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Small-Strain Deformations Superposed on Finite Deformations of Highly Elastic Incompressible Materials

Part 1. Constitutive Equations

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Small-Strain Deformations Superposed on Finite Deformations of Highly Elastic and Incompressible Materials*

Part I Constitutive Equations

By Tatsuzo KOGA**

ABSTRACT

A system of linear constitutive equations for a class of deformations added to a known state of finite deformation of thin elastic membranes is derived. Attension is restricted to the class of deformation defined such that lines of curvature of the undeformed membranes remain as lines of curvature in any state of deformation. Strains are assumed small in the additional deformation. The constitutive equations derived in the present paper turn out to be identical in form to those obtained by Fernandez-Sintes and Nachbar for axisymmetric deformations and to those obtained by Corneliussen and Shield for a circular cylindrical membrane.

NOTATIONS

a, A	deteminants of $a_{\alpha\beta}$ and $A^{\alpha\beta}$, respectively
$a_{\alpha\beta}$, $a^{\alpha\beta}$	covariant and contravariant components of the metric tensor associated with the midsurface of deformed membrane, α, $\beta=1, 2$
$A_{\alpha\beta}$, $A^{\alpha\beta}$	covariant and contravariant components of the metric tensor associated with the midsurface of deformed membrane, α, $\beta=1, 2$
$\bar{a}^{\alpha\beta}$, $\bar{D}^{\alpha\beta}$	quantities defined in Eqs. (26), α, $\beta=1, 2$
$\boldsymbol{a}_1$, $\boldsymbol{a}_2$	base vectors tangent to the midsurface of undeformed membrane
$\boldsymbol{a}_3$	normal unit vector to the midsurface
c^{ij}, C^{ij}	cofactors in the determinants of g_{ij} and G_{ij}, respectively, i, $j=1, 2, 3$
g, G	determinants of g_{ij} and G_{ij}, respectively
g_{ij}, g^{ij}	covariant and contravariant components of the metric tensor associated with the undeformed body, i, $j=1, 2, 3$
G_{ij}, G^{ij}	covariant and contravariant components of the metric tensor associated with the deformed body, i, $j=1, 2, 3$
$\boldsymbol{g}_i$, $\boldsymbol{G}_j$	base vectors associated with the undeformed and deformed body, respectively, i, $j=1, 2, 3$
h_0	wall-thickness of undeformed membrane
I_1, I_2, I_3	strain invariants
K^{ij}	coefficients of the constitutive equations, i, $j=1, 2, 3$
$n^{\alpha\beta}$	stress resultant, α, $\beta=1, 2$
P, D^{ij}	quantities defined in Eqs. (14)
R_0	radius of the undeformed circular cylindrical membrane
$\boldsymbol{r}$, $\boldsymbol{R}$	position vectors
$\boldsymbol{r}_0$, $\boldsymbol{R}_0$	position vectors to points on the midsurface
ds_0, ds	line elements in the undeformed and deformed states
Δu, Δv, Δw	components of the secondary-state displacement
W	strain enery function
$W_{,\alpha\beta}$, $W_{,\alpha}$	quantities defined in Eqs. (42)
x_i, y_i	rectangular Cartesian coordinates, $i=1, 2, 3$
$\Delta\varepsilon^{ii}$	small change in principal strains in the secondary state, $i=1, 2, 3$
θ^i	general curvilinear coordinates, $i=1, 2, 3$
λ_i	principal extension ratios, $i=1, 2, 3$
τ^{ij}	contravariant components of stress tensor, i, $j=1, 2, 3$
Φ, Ψ	quantities defined in Eqs. (14)
*	to indicate phycical components

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Δ to indicate quantities of the secondary state

p subscript to indicate quantities of the primary state

1. INTRODUCTION

1.1 Technical Background

Application of pneumatic structures to architecture and engineering has been rapidly expanding. Many examples of pneumatic structures can be found in the book edited by F. Otto[1]. Pneumatic structures are structural forms stabilized mainly by differences in pressure acting upon each face of thin structural components. The main advantage of pneumatic structures lies in weight-saving. Using modern synthetic materials in the form of thin membranes, we can, in principle, construct a huge dome covering such a wide area that any conventional structural form, even according to optimum design, can hardly sustain its own weight. The American pavilion built for Expo '70 in Osaka, Japan, was a typical example of a pneumatic structure covering a huge area of the exhibition floor. Its roof was made of a textile having an elliptical shape of 142 m in length and 84 m in width. The Fuji group pavilion was another example of a pneumatic structure built at Expo '70. The dome was made of cluster of sixteen pneumatic beams, each of which had a length of 78 m and a diameter of 4 m, and was bent into the form of a horseshoe.

