Near-Linear Orbit Uncertainty Propagation in Low-Earth Orbit

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A generalized formulation of the equinoctial elements is applied to the orbital uncertainty propagation of an object in Low-Earth orbit. This formulation, recently published in Celestial Mechanics and Dynamical Astronomy, absorbs the effect of the dominating perturbation term (e.g. J2) in the definition of the orbital elements, introducing the total energy as opposed to the classical Kepler orbital energy. Additionally, the fast variable is chosen as a time element following the definition of Stiefel and Scheifele thereby making the orbit equations of motion more linear. In order to test how long the orbit uncertainty propagation preserves its linearity, covariance realism tests are performed comparing the new formulation with classical ones (Cartesian and classical equinoctial). Ballistic and low-thrust maneuvering satellites are considered. The results show that the proposed generalized equinoctial elements preserve linearity for several more revolutions compared to other formulations.

Key Words: Orbit uncertainty, Equinoctial elements

1. Introduction

The propagation of the state uncertainty is a key concept in stochastic dynamical systems. This uncertainty arises from modeling errors in the dynamical model and the observations used to determine the state. Typically, the state uncertainty grows over time until eventually it is not possible to accurately predict anymore.

In the astrodynamics domain, the state uncertainty is routinely propagated in applications such as orbit determination and navigation of spacecraft, and space situational awareness for monitoring of space debris and Near-Earth asteroids.

A commonly used simplification is to assume that the orbital uncertainty follows a Gaussian distribution, arising from the Central Limit Theorem which states that a sum of random variables tends to be normally distributed, even if the individual random variables are not. This simplification has several beneficial analytical properties, being the most important one that a Gaussian distribution remains Gaussian under linear transformations. Thus, if a state transition matrix is employed to propagate an initially Gaussian distribution, the resulting distribution will also be Gaussian under this assumption.

However, the laws governing the orbital motion are nonlinear. Even if the initial distribution can be reasonably considered Gaussian, its probability density function (pdf) will evolve with time and the assumption of Gaussianity will break down. There are several techniques to mitigate the error of approximating the real pdf by a Gaussian distribution.

One possibility is to feed more observations to the navigation filter, which is easy if the body whose orbit is under study is an active spacecraft. However, this is not a trivial task for space debris for several reasons. In the first place, active ranging and GPS measurements are not possible since they are passive objects. Furthermore, if the object is small, optical observations may be challenging. Instead, radar observations may be required, which strongly depend on the ground track and may be widely separated on time. In any case, the number of objects in LEO may increase exponentially owing to the increase of space activity in the next years as can be seen in Fig. 1 which was taken from Krag et al. (2012) [1)]. If such an increase occurs, it may become difficult to obtain frequent measurements for all objects orbiting our planet.

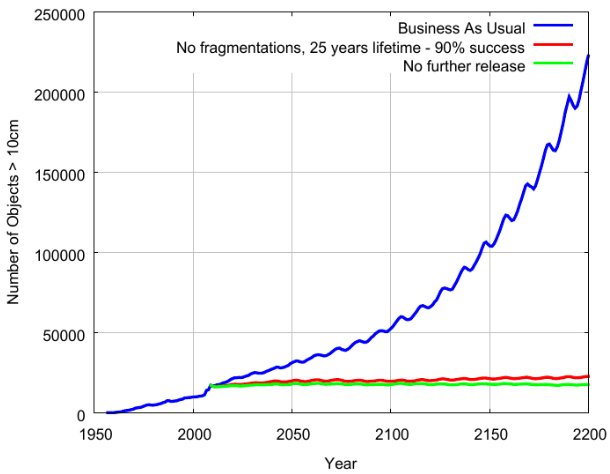

Fig. 1. Evolution of the number of objects in LEO [1)]

In parallel, one can employ advanced mathematical methods to account for the nonlinear nature of the dynamics, ranging from Monte-Carlo methods [2)] to solving the Fokker-Planck Equation which governs the time evolution of the pdf of the state [3)]. In between, a myriad of methods of different level of sophistication have been developed, like the use of unscented transform [4)], state transition tensors, [5)] differential algebra, [6)] polynomial chaos expansion, [7)] Gaussian Mixture models [8)] or Kriging, [9)] among others.

