Equilibria and Dynamical Structures with Quadratic Optimal Control

Tsuruta Ayano[1)], Mai Bando[1)], Daniel J. Scheeres[2)], and Shinji Hokamoto[1)]

1) Department of Aeronautics and Astronautics, Kyushu University, Japan

2) Department of Aerospace Engineering Sciences, University of Colorado Boulder, U.S.

33rd Workshop on JAXA Astrodynamics Symposium and Flight Mechanics

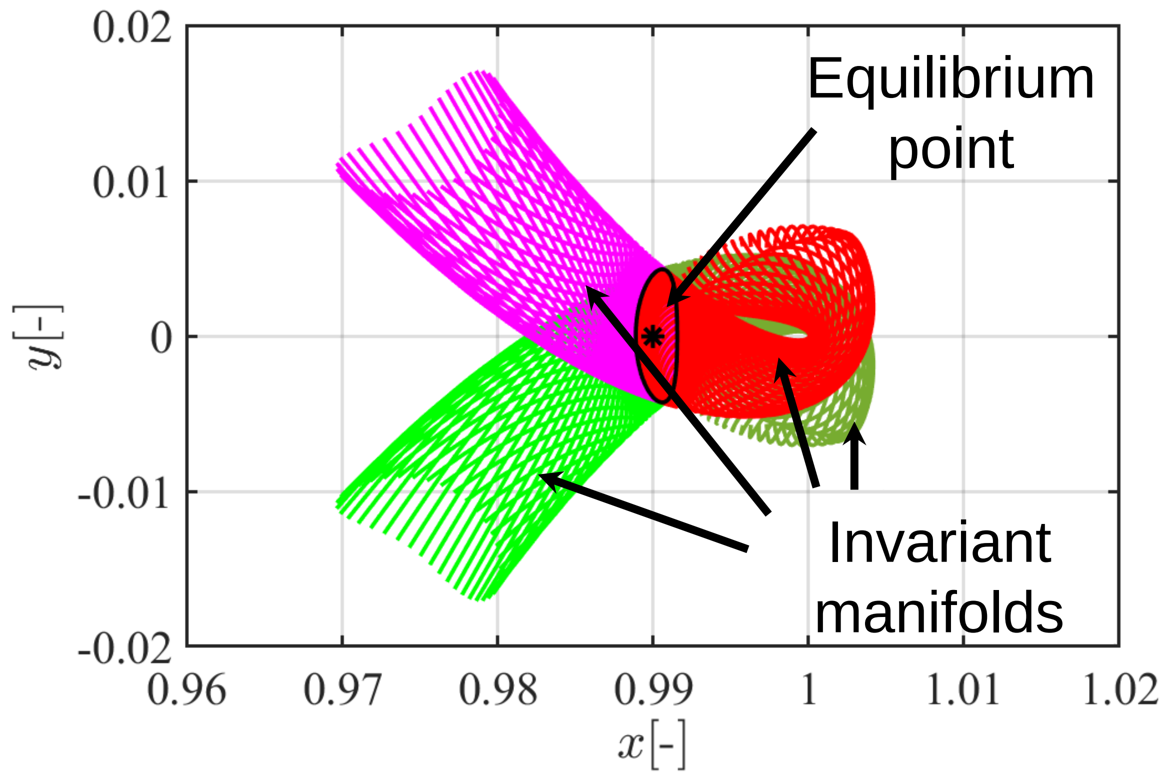
ASTRO-2023-C001(010)

Background (1/2)

Current trajectory design

Limited fuel → Equilibrium points and invariant manifolds are used

Equilibrium points · Invariant manifolds



Equilibrium points :

- The gravity and centrifugal force are balanced in the rotational coordinate system
- In CR3BP → Lagrangian points

Invariant manifolds :

- Dynamical structure around unstable equilibrium points
- Transition trajectories using invariant manifolds do not require inputs