Another advantage of pneumatic structures is that they can be folded and packaged into a compact size so that they can easily be delivered to the desired place, where they are erected into the desired shape. This makes pneumatic structures a very efficient form of structure in transportation engineering. The sail is probably the oldest application of pneumatic structures in the history of transportation engineering. Pressure differences give it a shape to produce aerodynamic forces, which are transmitted to the hull through the booms and the masts. The inflatable lifeboat is also a good example of a pneumatic structure. In an emergency, the lifeboat is erected into the desired shape by inflation. In this case, the pressure difference provides not only the shape of the lifeboat but also the load-carrying capacity to the otherwise collapsible lifeboat. The lifeboat in the inflated state can, thus, carry the weight of crew and passengers.

Application of pneumatic structures for surface-bound vehicles, such as the sail and the lifeboat on the ship, is not essential and could be replaced by conventional structures, or by some other devices. It is, however, essential for airborne and space-bound vehicles. The infiatable lifeboats carried on the ship may be replaced by conventional wooden boats, but those carried on airplanes can hardly be replaced by rigid lifeboats because of their weight and volume. Pneumatic structures, or any other form of expandable structures, become inevitable in space engineering, because exceptionally high expenditure is required for launching space vehicles, and also because the payloads should be packaged into a small size so that they can be fitted into the launch vehicles. For these reasons, designers should make every effort to decrease the structural weight of space vehicles and to design them so that they can be folded and expanded. Another important factor, which justifies the use of light-weight, thin-walled pneumatic structures in extraterrestrial regions, is that the loads acting on the structures outside the gravitational field of Earth are small. An article of N. J. Hoff[2] discusses some of the examples of pneumatic structures applied for space engineering.

The propellant tank of the Atlas intercontinental ballistic missile has been made of thin sheet of easily weldable, stainless steel. The tank has a circular cylindrical shape of 3 m in diameter and about 18 m in length, and its wall-thickness is 0.04~0.1 cm. It is so thin that it can not carry its own weight. It is stabilized and maintained in its shape of a circular cylindrical shell only by internal pressure of helium gas.

The gigantic balloon satellite Echo I was successfully launched into orbit by a three-stage Thor-Delta rocket. The diameter of the satellite was 30 m when inflated. It was made of Mylar plastic panels 0.01 mm thick with aluminum coating. The satellite was initially folded into a ball 65 cm in diameter and was then ejected into space and inflated to a spherical shape through the expansion of the residual air in it.

Research has been carried out on the possibility of the use of pneumatic structures as a decelerating device for atmospheric re-entry. The experimental paraglider Paresev was a kind of hybrid between parachute and sail. It consisted of a sail-like structure made of Dacron sheet and bounded by booms in an arrowhead shape. Another experimental re-entry vehicle, Ballute, was a hybrid between balloon and parachute. It was made of a textile and it contained an opening at the bottom, through which the air rammed in. Ballute was tested successfully even at supersonic speed.

Though there are numerous problems yet to be solved, pneumatic structures will certainly play the most important role in constructing extraterrestrial stations, because they seem to be the only economically feasible form of building in extraterrestrial regions. The problems to be solved include insulation against cosmic rays and heat and the requirement to withstand the large pressure differences.

1.2 Present Research

In the present series of papers, deformations of inflated thin membranes of rubbery materials will be investigated. There is no doubt that membranes made of a single rubbery material are least likely to be used in practice, particularly in space application, and that they are most likely to be used in the form of elaborate composite materials. It is, however, worthwhile to investigate the idealized membrane structures of noncomposite rubbery material, because of relative ease in analytical treatment and in experimental handling.

Rubbery materials, being highly elastic and incompressible, readily exhibit large deformations. In analysis, therefore, the theory of finite deformations has to be used.

The general theory of finite deformations of elastic bodies has been considered by many authors (see for example the book of A. E. Green and W. Zerna[3]). The equations involved in the general theory are so highly nonlinear that their exact solutions have been obtained only for a limited number of practical problems dealing with elastic bodies of the most simple geometric shapes before and after deformation. Summary of these solutions is given in Ref. 3 and in the book of A. E. Green and J. E. Adkins[4]. The general theory has been specialized for thin elastic membranes and several problems of finites deformations of membranes have been solved by R. S. Rivlin[5] and by J. E. Adkins and R. S. Rivlin[6]. Although the equations are considerably simplified in the membrane theory, their nonlinearity still rejects straightforward solution for most of the practical problems.