However, before applying any of these techniques, there is a powerful tool that can be exploited to reduce the nonlinear effects: using a different set of variables to represent the state vector. [10, 11, 12, 13, 14)]

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2. State Representation

Consider the equations of motion in Cartesian coordinates:

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v}, \tag{1}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\mu\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} + \boldsymbol{f}, \tag{2}$$

where $\boldsymbol{r}$ and $\boldsymbol{v}$ are the position and velocity vectors, respectively, μ is the central body gravitational constant and $\boldsymbol{f}$ is the non-Keplerian perturbation. In this problem, typically the central body acceleration is dominant, while the non-Keplerian effects are usually several orders of magnitude smaller. This system quickly becomes non-linear unless frequent measurements are included, both in the perturbed and unperturbed cases.

The question we must ask ourselves is: can we propose a set of variables that better describe the true propagated uncertainty? This is sketched in Fig. 2.

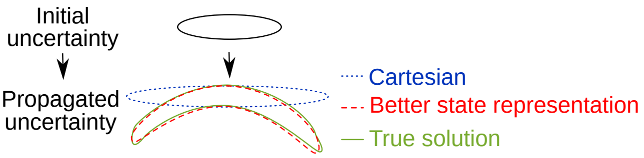

Fig. 2. Effect of the state representation in the orbit uncertainty

2.1. Conserved quantities

As Junkins et al. [10] pointed out, the use of orbital elements seem to reduce the nonlinearities of the pdf evolution.

When employing orbital elements, usually one is an angle which can be considered as a *fast variable* because its characteristic time is much shorter than the other variables. Examples of fast variables are true, eccentric and mean anomalies.

The other five quantities are usually *slow variables*, because they are conserved quantities in the Keplerian motion and, evolve slowly with the perturbations.

Then, by employing orbital elements, we have replaced six fast variables (position and velocity components) by five slow variables and a fast variable, improving the linearity of the system. This applies to any flavor of classical orbital elements, including equinoctial elements.

One example is represented by the equinoctial elements (EqOE) proposed by Broucke and Cefola [15]. The first element is the semi-major axis a. Then, P_1 and P_2 are introduced as the projections of the eccentricity vector $\boldsymbol{e}$ into two directions of an intermediate frame, which is called equinoctial frame:

$$P_1 = e\sin\bar{\omega}, \qquad P_2 = e\cos\bar{\omega}, \tag{3}$$

where $\bar{\omega}$ is the longitude of periapsis. The orientation of this reference frame is fixed by Q_1 and Q_2:

$$Q_1 = \sin\Omega\tan\frac{i}{2}, \qquad Q_2 = \cos\Omega\tan\frac{i}{2}, \tag{4}$$

where i is the inclination. Finally, the mean longitude ℓ constitutes the fast variable:

$$\ell = \bar{\omega} + M, \tag{5}$$

where M is the mean anomaly. This set of elements are non-singular for zero eccentricity or inclination, which makes them attractive for working with circular and/or equatorial orbits.

2.2. Time elements

It is possible to choose the fast variable of an orbital element set to furthermore reduce the error. One straightforward choice would be to replace the angular variable by the time of periapsis passage, which is constant in the Kepler problem. This can be generalized by using the definition of *time element* given by Stiefel and Sheifele: [16]

any quantity φ which, during a pure Kepler motion, is a linear function of the independent variable

(that is, time). The general form of the time element φ is:

$$\varphi := \alpha t + \beta. \tag{6}$$

Note that if $\alpha = 0$, the time element coincides with the aforementioned time of periapsis passage. The time derivative of the time element under Keplerian motion is constant and equal to α, which suggests that if α is also a state variable, the evolution of the uncertainty in φ will also be linear.

Using this concept, Horwood et al. proposed an alternate formulation of the equinoctial elements (AEqOE) [17]. Their relation to EqOE is shown in Fig. 3. A close inspection of Eq. (5) reveals that ℓ is actually a time element in the sense of Stiefel and Scheifele, because $M = nt$ where n is the mean motion, and $\bar{\omega}$ does not explicitly depend on time. This yields a more efficient orbit uncertainty propagation method after simply replacing the semi-major axis by the mean motion.

$$\begin{array}{rccccc}
\text{EqOE:}(a & P_1 & P_2 & Q_1 & Q_2 & \ell) \\
\Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\
\text{AEqOE:}(n & P_1 & P_2 & Q_1 & Q_2 & \ell)
\end{array}$$

Fig. 3. Relation between EqOE and AEqOE

Unfortunately, in LEO the assumption of a Gaussian distribution quickly becomes invalid due mainly to the J2 effect. This is because when considering this perturbation, which is relatively strong, the Keplerian energy is no longer a conserved quantity.

