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A Second-Order Accurate Procedure for Solving the Boundary Layer Equations Based on the Predictor-Corrector Form of the Crank-Nicolson Scheme

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A Second-Order Accurate Procedure for Solving the Boundary Layer Equations Based on the Predictor-Corrector Form of the Crank-Nicolson Scheme*

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ABSTRACT

This paper is concerned with the numerical accuracy of a predictor-corrector form of the Crank-Nicolson Scheme for solving boundary layer equations.

According to Blottner, the Crank-Nicolson Scheme, with a predictor-corrector step to deal with nonlinearity, exhibits only first-order accuracy unless the boundary layer continuity and momentum equations are solved in a coupled manner. In the present paper, a predictor-corrector form of the Crank-Nicolson Scheme, based on a coupled solution for the continuity and momentum equations, is presented for both the incompressible and compressible flows. The present scheme is then subjected to a computer experiment using the problem of the laminar boundary layer development in a linearly retarded edge velocity field. The results are then compared with the Davis Coupled Scheme. It is shown that the present scheme posesses second-order accuracy and is more efficient than the Davis Coupled Scheme.

概　要

本論文では，非圧縮性及び圧縮性層流境界層方程式に対する差分解法として予測子・修正子型クランク・ニコルソンスキームに基づく差分スキームを提案し，それが，実質的に二次の数値精度を持ち，かつ，既存の境界層方程式差分解法の中で最も高能率といわれるDavis Coupled Schemeと比較しても遜色なく高能率であることを示す。

Blottner によると，標準のクランク・ニコルソンスキームを境界層方程式に適用した場合，連続・運動量両式は，非連形態(uncoupled manner)で取扱われるので，非線型項に対する線型化・反復処理過程に於ける収束率が悪く，もし反復回数が少ないまま用いられると，例えば，反復回数が唯一回の場合，たとえ差分式上は二次精度差分スキームであっても実質の数値精度は，一次精度分スキームのそれと同じ振舞いをすることが指摘されている。Blottner

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は，更に推論して，クランク・ニコルソンスキームの一変形であるDouglasとJonesによって提案された予測子・修正子型クランクニコルソンスキーム（ＰＣ－ＣＮＳと略す）についても言及し，このスキームは，一格子点の計算が予測子，修正子の二段階で完結し，これはアルゴリズム上は唯一回しか反復処理しない標準のクランク・ニコルソンスキームと対比されるので，もし連続・運動量の両式を非連成態で取扱った場合，数値精度は実質的に一次になると述べている。ＰＣ－ＣＮＳは，広範囲の物理現象に成功裡に適用されているクランク・ニコルソンスキームの一種であること，及び非線型方程式に対しても反復処理過程が不要なので高能率性が期待されること等を考慮すると，境界層方程式の差分解法として，実質的に二次精度をもつスキームとして構成することが望ましいが未だ見当らない。

本論文の主旨は，境界層方程式に対する実質的に二次精度を有する予測子・修正子型クランク・ニコルソンスキームを構成し，それが二次精度及び高能率性をもつことを数値実験により示すことである。本論文では，まずＰＣ－ＣＮＳは，境界層連続・運動量両式を非連成態で取扱った場合，実際に一次精度スキームに退化することを示しBlottner の推論を実証する。次に非圧縮性及び圧縮性境界層方程式に対する二次精度のＰＣ－ＣＮＳを構成し提示する。この場合連続・運動量両式は連成態で取扱われるが，それは流れ関数及び境界層座標変換を通じて直接行なわれる。 本論文で提案された差分スキームは， 境界層外縁の速度が線形減速する流れ場を例題として数値実験にかけられ，かつ，計算能率についても他の差分スキームと比較される。その結果，本差分スキームは，実質数値精度が二次であることが確認されるとともに，現在最も高能率差分法のひとつといわれるDavis Coupled Scheme と比較しても，遜色ない高能率性をもつことが確かめられる。

1. Introduction

With the emergence of high-speed digital computers, numerical solution of partial differential equations governing boundary layer flows has become quite commonplace. A number of schemes have been proposed. Implicit finite difference schemes of second-order accuracy are now widely used from several points of view such as the accuracy, the efficiency and the simplicity. Among them, the Crank-Nicolson Scheme (hereafter designated by the abbreviation "CNS") has been successfully applied to various types of problem: transient heat conductions, chemically reacting flows, boundary layer flows and so on. Since the boundary layer equations in which we are interested are the system of nonlinear partial differential equations, the corresponding Crank-Nicolson finite difference equations are also nonlinear and an iteration procedure is necessary at each step along the surface. It has been demonstrated by Blottner[1] that the only one iteration for solving the nonlinear algebraic equations gives first-order accuracy in the flow direction and a number of iterations are necessary to achieve second-order accuracy when the boundary layer continuity and momentum equations are handled in an uncoupled manner. According to Blottner this fact is due to the convergency of the iteration procedure and the strong coupling between the boundary layer continuity and momentum equations. This also holds true for the predictor-corrector method proposed by Douglas and Jones[2] which is a modification of the CNS. Blottner also states that the predictor-corrector form of the CNS with the continuity and momentum equations handled in an uncoupled manner is not appropriate for solving the boundary layer equations if second-order accuracy is desired. However it seems that the predictor-corrector form of the Crank-Nicolson Scheme (PC-CNS) is a form of the simplest and most efficient schemes since it is non-iterative. It is desirable to devise a second-order accurate procedure based on this PC-CNS.