To overcome the difficulties, A. E. Green, R. S. Rivlin and R. T. Shield[7] developed a general theory of small deformations superposed on a known finite deformation. A. H. Corneliussen and R. T. Shield[8] reformulated the general theory developed in Ref. 7 to specialize it for the treatment of thin elastic membranes. In the theories developed in Ref. 7 and reformulated for thin membranes in Ref. 8, it is assumed that the final state of stress and displacement is a result of superposition of perturbed state of stress and small displacement caused by the application of additional load on the known state of stress and finite displacement. They could, thus, obtain the linear stress-displacement relations, namely, the relations between the additional stresses and the additional small displacements.

J. Fernandez-Sintes and W. Nachbar[9] modified the theory developed in Ref. 8 by taking perturbation of the principal extension ratios to apply it for the rotationally symmetric problems of inflated thin membranes. The principal extension ratio is defined as the ratio of the length of a line element ds, which lies along the principal axis of the deformed body, to the length of the corresponding line element ds_0 before deformation. Perturbation of the principal extension ratios, thus implies perturbation of the principal strains. The theory of Fernandez-Sintes and Nachbar, therefore, holds for the case where the strains of the additional deformations are small regardless of the magnitude of the additional displacements, whereas Corneliussen and Shield's is valid only when the additional displacements are small. Since the axisymmetric deformation is realized only when the initial shape of the membrane is also axisymmetric, application of Fernadez-Sintes and Nachbar's theory is limited to the axisymmetric deformation of the initially axisymmetric membranes. The theory of Corneliussen and Shield holds for arbitrary pattern of small displacement of the additional deformation. One must, however, specify the initial shape of the membrane in order to obtain the explicit formulae of the stress-displacement relations.

In the present paper, the linear constituve equations, namely, the linear relations between the additional stress resultants and the additional strains, are obtained in the similar manner of Fernandez-Sintes and Nachbar by taking perturbation of the principal extension ratios. The deformation considered in the present paper is not, however, restricted to the axisymmetric one but to a class of deformation defined such that the lines of cnrvature of the initial, undeformed membrane remain as lines of curvature in any state of deformation. The class of deformation thus defined includes the axisymmetric deformation as a special case. Furthermore, the theory developed in the present paper doesn't require any specification of the initial shape of the membrane.

In order to make comparison of the present result with that of Corneliussen and Shield, derivation is first made on tensor components. It turns out that the both results are identical in form, if the linear strain-displacement relations for circular cylindrical shells hold for the additional small displacement. The constitutive equations are then written in terms of physical components. They turn out to be identical in form with those obtained by Fernandez-Sintes and Nachbar for axisymmetric deformations.

2. BASIC EQUATIONS

Let us consider a continuous three-dimentional body in the three-dimensional Euclidean space. The body is assumed elastic, homogeneous and, in its undeformed state, isotopic.

A typical point of the body before deformation can be described by a rectangular Cartesion coordinate system (x_1, x_2, x_3) fixed in the space. It may also be specified by a general curvilinear coordinate system (θ^1, θ^2, θ^3) fixed in the body, so that

$$x_i = x_i(\theta^1, \theta^2, \theta^3) \tag{1}$$

Throughout the present paper, the convention that Latin indices take the integer values 1, 2 and 3 and that the repeated indices indicate summation will be used, unless otherwise specified.

If the position vector to a typical point is denoted by $\boldsymbol{r}$, the covariant base vector $\boldsymbol{g}_i$ associated with the coordinate θ^i in the undeformed body is

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{r}}{\partial \theta^i} \tag{2}$$

The covariant components of the metric tensor associated with the deformed body is defined by

$$g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j \tag{3}$$

Then, the line element ds_0 of the undeformed body is given by

$$(ds_0)^2 = dx_k dx_k = g_{ij} d\theta^i d\theta^j \tag{4}$$

The contravariant components of the metric tensor are given by

$$g^{ij} = c^{ij}/g \tag{5}$$

where c^{ij} is the cofactor of g_{ij} in the determinant g.

Let the coordinates of a typical point of the body after deformation be (y_1, y_2, y_3) in the rectangular Cartesian coordinate system fixed in the space. Since the curvilinear coordinates (θ^1, θ^2, θ^3) are fixed in the body and moves with it as it is deformed, the point can also be described by (θ^1, θ^2, θ^3) so that

$$y_k = y_k(\theta^1, \theta^2, \theta^3) \tag{6}$$

The position vector to a typical point of the deformed body is denoted by $\boldsymbol{R}$. The covariant base vector $\boldsymbol{G}_i$ associated with the coordinate θ^i is now

$$\boldsymbol{G}_i = \frac{\partial \boldsymbol{R}}{\partial \theta^i} \tag{7}$$

The line element ds of the deformed body is given by

$$(ds)^2 = dy_k dy_k = G_{ij} d\theta^i d\theta^j \tag{8}$$

where G_{ij} is the covariant component of the metric tensor defined by

$$G_{ij} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j \tag{9}$$

The contravariant components of the metric tensor are given by

$$G^{ij} = C^{ij}/G \tag{10}$$

where C^{ij} is the cofactor of G_{ij} in the determinant G.

