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An Asymptotic Solution of the Nonlinear Equations of Motion of an Airplane

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An Asymptotic Solution of the Nonlinear Equations of Motion of an Airplane*

By Hiroshi ENDŌ**

Summary

A method of solving the nonlinear equations of motion of an airplane is studied assuming that the nonlinear effect is small, and that the damping of a single oscillatory mode is weak. This is essentially a practical extension of the theory of nonlinear oscillation by Bogoljubov and Mitropol'skii, to airplane dynamics.

The procedures of constructing the asymptotic solution of the critical mode are illustrated. A numerical example is given in respect to the Duth roll stability in a low speed flight condition, in which the nonlinearity is introduced by the transient stall at the wing tip due to the disturbance in roll. Especially estimated is the time to half, or otherwise the time to double, of the amplitude of perturbed motions as a criterion of stability, which is hereby proved to be dependent on the initial amplitude of disturbance. This makes a distinct feature of nonlinear oscillation.

1. Introduction

In the ordinary linearized theory on airplane dynamics, the perturbations are assumed to be small and the nonlinear terms are usually neglected. In some case, however, the neglected terms may be of some importance, and then we have to deal with a system of nonlinear equations, the general solution of which can hardly be obtained. In the following, an asymptotic method of solving them, which applies to the significant-although limited-cases, will be discussed.

There appear a number of modes in the motion of a system with many degrees of freedom such as an airplane. These modes couple one another in nonlinear cases. This gives rise to a great number of higher order terms in the equations, and makes it difficult to solve them.

Bogoljubov and Mitropol'skii treated the natural oscillation of a nearly linear system with many degrees of freedoms in their book. They show that the solution can be constructed asymptotically in a similar manner as a system of a single degree of freedom, provided that there exists a *fundamental tone* of the oscillation, that is, the state is realized in which all degrees of freedom are excited with one and the same frequency. The motion of an airplane is composed of many modes of oscillation, and the state in which only one of them is exclusively excited is not realized in general. However,

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when a specified mode is on the margin of instability, this mode survives the others, and as the result, except at the very outset of motion, the *state of a single periodicity* comes out. Dutch roll, for example, is damped comparatively more weakly than the other modes of an airplane, and its stability sometimes comes into question. Thus, it is not pointless to treat the case where the perturbed motion can be approximately expressed by a single mode of Dutch roll. We shall study in the following how to estimate the nonlinear effect on the stability of an airplane in such a flight condition.

2. Symbols

b: wing span
c: wing chord
c_d: local coefficient of drag
c_l: local coefficient of lift
D: differential operator $(=d/d\tau)$
i_A, i_C: moment of inertia coefficients for rolling and yawing axis, respectively
i_E: product of inertia coefficient about rolling and yawing axes
L: rolling moment
l_v, l_p, l_r: nondimensional rolling moment derivatives due to side slip, rolling and yawing, respectively
N: yawing moment
n_v, n_p, n_r: nondimensional yawing moment derivatives due to side slip, rolling and yawing, respectively
p, r angular verocity in roll and yaw, respectively
s: wing semispan
S: wing area
v: increment of side slip velocity in disturbed flight
V: resultant velocity of aircraft in the steady flight condition
y_v: nondimensional side force derivative due to side slip
α: angle of attack
α_i: induced angle of attack
ε: parameter of smallness of perturbation terms
ϕ: angle of bank
μ: lateral relative density
τ: aerodynamic time

3. Estimation of the nonlinear terms

When an airplane flying at a high angle of attack undergoes a disturbance including rolling oscillation, the change in the local angle of attack may result in the momentary stall of the wing tip if the disturbance is large enough. Then the spanwise distributions of lift and drag change irregularly, which in turn result in the nonlinear rolling and yawing moments.

The load distribution on a wing is determined by the local angle of attack at each spanwise position. It changes mainly due to rolling in case of a lateral motion; while the side slip and yawing, which are the motions in the plane containing the wing, do not cause any appreciable change in the angle of attack. So we may consider that the nonlinearity originates from the rolling motion alone. To estimate the linear terms, however, we have to consider the change in the angle of attack due to side slip and yawing because the wing dihedral, sweep etc. have to be taken into account. Now we proceed to estimate the nonlinearity in the aerodynamic forces and moments due to rolling. Then, we can put

$$\alpha(y)=\alpha_0-\alpha_i(y)+py/V. \tag{1}$$

$\alpha_i(y)$ on the right hand side is expressed after Prandtl

$$\alpha_i(y)=\frac{b}{2\pi}\int_{-b/2}^{b/2}\frac{d}{d\eta}\left(\frac{cc_l}{4b}\right)\frac{d\eta}{\eta-y} \tag{2}$$

Although it is questionable whether the above expression is valid for the case where the rolling and the tip stall are involved, we assume it because we have no appropriate alternative. It may be necessary to improve the estimation by means of a more appropriate theory or, if any, experimental results.

The lift and drag acting on the wing elements are multiplied by the spanwise arm length, and then integrated to give the rolling and yawing moments. Thus we have

$$L=-\frac{1}{2}\rho V^2\int_{-b/2}^{b/2}(c_l+c_d py/V)cy\,dy \tag{3}$$

$$N=-\frac{1}{2}\rho V^2\int_{-b/2}^{b/2}(c_l py/V-c_d)cy\,dy \tag{4}$$

To carry out the above integrals, we have to know c_l and c_d as functions of y, which can be obtained from the experimental data of two-dimensional wing section if α is known. So we need to solve simultaneously (1) and (2) for α using the numerical relations between α and c_l. The integrand of the right hand side of (2) is also a function of α, and then this system of equations forms a sort of integral equation. We solve it by means of iteration, that is, starting from the tentative values of $\alpha_i(y)$, we repeat the calculation according to the following flow chart until it converges on definite values.

