TOIPCS ON VARIATIONAL PRINCIPLES IN ENGINEERING

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More or less, the objectives of the engineering are to find optimal solution for the design of the products. Classically the variational methods have been used to find the optimal design in the engineering fields. So many numerical solutions are based upon the variational methods or its similar approaches even if the variational principle does not exist. Variational principles in the fluid dynamics has not been established yet, while those in elasticity and plasticity and optimal control theory have been highly confirmed and applied to actual problems.
In this presentation, the methods of calculus variations are reviewed in the fields of structure and elasticity, optimal control theory and fluid dynamics. Firstly the fields of elasticity and optimal control theory are briefly reviewed. The details of them are not discussed here. Focusing on the fluid dynamics, a process of deriving the Hamilton's principle is shown toward the applications for fluid-structure interaction problems like flutter or other interdisciplinary problems in engineering. The principle of virtual work is applied to the governing equations with boundary conditions. The conservation laws of mass and energy as well as the geometrical boundary conditions are treated as subsidiary conditions. In the present approach, a new concept of substantial variations of the state variables is introduced in order to describe the transportation terms and subsidiary conditions in a simple manner.
It is a classical approach but might be useful for the development of computer aided design tools for aeroelasticity system and its optimal design.

Keyword: fluid dynamics, variational principle, fluid- structure interactions

1. INTRODUCTION [1,2,3,5,8,9)]

Calculus of variations takes an important roles in a variety of engineering fields. This is due to the fact that a physical system often behaves in a manner such that some functional (function of functions) depending on its behavior assumes a stationary value. It is actually employed to find the optimal solutions of dynamic problems as well as static ones, and mechanical systems or physical phenomena, through the procedure of making properly defined functional stationary by calculus of variations.

The variational formulation has advantages; i)it is helpful in carrying out a common mathematical procedure. ii) it provides transformation of a given problem into an equivalent one that can be solved more easily than original, and iii) when a problem can not be solved exactly, it often provides an approximate formulation for the problem which yields a solution compatible with assumed degree of approximation. If it is applied a discretized model or a distributed parametered system, the problems are reduced finally to solve a set of algebraic equations which are directly obtained by making the discretized functional minimum or maximum, instead of solving a set of differential or integral equations with the boundary or initial conditions analytically. Moreover, in many cases, the functional consists of quantities which have definite physical meanings such as kinetic energy, potential energy and so on. This fact helps us to observe the physical phenomena from the viewpoint of the basic physical principles.

The variational principles for elasticity are highly and phylogenically ordered as a system in which several variational principles are clearly derived based upon the principle of virtual work and they can be mutually converted from one to another through systematic procedures. The reduced principles are applied to analyses of beams, plates, membranes and their combined structures. The Rayleigh–Ritz Method

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and the Finite Element Method (FEM) are the typical examples of its applications.

In the optimal control theory, calculus of variations is inevitable to obtain the governing equations and boundary conditions for the solutions of minimum/maximum problems. In the optimization problem, whichever it is a static or a dynamical problem, an artificial functional is defined depending on the purpose of the analyses, which is so called objective function or performance index. If the functional is accompanied by constraints, it will be augmented by introducing those constraints with the Lagrange multipliers. A set of necessary equations including a part of boundary conditions (or initial and terminal conditions) can be obtained by making the objective function or performance index stationary. It should be noted that there is a big difference between the variational principles in the theory of elasticity and those in the optimal control theory because those in elasticity are conservative while not in the optimal control theory. The Lagrange multipliers will take important roles to easily manipulate the imposed constraints in the non-conservative system as is found later.

As long as the author knows, the calculus of variations was initially applied to fluid dynamics by by D'Alembert (1717～83). A contribution of Clebsh in 1800's should be noted, but in 1900's a remarkable progresses were made by Hargreaves, Herivel and Lin. Hargreaves suggested that in the inviscid and irrotational fluid dynamics, the governing equations would be obtained as stationary conditions of a functional which is defined by an integral of static pressure in the whole flow field. After the Hargreaves achievements, Bateman also insisted that the functional is written by use of velocity potential if the flow is inviscid and irrotational. Batemans' principle is used frequently for the solution of potential flow even today. Herivel and Lin showed that the Hamilton's principle in kinetics is valid for inviscid and compressible flow and that the stationary conditions of a functional expressed by kinetic energy minus internal energy are equivalent to the governing equations and the boundary conditions of the flow fields.