When an elastic body is homogeneous and isotropic, a strain energy function W exists, which depends on three strain invariants I_1, I_2 and I_3, so that

$$W = W(I_1, I_2, I_3) \tag{11}$$

where the strain invariants are given in the form

$$\left.\begin{aligned} I_1 &= g^{ij} G_{ij} \\ I_2 &= G^{ij} g_{ij} I_3 \\ I_3 &= G/g \end{aligned}\right\} \tag{12}$$

The contravariant components of the stress tensor are given in the form

$$\tau^{ij} = \Phi g^{ij} + \Psi D^{ij} + P G^{ij} \tag{13}$$

where

$$\left.\begin{aligned} \Phi &= \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_1} \\ \Psi &= \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_2} \\ P &= \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_3} \\ D^{ij} &= I_1 g^{ij} - g^{ir} g^{js} G_{rs} \end{aligned}\right\} \tag{14}$$

A membrane theory is formulated by simplifying the general theory for three-dimensional elastic bodies on the basis of the assumption that the transverse tress resultants and couples are small in comparison with the stress resultants acting in the tangent plane to the deformed midsurface of the membrane. Let the general curvilinear coordinates θ^1 and θ^2 be defined in the midsurface and the θ^3 coordinate be chosen such that it measures the normal distance from the tangent plane of the midsurface of the membrane. Then the covariant base vector $\boldsymbol{a}_\alpha$ which is tangent to the midsurface of the undeformed membrane is defined by

$$\boldsymbol{a}_\alpha = \frac{\partial \boldsymbol{r}_0}{\partial \theta^\alpha} \tag{15}$$

where $\boldsymbol{r}_0$ is the position vector to a typical point on the midsurface. Here, and in what follows, Greek indices take integer values 1 and 2.

A unit vector $\boldsymbol{a}_3$ is defined such that it is normal to the tangent plane containing $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, namely,

$$\boldsymbol{a}_3=\frac{\boldsymbol{a}_1\times\boldsymbol{a}_2}{|\boldsymbol{a}_1\times\boldsymbol{a}_2|} \tag{16}$$

The position vector $\boldsymbol{r}$ to an arbitrary point in the membrane is written in the form

$$\boldsymbol{r}=\boldsymbol{r}_0+\theta^3\boldsymbol{a}_3 \tag{17}$$

The covariant components of the metric tensor g_{ij} are now calculated. The result is

$$\left.\begin{aligned} &g_{\alpha\beta}=a_{\alpha\beta}+O(\theta^3)\\ &g_{\alpha3}=0, \qquad g_{33}=1 \end{aligned}\right\} \tag{18}$$

where $a_{\alpha\beta}$ is the covariant components of the surface metric tensor associated with the midsurface of the undeformed membrane, and the second term in the right-hand members of the first equation indicates the term of order of magnitude θ^3. The contravariant components of the surface metric tensor are given by

$$a^{\alpha\beta}=c^{\alpha\beta}/a \tag{19}$$

where a is the determinant of the elements of $a_{\alpha\beta}$ and $c^{\alpha\beta}$ is the cofactor in a.

If terms of order of magnitude θ^3 are neglected, the following relations are obtained:

$$\left.\begin{aligned} &g_{\alpha\beta}=a_{\alpha\beta}, \qquad g_{\alpha3}=0, \qquad g_{33}=1\\ &g^{\alpha\beta}=a^{\alpha\beta}, \qquad g^{\alpha3}=0, \qquad g^{33}=1\\ &g=a \end{aligned}\right\} \tag{20}$$

Since the coordinate θ^3 has been so chosen that it measures the normal distance from the midsurface in the undeformed state, and that it is not fixed in the body, the normal distance from the midsurface in the deformed state may be described by $\lambda_3\theta^3$ provided that λ_3 is the principal extension ratio in the direction of θ^3. The position vector $\boldsymbol{R}$ to a typical point in the deformed membrane may now be written in the form

$$\boldsymbol{R}=\boldsymbol{R}_0+\lambda_3\theta^3\boldsymbol{a}_3 \tag{21}$$

where $\boldsymbol{R}_0$ is the position vector to an arbitrary point on the midsurface of the deformed membrane.

