SOME CONFIRMATIONS RELATED WITH PRANDTL'S LIFTING LINE THEORY

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The Lifting Line Theory is based upon the fact that the local lift at a wing section depends not only on the local section angle of attack but also on the downwash induced by the tailing vortexes shed from the next segments of the wing. It predicts quantitatively the spanwise lift distributions of 3D wing, the induced drag and the load distribution of minimum- drag 3D wing. It is also well consistent with the momentum theory about 3D wing. There are two puzzles for the engineers in the aeronautics, especially for aeroelasticians and aerodynamicians in the world. (i) the collocation point for the numerical solution of lifting surface theory (DLM or VLM) must be at the 3-quarter-chord point of each element. Why? (ii) in the momentum theory about 3D wing, what should be the reference mass to exactly explain the lift and drag? In the present note, the two points are clarified with the help of some mathematical manipulations, using the models of an isolated horseshoe vortex and an elliptically load distributed bound vortex with a trailing vortex sheet.

Keyword: lifting line theory, induced drag, momentum theory

1. INTRODUCTION[1~5)]

Since the earlier times than the first flight of powered aircraft, wing theories have been rigorously developed. Prandtl's lifting line theory developed in 1918 is one of the greatest achievements in the history of the aeronautics. It is a simple but a great theoretical interpretation of physical phenomena around lifting surface, which properly explains the relationships among the lift and the induced drag as well as the momentum theory about 3D wing. It was also the first step toward the sophisticated lifting surface theory established in 1950's.

There are two facts which are well known among aeroelasticians and aerodynamicians, those are;

i) The collocation point must be specified at the 3-quarter- chord point in divided each wing panel when we employ the DLM(Doublet Lattice Method) or VLM(Vortex Lattice Method) for the solution of lifting surface theory.

ii)When we apply the momentum theory to 3D wing, the reference mass should be the mass of the air which flows through a circle with a diameter of the wing span.

The theoretical explanations on the above two points are given in the present note by further studying the lifting line theory.

1. OUTLINE OF LIFTING LINE THEORY[3,4)]

The mathematical model of the lifting line theory consists of a bound vortex $\Gamma(y)$ supposed to be on the wing surface and trailing vortexes $d\Gamma/d\eta$ shed from the bound vortex depending on the variations in its magnitude. Introducing the concept of the Kutta–Joukowski Theorem and the relation between lift and downwash (the angle-of-attack) at a wing section y, we obtain an integral-differential equation with respect to the distributed circulation $\Gamma(y)$ in a form as

$$\rho U\Gamma(y) = \frac{1}{2}\rho U^2 a_0 C(y)\left[\alpha(y) - \frac{1}{4\pi U}\oint_{-l}^{+l}\frac{d\Gamma}{d\eta}\frac{d\eta}{y-\eta}\right] \quad (1)$$

where l is a half-span length of the 3D wing; $\alpha(y)$ is local angle of attack; $a_0(=2\pi)$ is the theoretical

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value of lift-curve slope of 2D wing; $C(y)$ is the local chord-length, respectively.

The second term in the right-hand side of Eq.1 is the downwash induced by trailing vortexes. It describes mathematically that the local lift at a wing section is determined not only due to the section local angle-of- attack but also due to the influence of the tailing vortexes shed from the next segments of the wing. The spanwise lift distributions on 3D wing can be predicted by solving Eq.1 and 2 with boundary condition at every spanwise section. The Eq.1 can be rewritten as a relationship among the boundary condition, the downwashes due to bound vortex and the trailing vortex at a section y as

$$U\alpha(y) = \frac{1}{2\pi}\frac{\Gamma(y)}{\frac{1}{2}C(y)} + \frac{1}{4\pi}\oint_{-l}^{+l}\frac{d\Gamma}{d\eta}\frac{d\eta}{y-\eta} \quad . \tag{2}$$

$$w_{3/4}^{(B)} = \frac{1}{2\pi}\frac{\Gamma(y)}{\frac{1}{2}C(y)} \quad , \qquad w_{1/4}^{(T)} = \frac{1}{4\pi}\oint_{-l}^{+l}\frac{d\Gamma}{d\eta}\frac{d\eta}{y-\eta} \qquad (3),\ (4)$$