$$\alpha_i(\text{starting values}) \xrightarrow{\text{Eq. (1)}} \alpha(y;\ p) \longrightarrow \alpha(y;\ p) \xrightarrow{\text{Eq. (2)}} \alpha_i(y;\ p)$$

The above procedure is repeated for various values of the parameter p. In the numerical example which follows, we use the airfoil section data of NACA 65-209, and some extrapolation is made in order to get values beyond the stalling angle of attack. The

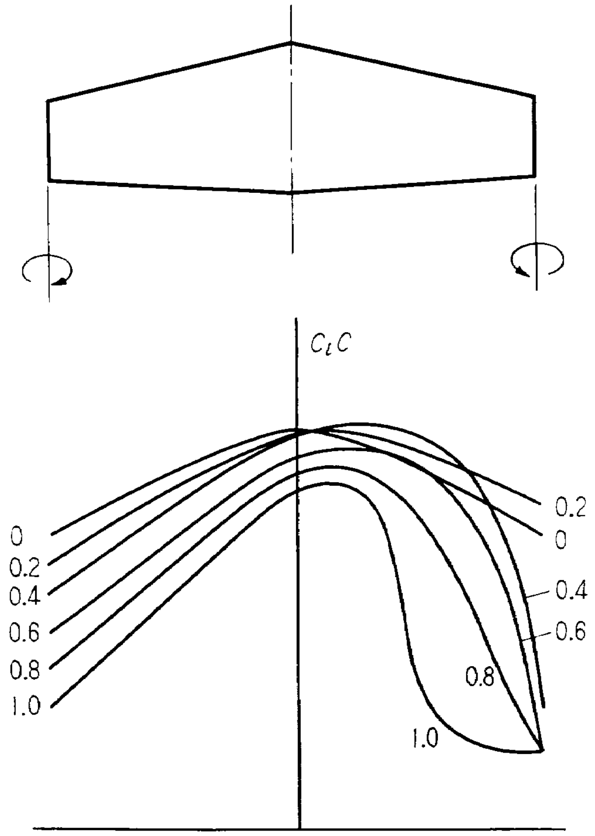

Fig. 1. Spanwise load distribution on a rolling wing. The numerals by theside of each curves are the rate of roll p.

wing is assumed to have the aspect ratio of 4, the taper ratio of 0.6, no sweep and no twits. The lift coefficient C_L of unperturbed flight is taken as 1.0.

Using the angle $\alpha(y)$ thus obtained, the spanwise distributions of c_l and c_d were calculated. The load distribution curves are shown in Fig. 1, where the values of p are inserted by the side of corresponding curves. It is seen in this figure that, the load at the tip of the descending wing decreases abnormally as p increases beyond 0.2. The corresponding change in rolling and yawing moments are shown in Fig. 2, and the p-derivatives of them in Fig. 3. In the range where p is small, they are practically constant: this satisfies the requirement of the linearized theory. Nonlinearity appears as p increases over 0.2. On account of the symmetry of the lateral motion, both l_p and n_p are even functions of p. It follows that the nonlinear terms begin with the quadratic terms of p. From the curves in Fig. 3, we approximate these derivatives by the following expressions.

$$l_p = -0.354 + 0.210p^2 \tag{5}$$

$$n_p = -0.0643 + 0.305p^2 \tag{6}$$

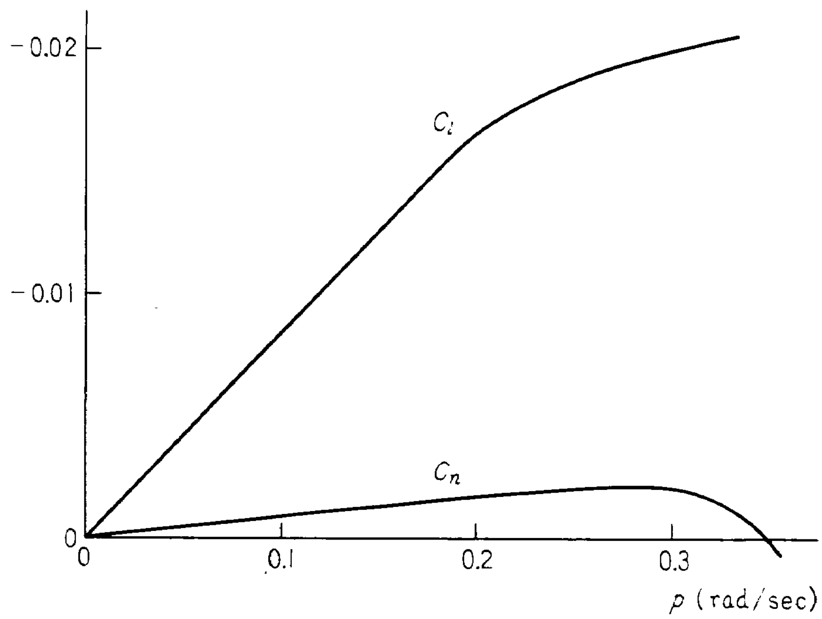

Fig. 2. Rolling and yawing moments of a rolling wing.

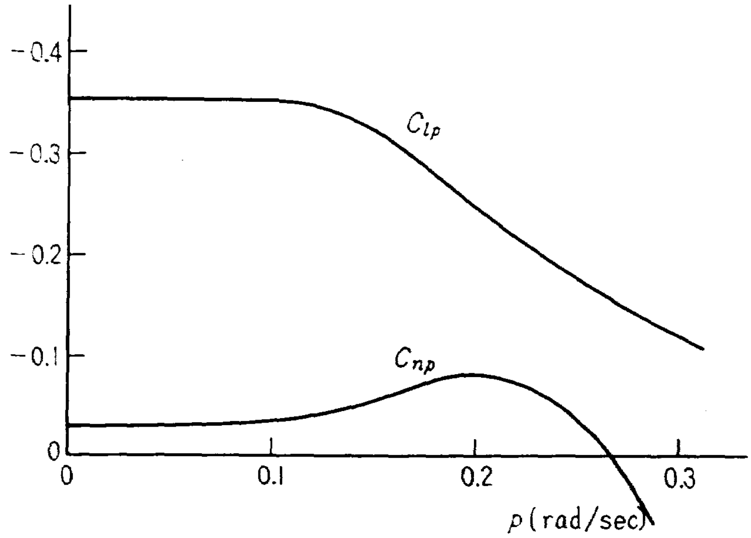

Fig. 3. p-derivatives of the rolling and yawing moments of a rolling wing.