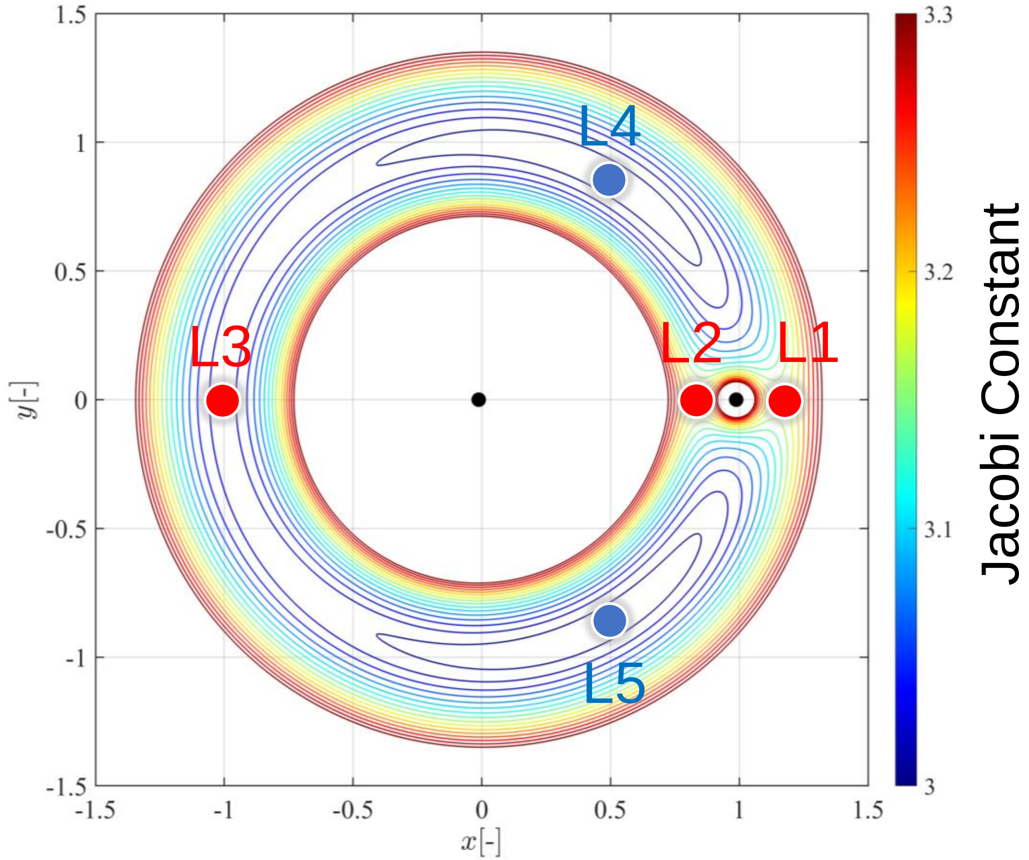
Background (2/2)

Problem

- Lagrangian points are not always in the best position for the mission.
- Invariant manifolds used as transport structures are limited.

Previous Research[1)]

Artificial equilibrium points with Low-thrust continuous inputs



Lagrangian points and zero-velocity curve of CR3BP

This research

- Artificial equilibrium points with continuous optimal control inputs
- Research including analysis of invariant manifolds around the artificial equilibrium point

1) Morimoto, M. Y., Yamakawa, H. and Uesugi, K.: Artificial equilibrium points in the low-thrust restricted three-body problem, Journal of Guidance, Control, and Dynamics, 30(5) (2007), pp. 1563-1568.

Method : Optimal control problem (1/3)

Deal with continuous optimal control inputs

Optimal Control Problem[2)] **:**

The problem of minimizing the cost function

$$J = \int_{t_0}^{t_f} L(\boldsymbol{x}(t), \boldsymbol{u}(t), t)\, dt$$

Subject to $\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{u}$

$f(\boldsymbol{x})$: natural dynamics

$\boldsymbol{u}$: control inputs

2) Ohtsuka, T.: Introduction to Nonlinear Optimal Control, Corona Publishing, Tokyo, 2011 (in Japanese).

Method : Optimal control problem (2/3)

Cost function $J = \int_{t_0}^{t_f} L(\boldsymbol{x}(t), \boldsymbol{u}(t), t)\, dt$ Subject to $\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{u}$

Hamilton function

$$H(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}) = L(\boldsymbol{x}, \boldsymbol{u}) + \boldsymbol{p}^T \boldsymbol{f}(\boldsymbol{x})$$

Euler Lagrange equation

$$\begin{cases} \dot{\boldsymbol{x}} = \left(\dfrac{\partial H}{\partial \boldsymbol{p}}\right)^T, \ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \dot{\boldsymbol{p}} = -\left(\dfrac{\partial H}{\partial \boldsymbol{x}}\right)^T \\ \dfrac{\partial H}{\partial \boldsymbol{u}} = 0 \end{cases}$$

Regard as the equations of motion of a dynamical system with optimal control input

Method : Optimal control problem (3/3)

Euler Lagrange equation

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{u}$$

(n – dim)

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{\partial H(\boldsymbol{x},\boldsymbol{p})}{\partial \boldsymbol{p}}\right)^T \\ -\left(\dfrac{\partial H(\boldsymbol{x},\boldsymbol{p})}{\partial \boldsymbol{x}}\right)^T \end{bmatrix}$$

(2n – dim)

- Can analyze the dynamical structure around the equilibrium point with optimal control inputs using conventional methods of trajectory design for systems with no added inputs,
- Can explain optimal control in terms of dynamics

Research objective

Research objective

Investigate the equilibrium point with continuous optimal control inputs and its dynamical structures

Research flow

1. Derive the equations of motion of a dynamical system with optimal control inputs
2. Derive the conditions for the equilibrium points
3. Analyze the stability of equilibrium points
4. Investigate the dynamical structure around the equilibrium points