It is reasonably considered that there are no differences between elasticity and fluid dynamics from the view point that both are continuums. However, a small particle in elasticity is assumed to be steady while in fluid dynamics a particle is moving even if the phenomena is steady. Therefore, we encounter difficulties in deriving the variational principles for the fluid dynamics. In order to avoid this difficulties, it is valid to adopt a procedure similar to that used in the kinetics for the fluid dynamics as well. In the present studies, we use the Lagrangian description of the fluid flow. We will apply the principle of virtual work to the equations of motion of the fluid, with the conservation laws of mass and energy as subsidiary conditions. Finally, we will reach a functional of which stationary conditions are identical to the governing equations and the boundary conditions for the flow field.

The considerations is limited within the inviscid and irrotational flows throughout present paper.

2.CALCULUS OF VARIATIONS IN IN ELASTICITY [1)]

There are numerous texts and literatures on the Calculus of variations in elasticity and plasticity. The principle of virtual work, the principle of minimum potential energy, the principle of complementary energy and their mutual theoretical relations are detailed, for example, in Ref.[1]. Those variational principles can be converted from one to another (Fig.1) by rigorous manipulations with the help of Lagrange multipliers.

It is well known that the associate theories are actually applied to the formulation for Finite Element Method (FEM) , a computer software such as NASTRAN. The FEM was originally invented in the middle of 1950's in the fields of aeronautics for the structural analyses of a swept-back wing which could not be done by beam theory. However, in a short time, it became to be widely utilized not only for the aircraft structural analyses but also for the structural analyses in various engineering fields. In Fig.1, the inter-relationships among various variational principles in elasticity and plasticity are shown. When we must consider the problems with subsidiary conditions, the functional is augmented by introducing those subsidiary conditions with Lagrange multipliers. The augmented functional will take a very important roles in converting a variational principle to another in a common mathematical procedure. The details of their relationships are so beautifully and clearly summarized in Ref.[1]. We are not going further into the details here.

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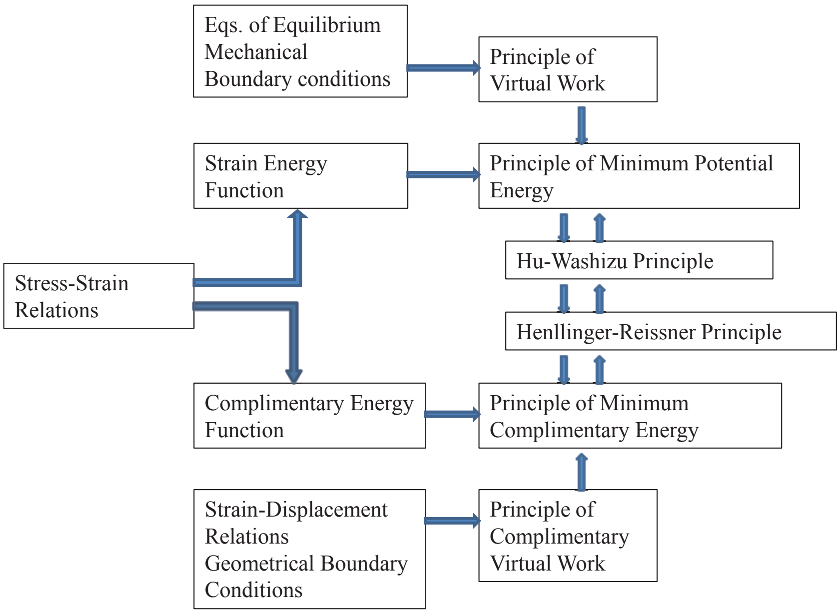

Figure 1: Relationships among vriational principles in elasticity and plasticity

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3.CALCULUS OF VARIATIONS IN OPTIMAL CONTROL THEORY [2,5)]

The Euler equations in the optimal control theory are briefed bellow. As an example to show the outline of the optimal control theory, a problem with the terminal time t_f relaxed is considered. The necessary conditions to obtain the optimal solution are generally derived as follows.

The equations of motion and the initial condition can be expressed as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \quad , \tag{3.1}$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x_0} \quad . \tag{3.2}$$

Where $\boldsymbol{x}$ is the state variable vector; $\boldsymbol{u}$ is the control variable vecter. The terminal condition may be imposed by functions of state variables at $t = t_f$ as

$$\boldsymbol{\psi}\left[\boldsymbol{x}(t_f), t_f\right] = 0 \quad . \tag{3.3}$$

If the objective function which should be minimized (or maximizes) is defined by

$$J = \phi\left[\boldsymbol{x}(t_f)\right] + \int_{t_0}^{t_f} L[\boldsymbol{x}, \boldsymbol{u}, t]dt \quad , \tag{3.4}$$

the problem is to obtain the optimal control $\boldsymbol{u}$ which minimizes(or maximizes) the quantity defined by Eq.3.4 under the subsidiary conditions Eqs.3.1~3.3.