The bound vortex should be located at the quarter-chord point since that is the location of the aerodynamic center of a thin flat plate airfoil. The right-side term of Eq.3 is identical with the downwash at a half-chord downstream from the bound vortex, that means the downwash at the 3-quarter-chord point of the section. On the other hand, the integral term in Eq.4 expresses the totally integrated downwash at a quarter-chord-point due to the trailing vortex sheet.

If we let the angle-of-attack at the 3-quarter-chord point represent the angle-of-attack of the section, there is a small difference in evaluation points between $w_{3/4}^{(B)}$ and $w_{1/4}^{(T)}$. However, in the case of high-aspect-ratio wing, the difference is small enough because

$$|y-\eta| \gg |c(y)| \tag{5}$$

is true in most part over the 3D wing [1].

3. INDUCED VELOCITY DISTRIBUTION BY A HORSESHOE VORTEX[1,2)]

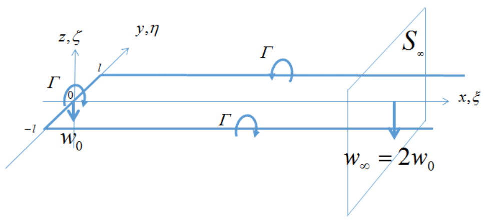

Figure 1: Schematic View of a Horseshoe Vortex

A horseshoe vortex is one of the simplest models of wing circulation system. The coordinates system used here is defined as is shown in Fig.1. It assumes that a bound vortex of which magnitude is Γ is located along y-axis and the trailing vortexes from tips are modeled as is show in Fig.1. The spatial distributions of the induced velocity due to the horseshoe vortex are calculated by use of the law of Biot-Savart. The downwash at a point (x,y,z) becomes

$$w(x,y,z) = \frac{\Gamma}{4\pi}\left[\frac{x}{x^2+y^2}\left(\frac{l+y}{r_+}+\frac{l-y}{r_-}\right) + \frac{l+y}{(l+y)^2+z^2}\left(\frac{x}{r_+}+1\right) + \frac{l-y}{(l-y)^2+z^2}\left(\frac{x}{r_-}+1\right)\right] \tag{6}$$

where $\quad r_{\pm} = \sqrt{x^2+(l\pm y)^2+z^2}$.

From Eq.1, the downwashes at the origin and at an infinite downstream point are evaluated as,

$$w_0 = w_{x=0,y=0,z=0} = \frac{\Gamma}{2\pi l} \qquad \text{and} \qquad w_\infty = w_{x=\infty,y=0,z=0} = \frac{\Gamma}{\pi l} \tag{7}$$

respectively. Though w_0 and w_∞ are not constant in the *yz*-plane, if we consider them as the representative magnitudes of the downwash at $x = 0$ and $x = \infty$, then the well-known relationship between w_0 and w_∞ of

$$w_\infty = 2w_0 \tag{8}$$

is maintained.

The change in the z-direction velocity components of the upstream and the downstream is related to the change in the z-direction momentum of the total flow which is passing through a yz plane per unit time. The total change of the z direction momentum of the flow can be counted by integrating the downwash *w(∞,y,z)* in the plane S_∞ and if we use w_∞ (Eq.7) as a representative velocity at $x = \infty$,

$$\rho U \int_{S_\infty} w(\infty, y, z) dydz = \rho U(2\pi l^2)\, w_\infty \tag{9}$$

is obtained. Therefore the total mass of the air which changes the flow direction is apparently to be

$$S_B = \rho U(2\pi l^2) \quad . \tag{10}$$

This is recognized as the total mass flow which passes per unit time through a circular with diameter $2\sqrt{2}l$.