Dynamical model

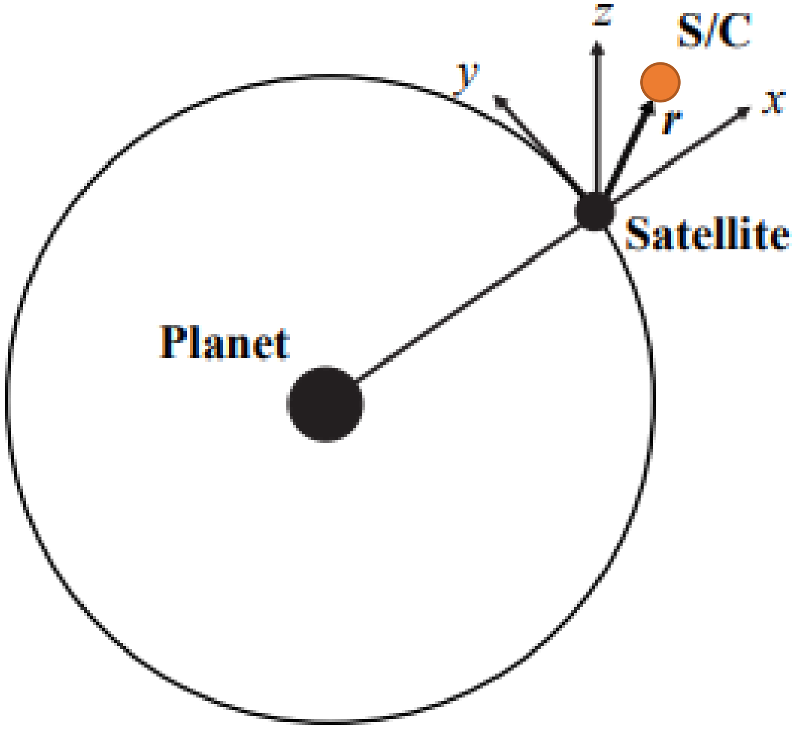
Hill three-body problem (Hill3BP) [3)]

Equations of motion of natural dynamics

$$\begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dfrac{\partial U}{\partial x} + 2\dot{y} \\ \dfrac{\partial U}{\partial y} - 2\dot{x} \\ \dfrac{\partial U}{\partial z} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{U_r} + 2\boldsymbol{J_a}\boldsymbol{v} \end{bmatrix} = \boldsymbol{f}(\boldsymbol{x})$$

$$U = \frac{1}{|\boldsymbol{r}|} + \frac{1}{2}(3x^2 - z^2)$$

$$\boldsymbol{J_a} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



3) Scheeres, D. J.: Orbital motion in strongly perturbed environments: applications to asteroid, comet and planetary satellite orbiters, Springer, 2016.

Dynamical model

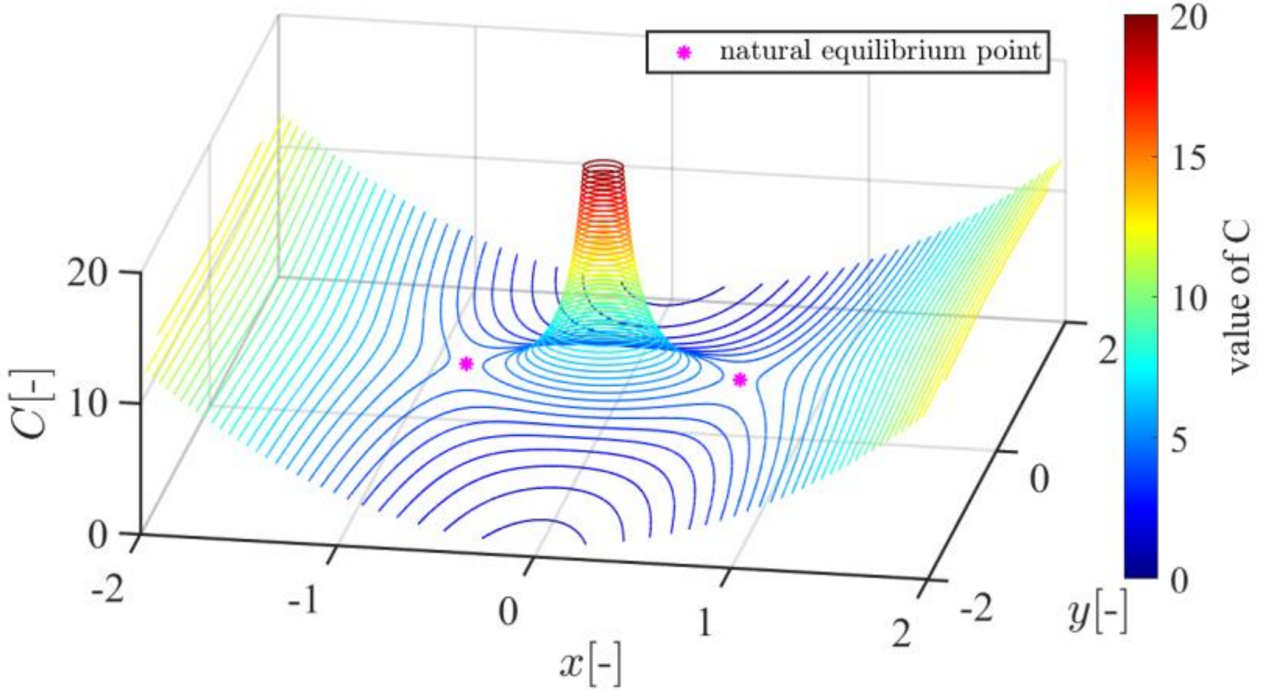
Hill three-body problem (Hill3BP)

Equations of motion of natural dynamics

$$\begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \frac{\partial U}{\partial x} + 2\dot{y} \\ \frac{\partial U}{\partial y} - 2\dot{x} \\ \frac{\partial U}{\partial z} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{U_r} + 2\boldsymbol{J_a}\boldsymbol{v} \end{bmatrix} = \boldsymbol{f}(\boldsymbol{x})$$

$$U = \frac{1}{|\boldsymbol{r}|} + \frac{1}{2}(3x^2 - z^2)$$

$$\boldsymbol{J}_a = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



There are two natural equilibrium points on the x-axis.