It must be noted that the rolling and yawing moment derivatives, l_v and n_v, due to the side slip may be affected by the stall of the wing tip. This effect is, however, not taken into account, for it is so complicately masked by the side wash over the fuselage and by the increased contribution due to the fin, that their analytical expressions are not available in this stage of our study. Therefore, the discussion which follows should be regarded not as a perfect treatment of the Dutch roll stability, but as an illustration of the mathematical procedures and the distinctive features of nonlinear dynamics.

4. Asymptotic solutions of the nonlinear equations of motion

The separation of the equations of motion into the longitudinal and lateral groups, which is usually done in treating the stability of an airplane, is based on the conditions that:

i) there exists a plane of symmetry, and
ii) the longitudinal and lateral motions are not coupled together through the aerodynamic and inertial nonlinear effects.

The system of equations of motion including nonlinear terms can not be separated into two grouys because of the limitation ii). In our approximation, however, only the nonlinear effects of aerodynamic origin are considered, which contain no coupling terms as seen in Eqs. (5) and (6). Consequently, we can treat the longitudinal or lateral equations of motion separately as in the ordinary linearized theory. When the coupling is to be involved, we can deal with the complete system of equations of motion.

We shall be concerned merely for simplicity with the motion with control fixed. Then the lateral equations of motion are

$$\left.\begin{aligned}&\mu(D-y_v)\hat{v}-y_p\hat{p}+(\mu-y_r)\hat{r}-\frac{1}{2}\mu C_L\phi=0\\&-\mu l_v\hat{v}+(i_AD-l_p)\hat{p}-(i_ED+l_r)\hat{r}\qquad=\varepsilon l_{pp}\hat{p}^3\\&-\mu n_v\hat{v}-(i_ED+n_p)\hat{p}+i_cD-n_r)\hat{r}\qquad=\varepsilon n_{pp}\hat{p}^3\\&\qquad\hat{p}\qquad\qquad -D\phi=0\end{aligned}\right\}\tag{7}$$

where the higher order coefficients l_{pp} and n_{pp} originate from the higher order terms in Eqs. (5) and (6) respectively, and ε is the parameter of smallness representing the order of perturbation terms. As p is of the order of 0.1, ε is taken to be 0.1.

The airplane in question has a weakly damped Dutch roll mode as previously mentioned, and the homogeneous system of Eq. (7) has a nearly harmonic solution. Then, modifying one or more of the stability derivatives in Eq. (7), we can make an harmonic oscillation system of it. We execute this by increasing l_p by a small amount $\varepsilon\Delta l_p$, thus

$$l_{po}=l_p+\varepsilon\Delta l_p\tag{8}$$

We regard this *imaginary* harmonic system as the unperturbed one, and try to approximate the real one by adding to it the perturbations consisting of the nonlinear effects and the damping which is introduced by the excess damping in roll derivative $\varepsilon\Delta l_p$.

Substituting Eq. (8) into Eq. (7), and putting the perturbation terms together, we have

$$\begin{aligned}&\mu(D-y_v)\hat{v}-y_p\hat{p}+(\mu-y_r)\hat{r}-\frac{1}{2}\mu C_L\phi=0\\&-\mu l_v\hat{v}+(i_AD-l_{po})\hat{p}-(i_ED+l_r)\hat{r}\qquad=\varepsilon(-\Delta l_p\hat{p}+l_{pp}\hat{p}^3)\\&-\mu n_v\hat{v}-(i_ED+n_p)\hat{p}+(i_cD-n_r)\hat{r}\qquad=\varepsilon n_{pp}\hat{p}^3\\&\qquad\hat{p}\qquad\qquad -D\phi=0\end{aligned}\tag{9}$$

This system is manipulated to have the general form as follows,

$$Dx_k-\sum_{l=1}^{4}c_{kl}x_l=\varepsilon f_k(x_1,\ x_2,\ x_3,\ x_4,\ \tau)\qquad(k=1,\ 2,\ 3,\ 4)\tag{10}$$

where x_1, x_2, x_3, and x_4 stand for $\hat{v}$, $\hat{p}$, $\hat{r}$ aud ϕ respectively, and

$$\begin{aligned}&c_{11}=y_v,\quad c_{12}=y_p/\mu,\quad c_{13}=y_r/\mu-1,\quad c_{14}=C_L/2,\\&c_{21}=\mu(i_cl_v+i_En_v)/(i_Ai_C-i^2{}_E),\quad c_{22}=(i_cl_{po}+i_En_p)/(i_Ai_C-i^2{}_E),\\&c_{23}(il_r+i_En_r)/(i_Ai_C-i^2{}_E),\quad c_{24}=0,\\&c_{31}=\mu(i_El_r+i_An_r)/(i_Ai_C-i^2{}_E),\quad c_{32}=(i_El_p+i_An_p)/(i_Ai_C-i^2{}_E),\\&c_{33}=(i_El_r+i_An_r)/(i_Ai_C-i^2{}_E),\quad c_{34}=0,\quad c_{41}=c_{43}=c_{44}=0,\\&c_{42}=1,\\&f_2=-\{i_C\Delta l_p\hat{p}+(i_Cl_{pp}+i_En_{pp})\hat{p}^3\}/(i_Ai_C-i^2{}_E),\\&f_3=-\{i_E\Delta l_p\hat{p}+)i_El_{pp}+i_An_{pp})\hat{p}^3\}/(i_Ai_C-i^2{}_E).\\&f_1=f_4=0.\end{aligned}$$