Proceeding as before, one obtains the following relations:

$$\left.\begin{aligned} &G_{\alpha\beta}=A_{\alpha\beta}, \qquad G_{\alpha3}=0, \qquad G_{33}=\lambda_3^2\\ &G^{\alpha\beta}=A^{\alpha\beta}, \qquad G^{\alpha3}=0, \qquad G^{33}=1/\lambda_3^2\\ &G=\lambda_3^2A \end{aligned}\right\} \tag{22}$$

where $A_{\alpha\beta}$ and $A^{\alpha\beta}$ are, respectively, the covariant and contravariant components of the surface tensor associated with the midsurface of the deformed membrane, and A is the determinant of the elements of $A_{\alpha\beta}$. The superscript upon λ_3 indicates power.

The components of the stress resultant are now obtained in the form

$$n^{\alpha\beta}=h_0\lambda_3(\Phi a^{\alpha\beta}+\Psi D^{\alpha\beta}+PA^{\alpha\beta}) \tag{23}$$

where h_0 is the thickness of the undeformed membrane.

The transverse stresses are zero due to the approximation made in formulating the membrane theory. It can be shown with the aid of Eqs. (20) and (22) that $\tau^{\alpha3}=0$ are identically satisfied, whereas $\tau^{33}=0$ leads to the following equation:

$$\Phi+\Psi D^{33}+P/\lambda_3^2=0 \tag{24}$$

Quantity P can now be eliminated from Eq. (23) with the aid of Eq. (24). The result may be written in the form

$$n^{\alpha\beta}=h_0\lambda_3(\bar{a}^{\alpha\beta}\Phi+\bar{D}^{\alpha\beta}\Psi) \tag{25}$$

where

$$\left.\begin{aligned} &\bar{a}^{\alpha\beta}=a^{\alpha\beta}-\lambda_3^2A^{\alpha\beta}\\ &\bar{D}^{\alpha\beta}=D^{\alpha\beta}+\lambda_3^2A^{\alpha\beta}D^{33} \end{aligned}\right\} \tag{26}$$

The coordinates θ^1 and θ^2 have thus far been left arbitrary except they are material coordinates defined on the midsurface of the membrane. They are now specified such that they coincide with the lines of curvature of the undeformed membrane. It is, then, a well-known result of differential geometry of surface in the three-dimensional Euclidean space that the components a_{12} and a^{12} are zero, namely,

$$a_{12}=a^{12}=0 \tag{27}$$

The line element ds_0 in the midsurface of the undeformed membrane is given by

$$(ds_0)^2=(d\theta^1)^2+(d\theta^2)^2 \tag{28}$$

The components of the metric tensor are now obtained in the form

$$\left.\begin{aligned} &a_{11}=a_{22}=a^{11}=a^{22}=1, \quad a_{12}=a^{12}=0\\ &a=1 \end{aligned}\right\} \tag{29}$$

Attention is restricted in the present analysis to a class of deformations defined such that the θ^1 and θ^2 coordinates coincide with the lines of curvature of the membrane in any state of deformation. Then, again from the result of differential geometry, the following relations hold for the metric tensor associated with the midsurface of the deformed membrane:

$$A_{12}=A^{12}=0 \tag{30}$$

These imply that the θ^1 and θ^2 coordinates coincide with the principal axes of the deformed membrane, and that the intrinsic properties are completely determined by introducing the principal extension ratios λ_1 and λ_2 associated with θ^1 and θ^2, respectively. The line element ds of the midsurface of the deformed membrane is then given by

$$(ds)^2=(\lambda_1d\theta^1)^2+(\lambda_2d\theta^2)^2 \tag{31}$$

Hence the components of the surface metric tensor associated with the deformed midsurface are obtained

$$\left.\begin{array}{lll} A_{11}=\lambda_1^2, & A_{22}=\lambda_2^2, & A_{12}=0 \\ A^{11}=1/\lambda_1^2, & A^{22}=1/\lambda_2^2, & A^{12}=0 \end{array}\right\} \quad (32)$$

where the superscripts upon λ_1 and λ_2 indicate power.

It is further assumed that the membrane is made of an incompressible material. The assumption of incompressibility of the material is equivalent to imposing the following condition:

$$I_3=1 \quad (33)$$

which may also be written in terms of the principal extension ratios

$$\lambda_1\lambda_2\lambda_3=1 \quad (34)$$

The strain invariants I_1 and I_2 are now written, with the aid of Eqs. (12) and (34), in the form

$$\left.\begin{array}{l} I_1=\lambda_1^2+\lambda_2^2+1/(\lambda_1\lambda_2)^2 \\ I_2=1/\lambda_1^2+1/\lambda_2^2+(\lambda_1\lambda_2)^2 \end{array}\right\} \quad (35)$$

More detailed derivation of the equations formulated in the present section can be found in Ref. 3 and 8.