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By use of calculus of variations, the necessary conditions for J to be stationary are obtained by making the following augmented functional stationary.

$$J = [\phi + \boldsymbol{\nu\psi}]_{t=t_f} + \int_{t_0}^{t_f} \{L[\boldsymbol{x},\boldsymbol{u},t] + \boldsymbol{\lambda}[\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t) - \dot{\boldsymbol{x}}]\}\, dt \tag{3.5}$$

where, $\boldsymbol{\lambda}$, $\boldsymbol{\nu}$ are the Lagrange multipliers to introduce the Eq.3.1 and 3.3 .
and where the independent variables subject to variation in Eq. 3.5 are $\boldsymbol{x}$, $\boldsymbol{u}$, t_f and Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$. The initial conditions Eq.3.2 still remains as subsidiary conditions.

By taking the first variation and setting it to be zero,

$$\delta J = 0 \tag{3.6}$$

the adjoint system and its initial and terminal conditions are obtained as

$$\delta\boldsymbol{x}: \quad \dot{\boldsymbol{\lambda}} = -\frac{\partial L}{\partial x} - \boldsymbol{\lambda}\frac{\partial \boldsymbol{f}}{\partial x} \quad \text{or} \quad \dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial x} \quad , \tag{3.7}$$

$$\delta^*\boldsymbol{x}_f: \quad \boldsymbol{\lambda}(t_f) = \left(\frac{\partial \phi}{\partial x} + \boldsymbol{\nu}\frac{\partial \boldsymbol{\psi}}{\partial x}\right)_{t_f} \quad , \tag{3.8}$$

$$\delta\boldsymbol{x}_0 = 0: \quad \boldsymbol{\lambda}(t_0) = \text{free} \tag{3.9}$$

And the optimality condition is led as

$$\delta\boldsymbol{u}: \quad \frac{\partial H}{\partial \boldsymbol{u}} = \frac{\partial L}{\partial \boldsymbol{u}} + \boldsymbol{\lambda}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} = 0 \quad . \tag{3.10}$$

$$\delta t_f: \quad \left(\frac{d\Phi}{dt} + L + \boldsymbol{\lambda}\dot{\boldsymbol{x}}\right)_{t_f} = 0 \quad . \tag{3.11}$$

Where H is called Hamiltonian in the optimal control theory and defined by

$$H = L[\boldsymbol{x},\boldsymbol{u},t] + \boldsymbol{\lambda f}(\boldsymbol{x},\boldsymbol{u},t) \tag{3.12}$$

If the terminal time t_f is free (not specified priori), then the Eq.3.11 is led with the help of transversality condition with respect to terminal time t_f, which is very important necessary condition to solve such problems that the boundaries are not priori specified.

Among the admissible trajectories, the exact solution makes the function J stationary. Then we find that the original problem is reduced to the one to solve the set of differential equations defined Eq.3.1 and Eqs.3.7 with the initial conditions and the terminal conditions defined by Eqs.3.2~3.3, Eq.3.8~3.9 and Eq.3.11 . The optimal control law, that is , the control variable $\boldsymbol{u}$ is obtained as a solution from Eq.3.10. It is of interest that the optimal solution will keep the Hamitonian H constant on the optimal trajectory. Actually the fact is derived from the stationary conditions Eqs.3.7 and Eq.3.10.

Usually the equations of motion and the adjoint equations are iteratively integrated by use of the Runge-Kutta algorithm and a shooting method so that the initial and terminal conditions are satisfied.
For another choice, direct solution method based upon the variational technique can be found in Ref.[2,3].

4. CALCULUS OF VARIATIONS IN FLUID DYNAMICS [5~9)]

(1)Governing equations for inviscid and irrotational fluid dynamics

We consider a fluid particle moving with the flow. Supposing that a particle which is initially at $\boldsymbol{X} = (X_1, X_2, X_3)$ moves to $\boldsymbol{x} = (x_1, x_2, x_3)$ at *t=t* (Fig.2), then the vector $\boldsymbol{x}$ is defined as a function of the initial location coordinates vector $\boldsymbol{X}$ and time *t* as,

$$\boldsymbol{x} = \boldsymbol{\theta}(\boldsymbol{X},t) \quad . \tag{4.1}$$

We postulate a cluster which consists of given fluid particles at *t=0* will be still together with each other at *t=t* even if its shape, volume and location are changed. Then the inversion relation of Eq.4.1 can be hold in a form as

$$\boldsymbol{X} = \boldsymbol{\Theta}(\boldsymbol{x},t) \quad . \tag{4.2}$$

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Then an arbitrary state variable is expressed by

$$F = F(\boldsymbol{x}, t) \tag{4.3}$$

or by use of Eq.4.1,

$$\begin{aligned} F &= F[\boldsymbol{\theta}(\boldsymbol{x}, t), t] \\ &= F(\boldsymbol{X}, t) \end{aligned} \tag{4.4}$$

The state variables expressed by Eq.4.3 and 4.4 are called space state variable and material state variable, respectively. It is well known that if we describe the fluid motion by use of $\boldsymbol{x}$ and t, it is called Eulerian description of fluid motion. Otherwise Lagrangian description by use of vector $\boldsymbol{X}$ and t .