The purpose of this paper is to present a second-order accurate scheme based on the PC-CNS for solving both the incompressible and compressible boundary layer equations. The present scheme is a coupled one in which the coupling of the boundary layer continuity and momentum equations is made by introducing a stream function together with the appropriate transformation.

In section 2 is given a general description of the PC-CNS for a scalar parabolic equation. In section 3 is given an example that the PC-CNS degenerates to a first-order accurate scheme when the boundary layer continuity and momentum equations are handled in an uncoupled manner. The second-order accurate procedure based on the PC-CNS are given for incompressible flows in section 4 and for compressible flows in section 5. The accuracy of the present scheme is tested for the linearly retarded edge velocity flow, of which the results are given in both sections 4 and 5.

2. General description of the predictor-corrector forms of the Crank-Nicolson Scheme (PC-CNS).

For a scalar parabolic equation:

$$\frac{\partial^2 u}{\partial y^2} = F\left(x, y, u, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}\right),$$

$$0 \le y \le 1, \; 0 \le x \tag{1}$$

$u(0, y)$, $u(x, 0)$, $u(x, 1)$ specified,

$$0 \le y \le 1, \; 0 \le x,$$

the standard Crank-Nicolson finite difference equation is

$$\frac{1}{2}(\Delta_y^2 u_{i+1,j} + \Delta_y^2 u_{i,j}) = F\Big(x_{i+1/2}, y_j, \frac{1}{2}(u_{i+1,j} + u_{i,j}), \frac{1}{2}(\delta_y u_{i+1,j} + \delta_y u_{i,j}), \frac{u_{i+1,j} - u_{i,j}}{\Delta x}\Big), \tag{2}$$

where

$$x_i = i\Delta x, \; y_j = j\Delta y, \; u_{i,j} = u(x_i, y_j),$$
$$0 < \Delta x, \Delta y$$
$$\Delta_y^2 u_{i,j} = (u_{i,j+1} - 2u_{i,j} + u_{i,j-1})/\Delta y^2$$
$$\delta_y u_{i,j} = (u_{i,j+1} - u_{i,j-1})/(2\Delta y). \tag{3}$$

Provided the Eq. (1) is nonlinear, the algebraic equation (2) is also nonlinear.

The predictor-corrector forms of the CNS proposed by Douglas and Jones[2] are modifications of the CNS and written as follows:

Predictor

$$\Delta_y^2 u^*_{i+1/2,j} = F\,(x_{i+1/2},\, y_j,\, u_{i,j},\, \delta_y u_{i,j},\, \frac{u^*_{i+1/2,j} - u_{i,j}}{\Delta x/2}), \tag{4}$$

Corrector

$$\frac{1}{2}(\Delta_y^2 u_{i+1,j} + \Delta_y^2 u_{i,j}) = F\,(x_{i+1/2},\, y_j,\, u^*_{i+1/2,j},\, \delta_y u^*_{i+1/2,j},\, \frac{u_{i+1,j} - u_{i,j}}{\Delta x}) \tag{5}$$

or

$$\frac{1}{2}(\Delta_y^2 u_{i+1,j} + \Delta_y^2 u_{i,j}) = F\,(x_{i+1/2},\, y_j,\, u^*_{i+1/2,j},\, \frac{1}{2}(\delta_y u_{i+1,j} + \delta_y u_{i,j}),\, \frac{u_{i+1,j} - u_{i,j}}{\Delta x}). \tag{6}$$

Clearly, the predictor-corrector forms of the Crank-Nicolson equations [(4), (5)] or [(4), (6)] combined with initial and boundary data

$$u_{0,j} = u\,(0, y_j),\; u_{i,0} = u\,(x_i, 0),\; u_{i,J} = u\,(x_i, 1) \tag{7}$$

lead to linear algebraic equations if either F is linear in $\partial u/\partial x$ or F is linear in both $\partial u/\partial x$ and $\partial u/\partial y$. In both cases the unconditional stability of the CNS is retained.