Optimal control problem

Deal with continuous optimal control inputs

Optimal Control Problem :
The problem of minimizing the cost function

$$J = \int_{t_0}^{t_f} L(\boldsymbol{x}(t), \boldsymbol{u}(t), t)\, dt$$

Subject to $\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{u}$

$f(\boldsymbol{x})$: natural dynamics
$\boldsymbol{u}$: control inputs

In this study

Quadratic cost function

$$J = \int_{t_0}^{t_f} \left(\frac{1}{2} \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \frac{1}{2} \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u} \right) dt$$

$\boldsymbol{Q} (\geq \boldsymbol{0})$: weight on state
$\boldsymbol{R} (> \boldsymbol{0})$: weight on control inputs

Quadratic cost function

$$J = \int_{t_0}^{t_f} \left(\frac{1}{2} \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \frac{1}{2} \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u} \right) dt$$

$\boldsymbol{Q}$: weight on state
$\boldsymbol{R}$: weight on control inputs

Hamilton function H

$$H(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}) = \left(\frac{1}{2} \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \frac{1}{2} \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u} \right) + \boldsymbol{p}^T (\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{u})$$

Euler Lagrange equation

$$\begin{cases} \dot{\boldsymbol{x}} = \left(\dfrac{\partial H}{\partial \boldsymbol{p}} \right)^T = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{u} \ , \ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \dot{\boldsymbol{p}} = -\left(\dfrac{\partial H}{\partial \boldsymbol{x}} \right)^T \\ \boldsymbol{u} = -\boldsymbol{R}^{-1} \boldsymbol{B}^T \boldsymbol{p} \end{cases}$$

Quadratic cost function

$$J = \int_{t_0}^{t_f} \left(\frac{1}{2} \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \frac{1}{2} \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u} \right) dt$$

$\boldsymbol{Q}$: weight on state

$\boldsymbol{R}$: weight on control inputs

Hamilton function H

$$H(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}) = \left(\frac{1}{2} \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \frac{1}{2} \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u} \right) + \boldsymbol{p}^T (\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{u})$$

Euler Lagrange equation

$$\begin{cases} \dot{\boldsymbol{x}} = \left(\frac{\partial H}{\partial \boldsymbol{p}} \right)^T = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{u} \ , \ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \dot{\boldsymbol{p}} = -\left(\frac{\partial H}{\partial \boldsymbol{x}} \right)^T \\ \boldsymbol{u} = -\boldsymbol{R}^{-1} \boldsymbol{B}^T \boldsymbol{p} \end{cases}$$

Euler Lagrange equation

$$\begin{cases} \dot{\boldsymbol{x}} = \left(\dfrac{\partial H(\boldsymbol{x},\boldsymbol{p})}{\partial \boldsymbol{p}}\right)^T \\ \dot{\boldsymbol{p}} = -\left(\dfrac{\partial H(\boldsymbol{x},\boldsymbol{p})}{\partial \boldsymbol{x}}\right)^T \end{cases}$$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{v} \end{bmatrix}$$

$$\boldsymbol{p} = \begin{bmatrix} \boldsymbol{p}_r \\ \boldsymbol{p}_v \end{bmatrix}$$

Equations of motion with optimal control inputs

$$\begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v} \\ 2\boldsymbol{J}_a\boldsymbol{v} + \boldsymbol{U}_r - \boldsymbol{R}\boldsymbol{p}_v \end{bmatrix}$$

$$\begin{bmatrix} \dot{\boldsymbol{p}}_r \\ \dot{\boldsymbol{p}}_v \end{bmatrix} = \begin{bmatrix} -\boldsymbol{Q}_{1:3}\boldsymbol{x} - \boldsymbol{U}_{rr}\boldsymbol{p}_v \\ -\boldsymbol{Q}_{4:6}\boldsymbol{x} - \boldsymbol{p}_r + 2\boldsymbol{J}_a\boldsymbol{p}_v \end{bmatrix}$$