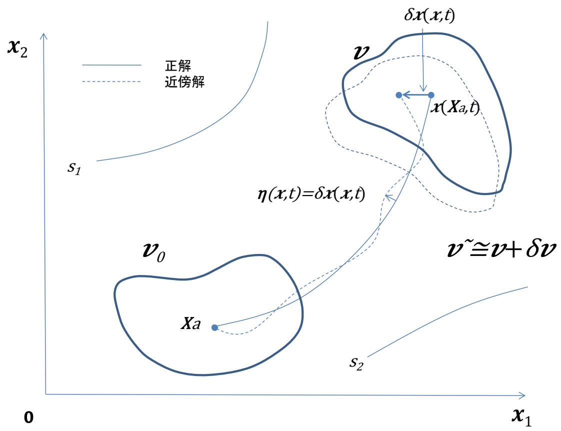

Figure 2: Lagrangian expression of fluid motion

The conservation law of mass is written as

$$\int_v \rho dv = \text{const.} \qquad , \tag{4.5}$$

where v is a given volume occupied by the same particles moving with the flow, ρ is the density of fluid.
If the flow is assumed to be isentropic, the conservation law of energy is expressed by entropy per unit volume S as

$$S = \text{const.} \qquad \text{along} \quad \text{streamline} \quad . \tag{4.6}$$

The equation of motion becomes

$$\rho \boldsymbol{a} = \rho \bar{\boldsymbol{f}} - \text{grad}\, p \tag{4.7}$$

Here the vector $\boldsymbol{a}$ is acceleration and defined with the velocity vector $\boldsymbol{q}$

$$\boldsymbol{a} = \frac{d\boldsymbol{q}}{dt} = \frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{q}\text{grad}\boldsymbol{q} \qquad . \tag{4.8}$$

and where p is pressure, $\bar{\boldsymbol{f}}$, a body force vector per unit mass, respectively.
The boundary conditions are treated in the same manner to the case of elasticity theory,

$$\boldsymbol{t} = \bar{\boldsymbol{t}} \qquad \text{on} \quad s_1 \tag{4.9}$$

$$\boldsymbol{x} = \bar{\boldsymbol{x}} \qquad \text{on} \quad s_2 \tag{4.10}$$

Where, the vector $\boldsymbol{t}$ is a kind of stress vector defined by $\boldsymbol{t} = -p\boldsymbol{n}$ on the boundary s_1. $\overline{(\ )}$ denotes that the value of the state variable parenthesized is specified.
Eq.4.9 is mechanical boundary condition which means the stress on the surface s_1 is specified and Eq.4.10 is geometrical boundary condition which denotes that the motion of the fluid is geometrically constrained on the boundary s_2.

(2)Principle of virtual work in Lagrangian descriptions

The principle of the virtual work is applied to the equations of motion Eqs.4.7 and the boundary conditions 4.9. In the Lagrangian description, a flow particle identified by the initial condition $\boldsymbol{X}$, is moving on a track expressed by $\boldsymbol{x} = \boldsymbol{\theta}(\boldsymbol{X}, t)$. If the state variables, $p(\boldsymbol{x}, t)$, $\boldsymbol{q}(\boldsymbol{x}, t)$ and $S(\boldsymbol{x}, t)$ satisfy all

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governing equations and boundary conditions, the corresponding $\boldsymbol{x}$ are obviously the solutions of the problem which is defined in the previous section.
Now, we give a set of arbitrary infinitesimal virtual displacements $\boldsymbol{\eta}(\boldsymbol{x},t)=\delta\boldsymbol{x}$ to the nominal solution $\boldsymbol{x}$. Then,

$$-\int_v(\rho\bar{\boldsymbol{f}}-\mathrm{grad}\,p-\rho\boldsymbol{a})\delta\boldsymbol{x}dv+\int_{s1}(\boldsymbol{t}-\bar{\boldsymbol{t}})\,\delta\boldsymbol{x}ds=0 \quad , \tag{4.11}$$

is obtained. The subsidiary conditions are the geometric boundary conditions,

$$\boldsymbol{n}\delta\boldsymbol{x}=0 \qquad \text{on} \quad s_2 \tag{4.12}$$

and the conservation laws of mass,4.5 and energy, 4.6.
Eq.4.11 can be easily rewritten with the help of Gaussian integral theorem as

$$\int_v\rho\boldsymbol{a}\,\delta\boldsymbol{x}dv-\int_v \mathrm{pdiv}(\delta\boldsymbol{x})dv-\int_v\rho\bar{\boldsymbol{f}}\,\delta\boldsymbol{x}dv \quad -\int_{s_1}(\boldsymbol{t}-\bar{\boldsymbol{t}})\,\delta\boldsymbol{x}ds=0 \tag{4.13}$$

