. The largest reflector was the laboratory's wall, which lay about *4m* to the south of the antenna array.

Some kinematical experiments have been done to examine AEMS. All results demonstrate that AEMS is very effective in the kinematical environment. A result of experiments is shown in Fig.4.

c) Comparison of Computational Issue

The recorded experiment data persist for *1* hour and contain the total number of epochs of *1698*. The data processing is carried out in an IBM ThinkPad portable computer with intel pentium III CPU. The computational time of *7* algorithms is listed in Table 2 where "Total" column means the total spending time for processing all epochs, and "One epoch" column records the average spending time for processing data of one epoch, and "Percentage" column lists the spending percentage of an algorithm compared with that of NLLSFit algorithm. The absolute spending of an algorithm is dependent on many factors such as the coding, CPU, and the accuracy constraint if it is an iterative procedure. Here, the iterative accuracy of NLLSFit is set to less than 10^{-3} radians with the maximum iteration of *5*. Table 2 clearly shows that the computation loading of AEMS for *3* baselines is only about 36.5% of that of NLLSFit, and that of AEMS for 2 baselines is 29.4% of NLLSFit. The reduction by almost 2/3 is very obvious. The computationally spending percentage of *5* algorithms is also shown in Fig.5. Here, NLLSFit is the reference which percentage is set to 100%.

Table 2 Comparison of computional time for coplanar configuration

Algorithms	Total (s)	One epoch (ms)	Percentage (%)
AEMS(b_1,b_2 and b_3)	0.052	0.031	36.5
NLLSFit	0.143	0.084	100
TRIAD(b_1,b_2)	0.076	0.041	48.2
TRIAD(b_1,b_3)	0.060	0.035	42.0
TRIAD(b_2,b_3)	0.054	0.032	37.7
AEMS(b_1,b_3)	0.042	0.025	29.4
AEMS(b_2)	0.025	0.015	17.5

d) Non-coplanar Baseline Configuration

AEMS is compared with PWP and NLLSFit in this section. A simulation method named semi-mathematical simulation is used to generate single-differential carrier phase measurements for a non-coplanar baseline configuration. The measurement errors and LOS vectors are completely the same as the actual data taken from an experiment with the coplanar baseline configuration (see Fig.2). Since the baseline length is short enough, errors in the measurements

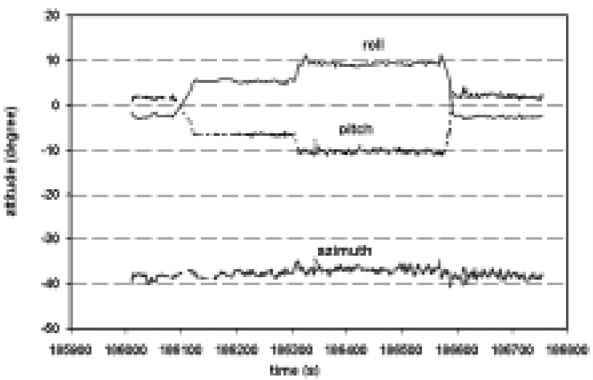

Fig. 4 kinematic Results of AEMS

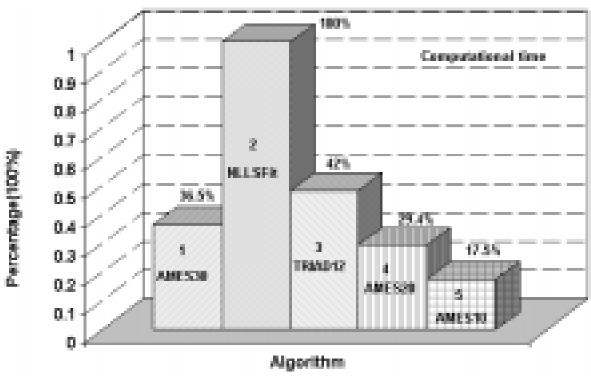

Fig. 5 Computational loads for coplanar configuration

Table 3 Comparison of algorithms for non-coplanar configuration

	STD(deg)			
Algorithm	Azimuth	Pitch	Roll	computional time (ms)
AMES	0.186	0.138	0.153	0.036
NLLSFit	0.182	0.156	0.172	0.190
PWP	0.174	0.115	0.111	0.078