3. First-order accurate formulation of the PC-CNS.

In this section, it will be demonstrated that the PC-CNS is first-order accurate in x-direction if the boundary layer continuity and momentum equations are handled in an uncoupled manner. For this purpose only the incompressible flow is considered.

The steady, incompressible, laminar boundary layer equations for two-dimensional flows are:

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8}$$

Momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2}\,. \tag{9}$$

The boundary conditions for the above equations are as follows:

$$y = 0 : u = v = 0,\; y = \delta : u = u_e. \tag{10}$$

Following Blottner[1], the Levy-Lees transformation is introduced. The new independent variables are

$$\xi\,(x) = \kappa\,(\rho\mu)_{ref}\int_0^x u_e\,dx,$$
$$\zeta\,(x, y) = \rho u_e \sqrt{\kappa/2\xi}\; y. \tag{11}$$

The boundary layer equations become the followings in the transformed plane:

Continuity

$$2\xi F_\xi + V_\zeta + F = 0, \tag{12}$$

Momentum

$$2\xi F F_\xi + V F_\zeta + \beta (F^2 - 1) = F_{\zeta\zeta}, \qquad (13)$$

where

$$\beta = 2\xi/u_e\,(du_e/d\xi),\ F = u/u_e,$$
$$V = 2\xi\,(F\zeta_x + \rho v\sqrt{\kappa/2\xi}\,)/\{\kappa\,(\rho\mu)_{ref}u_e\}. \qquad (14)$$

The boundary conditions become

$$\zeta = 0 : F = V = 0,\ \zeta = \zeta_e : F = 1. \qquad (15)$$

Applying Eqs. (4) and (6), to Eqs. (12) and (13), then the resultant finite difference equations may be written as:

Predictor
(Momentum)

$$\Delta_\zeta^2 F^*_{i+1/2,j} - V_{i,j}\delta_\zeta F_{i,j} + \beta(1 - F^2_{i,j}) = 2\xi_{i+1/2}F_{i,j}\,\frac{F^*_{i+1/2,j} - F_{i,j}}{\Delta\xi/2}, \qquad (16)$$

(Continuity)

$$2\xi_{i+1/2}\frac{1}{2}\Big(\frac{F^*_{i+1/2,j} - F_{i,j}}{\Delta\xi/2} + \frac{F^*_{i+1/2,j-1} - F_{i,j-1}}{\Delta\xi/2}\Big) + \frac{V^*_{i+1/2,j} - V^*_{i+1/2,j-1}}{\Delta\zeta} + \frac{1}{2}(F^*_{i+1/2,j} + F^*_{i+1/2,j-1}) = 0 \qquad (17)$$

Corrector
(Momentum)

$$\frac{1}{2}(\Delta_\zeta^2 F_{i+1,j} + \Delta_\zeta^2 F_{i,j}) - V^*_{i+1/2,j}\frac{1}{2}(\delta_\zeta F_{i+1,j} + \delta_\zeta F_{i,j}) + \beta(1 - F^{*2}_{i+1/2,j}) = 2\xi_{i+1/2}F^*_{i+1/2,j}\,\frac{F_{i+1,j} - F_{i,j}}{\Delta\xi}, \qquad (18)$$

(Continity)

$$2\xi_{i+1/2}\frac{1}{2}\Big(\frac{F_{i+1,j} - F_{i,j}}{\Delta\xi} + \frac{F_{i+1,j-1} - F_{i,j-1}}{\Delta\xi}\Big) + \frac{1}{2}\Big(\frac{V_{i+1,j} - V_{i+1,j-1}}{\Delta\zeta} + \frac{V_{i,j} - V_{i,j-1}}{\Delta\zeta}\Big) + \frac{1}{4}(F_{i+1,j} + F_{i+1,j-1} + F_{i,j} + F_{i,j-1}) = 0. \qquad (19)$$

At the predictor step, the $F^*_{i+1/2,j}$'s are determined using Eq. (16). Then Eq. (17) is used to solve for $V^*_{i+1/2,j}$. The similar procedure is used at the corrector step. These uncoupled formulas have the trancation error of $O(\Delta\xi^2) + O(\Delta\zeta^2)$, which is derived by expanding F and V in Taylar series and substituting them into Eqs. (18) and (19), noting $F^*_{i+1/2,j} = F_{i,j} + (\Delta\xi/2)(F_\xi)_{i,j} + O(\Delta\xi^2)$ and $V^*_{i+1/2,j} = V_{i,j} + (\Delta\xi/2)(V_\xi)_{i,j} + O(\Delta\xi^2)$.