As Eq.4.11 or Eq.4.13 is definitely held to be zero at $t=t$, the following equation which is integrated Eq.4.13 with respect to time in the interval of $t_1 \le t\le t_2$,

$$\iint_{t_1\,v}^{t_2}\rho\boldsymbol{a}\,\delta\boldsymbol{x}dvdt-\iint_{t_1\,v}^{t_2}\mathrm{pdiv}(\delta\boldsymbol{x})dvdt\;-\iint_{t_1\,v}^{t_2}\rho\bar{f}\,\delta\boldsymbol{x}dvdt-\iint_{t_1\,s_1}^{t_2}\bar{\boldsymbol{t}}\,\delta\boldsymbol{x}dsdt=0\;. \tag{4.14}$$

is confirmed.

The expression of 4.13 is so called the Principle of Virtual Work, which is similar to that in the fields of elasticity or mechanical dynamics.

Except for simple case like steady incompressible flow, it is extremely difficult to argue the variational principles and to explore the physical meaning, as long as we use Eulerian descriptions of the fluid.
On the other hand, by use of Lagrangian description, the functional to be stationary can be easily led with proper subsidiary conditions, in a similar manner in elasticity and dynamical problems. The procedures will be shown in the following sections.

(3)Introduction of substantial variation

In calculus of variations in elasticity and plasticity, the independent variables subject to variation are defined as functions of space coordinates which are not subject to variation. In the present formulation, however, all of the state variables are the functions expressed by use of the coordinates of fluid particles, $\boldsymbol{x}$. When we consider the variation of a given state variable $F(\boldsymbol{x},t)$, attention should be paid to the variation of $F(\boldsymbol{x},t)$.
In Fig.3, it is shown that if variations, $\delta\boldsymbol{x}$ are given to $\boldsymbol{x}$. $F(\boldsymbol{x},t)$ will change from nominal solution $F(\boldsymbol{x},t)$ to neighboring solution $\tilde{F}(\boldsymbol{x},t)$, then the substantial variation of F, becomes,

$$\begin{aligned}\delta^*F(\boldsymbol{x},t)&=\tilde{F}(\boldsymbol{x}+\delta\boldsymbol{x},t)-F(\boldsymbol{x},t)\\&\cong\tilde{F}(\boldsymbol{x},t)-F(\boldsymbol{x},t)+\delta\boldsymbol{x}\,\mathrm{grad}F\\&=\delta F(\boldsymbol{x},t)+\delta\boldsymbol{x}\,\mathrm{grad}F\end{aligned} \tag{4.15}$$

Where,

$$\delta F(\boldsymbol{x},t)=\tilde{F}(\boldsymbol{x}+\delta\boldsymbol{x},t)-F(\boldsymbol{x},t). \tag{4.16}$$

and δF is the variation of F at $\boldsymbol{x}=\boldsymbol{x}$.

The similar concept configured by Eq.(4.15) is frequently encountered in the cases, for example, minimum time optimization problem, fluid dynamics with free surfaces and so on, where the boundaries are not defined priori the problem is solved.

As far as the present case concerns, the definition 4.15 well corresponds to the substantial derivative which is used in the fluid dynamics, if perfunctorily replace δ^*, δ and $\delta\boldsymbol{x}$ by *d/dt*, *∂/∂t* and $\boldsymbol{q}$, respectively.

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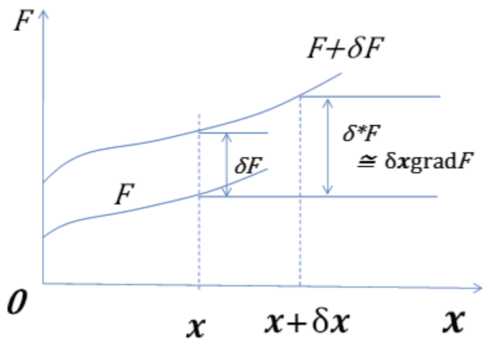

Figure 3: Concept of substantial variation

Then, we distinguish the quantity defined by 4.15 from usual variation defined by the expression 4.16 and call it "substantial variation" in this text.