The system of Eq. (10) is formally identical to the one treated by Bogoljubov and Mitropol'skii in the theory of nonlinear oscillation with many degrees of freedom. So we can follow the mathematical procedures developed by them. Since the unperturbed system

$$Dx_k - \sum_{l=1}^{4} c_{kl} x_l = 0 \quad (k=1,\ 2,\ 3,\ 4) \tag{12}$$

has an harmonic solution, the eigen values λ's of the homogeneous system of the algebraic equations

$$\sum_{l=1}^{4} (\lambda \delta_{kl} - c_{kl}) x_k = 0 \quad (k=1,\ 2,\ 3,\ 4) \tag{13}$$

($\delta_{kl}=1$ if $k=l$, $\delta_{kl}=0$ if $k \neq l$), that is, the roots of the characteristic equation

$$\Delta(\lambda) = 0 \tag{14}$$

where

$$\Delta(\lambda) = \begin{vmatrix} c_{11}-\lambda & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22}-\lambda & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33}-\lambda & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44}-\lambda \end{vmatrix}$$

contain a single pair of purely imaginary ones $\lambda = \pm i\omega$, Let φ_k and $\varphi_k{}^*$ $(k=1,\ 2,\ 3,\ 4)$ denote the nontrivial solutions of Eq. (13) corresponding to $\lambda = i\omega$ and $\lambda = -i\omega$ respectively, then the solutions of the unperturbed system (12) are

$$x_k^{(0)} = a(\varphi_k e^{i\omega\tau} + \varphi_k^* e^{-i\omega\tau}) \qquad (k=1,\ 2,\ 3,\ 4) \tag{15}$$

where a is the amplitude, and φ_k and φ_k^* are the complex constants specifying the shape of the mode, that is, the relative amplitudes and phases of the disturbed motion, Dutch roll in our case.

In the perturbed system (10), the energy of motion is consumed or accumulated incessantly through the perturbations. Consequently, the amplitude a is not constant, and the momentary angular frequency is no longer independent of the amplitude a because of the nonlinear effect. Considering that the solutions of Eq. (10) tend in the limit of $\varepsilon \to 0$, to the solutions of the unperturbed system (15), we expand it in the power series of ε as follows;

$$\begin{aligned} x_k = a(\tau)(\varphi_k e^{i\psi} + \varphi_k^* e^{-i\psi}) + \varepsilon x_k^{(1)}(a,\ \psi) + \varepsilon^2 x_k^{(2)}(a,\ \psi) + \cdots\cdots \\ + \varepsilon^m x_k^{(m)}(a,\ \psi) + \cdots\cdots \qquad (k=1,\ 2,\ 3,\ 4) \end{aligned} \tag{16}$$

where the first term of the right hand side is formally identical with the solution (12) of the unperturbed system, but the amplitude a and the total phase ψ are functions of time τ. Taking into consideration the fact that the development of the energy of the system is not independent on the amplitude because of the nonlinear effect, their time variations are again expanded in the power series of ε as follows;

$$Da=A_1(a)+\varepsilon A_2(a)+\cdots\cdots \tag{17}$$
$$D\psi=\omega+\varepsilon B_1(a)+\varepsilon^2 B_2(a)+\cdots\cdots$$

where A_1, A_2, $\cdots\cdots$; B_1, B_2, $\cdots\cdots$are constants to be determined. In this way, our problem has been reduced to that of determining $x_1(a, \psi)$, $x_2(a, \psi)$, $\cdots\cdots$; $A_1(a)$, $B_1(a)$, $A_2(a)$, $B_2(a)$, $\cdots\cdots$that x'_ks of Eq. (16) with Eq. (17) may satisfy Eq. (10) as functions of ψ, with period 2π. Substituting Eq. (16) for x'_ks in Eq. (10), we express both sides in power series of ε by means of Eq. (17). Equating the terms of the same power both sides, we have from the first power terms,

$$\omega\frac{\partial x_k^{(1)}}{\partial\psi}-\sum_{l=1}^{4}c_{kl}x_l^{(1)}=f_k(x_1^{(0)},\ x_2^{(0)},\ x_3^{(0)},\ x_4^{(0)})-\frac{\partial x_k^{(0)}}{\partial a}A_1-\frac{\partial x_k^{(0)}}{\partial\psi}B_1$$
$$(k=1,\ 2,\ 3,\ 4) \tag{18}$$

and from the second power terms,

$$\omega\frac{\partial x_k^{(2)}}{\partial\psi}-\sum_{l=1}^{4}c_{kl}x_l^{(2)}=\sum_{l=1}^{4}\frac{\partial f_k}{\partial x_l}(x_1^{(0)},\ x_2^{(0)},\ x_3^{(0)},\ x_4^{(0)})\,x_l^{(1)}$$
$$-\frac{\partial x_k^{(1)}}{\partial a}A_1-\frac{\partial x_k^{(1)}}{\partial\psi}B_1-\frac{\partial x_k^{(0)}}{\partial a}A_2-\frac{\partial x_k^{(0)}}{\partial\psi}B_2$$
$$(k=1,\ 2,\ 3,\ 4) \tag{19}$$