The numerical test, however, shows that the uncoupled formulas, Eqs. (16) to (19), have the first-order behavior in the ξ-direction. The Howarth flow is used as the test problem. This flow has a linearly retarded velocity field:

$$u_e/U_\infty = 1 - x/L,\ (L = 8\ \text{(reference length)}). \qquad (20)$$

The result is shown in Fig. 1, where the error of the velocity gradient at the wall is shown as a function of the step size $\Delta\xi$ for $\Delta\zeta = 0.2$. The assumed "exact solu-

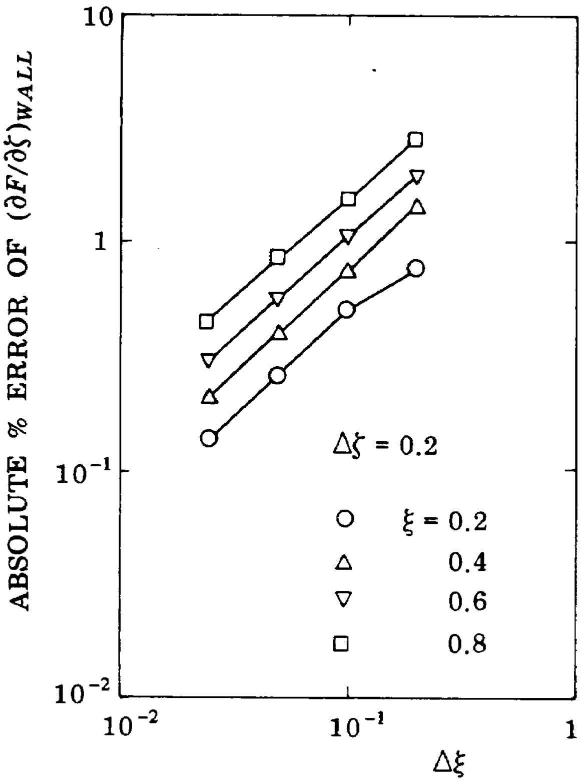

Fig. 1 Accuracy of uncoupled PC-CNS for Howarth problem

tion" which is used to evaluate the absolute % error is obtained by the Richardson extraporation:

$$F'_{\Delta\xi\to 0} = F'(\frac{1}{2}\Delta\xi) + \frac{1}{3}\{F'(\frac{1}{2}\Delta\xi) - F'(\Delta\xi)\}, \quad (21)$$

where $F' = (\partial F/\partial \zeta)_{wall}$. The velocity gradient is determined from the following second-order expression:

$$(\partial F/\partial \zeta)_{wall} = \{-(3/2)F_1 + 2F_2 - (1/2)F_3\}/\Delta\zeta \quad \text{(1: wall)}, \quad (22)$$

where a uniform grid is assumed. As can be seen in Fig. 1, for the smaller step sizes the slope is approaching a value of unity which it should have as a first-order scheme.

4. Formulation of a second-order accurate coupled PC-CNS for the incompressible flows.

In this section, the second-order accurate PC-CNS for the incompressible flows will be presented. The boundary layer continuity and momentum equations are handled in a coupled manner. We introduce the transformation due to Cebeci[3], because this transformation eliminates the continuity equation through a stream function and results in the straightforward coupling of these equations.

Defining the stream function by

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x} \quad (23)$$

and introducing the following transformation

$$x = x, \ \eta = (u_e/\nu x)^{1/2} y \quad (24)$$

together with the dimensionless stream function $f(x, \eta)$ defined by

$$\Psi = (u_e \nu x)^{1/2} f(x, \eta), \quad (25)$$

then the boundary layer equations, (8) and (9), become

$$F = f_\eta, \quad (26)$$

$$F_{\eta\eta} + P_1 f F_\eta + P_2(1 - F^2) = x(FF_x - F_\eta f_x), \quad (27)$$

where

$$F = u/u_e, P_1 = (1 + P_2)/2,$$
$$P_2 = (x/u_e)(du_e/dx). \quad (28)$$

Here F is introduced to reduce the order of the equation from three to two. The boundary conditions are

$$\eta = 0 : f = F = 0, \eta = \eta_e : F = 1. \quad (29)$$

Applying Eqs. (4) and (5) to Eqs. (26) and (27), we have

Predictor

$$\frac{1}{2}(F^*_{i+1/2,j} + F^*_{i+1/2,j-1}) = \frac{f^*_{i+1/2,j} - f^*_{i+1/2,j-1}}{\Delta\eta}, \quad (30)$$

$$\Delta^2_\eta F^*_{i+1/2,j} + P_1 f_{i,j}\delta_\eta F_{i,j} + P_2(1 - F^2_{i,j}) = x_{i+1/2}\left(F_{i,j}\frac{F^*_{i+1/2,j} - F_{i,j}}{\Delta x/2} - \delta_\eta F_{i,j}\frac{f^*_{i+1/2,j} - f_{i,j}}{\Delta x/2}\right), \quad (31)$$