The substantial variations of velocity $\boldsymbol{q}$, density ρ and entropy S are expressed as

$$\delta^*\boldsymbol{q} = \delta\boldsymbol{q} + \delta\boldsymbol{x}\,\mathrm{grad}\boldsymbol{q} \quad , \tag{4.17}$$

$$\delta^*\rho = \delta\rho + \delta\boldsymbol{x}\,\mathrm{grad}\rho \tag{4.18}$$

and

$$\delta^* S = \delta S + \delta\boldsymbol{x}\,\mathrm{grad}S \quad , \tag{4.19}$$

respectively. In case the vector $\boldsymbol{x}$ is not subject to variation, that is, $\delta\boldsymbol{x} = 0$, they are identical with the usual variations and obviously there is a relation between $\delta^*\boldsymbol{q}$ and $\delta\boldsymbol{x}$ written as

$$\delta^*\boldsymbol{q} = \frac{d}{dt}\,\delta\boldsymbol{x} \quad . \tag{4.20}$$

(4)Variational expression of mass conservation

In the present case, if the variation vector $\delta\boldsymbol{x}$ are given to the particle coordinates $\boldsymbol{x}$, the density of fluid must be relaxed because $\delta\boldsymbol{x}$ and $\delta\rho$ can not change independently due to the constraint of conservation law of mass. The constraints between $\delta\boldsymbol{x}$ **and** $\delta\rho$ are explicitly derived as follows.

Consider a given small volume v closed by a surface moving with the flow as is shown in Fig.2. Supposing that the variation vector $\delta\boldsymbol{x}$ is given to the volume v at $t = t$ and that the volume will move and also be deformed, the mass in the volume is still conserved. Therefore,

$$\int_{\tilde{v}} (\rho + \delta\rho)\, dv = \int_v \rho\, dv \tag{4.21}$$

is satisfied. Assuming that the variation vector $\delta\boldsymbol{x}$ is small enough and that the second powers of $\delta\boldsymbol{x}$ can be neglected, then Eq.4.21 leads to

$$\int_v \delta\rho\, dv + \int_s \rho\delta\boldsymbol{x}\boldsymbol{n}\, ds = 0 \quad , \tag{4.22}$$

And applying the Gaussian integral theorem, it can be rewritten as

$$\int_v [\delta\rho + \mathrm{div}(\rho\delta\boldsymbol{x})]\, dv = 0 \ . \tag{4.23}$$

Here v is arbitrary, then,

$$\delta\rho + \mathrm{div}(\rho\delta\boldsymbol{x}) = 0 \ , \tag{4.24}$$

or using Eq.4.18, we obtain

$$\delta^*\rho = -\rho\,\mathrm{div}(\delta\boldsymbol{x}) \quad . \tag{4.25}$$

Eq.4.24 is the explicit form of the constraints between $\delta\rho$ and $\delta\boldsymbol{x}$**.**

Next, when the variation vector $\delta\boldsymbol{x}$ and δv are given, the first variation of quantity $\int_v \rho F\, dv$ is expressed as

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$$\delta^*[\int_v \rho F\, dv] = \int_{\tilde{v}}(\rho+\delta\rho)(F+\delta F)dv - \int_v \rho F\, dv \qquad , \tag{4.26}$$

which deduces a form using the relationship 4.25,

$$\delta^*[\int_v \rho F\, dv] = \int_v \rho\delta^* F\, dv \ . \tag{4.27}$$

It is correctly concluded that when the first variation of $\int_v \rho F\, dv$ is evaluated, the density ρ can be treated as if it were not subject to variation, as long as the constraint 4.24 is maintained as subsidiary condition.

(5) Variational expression of equations of energy conservation

The entropy S of the fluid is one of the thermodynamic state variables and a function of absolute temperature T which has a relationship with the internal energy per unit mass e and ρ as,

$$TdS = de + pd(\frac{1}{\rho}) \quad . \tag{4.28}$$

If we choose S and ρ as independent variables in Eq.4.28, the internal energy e is expressed as

$$e = e(S,\rho) \quad . \tag{4.29}$$

Eq.4.28 is the first law of thermodynamics and can be applied to a given small volume v of fluid which always consists of same particles. Therefore, δ^*S, δ^*e and $\delta^*\rho$ are governed by the first law of thermodynamics in a form of

$$T\delta^*S = \delta^*e + p\delta^*(\frac{1}{\rho}) \quad . \tag{4.30}$$

For the adiabatic flow, the conservation law of energy is expressed as Eq.4.6 using entropy S.
Maintaining the entropy to be constant, the variation vector $\boldsymbol{\delta x}$ are given to a small volume moving with flow. This constraint will be expressed as

$$\delta^*S = 0 \qquad , \tag{4.31}$$

or in another form by use of Eq.4.30,

$$\delta^*e + p\delta^*(\frac{1}{\rho})=0 \ . \tag{4.32}$$

Furthermore, Eq.(4.32) is reformed with the help of Eq.4.25 as

$$\rho\delta^*e + p\mathrm{div}(\delta\boldsymbol{x}) = 0 \qquad , \tag{4.33}$$

which is the relationship between δ^*e and $\delta\boldsymbol{x}$ and can be applied to a small volume v in an integral form;

$$\int_v[\rho\delta^*e + p\mathrm{div}(\delta\boldsymbol{x})]dv = 0 \qquad . \tag{4.44}$$