Similar expressions follow as to the higher order terms. Substituting Eq. (15) for $x_k^{(0)}$ tn the right hand side of Eq. (18), and after rearranging, we have

$$\omega\frac{\partial x_k^{(1)}}{\partial\psi}-\sum_{l=1}^{4}c_{kl}x_l^{(1)}=\{\Phi_k^{(1)}(a)-A_1\varphi_k-iB_1a\varphi_k\}\,e^{i\psi}$$
$$+\{\Phi_k^{(-1)}(a)-A_1\varphi_k^*+iB_1a\varphi_k^*\}\,e^{-i\psi}$$
$$+\sum_{\substack{m\neq\pm1\\-\infty<m<\infty}}\Phi_k^{(m)}(a)e^{im\psi}\qquad(k=1,\ 2,\ 3,\ 4) \tag{20}$$

where

$$\Phi_2^{(1)}=-\{i_C\Delta l_p a\varphi_2+3(i_C l_{pp}+i_E n_{pp})a^3\varphi_2\varphi_2^{*2}\}/(i_A i_C-i_E^2),$$

$$\Phi_2^{(-1)}=-\{i_C\Delta l_p a\varphi_2^*+3(i_C l_{pp}+i_E n_{pp})a^3\varphi_2\varphi_2^{*2}\}/(i_A i_C-i_E^2),$$

$$\Phi_3^{(1)}=-\{i_E\Delta l_p a\varphi_2^*+3(i_E l_{pp}+i_A n_{pp})a^3\varphi_2\varphi_2^{*2}\}/(i_A i_C-i_E^2),$$

$$\Phi_2^{(3)}=-(i_C l_{pp}+i_E n_{pp})a^3\varphi_2^2\varphi_2^*/(i_A i_C-i_E^2),$$

$$\Phi_2^{(-3)}=-(i_C l_{pp}+i_E n_{pp})a_3\varphi_2\varphi_2^{*2}/(i_A i_C-i_E^2),$$

$$\Phi_3^{(3)}=-(i_E l_{pp}+i_A n_{pp})a^3\varphi_2^2\varphi_2^*/(i_A i_C-i_E^2),$$

$$\Phi_3^{(-3)}=-(i_E l_{pp}+i_A n_{pp})a^3\varphi\varphi_2^{*2}/(i_A i_C-i_E^2), \tag{21}$$

and other $\Phi_k^{(m)}$'s are all zero.

The terms on the right hand side of Eq. (20) are all known except A_1 and B_1. These unknown quantities will be determined in the following. Replacing ψ with $\omega\tau$ on the left hand side of Eq. (20), it becomes identical with the unperturbed system (12). Thus, our problem has been reduced to that of forced oscillation of the harmonic system (12) by the perturbation forces of the frequencies ω and $m\omega$, where ω is the natural frequency of the system, and m is any integer. In case of the resonant excitation with frequency ω, there is no solution in general. We have the solutions only if the perturbation terms on the right hand side of Eq. (18) satisfy the following condition,

$$\sum_{l=1}^{4}\chi_l\{\Phi_l^{(1)}(a)-A_1\varphi_l-iaB_1\varphi_l\}=0 \tag{22}$$

where χ_l's ($l=1, 2, 3, 4$) are the nontrivial solutions for $\lambda=-i\omega$ of the system of equations adjoint to Eq. (13), viz.

$$\sum_{l=1}^{4}(\lambda\delta_{kl}+c_{lk})x_l=0 \tag{23}$$

From the above condition, we have the expressions for A_1 and B_1 as follows;

$$\begin{aligned}A_1+iaB_1&=\sum_{l=1}^{4}\chi_l\Phi_l^{(1)}/\sum_{l=1}^{4}\chi_l\varphi_l\\&=-[\chi_2\{i_C\Delta l_p a\varphi_2+3(i_C l_{qq}+i_E n_{pp})a^3\varphi_2^2\varphi_2^*\}\\&\quad+\chi_3\{i_E\Delta l_p a\varphi_2+3(i_E l_{pp}+i_A n_{pq})a^3\varphi_2^2\varphi_2^*\}]/(i_A i_C-i_E^2)\sum_{l=1}^{4}\chi_l\varphi_l\end{aligned} \tag{24}$$

Comparing the real and imaginary parts of both sides, we can readily estimate A_1 and B_1. With the values of A_1 and B_1 thus obtained, the solution of Eq. (20) is written down as

$$\begin{aligned}x_k^{(1)}(a,\psi)&=C_1(a)\varphi_k e^{i\psi}+C_1^*(a)\varphi_k^*e^{-i\psi}+e^{i\psi}\sum_{l=1}^{4}S_{kl}\{\Phi_l^{(1)}(a)-(A_1+iaB_1)\varphi_l\}\\&\quad-e^{-i\psi}\sum_{1=l}^{4}S_{kl}\{\Phi_l^{(-1)}(a)-(A_1-iaB_1)\varphi_l^*\}+\sum_{m\neq\pm1}e^{im\psi}\{\sum_{l=1}^{4}Z_{kl}(im\omega)\Phi_l^{(m)}(a)\}\\&\qquad(k=1,\ 2,\ 3,\ 4)\end{aligned} \tag{25}$$

where $Z_{kl}(im\omega)=\Delta_{kl}(im\omega)/\Delta(im\omega)$, $S_{kl}(i\omega)=\left\{\dfrac{\partial\Delta_{kl}(\lambda)}{\partial\lambda}\Big/\dfrac{\partial\Delta(\lambda)}{\partial\lambda}\right\}_{\lambda=i\omega,}$ and $\Delta(\lambda)$ is the determinant defined by (14), and Δ_{kl} is its minor. $C_1(a)$ and $C_1^*(a)$ are the constants to be determined by the initial conditions. Substituting Eq. (25) for $x_k^{(1)}(a,\ \psi)$ on the right hand side of Eq. (19), we have the equation for $x_k^{(2)}(a,\ \psi)$, which can be solved in the same manner as for $x_k^{(1)}(a,\ \psi)$. A_2 and B_2 are again determined by the condition similar to Eq. (22). Thus we can determine $x_k^{(1)}$. $x_k^{(2)}$, $x_k^{(3)}$, and so forth successively.