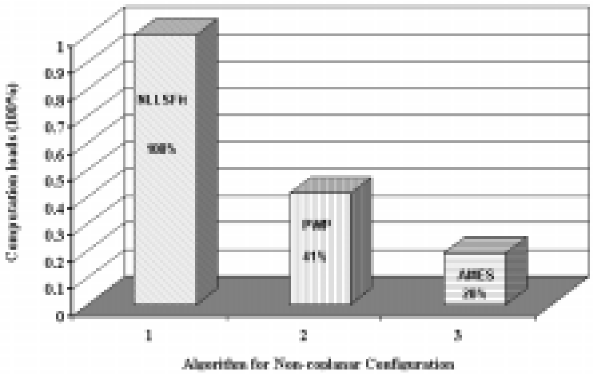

Fig. 6 Computational loads for non-coplanar configuration

can be assumed to be almost equal when the baseline configuration is changed. With this assumption we can simulate errors by

$$\boldsymbol{\varepsilon}_{ij}=\Delta\boldsymbol{\phi}_{ij}+n_{ij}\cdot\boldsymbol{\lambda}-\mathbf{s}_i^T\mathbf{A}^T\mathbf{b}_i.$$

Thus, the measurement data are computed by the sum of a new baseline's projection on a LOS vector and the measurement errors as

$$\Delta\boldsymbol{\phi}_{ij}^{new}=\mathbf{s}_i^T\mathbf{A}^T\mathbf{b}_i^{new}-n_{ij}^{new}\cdot\boldsymbol{\lambda}+\boldsymbol{\varepsilon}_{ij}.$$

The non-coplanar baseline configuration employed in the simulation is shown in Fig.1(e). Three baselines are defined as follows:

$$\mathbf{b}_1=[-b\ \ -h/3\ \ -2b]^T,\ \mathbf{b}_2=[0\ \ 2h/3\ \ -2b]^T,\ \mathbf{b}_3=[b\ \ -h/3\ \ -2b]^T$$
$$\text{where } b=\sqrt{2}/2m,\ \ h=\sqrt{6}/2m$$

It is easy to understand that this configuration satisfies either the balanced condition or the innately symmetric condition. Results of a simulation are listed in Table 3. The computation loads of three algorithms are compared in Fig.6. It clearly shows that AMES greatly reduces the computational time.

9. Conclusions

By converting the attitude matrix into a state vector, a new approach is presented which efficiently resolves the problem of attitude determination using GPS. This new algorithm has a number of advantages which distinguish it considerably from traditional methods. First, it is just as easy to realize in the computer as to create a program for a standard weighted least squares method or a standard least squares method. Second, it does not require any initial values of attitude and avoids computationally expensive iteration. Lastly, it can be applied not only to both the coplanar and the non-coplanar baseline configurations, but also to the one-baseline configuration as the compass algorithm shows. Due to the symmetric baseline configuration which can be regarded as the generalization of the balanced condition and is no longer the necessary condition for the optimality and just acts as the role to simplify the derived solution, the 9-dimensional state vector can be decomposed into three non-independent, 3-dimensional vectors, which allows 3 row vectors of attitude matrix to be estimated separately. This greatly reduces the computational burden.

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Appendix A: The Definition of Wahba's Problem

The original definition of Wahba's problem is in a form of least squares as follows. [16],

Given two sets of n points $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$, and $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$, where $n \geq 2$, find the rotation matrix A (i.e., the orthogonal matrix with determinant +1) which brings the first set into the best least squares coincidence with the second. That is, find A which minimizes

$$\sum_{j=1}^{n} \| \mathbf{u}_j - \mathbf{A}\mathbf{v}_j \|^2 \tag{A1}$$

This problem has arisen in the estimation of attitude of a satellite by using direction cosines $\{\mathbf{u}_k\}$ of objects as observed in a satellite fixed frame of reference and direction cosines $\{\mathbf{v}_k\}$ of the same objects in a known frame of reference. A is then a least squares estimate of the rotation matrix which carries the known frame of reference into the satellite fixed frame of reference.

The weighted least squares form of Wahba's problem can be expressed as finding A which minimizes the following cost function [17]

$$J(\mathbf{A}) = \sum_{j=1}^{n} w_j \| \mathbf{u}_j - \mathbf{A}\mathbf{v}_j \|^2 \tag{A2}$$

where $\{w_1, w_2, ..., w_n\}$ is a set of positive weighted coefficients. There are not any effects to the resulting solution if set w_j satisfy the following condition:

$$\sum_{j=1}^{n} w_j = 1 \tag{A3}$$

Now define a new cost function as follows,

$$g(\mathbf{A}) = 1 - J(\mathbf{A}) = \sum_{j=1}^{n} w_j \mathbf{u}_j^T \mathbf{A}\mathbf{v}_j \tag{A4}$$

It is easy to see that the optimal solution that minimizes $J(\mathbf{A})$ makes $g(\mathbf{A})$ maximum. We can rewrite $g(\mathbf{A})$ as in the following form,

$$g(\mathbf{A}) = \sum_{j=1}^{n} w_j \, tr\,[\mathbf{u}_j^T \mathbf{A}\mathbf{v}_j] = tr\,[\mathbf{A}\mathbf{b}^{\mathrm{T}}] \tag{A5}$$

where

$$\mathbf{B} = \sum_{j=1}^{n} w_j \mathbf{u}_j \mathbf{v}_j^T \tag{A6}$$

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