3. LINEAR CONSTITUTIVE EQUATIONS

It is now assumed that the final state of deformation is a result of superposition of a known state of deformation, namely, a state of deformation whose equilibrium configuration can be determined either numerically or analytically without much simplifying assumptions, and the state of deformation produced by the addition of external loads to the known state of deformation. In the present section the constitutive equations, namely, the relations between the stress and the strain, for the unknown state of deformation due to the additional external loads are derived by introducing certain essential assumptions to make the resulting formulae feasible for the solution of the unknown state of deformation.

First, a known state of finite deformation is considered, which is referred to as the primary state of deformation. The quantities pertaining to the primary state are designated by the subscript p. The intrinsic properties of the primary state of deformation is, thus, completely determined by λ_{1p} and λ_{2p}. The equilibrium of the primary state is then slightly perturbed by applying the additional external load. The state of the perturbed deformation is referred to as the secondary state. The quantities pertaining to the secondary state are designated with the symbol Δ. The final state of deformation is, again, characterized by the principal extension ratios λ_1 and λ_2, which now take the form

$$\left.\begin{array}{l} \lambda_1=\lambda_{1p}+\Delta\lambda_1 \\ \lambda_2=\lambda_{2p}+\Delta\lambda_2 \end{array}\right\} \quad (36)$$

The change in the extension ratios, $\Delta\lambda_1$ and $\Delta\lambda_2$, is assumed so small that the quantities depending on λ_1 and λ_2 can be approximated as a result of superposition of these depending linearly on $\Delta\lambda_1$ and $\Delta\lambda_2$ upon those completely characterized by λ_{1p} and λ_{2p}. This assumption is equivalent to retaining only the first order terms in $\Delta\lambda_1$ and $\Delta\lambda_2$ in the Taylor expansion of the functions depending on λ_1 and λ_2. Accordingly, the stress reseltants now assume the form

$$n^{\alpha\beta}=n_p^{\alpha\beta}+\Delta n^{\alpha\beta} \quad (37)$$

where $\Delta n^{\alpha\beta}$ is the part of $n^{\alpha\beta}$ pertaining to the secondary state and linearly depending on $\Delta\lambda_1$ and and $\Delta\lambda_2$.

The principal strain components of the secondary state $\Delta\varepsilon^{11}$, $\Delta\varepsilon^{22}$ and $\Delta\varepsilon^{33}$ may be defined by

$$\Delta\varepsilon^{ii}=\Delta\lambda_i/\lambda_{ip} \quad (i \text{ not summed}) \quad (38)$$

It can then be shown with the aid of Eq. (34) that the following relation holds:

$$\Delta\varepsilon^{33}=-(\Delta\varepsilon^{11}+\Delta\varepsilon^{22}) \quad (39)$$

The Taylor expansion of the right-hand members of Eq. (25) about the primary state and retension of only the first order terms in $\Delta\lambda_1$ and $\Delta\lambda_2$ in the expansion yield the following linear relations between the stress resultants and the principal strain components of the secondary state:

$$\left.\begin{array}{l} \Delta n^{11}=K_p^{11}\Delta\varepsilon^{11}+K_p^{12}\Delta\varepsilon^{22} \\ \Delta n^{22}=K_p^{21}\Delta\varepsilon^{11}+K_p^{22}\Delta\varepsilon^{22} \end{array}\right\} \quad (40)$$

where the coefficients K_p^{11}, K_p^{12}, K_p^{21} and K_p^{22} are completely determined if the primary state is known. A lengthy algebraic manipulation yields the following expression of these coefficients:

$$\begin{aligned} K_p^{11}=&2h_0(\lambda_{1p}\lambda_{2p})^{-1}[2\lambda_1^2(1-\lambda_1^{-4}\lambda_2^{-2})^2 \\ &\times(W_{,11}+2\lambda_2^2W_{,12}+\lambda_2^4W_{,22}) \\ &-(1-5\lambda_1^{-4}\lambda_2^{-2})(W_{,1}+\lambda_2^2W_{,2})]_p \quad (41\cdot a) \end{aligned}$$

$$\begin{aligned} K_p^{12}=&2h_0(\lambda_{1p}\lambda_{2p})^{-1}\{2\lambda_2^2(1-\lambda_1^{-4}\lambda_2^{-2}) \\ &\times(1-\lambda_1^{-2}\lambda_2^{-4})[W_{,11}+(\lambda_1^2+\lambda_2^2)W_{,12} \\ &+\lambda_1^2\lambda_2^2W_{,22}]-(1-3\lambda_1^{-4}\lambda_2^{-2})W_{,1} \\ &+\lambda_2^2(1+\lambda_1^{-4}\lambda_2^{-2})W_{,2}\}_p \quad (41\cdot b) \end{aligned}$$