Summarizing, the zeroth approximation of our solution is that of the unperturbed harmonic system (12), viz.

$$x_k^{(0)}=a(\varphi_k e^{i\omega\tau}+\varphi_k^* e^{-i\omega\tau}) \quad (k=1,\ 2,\ 3,\ 4) \tag{15}$$

In the first approximation, the amplitude a and the angular frequency ω are regarded as a function of time. Thus, we have

$$x=a(\tau)(\varphi_k e^{i\psi}+\varphi_k^* e^{-i\psi}) \quad (k=1,\ 2,\ 3,\ 4) \tag{26}$$

with

$$\left.\begin{aligned} Da&=\varepsilon A_1(a) \\ D\psi&=\omega+\varepsilon B_1(a) \end{aligned}\right\} \tag{27}$$

5. Stability of motion: Numerical example

The criterion of stability of motion we are most interested in, is obtained from Eq. (27). The time to half or the time to double, which are customarily referred to as the criteria of stability of motion, are estimated from Eq. (27) through direct integration. The results are as follows:

$$\left.\begin{aligned} t_{\mathrm{half}}(a_0)&=\hat{t}\int_{a_0}^{a_0/2}\frac{da}{\varepsilon A_1(a)} \\ t_{\mathrm{double}}(a_0)&=\hat{t}\int_{a_0}^{2a_0}\frac{da}{\varepsilon A_1(a)} \end{aligned}\right\} \tag{28}$$

While these values are constant in the linearized theory, they are now dependent on the initial value a_0 of the amplitude, that is, the stability of motion depends on the intensity of the initial disturbance. Even if a system is stable for a small disturbance, we can not presume that it remains so for a large one, and vice versa. We shall illustrate this in the following.

We are now concerned with the Dutch roll stability of the airplane, the aerodynamic moments of which have such nonlinearity as described in section 3. The other stability derivatives and the inertial parameters are, as in the linearized theory,

$$\left.\begin{aligned} &\mu=25.6 && C_L=1.0 && y_v=-0.39 \\ &i_A=0.124 && l_v=-0.201 && n_v=0.043 \\ &i_C=0.18 && l_p=-0.354 && n_p=-0.0643 \\ &i_E=-0.02 && l_r=0.199 && n_r=-0.123 \end{aligned}\right\} \tag{29}$$

The airplane is assumed to be in a level flight condition at the sea level. The moment of inertia about the roll axis has been taken larger than usual, with the result that the damping in roll is reduced appreciably.

The stability of this airplane informed by the ordinary linearized theory is as follows:

$$\left.\begin{array}{ll}\text{spiral mode} & t_{\text{half}}=11.9\ \text{sec} \\ \text{rolling convergence} & t_{\text{half}}=0.55\ \text{sec} \\ \text{lateral oscillation (Dutch roll)} & t_{\text{half}}=41.3\ \text{sec} \\ & (T=5.2\ \text{sec})\end{array}\right\} \tag{30}$$

As is seen in these data, Dutch roll is slightly stable and the damping is by far weaker than that of the other modes. It may be observed that the former mode dominates in the motion except in the very early stage of the disturbance, and so the condition of single periodicity is approximately fulfilled. Dutch roll becomes neutrally stable if l_p is increased by a small amount of 0.033. Thus putting $l_{po}=-0.321$, and using the values of (31), the harmonic system (12) becomes

$$\begin{array}{ll} D\hat{v}+0.39\hat{v}+\hat{r}-0.500\phi & =0 \\ D\hat{p}+43.3\hat{v}+2.58\hat{p}-1.75\hat{r} & =0 \\ D\hat{r}-10.93v+0.00704\hat{p}+0.878\hat{r} & =0 \\ D\phi \qquad\qquad -\hat{p} & =0 \end{array} \tag{31}$$

The constants specifying the shape of modes φ_k and φ_k^* are obtained by solving the above equations with $\lambda=3.74i$, the root corresponding to Dutch roll mode. Regarding a as the amplitude of the angle of bank ϕ, they become

$$\left.\begin{array}{l}\varphi_1=0.299-0.264i \\ \varphi_2=3.74i \\ \varphi_3=-0.604-1.014i \\ \varphi_4=1.0\end{array}\right\} \tag{32}$$

These quantities indicate the state of excitation in the lateral three degrees of freedom, which are shown in Fig. 4 in the vectorial form. By the way, the solutions χ's of the adjoint system are as follows:

$$\left.\begin{array}{l}\chi_1=7.48i \\ \chi_2=0.148-0.203i \\ \chi_3=-2.01-0.425i \\ \chi_4=1.0\end{array}\right\} \tag{33}$$

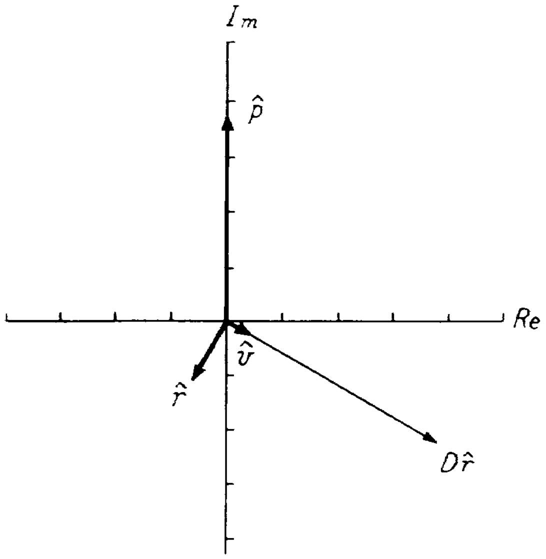

Fig. 4. Shape of mode of Dutch roll.