Eq.4.44 will be led by use of relation 4.27 and Gaussian Integral theorem to

$$\delta^*[\int_v \rho e\, dv] = \int_v(\mathrm{grad}p)\delta\boldsymbol{x}\, dv - \int_{s_1}\boldsymbol{t}\,\delta\boldsymbol{x}ds \quad . \tag{4.45}$$

Eq.4.45 means that the substantial variation of internal energy in v due to $\delta\boldsymbol{x}$**,** which equals the sum of the work done by pressure p and the work done by the adjacent fluid volume having common boundary s_1. By integration Eq.4.33 or Eq.4.44 with respect to t,

$$\delta^* \iint_{t_1 v}^{t_2} \rho e\, dvdt \ = -\iint_{t_1 v}^{t_2} \mathrm{pdiv}(\delta\boldsymbol{x})\, dvdt \tag{4.46}$$

is obtained.

The second term in the right-hand side of Eq.4.45 will take a very important role when we consider fluid and structure interference problems in future because there are mutual transfers of the kinetic energy and the potential energy through the boundaries between the fluid and the structure.

(6) Variational expression of energy conservation

In this section, the substantial variation of the kinematic energy is considered, which are due to the

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variation vector $\boldsymbol{\delta x}$ given to the nominal solutions of the fluid particle path vector $\boldsymbol{x}$. The similar procedures used in the classical kinetics can be properly applied. The virtual work done due to inertia force $\rho\boldsymbol{a}$ is calculated as

$$\boldsymbol{\rho a}\delta\boldsymbol{x} = \rho\frac{d}{dt}(\boldsymbol{q}\delta\boldsymbol{x}) - \rho\boldsymbol{q}\frac{d\delta x}{dt}$$

$$= \rho\frac{d}{dt}(\boldsymbol{q}\delta\boldsymbol{x}) - \rho\delta^*(\frac{1}{2}q^2) \quad . \tag{4.47}$$

By applying Eq.4.47 to a small volume v, and integrating it with respect to time in the interval of $t_0 \leq t \leq t_1$, then we obtain

$$\iint_{t_0\,v}^{t_1} \rho\boldsymbol{a}\delta\boldsymbol{x}\,dvdt = \iint_{t_0\,v}^{t_1} \rho\frac{d}{dt}(\boldsymbol{q}\delta\boldsymbol{x})dvdt - \iint_{t_0\,v}^{t_1} \rho\delta^*(\frac{1}{2}q^2)\,dvdt \quad . \tag{4.48}$$

By use of the relation of Eq.4.27

$$\frac{d}{dt}\int_v \rho F dv = \int_v \rho\frac{d}{dt}F dv \tag{4.49}$$

Eq.4.48 can be rewritten and the substantial variation of kinetic energy is expressed as

$$\delta^* \iint_{t_0 v}^{t_1} \frac{1}{2}\rho q^2 dvdt = -\iint_{t_0 v}^{t_1} \rho\boldsymbol{a}\delta\boldsymbol{x}dvdt + \int_v \rho\boldsymbol{q}\delta\boldsymbol{x}dv\,|_{t_0}^{t_1} \tag{4.50}$$

Here we employ the convention

$$\delta\boldsymbol{x}(\boldsymbol{x}, t_0) = 0 \quad , \tag{4.51}$$

$$\delta\boldsymbol{x}(\boldsymbol{x}, t_1) = 0 \quad . \tag{4.52}$$

Finally, Eq.4.47 leads

$$\delta^* \iint_{t_0 v}^{t_1} \frac{1}{2}\rho q^2 dvdt = -\iint_{t_0 v}^{t_1} \rho\boldsymbol{a}\delta\boldsymbol{x}dvdt \quad . \tag{4.52}$$

(7) Derivation of functional to be stationary

Based upon the arguments from section (2) to (6), we can easily reach a functional to be stationary in the present fluid dynamic problem.