$$\begin{aligned} K_p^{21}=&2h_0(\lambda_{1p}\lambda_{2p})^{-1}\{2\lambda_1^2(1-\lambda_1^{-2}\lambda_2^{-4}) \\ &\times(1-\lambda_1^{-4}\lambda_2^{-2})[W_{,11}+(\lambda_1^2+\lambda_2^2)W_{,12} \\ &+\lambda_1^2\lambda_2^2W_{,22}]-(1-3\lambda_1^{-2}\lambda_2^{-4})W_{,1} \\ &+\lambda_1^2(1+\lambda_1^{-2}\lambda_2^{-4})W_{,2}\}_p \quad (41\cdot c) \end{aligned}$$

$$\begin{aligned} K_p^{22}=&2h_0(\lambda_{1p}\lambda_{2p})^{-1}[2\lambda_2^2(1-\lambda_1^{-2}\lambda_2^{-4})^2 \\ &\times(W_{,11}+2\lambda_1^2W_{,12}+\lambda_1^4W_{,22}) \\ &-(1-5\lambda_1^{-2}\lambda_2^{-4})(W_{,1}+\lambda_1^2W_{,2})]_p \quad (41\cdot d) \end{aligned}$$

where

$$\left.\begin{array}{l} W_{,\alpha\beta}=\dfrac{\partial^2 W}{\partial I_\alpha \partial I_\beta} \\ W_{,\alpha}=\dfrac{\partial W}{\partial I_\alpha} \end{array}\right\} \quad (42)$$

It can easily be proved that the linear equations (40), together with Eqs. (42), are identical to the linear constitutive equations obtained by Corneliussen and Shield for circular cylindrical membranes, if the displacements are assumed so small that the linear strain-displacement relations for a circular cylindrical shell of radius R_p hold, namely, if the following relations hold:

$$\left.\begin{aligned}\Delta\varepsilon^{11}&=\frac{\partial(\Delta v)}{\partial\theta^1}+\frac{\Delta w}{R_p}\\ \Delta\varepsilon^{22}&=\frac{\partial(\Delta u)}{\partial\theta^2}\end{aligned}\right\}\qquad(43)$$

where the θ^1 and θ^2 coordinates are chosen such that they measure the length along the circumference and longitude, respectively, and Δu, Δv, and Δw are the components of the displacement in the secondary state in the direction of $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$, respectively.

Only the covariant components of stress tensors have so far been dealt with. The corresponding physical components of stress resultant, which are designated by *, are readily obtained by the formulas

$$n^{\alpha\alpha*}=n^{\alpha\alpha}\sqrt{\frac{A_{\alpha\alpha}}{A^{\alpha\alpha}}}\quad(\alpha\text{ not summed})\quad(44)$$

or,

$$n^{11*}=\lambda_1{}^2n^{11},\qquad n^{22*}=\lambda_2{}^2n^{22}\qquad(45)$$

The stress-strain relations (40) are now written in terms of the physical components

$$\left.\begin{aligned}\Delta n^{11*}&=K_p{}^{11*}\Delta\varepsilon^{11}+K_p{}^{12*}\Delta\varepsilon^{22}\\ \Delta n^{22*}&=K_p{}^{21*}\Delta\varepsilon^{11}+K_p{}^{22*}\Delta\varepsilon^{22}\end{aligned}\right\}\qquad(46)$$

where

$$\left.\begin{aligned}K_p{}^{11*}&=(K_p{}^{11}+2n_p{}^{11})\lambda_{1p}{}^2\\ K_p{}^{12*}&=\lambda_{1p}{}^2K_p{}^{12}\\ K_p{}^{21*}&=\lambda_{2p}{}^2K_p{}^{21}\\ K_p{}^{22*}&=\lambda_{2p}{}^2(K_p{}^{22}+2n_p{}^{22})\end{aligned}\right\}\qquad(47)$$