With $\varepsilon=0.1$, $l_{pp}=2.10$, and $n_{pp}=3.05$, Eq. (21) becomes

$$\left.\begin{array}{l}\Phi_2^{(1)}=\{-1.015(10a)+2.28(10a)^3\}i \\ \Phi_3^{(1)}=\{1.13(10a)+2.40(10a)^3\}i\end{array}\right\} \tag{34}$$

Substituting these in equation (24), and comparing the real or imaginary parts of both sides, we have

$$\left.\begin{array}{l}A_1(a)=-0.569a-349a^3 \\ B_1(a)=-0.195a-602a^3\end{array}\right\} \tag{35}$$

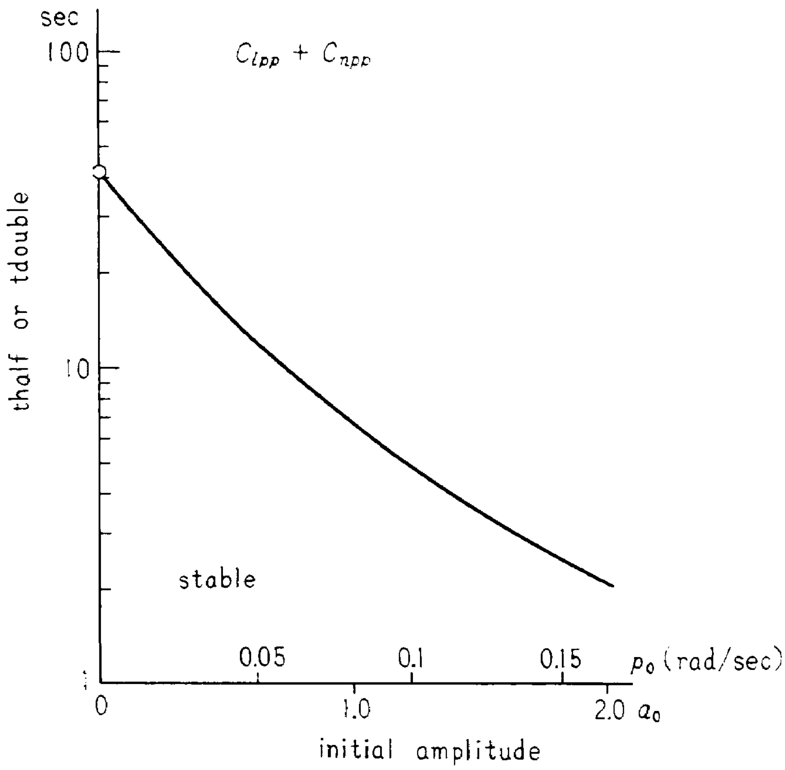

Fig. 5. Relation between the stability of Dutch roll and the initial amplitude of disturbance

We can now calculate t_{double} or t_{half} by Eq. (28) as functions of the initial amplitude a_0. The result is shown in Fig. 5. The limit of its value as $a_0 \to 0$ coincides with the one

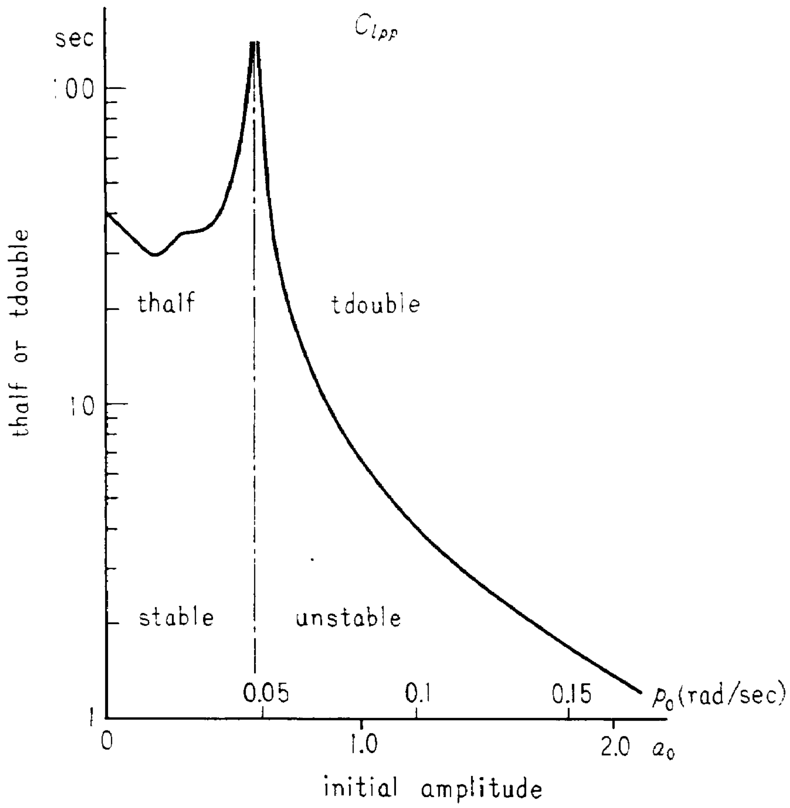

Fig. 6. Same as Fig. 5, but taking into consideration only the nonlinearity in l_p.

obtained from the linearized theory indicated by a circle on the ordinate. This coincidence certifies the accuracy of this asymptotic method.

We can see in this figure that t_{half} decreases as a increases, that is, the motion becomes more stable. This implies that the energy of oscillation is dissipated through the nonlinear effects. To examine the cause of this, we shall see what role the individual nonlinear terms play. In the first instance, we shall retain only l_{pp} out of the two nonlinear terms, and calculate t_{half} or t_{double} as before. The result is shown in Fig. 6, indicating that the oscillation damps when p_0 is smaller than 0.05, but that it becomes unstable as p_0 increases beyond that value. This is because of the decrease of damping in roll due to the momentary stall of the wing tip caused by increased rate of roll p. This case is a distinct example in which the stability of nonlinear oscillation depends upon the amplitude.

In the second place, we retain n_{pp} alone, and repeat the calculation, the result of which is given in Fig. 7. Any sign of destabilization is not revealed as a_0 increases.