Corrector

$$\frac{1}{2}(F_{i+1,j} + F_{i+1,j-1}) = \frac{f_{i+1,j} - f_{i+1,j-1}}{\Delta\eta}, \quad (32)$$

$$\frac{1}{2}(\Delta^2_\eta F_{i+1,j} + \Delta^2_\eta F_{i,j}) + P_1 f^*_{i+1/2,j}\delta_\eta F^*_{i+1/2,j} + P_2(1 - F^{*2}_{i+1/2,j}) = x_{i+1/2}\left(F^*_{i+1/2,j}\frac{F_{i+1,j} - F_{i,j}}{\Delta x} - \delta_\eta F^*_{i+1/2,j}\frac{f_{i+1,j} - f_{i,j}}{\Delta x}\right). \quad (33)$$

Here, P_1 and P_2 are evaluated at $x_{i+1/2} = x_i + (1/2)\Delta x$. The above mentioned finite difference equations are linear with respect to the unknowns $(f^*_{i+1/2,j}, F^*_{i+1/2,j})$ and $(f_{i+1,j}, F_{i+1,j})$. Moreover the tranca-tion error remains to be second-order: $O(\Delta x^2) + O(\Delta\eta^2)$.

The numerical test is also made by use of the Howarth problem as discussed in the previous section to evaluate the accuracy of the present scheme. The results for $(\partial F/\partial\eta)_{wall}$ at each x-step are given in Fig. 2. The curves have a slope of approximately 2 which shows that the scheme is second-order accurate in the x-direction. The effect of $\Delta\eta$ on the accuracy of the wall velocity gradient is shown in Fig. 3.