3. Generalized Equinoctial Elements

To palliate the limitations of AEqOE in real applications, we recently presented a generalized equinoctial elements formulation (GEqOE) [18].

The main concept behind GEqOE is to exploit the fact that some of the perturbing forces can be derived from a potential U. When this is the case, we can introduce the total energy ε as the sum of the Keplerian energy ε_K and the potential energy:

$$\varepsilon = \varepsilon_K + U, \tag{7}$$

and define the *generalized* angular momentum c as

$$c = \sqrt{h^2 + 2r^2 U} \tag{8}$$

where h and r are the the osculating angular momentum and orbital radius, respectively. Combining ε and c, we can construct generalized elements to describe the in-plane motion. These elements show a smoother behavior than their osculating counterparts when all the forces can be derived from a potential, and reduce to the osculating elements when the potential vanishes.

In this manner, we define the generalized mean motion ν as the first element. Then, we introduce p_1 and p_2 as the generalized version of (A)EqOE's P_1 and P_2:

$$p_1 = g \sin \Psi, \qquad p_2 = g \cos \Psi, \tag{9}$$

where g is the generalized eccentricity, and Ψ is the generalized longitude of periapsis.

The intermediate frame used to define some of the GEqOE is again the equinoctial frame, therefore we include two of the classical equinoctial elements q_1 and q_2:

$$q_1 = \sin \Omega \tan \frac{i}{2}, \qquad q_2 = \cos \Omega \tan \frac{i}{2}, \tag{10}$$

Finally, we introduce the generalized true longitude $\mathcal{L}$ as the fast variable:

$$\mathcal{L} = \mathcal{M} + \Psi \tag{11}$$

where $\mathcal{M}$ is the generalized mean anomaly. The derivation of the generalized mean anomaly and the rest of the elements is explained in detail in [18].

The relation between GEqOE and AEqOE is shown in Fig. 4.

$$\begin{array}{lcccccc} \text{AEqOE:}(n & P_1 & P_2 & Q_1 & Q_2 & \ell) \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ \text{GEqOE:}(\nu & p_1 & p_2 & q_1 & q_2 & \mathcal{L}) \end{array}$$

Fig. 4. Relation between AEqOE and GEqOE

4. Results

A propagated Gaussian distribution, represented by its mean and covariance matrix, is realistic if it represents the true distribution in the variables chosen to represent the state. In this section, we compare different state representations by linear propagation of the initial covariance. As mentioned in the introduction, once a "good" state representation is identified nonlinear methods can be used to improve the accuracy with a lower computational cost than a "bad" state representation would entail.

We linearly propagate the covariance matrix $\boldsymbol{P}$ in time using the state transition matrix obtained by numeric differentiation of the equations of motion. Subsequently, the initial covariance is sampled and the true state vector of each sample $\boldsymbol{x}_i$ is calculated using Cowell's method. The Mahalanobis distance of each sample $\mathfrak{M}_i$ is defined as the square of the sigma-level of the sample:

$$\mathfrak{M}_i = (\boldsymbol{x}_i - \boldsymbol{\mu})^\top \boldsymbol{P}^{-1} (\boldsymbol{x}_i - \boldsymbol{\mu}) \tag{12}$$

where $\boldsymbol{\mu}$ is the average state vector of the samples.

Following Aristoff et al. [19], we employ the Cramér-von Mises (CvM) test on the Mahalanobis distribution to assess the covariance realism. The test statistic Q is calculated as

$$Q = \frac{1}{12N} + \sum_{j=1}^{N} \left(\frac{2j-1}{2N} - F(\mathfrak{M}_j) \right)^2 \tag{13}$$

where the summation is performed in increasing order of the Mahalanobis distance, N is the number of samples, and $F(z)$ is the cumulative distribution function of a chi-squared distribution with 6 degrees of freedom:

$$F(z) = 1 - \frac{1}{8} \exp -\frac{z}{2} \left(z^2 + 4z + 8 \right). \tag{14}$$

The covariance is a good representation of the real distribution if Q is in a tabulated interval that depends on the number of samples; in practical terms, $Q \lessapprox 1.16$ must be satisfied.

We apply this test to a LEO scenario inspired by the one proposed by Aristoff et al. [19]. The initial state and covariance are given in Tables 1 and 2, respectively. Only the gravitational force of the Earth is considered, and the gravitational field is simplified to point mass with J_2. The initial covariance is sampled with 10000 points. We compare Cartesian coordinates, EqOE, AEqOE and GEqOE.