Conditions for equilibrium point

12-dim stationary conditions

$$\begin{bmatrix}\dot{\boldsymbol{r}}\\ \dot{\boldsymbol{v}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{v}\\ 2\boldsymbol{J}_a\boldsymbol{v} + \boldsymbol{U}_r - \boldsymbol{R}\boldsymbol{p}_v\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}\dot{\boldsymbol{p}}_r\\ \dot{\boldsymbol{p}}_v\end{bmatrix} = \begin{bmatrix}-\boldsymbol{Q}_{1:3}\boldsymbol{x} - \boldsymbol{U}_{rr}\boldsymbol{p}_v\\ -\boldsymbol{Q}_{4:6}\boldsymbol{x} - \boldsymbol{p}_r + 2\boldsymbol{J}_a\boldsymbol{p}_v\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Conditions for equilibrium point

$$\left(\boldsymbol{x}_0 = \begin{bmatrix}\boldsymbol{r}_0\\ \boldsymbol{v}_{\boldsymbol{0}}\end{bmatrix}, \boldsymbol{p}_0 = \begin{bmatrix}\boldsymbol{p}_{r0}\\ \boldsymbol{p}_{v0}\end{bmatrix}\right)$$

$$\boldsymbol{v}_{\boldsymbol{0}} = \boldsymbol{0}$$

$$\boldsymbol{p}_{v0} = \boldsymbol{R}^{-1}\boldsymbol{U}_r$$

$$-\boldsymbol{Q}_{1:3}\boldsymbol{x}_{\boldsymbol{0}} - \boldsymbol{U}_{rr}\boldsymbol{R}^{-1}\boldsymbol{U}_r = 0$$

$$\boldsymbol{p}_{r0} = -\boldsymbol{Q}_{4:6}\boldsymbol{x}_{\boldsymbol{0}} + 2\boldsymbol{J}_a\boldsymbol{R}^{-1}\boldsymbol{U}_r$$

Conditions for equilibrium point

12-dim stationary conditions

$$\begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v} \\ 2\boldsymbol{J}_a\boldsymbol{v} + \boldsymbol{U}_r - \boldsymbol{R}\boldsymbol{p}_v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\boldsymbol{p}}_r \\ \dot{\boldsymbol{p}}_v \end{bmatrix} = \begin{bmatrix} -\boldsymbol{Q}_{1:3}\boldsymbol{x} - \boldsymbol{U}_{rr}\boldsymbol{p}_v \\ -\boldsymbol{Q}_{4:6}\boldsymbol{x} - \boldsymbol{p}_r + 2\boldsymbol{J}_a\boldsymbol{p}_v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Conditions for equilibrium point

$$\left(\boldsymbol{x}_0 = \begin{bmatrix} \boldsymbol{r}_0 \\ \boldsymbol{v}_0 \end{bmatrix}, \boldsymbol{p}_0 = \begin{bmatrix} \boldsymbol{p}_{r0} \\ \boldsymbol{p}_{v0} \end{bmatrix} \right)$$

$$\boldsymbol{v}_0 = \boldsymbol{0}$$

$$\boldsymbol{p}_{v0} = \boldsymbol{R}^{-1}\boldsymbol{U}_r$$

$$-\boldsymbol{Q}_{1:3}\boldsymbol{x}_0 - \boldsymbol{U}_{rr}\boldsymbol{R}^{-1}\boldsymbol{U}_r = 0$$

$$\boldsymbol{p}_{r0} = -\boldsymbol{Q}_{4:6}\boldsymbol{x}_0 + 2\boldsymbol{J}_a\boldsymbol{R}^{-1}\boldsymbol{U}_r$$

Conditions for equilibrium point

12-dim stationary conditions

$$\begin{bmatrix}\dot{\boldsymbol{r}}\\ \dot{\boldsymbol{v}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{v}\\ 2J_a\boldsymbol{v} + \boldsymbol{U}_r - \boldsymbol{R}\boldsymbol{p}_v\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}\dot{\boldsymbol{p}}_r\\ \dot{\boldsymbol{p}}_v\end{bmatrix} = \begin{bmatrix}-\boldsymbol{Q}_{1:3}\boldsymbol{x} - \boldsymbol{U}_{rr}\boldsymbol{p}_v\\ -\boldsymbol{Q}_{4:6}\boldsymbol{x} - \boldsymbol{p}_r + 2J_a\boldsymbol{p}_v\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Conditions for equilibrium point

$$\left(\boldsymbol{x}_0 = \begin{bmatrix}\boldsymbol{r}_0\\ \boldsymbol{v}_0\end{bmatrix}, \boldsymbol{p}_0 = \begin{bmatrix}\boldsymbol{p}_{r0}\\ \boldsymbol{p}_{v0}\end{bmatrix}\right)$$

$$\boldsymbol{v}_0 = \boldsymbol{0}$$

$$\boldsymbol{p}_{v0} = \boldsymbol{R}^{-1}\boldsymbol{U}_r$$

$$-\boldsymbol{Q}_{1:3}\boldsymbol{x}_0 - \boldsymbol{U}_{rr}\boldsymbol{R}^{-1}\boldsymbol{U}_r = 0$$

$$\boldsymbol{p}_{r0} = -\boldsymbol{Q}_{4:6}\boldsymbol{x}_0 + 2J_a\boldsymbol{R}^{-1}\boldsymbol{U}_r$$

Equilibrium point with optimal control inputs

Equilibrium point on x-axis in Hill3BP

$$\boldsymbol{v_0} = \boldsymbol{0}$$

$$\boldsymbol{p}_{v0} = \boldsymbol{R}^{-1}\boldsymbol{U}_r$$

$$-\boldsymbol{Q}_{1:3} - \boldsymbol{U}_{rr}\boldsymbol{R}^{-1}\boldsymbol{U}_r = 0$$