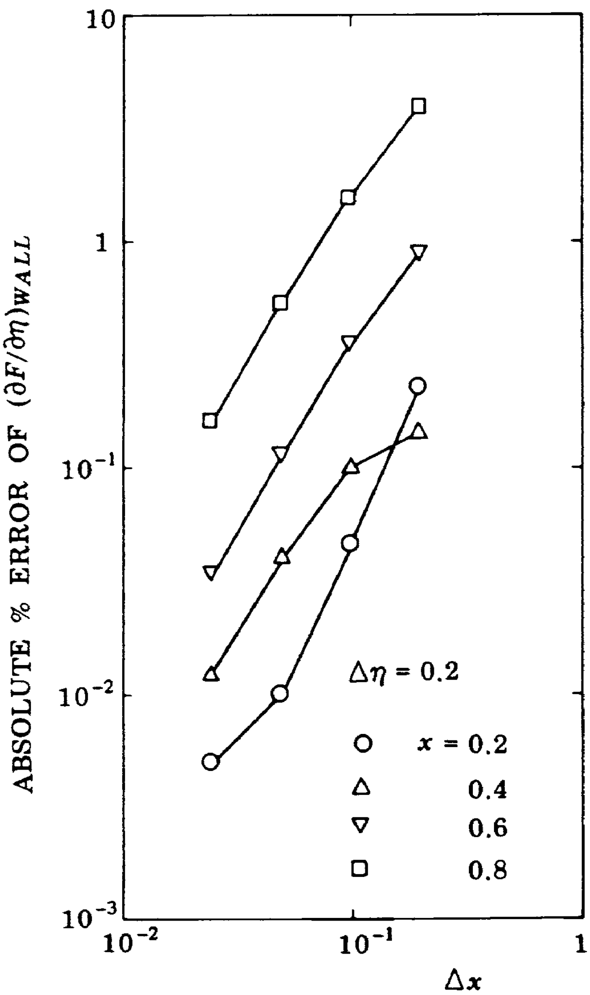

Fig. 2 Accuracy of coupled PC-CNS with various Δx for Howarth problem

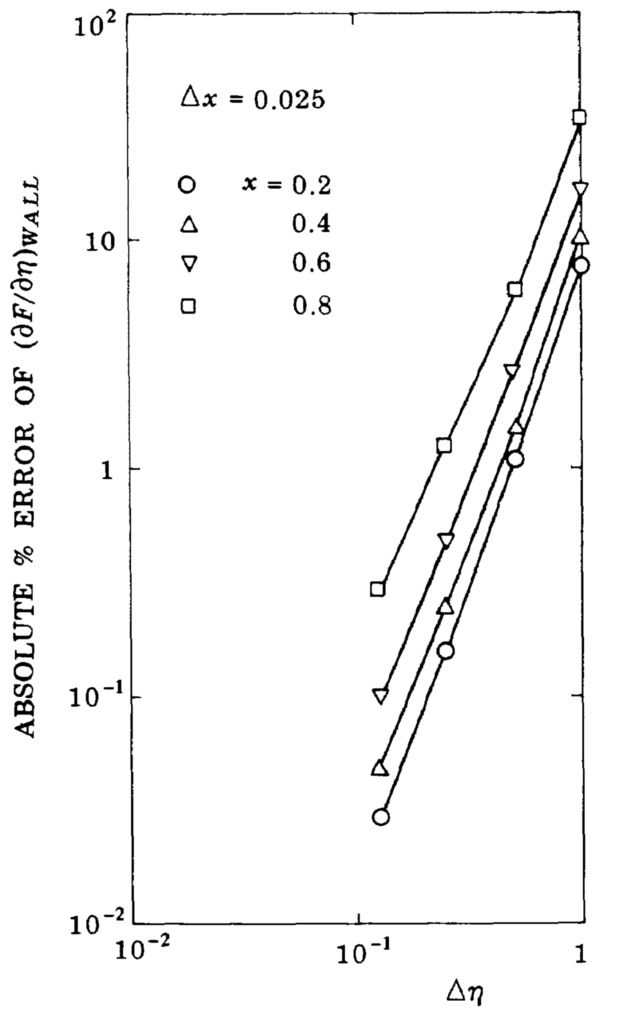

Fig. 3 Accuracy of coupled PC-CNS with various $\Delta\eta$ for Howarth problem

The results show that the scheme is second-order accurate in the η-direction too.

The present second-order scheme is a coupled one. Thus we call the scheme the coupled type Predictor-Corrector Crank-Nicolson Scheme (the coupled PC-CNS).

5. An extension of the coupled PC-CNS to the compressible flow case.

In this section, the scheme presented in the previous section is extended to the compressible boundary layers. The steady, laminar, compressible boundary layer equations for two-dimensional flows of a perfect gas are:

Continuity

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{34}$$

Momentum

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \rho_e u_e \frac{d}{dx}(u_e) + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \tag{35}$$

Energy

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left\{\frac{\mu}{P_r}\frac{\partial H}{\partial y} + \mu\left(1 - \frac{1}{P_r}\right)\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)\right\}, \tag{36}$$

Equation of state

$$p = \rho R T. \tag{37}$$

The above equations are completed with the following relations and assumptions:

$$H = c_p T + u^2/2,\ c_p = const.,$$
$$P_r = \mu c_p / k = const.,\ \mu = \mu(T). \tag{38}$$

Boundary conditions are

$$y = 0 : u = v = 0, H = H_{wall},$$
$$y = \delta : u = u_e,\ H = H_e. \tag{39}$$

We define the stream function such that

$$\rho u = \frac{\partial \Psi}{\partial y},\ \rho v = -\frac{\partial \Psi}{\partial x}, \tag{40}$$

then the continuity equation is satisfied. Introducing the transformation

$$x = x,\ \eta = (u_e/\nu_e x)^{1/2}\int_0^y \rho/\rho_e dy,$$

$$\Psi = (\nu_e u_e x)^{1/2}\rho_e f(x,\eta), \quad (41)$$

the boundary layer equations become

Momentum

$$F = f_\eta, \quad (42)$$

$$CF_{\eta\eta} + C_\eta F_\eta + m_1 f F_\eta + m_2(c - F^2) = x(FF_x - F_\eta f_x), \quad (43)$$

Energy

$$\sigma_1 E_{\eta\eta} + \sigma_{1,\eta} E_\eta + \sigma_2 E_\eta + \sigma_{3,\eta} = x(FE_x - E_\eta f_x), \quad (44)$$

where

$$F = u/u_e,\ E = H/H_e,\ c = \rho_e/\rho,\ C = \rho\mu/\rho_e\mu_e \quad (45)$$

and

$$m_1 = \{1 + m_2 + (x/\rho_e\mu_e)(d(\rho_e\mu_e)/dx)\}/2,$$
$$m_2 = (x/u_e)(du_e/dx),\ \sigma_1 = C/P_r,$$
$$\sigma_2 = m_1 f,\ \sigma_3 = C(u_e^2/H_e)(1 - 1/P_r)FF_\eta. \quad (46)$$

Boundary conditions become

$$\eta = 0 : f = F = 0, E = E_{wall},$$
$$\eta = \eta_e : F = 1, E = 1. \quad (47)$$

The coupled PC-CNS is now extended to the compressible boundary layer equations (42) to (44) with the energy equation handled in an uncoupled manner.