Table 1. Initial state vector

Element	Value
a	7163.6 km
e	0.00949
i	72.9 deg
Ω	116 deg
ω	−20 deg
M	0 deg

Table 2. Initial covariance expressed in EqOE

Element	Value
a	20 km
P_1	0.001
P_2	0.001
q_1	0.001
q_2	0.001
ℓ	0.01 deg

Figure 5 shows the CvM test statistic as a function of time for this scenario. In the gray area, the covariance is a good representation of the true pdf according to the CvM test. The Cartesian representation quickly becomes strongly non-Gaussian, as was predicted. Gaussianity is conserved with AEqOE longer than with EqOE, which can be justified by the arguments laid out in Section **2.2.**. The GEqOE formulation is able to remain Gaussian for several orbits more compared to the other sets, until eventually the nonlinear effects acting in the non-conserved quantities degrade the accuracy of the linear prediction.

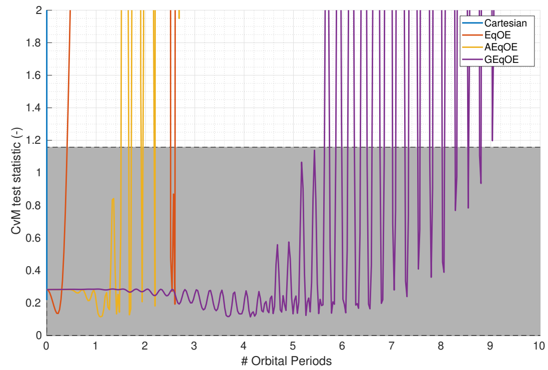

Fig. 5. Cramér-von Mises test statistic for the ballistic case

An interesting variation of this scenario is to assume that the tracked object is a small satellite of a mega-constellation and is maneuvering with a known tangential acceleration of 0.03 mN/kg. The results are shown in Fig. 6. The previous formulations (Cartesian, EqOE, AEqOE) are not strongly affected by the low-thrust because the J2 perturbation is dominant. However, since GEqOE can effectively mitigate the impact of the Earth's oblateness, the inclusion of additional non-conservative perturbations degrades the performance while still being better for longer times. This suggests that the GEqOE is likely to show a good performance even when considering a high fidelity dynamical model, since the nonlinear effects of the main perturbation are partially absorbed by the choice of the state vector.

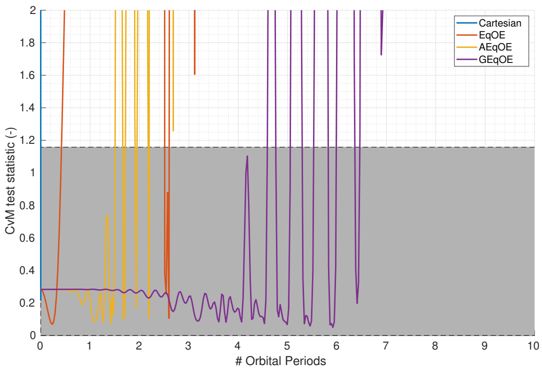

Fig. 6. Cramér-von Mises test statistic for the low-thrust case

5. Conclusions

The covariance realism of the orbital motion can be drastically improved by a careful selection of the state representation. It seems advantageous to select as state variables quantities that are conserved along the Keplerian motion, and show a slow variation in presence of perturbations. Moreover, it plays an important role the fact that the time derivative of the fast variable is a linear function of the other state variables and time.

These concepts have been exploited in the past to obtain more efficient state representations. In this work, we propose to employ an improved version of the Equinoctial Elements in which some of the forces can be derived from a potential. This formulation, which we call Generalized Equinoctial Elements, introduces a new conserved quantity in the main problem because the Keplerian energy is replaced by the total energy when considering the J2 perturbation, which is dominant in the Low Earth Orbit environment.

A Cramér-von Mises test is used to asses the performance of the proposed formulation in the LEO environment. Results show that the Generalized Equinoctial Elements can preserve covariance realism for a longer time span compared to previous formulations. Additionally, they are likely to outperform previous formulations when considering a high fidelity model because they are not limited by the dominant J2 effect.

In future works, analytical expressions for the variational equations needed to implement the numerical propagation of the state transition matrix will be derived and provided. Finally, a comparison with a recently developed formulation by Aristoff et al. (J2EqOe) will be performed.

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