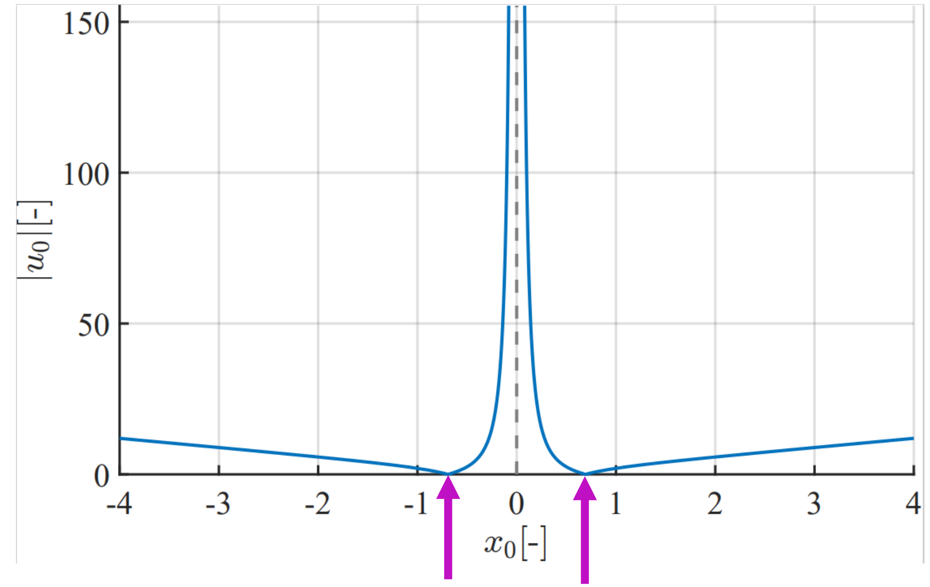
$$\boldsymbol{p}_{r0} = -\boldsymbol{Q}_{4:6} + 2\boldsymbol{J}_a\boldsymbol{R}^{-1}\boldsymbol{U}_r$$

If $\boldsymbol{Q}$ and $\boldsymbol{R}$ are diagonal matrices

$$\begin{cases} x_0 = \pm\left(\dfrac{-3R_{11} + \sqrt{81{R_{11}}^2 + 8R_{11}Q_{11}}}{2(Q_{11} + 9R_{11})}\right)^{\frac{1}{6}} \\ \boldsymbol{p}_{v0} = -2x_0\left(\dfrac{1}{|x_0|^3} - 3\right)\begin{bmatrix} 0 \\ -R_{11} \\ 0 \end{bmatrix} \\ \boldsymbol{p}_{v0} = -x_0\left(\dfrac{1}{|x_0|^3} - 3\right)\begin{bmatrix} R_{11} \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

Required optimal control inputs

$$\boldsymbol{u}_0 = x_0 \left(\frac{1}{|x_0|^3} - 3 \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$



Natural equilibrium point

$\boldsymbol{u}_0$ is only the function of position $\boldsymbol{r}_0$

Linearization

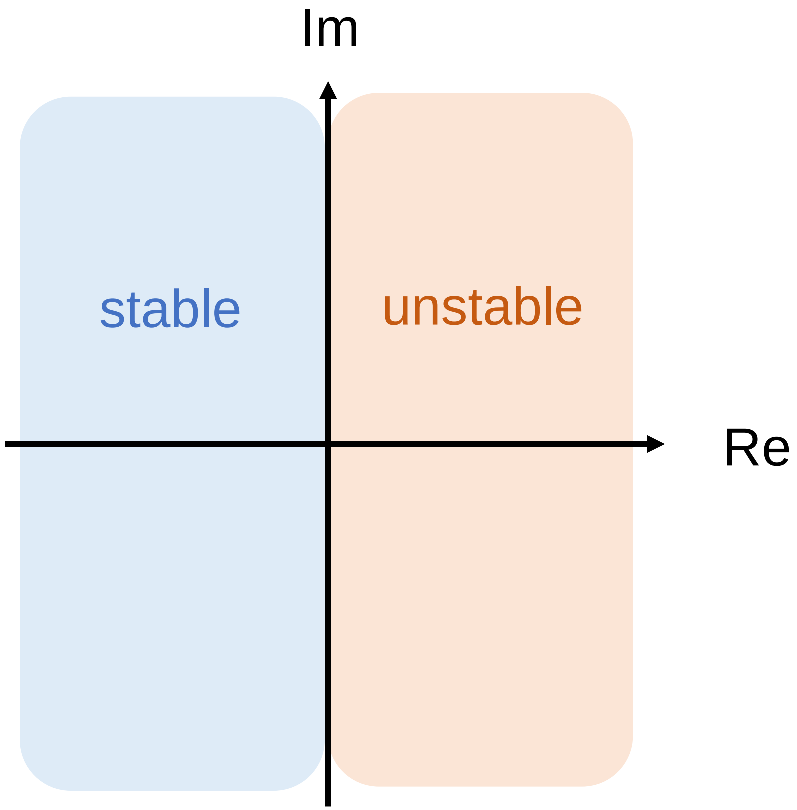
$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} \quad \boldsymbol{A} \quad \end{bmatrix}\Big|_{r0} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{bmatrix}$$