Predictor

$$\frac{1}{2}(F^*_{i+1/2,j} + F^*_{i+1/2,j-1}) = \frac{f^*_{i+1/2,j} - f^*_{i+1/2,j-1}}{\Delta\eta}, \quad (48)$$

$$C_{i,j}\Delta^2_\eta F^*_{i+1/2,j} + \delta_\eta C_{i,j}\delta_\eta F_{i,j} + m_1 f_{i,j}\delta_\eta F_{i,j} + m_2(c_{i,j} - F^2_{i,j}) = x_{i+1/2}\left(F_{i,j}\frac{F^*_{i+1/2,j} - F_{i,j}}{\Delta x/2} - \delta_\eta E_{i,j}\frac{f^*_{i+1/2,j} - f_{i,j}}{\Delta x/2}\right), \quad (49)$$

$$\sigma_{1,i,j}\Delta^2_\eta E^*_{i+1/2,j} + \delta_\eta\sigma_{1,i,j}\delta_\eta E_{i,j} + \sigma_{2,i,j}\delta_\eta E_{i,j} + \delta_\eta\sigma_{3,i,j} = x_{i+1/2}\left(F_{i,j}\frac{E^*_{i+1/2,j} - E_{i,j}}{\Delta x/2} - \delta_\eta E_{i,j}\frac{f^*_{i+1/2,j} - f_{i,j}}{\Delta x/2}\right), \quad (50)$$

Corrector

$$\frac{1}{2}(F_{i+1,j} + F_{i+1,j-1}) = \frac{f_{i+1,j} - f_{i+1,j-1}}{\Delta\eta}, \quad (51)$$

$$C^*_{i+1/2,j}\frac{1}{2}(\Delta^2_\eta F_{i+1,j} + \Delta^2_\eta F_{i,j}) + \delta_\eta C^*_{i+1/2,j}\delta_\eta F^*_{i+1/2,j} + m_1 f^*_{i+1/2,j}\delta_\eta F^*_{i+1/2,j} + m_2(c^*_{i+1/2,j} - F^{*2}_{i+1/2,j}) = x_{i+1/2}\left(F^*_{i+1/2,j}\frac{F_{i+1,j} - F_{i,j}}{\Delta x} - \delta_\eta F^*_{i+1/2,j}\frac{f_{i+1,j} - f_{i,j}}{\Delta x}\right), \quad (52)$$

$$\sigma^*_{1,i+1/2,j}\frac{1}{2}(\Delta^2_\eta E_{i+1,j} + \Delta^2_\eta E_{i,j}) + \delta_\eta\sigma^*_{1,i+1/2,j}\delta_\eta E^*_{i+1/2,j}$$

$$+ \sigma^*_{2,i+1/2,j}\delta_\eta E^*_{i+1/2,j} + \delta_\eta \sigma^*_{3,i+1/2,j}$$

$$= x_{i+1/2}\ (F^*_{i+1/2,j}\ \frac{E_{i+1,j} - E_{i,j}}{\Delta x}$$

$$- \delta_\eta E^*_{i+1/2,j}\ \frac{f_{i+1,j} - f_{i,j}}{\Delta x}). \qquad (53)$$

Here the coefficients m_1 and m_2 are evaluated at the midpoint $x_{i+1/2} = x_i + (1/2)\Delta x$. This system of equations is also linear and easily solved.

The numerical accuracy of the scheme is also tested. The linearly retarded edge velocity problem is solved for the case with $M_\infty = 4$ and $T_{wall} = T_\infty$. For this flow the edge velocity varies as

$$u_e/U_\infty = 1 - x/L, \qquad (54)$$

and additional edge properties and parameters are obtained from isentropic relations

$$T_e/T_\infty = 1 + (\gamma - 1)M^2_\infty/2\ [1 - (u_e/U_\infty)^2],$$

$$\rho_e/\rho_\infty = (T_e/T_\infty)^{\gamma/(\gamma-1)}, \qquad (55)$$

$\mu/\mu_\infty = T/T_\infty$ for linear viscosity low.

In this case we put $P_r = 1$, $L = 1$, $\gamma = 1.4$ and $\eta_e = 5$. The accuracy of the wall velocity gradient at $x = 0.06$ and 0.1 for various Δx with $\Delta\eta = 0.2$ is given in Fig. 4. The curves have a slope of approximately 2 which shows that the present scheme is second-order accurate. The results of the present scheme are compared with the numerical solution based on the Davis Coupled Scheme which is given by Blottner[1] in Fig. 4. The Davis Coupled Scheme is the Crank-Nicolson scheme with the continuity and momentum equations handled in a coupled manner, and was proposed by Davis. The coordinate transformation employed by Blottner is the same as in section 3, and does not agree with one employed in the present scheme. The difference between them, however, is not essential. For the present problem the error of the present scheme is nearly the same as that of the Davis Coupled Scheme.