A simple algebraic manipulation yields

$$\begin{aligned}K_p{}^{11*}=2h_0(\lambda_{1p}\lambda_{2p})^{-1}\{&(\lambda_1{}^2+3\lambda_1{}^{-2}\lambda_2{}^{-2})\\ &\times(W_{,1}+\lambda_2{}^2W_{,2})+2(\lambda_1{}^2-\lambda_1{}^{-2}\lambda_2{}^{-2})^2\\ &\times(W_{,11}+2\lambda_2{}^2W_{,12}+\lambda_2{}^4W_{,22})\}_p\quad(48\cdot a)\end{aligned}$$

$$\begin{aligned}K_p{}^{12*}=2h_0(\lambda_{1p}\lambda_{2p})^{-1}\{&(3\lambda_1{}^{-2}\lambda_2{}^{-2}-\lambda_1{}^2)W_{,1}\\ &+(\lambda_1{}^2+\lambda_1{}^{-2}\lambda_2{}^{-2})\lambda_2{}^2W_{,2}\\ &+2(\lambda_1{}^2-\lambda_1{}^{-2}\lambda_2{}^{-2})(\lambda_2{}^2-\lambda_1{}^{-2}\lambda_2{}^{-2})[W_{,11}\\ &+(\lambda_1{}^2+\lambda_2{}^2)W_{,12}+\lambda_1{}^2\lambda_2{}^2W_{,22}]\}_p\quad(48\cdot b)\end{aligned}$$

$$\begin{aligned}K_p{}^{21*}=2h_0(\lambda_{1p}\lambda_{2p})^{-1}\{&(3\lambda_1{}^{-2}\lambda_2{}^{-2}-\lambda_2{}^2)W_{,1}\\ &+(\lambda_1{}^2+\lambda_1{}^{-2}\lambda_2{}^{-2})\lambda_1{}^2W_{,2}\\ &+2(\lambda_1{}^2-\lambda_1{}^{-2}\lambda_2{}^{-2})(\lambda_2{}^2-\lambda_1{}^{-2}\lambda_2{}^{-2})[W_{,11}\\ &+(\lambda_1{}^2+\lambda_2{}^2)W_{,12}+\lambda_1{}^2\lambda_2{}^2W_{,22}]\}_p\quad(48\cdot c)\end{aligned}$$

$$\begin{aligned}K_p{}^{22*}=2h_0(\lambda_{1p}\lambda_{2p})^{-1}\{&(\lambda_2{}^2+3\lambda_1{}^{-2}\lambda_2{}^{-2})\\ &\times(W_{,1}+\lambda_1{}^2W_{,2})+2(\lambda_2{}^2+\lambda_1{}^{-2}\lambda_2{}^{-2})^2\\ &\times(W_{,11}+2\lambda_1{}^2W_{,12}+\lambda_1{}^4W_{,22})\}_p\quad(48\cdot d)\end{aligned}$$

It can easily be shown that the expression of $K_p{}^{\alpha\beta*}$ given in Eqs. (48) are identical in form to those obtained by Fernandez-Sintes and Nachbar for axisymmetric deformations.

It should be noted that the equalities

$$K_p{}^{11*}=K_p{}^{22*},\qquad K_p{}^{12*}=K_p{}^{21*}$$

do not hold in general, and that the initially isotropic membrane behaves nonisotropic in the secondary state of deformation.

4. CONCLUSION

A system of linear constitutive equations was derived in an explicit form based on the assumptions that the strains of the secondary state deformation are small and that the deformations are such that the lines of curvature of the initial, undeformed membrane remain lines of curvature in any state of deformation. It turned out to be identical in form with the results obtained by Fernandez-Sintes and Nachbar for axisymmetric deformations and by Corneliussen and Shield for small additional displacements of a circular cylindrical membrane. It should be emphasized that the result of the present analysis is valid for membranes of any initial configuration provided that the deformations are restricted to the class defined above. Since no assumption regarding the magnitude of the displacement in the secondary state has been made in the entire process of derivation of tne constitutive equations, they are valid even for finite deformations of the secondary state provided that the strans are small. Therefore, the strain-displacement relations are not necessarily linear, although the stress-strain relations are linear, which is a consequence of the assumption of small-strains. This distinguishes the present theory and Fernandez-Sintes and Nachbar's theory from that formulated by Corneliussen and Shield in which the secondary state displacement is assumed small and, consequently, the strain-displacement relations are always linear.

Deformations of many practically important structures are included in the class of deformations discussed in the present paper. Take an inflated circular cylindrical membrane for example. As the axisymmetric deformation of it due to axisymmetric line load, which has been investigated by Fernandez-Sintes and Nachbar and by N. J. Hoff and Nachbar[10], belongs to the class, so does the uniform deformation due to uniformly appled line load along a longitude. Thus, the important engineering problem of reinforcement of the inflated circular tube by orthogonal netting may be handled. Furthermore, a bending deformation of it also belongs to the present class, if the inflated circular cylindrical membrane can be assumed to behave like a solid beam so that the Bernoulli-Navier hypothosis for bending of beams holds. Bending of an inflated circular cylindrical membrane will be investigated in the forthcoming report.

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