λ_i : eigenvalue

$\boldsymbol{v}_i$: eigenvector

C_i : constant

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{bmatrix} = \sum_{i=1}^{12} C_i \, \boldsymbol{v}_i e^{\lambda_i t}$$

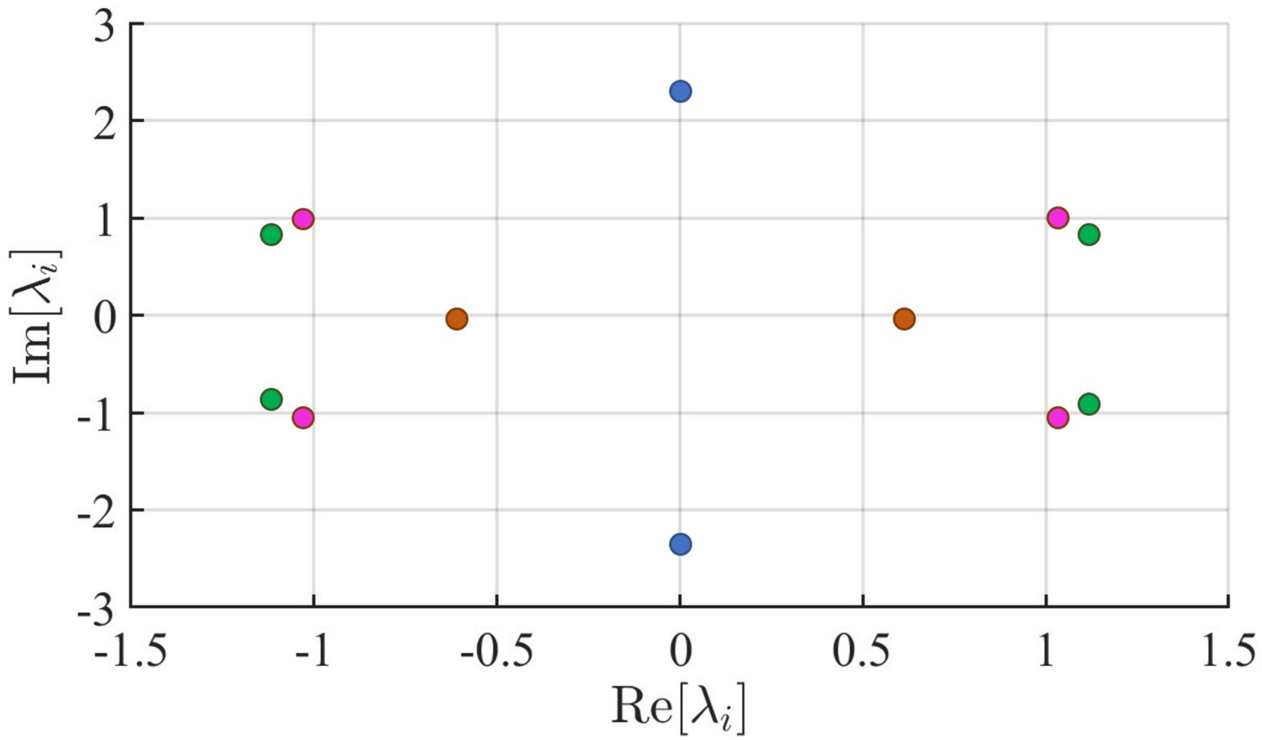


Linearization

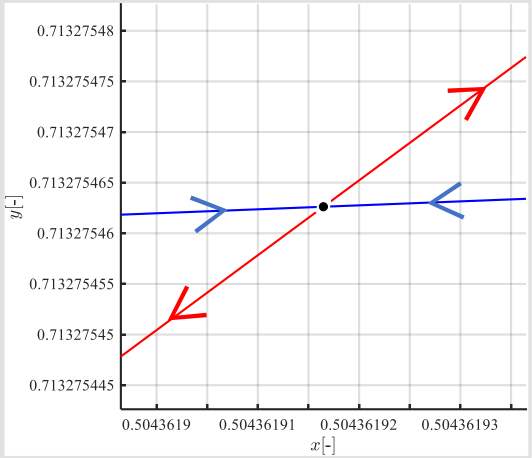
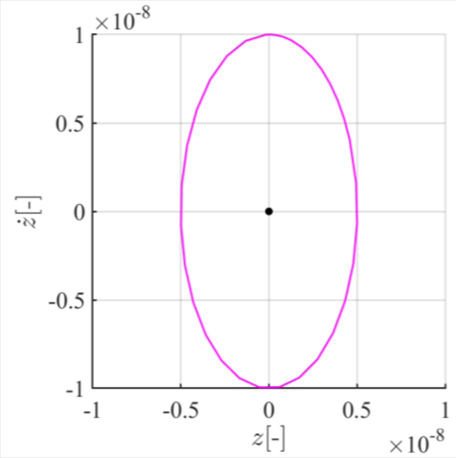
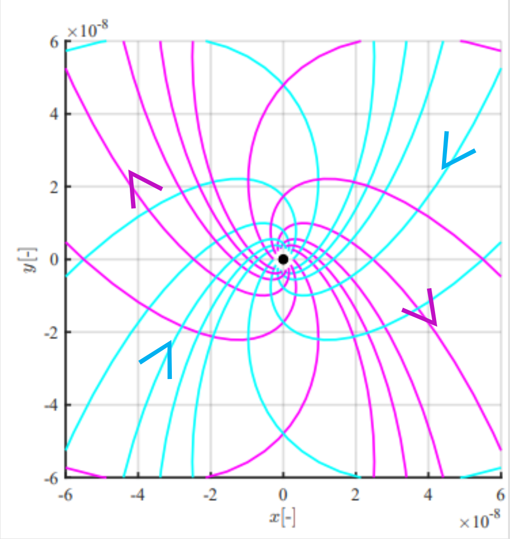
$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} & \boldsymbol{A} & \end{bmatrix}\Bigg|_{r0} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{bmatrix}$$