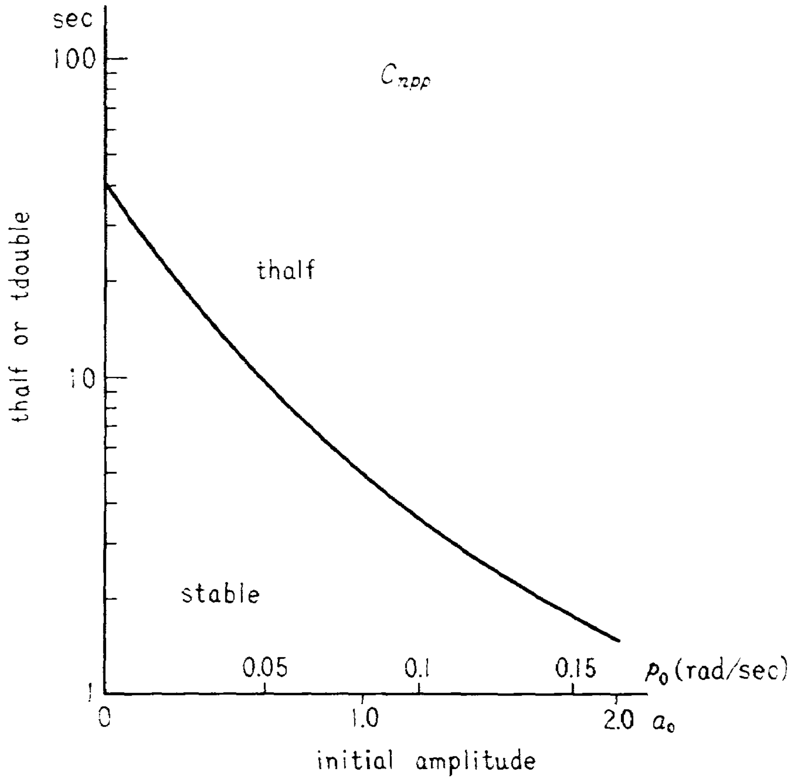

Fig. 7. Same as Fig. 5, but taking into consideration only the nonlinearity in n_p.

We shall examine this case in further detail. As is seen in Fig. 2 and Eq. (6), the nonlinear component of yawing moment N is proportional to the third power of p, and is thought to be in the same phase with p. Dr and p are almost out of phase each other (Fig. 4). Therefore, we can presume that Dr also is opposed to p^3, and thus to the nonlinear component of N. It follows that the nonlinearity in n_p has the damping effect on Dutch roll.

In case of Fig. 5, where both l_{pp} and n_{pp} are involved, we can observe that the stabilizing effect of n_{pp} is stronger than the destabilizing one of l_{pp}, resulting in stabilization of Dutch roll. This conclusion, however, has been deduced on the assumption that the nonlinear effects are weak. It is, therefore, evident that this does not hold in such a case when the stall developes over the greater part of the wing, and the stability derivatives are excessively affected.

6. Conclusions

1. An asymptotic theory on nonlinear oscillation, proposed by Bogoljubov and Mitropol'skii, is shown to be applicable in the dynamics of an airplane, to show how the nonlinearity in the aerodynamic forces affects the stability of an oscillatory mode, on the assumption that it is on the margin of instability.
2. The applicability of this method is of course limited because the damping of the mode in question is assumed to be weak. But this becomes a powerful tool when we examine whether the stability boundary of this mode shifts to the stable side or not owing to the nonlinearity in aerodynamic forces; for the stability is neutral on this boundary and the assumption is valid in the neighbourhood of it.
3. It is illustrated that the stability depends on the amplitude of disturbances. This makes a striking contrast to the linear oscillation, the stability of which is independent of it.

Acknowledgment

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An Asymptotic Solution of the Nonlinear Equations of Motion of an Airplane

Hiroshi ENDO

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Page	Line	Wrong	Correct
1	12	Duth roll stability	Dutch roll stability
3	30	$\rightarrow \alpha(y;\ p) \rightarrow \alpha(y;\ p) \rightarrow \alpha_i(y;\ p)$	$\rightarrow \alpha(y;\ p) \rightarrow c_l(y;\ p) \rightarrow \alpha_i(y;\ p)$
4	Fig. 1	Spanwise load	Spanwise load
4	Fig. 1	The numerals by theside	The numerals by the side
5	18	into two grouys	into two groups
6	3	$+i_cD-n_r)\hat{r}$	$+(i_CD-n_r)\hat{r}$
6	20	$+(i_cD-n_r)\hat{r}$	$+(i_CD-n_r)\hat{r}$
6	23	$c_{21}=\mu(i_cl_v+i_En_v)$	$c_{21}=\mu(i_Cl_v+i_En_v)$
6	26	$c_{22}=(i_cl_{po}+i_En_p)$	$c_{22}=(i_Cl_{po}+i_En_p)$
6	27	$c_{23}(il_r+i_En_r)$	$c_{23}=(i_Cl_r+i_En_r)$
6	32	$f_3=-\{i_E\Delta l_p\hat{p}+)i_El_{pp}+i_An_{pp})\hat{p}^3\}$	$f_3=-\{i_E\Delta l_p\hat{p}+(i_El_{pp}+i_An_{pp})\hat{p}^3\}$
8	16	$x_k^{(0)}$ tn the	$x_k^{(0)}$ in the
8	25	$\Phi_2^{(-3)}=-(i_Cl_{pp}+i_En_{pp})\alpha_3\varphi_2\varphi_2^{*2}$	$\Phi_2^{(-3)}=-(i_Cl_{pp}+i_En_{pp})\alpha^3\varphi_2\varphi_2^{*2}$
9	16	$+3(i_Cl_{gg}+i_En_{pp})$	$+3(i_Cl_{pp}+i_En_{pp})$
9	17	$+3(i_El_{pp}+i_An_{pg})$	$+3(i_El_{pp}+i_An_{pp})$
9	21	$-e^{-i\psi}\sum_{1=l}^{4}S_{kl}$	$-e^{-i\psi}\sum_{l=1}^{4}S_{kl}$
12~13	Fig. 5~7	thalf, tdouble	t_{half}, t_{double}

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