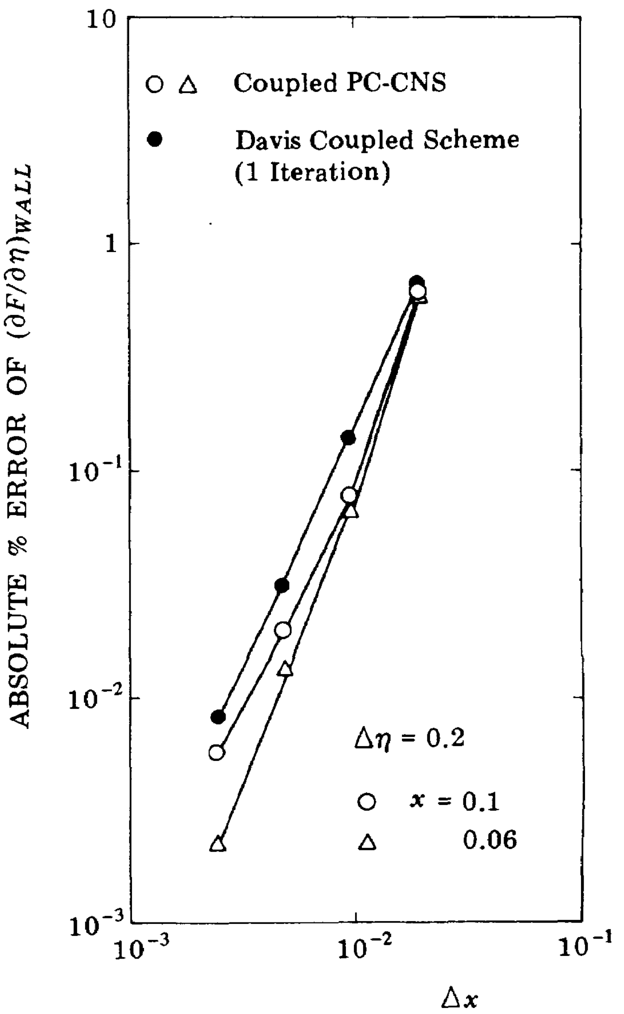

Fig. 4 Accuracy of Schemes with various Δx for linearly retarded flow at $M_\infty = 4$ and $T_{wall} = T_\infty$.

The computation time is very small because the present scheme is non-iterative. The solution with 161 × 26 grid points required 0.96 seconds of central processing time on the FACOM 230-75 computer — this gives 0.00023 sec./grid point for the computation time. On the other hand, the Davis Coupled Scheme required 0.00082 sec./grid point of central processing time on the CDC 6600 computer (370 × 88 grid points, total 26.8 seconds of CP time, Ref. 1). It is known that the FACOM 230-75 CP Unit provides nearly the same floating point computation rate as the CDC 6600 CP Unit. Hence it seems that the present scheme is more efficient than the Davis Coupled Scheme which is one of the most efficient finite difference schemes presently existing.

Concluding remarks

In this paper, it has been shown for the Howarth flow that the predictor-corrector method of Douglas and Jones is first-order accurate unless the continuity and momentum equations are handled in a coupled manner, thus giving support to Blottner's observation.

The second-order accurate scheme based on a predictor-corrector forms of the CNS has been presented for both the incompressible flows and compressible flows, where the coupling of the continuity and momentum equations is made by introducing the stream function together with the appropriate coordinate transformation. The present scheme, the coupled PC-CNS, has been shown to be second-order accurate for the linearly retarded edge velocity flows and seems to be more efficient than the Davis Coupled Scheme.

Notation

c_p	: specific heat at constant pressure
H	: total enthalpy
k	: thermal conductivity
M_∞	: freestream Mach number
P_r	: Prandtl number
p	: pressure
R	: gas constant
T	: temperature
U_∞	: freestream velocity
u, v	: velocity components tangential and normal to body surface
x, y	: coordinates along and normal to body surface
δ	: boundary layer thickness
γ	: ratio of specific heats
κ	: arbitrary constant (see Eq. (11))
μ	: viscosity
ν	: kinematic viscosity
ρ	: density
Δx, Δy, $\Delta\zeta$, $\Delta\xi$, $\Delta\eta$	: mesh width
Δ_y^2, Δ_ζ^2, Δ_η^2	: finite difference operators (see Eq. (3))
δ_y, δ_ζ, δ_η	: finite difference operators (see Eq. (3))
$(\ \ )_e$	: outer edge of the boundary layer
$(\ \ )_{i,j}$	: value at the grid point (x_i, η_j) [resp. (ξ_i, ζ_j)] where $x_i = i\Delta x$ and $\eta_j = j\Delta\eta$ [resp. $\xi_i = i\Delta\xi$ and $\zeta_j = j\Delta\zeta$].
$(\ \ )^*_{i+1/2,j}$	: value at the predictor step
$(\ \ )_{ref}$	: reference condition
$(\ \ )_\infty$	: freestream condition

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