4. INDUCED VELOCITY BY AN ELLIPTICALLY LOAD DISTRIBUTED BOUND VORTEX[1,3)]

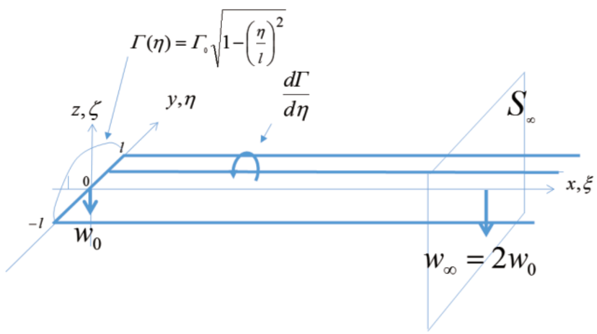

Figure 2: Schematic View of an Elliptically Load Distributed Bound Vortex

Elliptically load distributed bound vortex of which strength is defined by

$$\Gamma(\eta) = \Gamma_0\sqrt{1 - (\tfrac{\eta}{l})^2} \tag{11}$$

is considered. In this case, accounting the influence from the trailing vortex, the spatial distribution of the downwash behind the wing is derived by the Law of Biot-Savart and is obtained in a form of

$$w(x, y, z) = \frac{1}{4\pi}\frac{\Gamma(\eta)}{d\eta}\left[\frac{y-\eta}{(y-\eta)^2+z^2}\left(\frac{x}{r}+1\right)\right]\int_{-l}^{+l} d\eta + \frac{1}{4\pi}\Gamma(\eta)\frac{z}{r^3}\int_{-l}^{+l} d\eta \tag{12}$$

where $r = \sqrt{x^2 + (y-\eta)^2 + z^2}$.

Performing the integration in Eq.12 with respect to η at $x = 0$ and $x = \infty$, w_0 and w_∞ are obtained as

$$w_0 = w_{x=0,-l\le y\le l,=0,z=0} = \frac{\Gamma_0}{4l} \quad , \quad w_\infty = w_{x=\infty,-l\le y\le l,z=0} = \frac{\Gamma_0}{2l} \qquad -l \le y \le l \quad , \tag{13}$$

respectively. It is noted here that in the $z = 0$ plane which includes the wing surface, the magnitudes of the downwash do not depend on y.

The relationship between w_0 and w_∞,

$$w_\infty = 2w_0 \tag{14}$$

is also maintained here.

The momentum change of mass flow in z-direction can be counted in a similar manner to the previous case and leads

$$\rho U \int_{S_\infty} w_\infty(y,z)dydz = \rho U\left[\frac{1}{2}\pi l \Gamma_0\right] = \rho U(\pi l^2)\, w_\infty \tag{15}$$

Therefore the total mass which is used in the momentum theory about 3D wing becomes apparently to be

$$S_B = \rho U(\pi l^2) \tag{16}$$

This is the total mass of the air which flows per unit time through a circular with diameter $2l$. This is the reference mass which is usually used in the momentum theory about 3D wing and is well consistent with the related wing theories developed so far.

5. FINAL COMMENTS

Through the present study, followings are led.

(i) For the solutions of the lifting surface theory, the collocation point should be at a half-chord downstream from the assumed concentrated vortex. The bound vortex should be located at the quarter-chord point since that is the location of the aerodynamic center of a thin flat plate airfoil. It leads that the collocation point should be at the 3 quarter-chord point, where there is consistency between relationships of lift-circulation and lift-downwash.

(ii) In the momentum theory about lifting surface, reference mass exactly "the mass of air that goes through a circle per unit time, of which diameter equals the wing span length, in the case of 3D wing elliptically loaded in spanwise.

We hope the two points mentioned above are useful for those who analyze flutter or the unsteady aerodynamics to understand the basic theoretical backgrounds of DLM and VLM which are installed in the NASTRAN and other aerodynamic analyses tools.

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