For the sake of simplicity, the body forces and the forces applied at the boundaries are assumed to be conservative forces that can be introduced from the potential $\bar{\Omega}$ and $\bar{\Phi}$ in the forms as

$$-\int_v \rho\bar{f}\delta\boldsymbol{x}dv = \delta^*\int_v \rho\bar{\Omega}dv \quad , \tag{4.53}$$

$$-\int_{s_1} \boldsymbol{t}\delta\boldsymbol{x}ds = \delta^*\int_{s_1} \bar{\Phi}ds \quad . \tag{4.54}$$

In the previous sections, the considered integral area v is small and arbitrary, then we can apply the same concepts to the total area of interest $\Gamma = \sum v$ and we have the expressions of

$$-\iint_{t_0\Gamma}^{t_1} p\mathrm{div}(\delta\boldsymbol{x})\,dvdt = \delta^*\iint_{t_0\Gamma}^{t_1} \rho e\,dvdt \quad , \tag{4.56}$$

$$-\iint_{t_0\Gamma}^{t_1} \rho\boldsymbol{a}\delta\boldsymbol{x}dvdt = \delta^*\iint_{t_0\Gamma}^{t_1} \frac{1}{2}\rho q^2 dvdt \quad . \tag{4.57}$$

The principle of virtual work 4.11 can be written with the help of Eq.4.53~4.57 as

$$\delta^*\left[\iint_{t_0\Gamma}^{t_1} L\,dvdt - \iint_{t_0 s}^{t_1} \bar{\Phi}\,dsdt\right] = 0 \quad , \tag{4.58}$$

where,

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$$L = \frac{1}{2}\rho q^2 - \rho(e + \bar{\Omega}) \quad . \tag{4.59}$$

In Eq.4.58, the independent variables subject to variation are the coordinate vector $\boldsymbol{x}$ of the fluid particles. The subsidiary conditions are geometric boundary conditions 4.10, conservation equations of mass 4,5 and energy4.6.

They are summarized in variational forms as

$$\boldsymbol{n}\delta\boldsymbol{x} = 0 \qquad \text{on } s_2 \quad , \tag{4.12}$$

$$\delta^*\rho = -\rho \mathrm{div}(\delta\boldsymbol{x}) \qquad \text{in } v \quad , \tag{4.24}$$

$$\delta^* e = -\frac{p}{\rho}\mathrm{div}(\delta\boldsymbol{x}) \qquad \text{in } v \quad . \tag{4.32}$$

In the integrand L of Eq.4.59, the first term is kinetic energy; the second is internal energy; the third is potential energy; respectively, that is,

L=(kinetic energy)-(internal energy + potential energy)

It is found that the functional 4.59 is similar to the Hamiltonian which appears in the Hamilton's Principle in the kinetics.

If we define a functional Π by

$$\Pi = \iint_{t_0 \Gamma}^{t_1} L\, dvdt - \iint_{t_0 s}^{t_1} \bar{\Phi}\, dsdt \quad , \tag{4.60}$$

then for the inviscid and isentropic flow we may state a variational principle as follows; *Of all admissible functions* $\boldsymbol{x}$ *which satisfy Eq.4.12, Eq.4.24 and Eq.4.32, the actual solution makes the functional* Π *stationary.*

5. FINAL REMARKS

The topics are a little bit classical but it is noted that modern analytical tools in engineering which perform a huge amount of jobs on supercomputers are more or less based on the vriational methods. Therefore in order for engineers to understand a part of schemes of numerical methods used today, it has a meaning that the basic disciplines concerning the variational principles in engineering have been summarized in this paper.

By introducing the concept of substantial variation, it was shown that the Hamilton's principle is exactly derived from the principle of the virtual work in fluid dynamics. The orderly processes of conversion from the principle of virtual work to the Hamilton's principle provides a logical thread that links various variational principles in the fluid dynamics.

REFERENCES

1) Washizu, K.: Variational method in elasticity and plasticity, 3rd ed., Pergamon Press, Oxford,1982.
2) Bryson,A. and Ho,Y.: Applied optimal control, Tayler & Francis Group, 1975.
3) Hestenes, M.R.: Calculus of variations and optimal control theory, John Wiley & Sons, Inc., New York, 1966.
4) Nakamichi,J.: A direct approach using the finite element method for the solution of institute the linear optimal control problem with a quadratic criterion, The Franklin Institute Vol.304, No.4, 1978
5) Selinger,R.L, Whitham,G.B.: Variational principle in continuum mechanics, Proc.Roy.Society, A 305, 1-25, 1968.
6) Serrin, J.: Mathematical principles of classical fluid mechanics, Handbuch der Physik, BandVIII/1, Stromugsmechanik. Springer-Verlag, Berlin, 1959.
7) Bateman, H.: Partial differential equations of mathematical physics, Dover, New York, 1994
8) Selinger,R.L, Whitham,G.B.: Variational principle in continuum mechanics, Proc.Roy.Society, A 305, 1-25, 1968.
9) Wang, C.: Variational method in the theory of compressible fluid, J. Aeronaut, Sci., Vol. 15, No.11, pp.675-685, 1948.

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