λ_i : eigenvalue
$\boldsymbol{v}_i$: eigenvector
C_i : constant

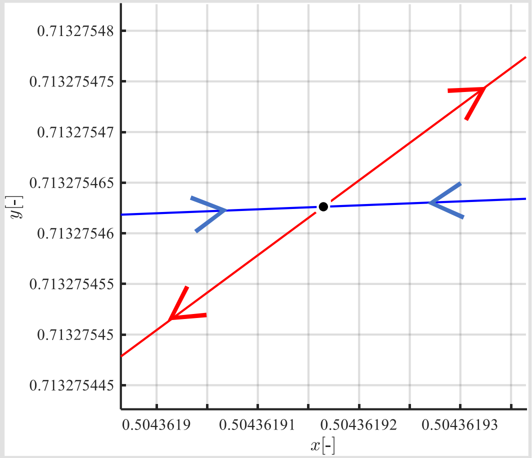
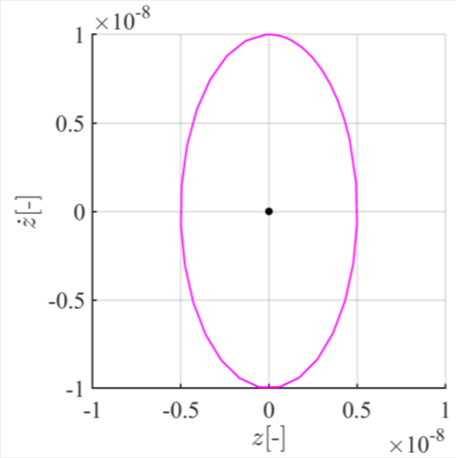
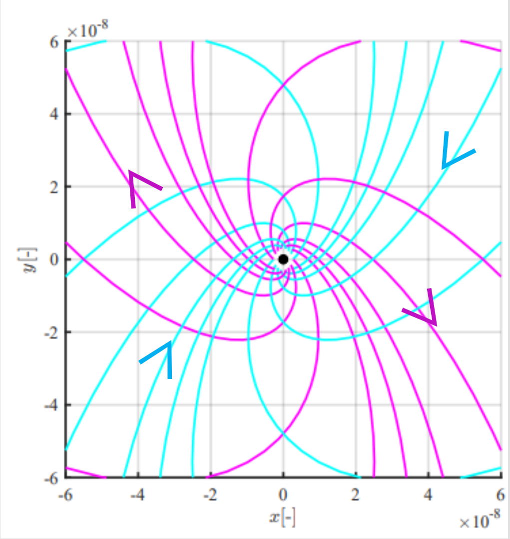
$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{bmatrix} = \sum_{i=1}^{12} C_i \, \boldsymbol{v}_i e^{\lambda_i t}$$



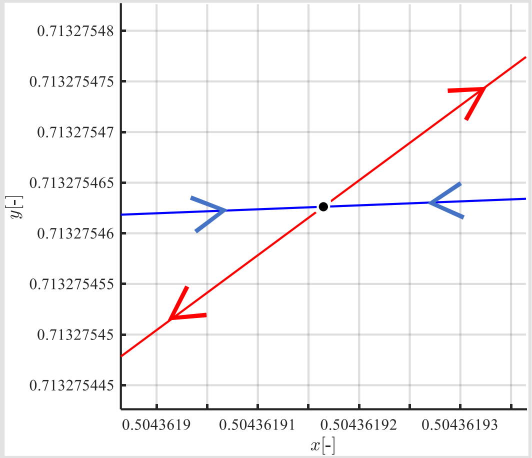
Types of invariant manifolds

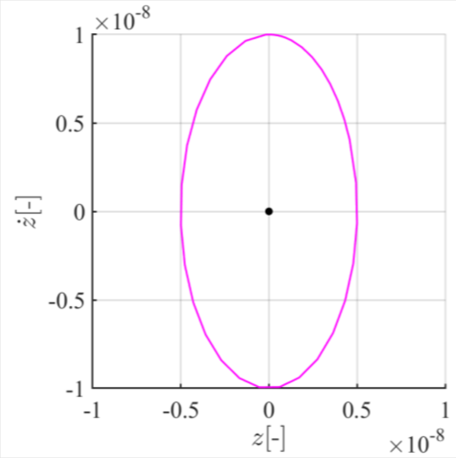
Eigenvalue	$\pm\lambda$	$\pm i\omega$	$\pm(\lambda \pm i\omega)$
Dynamical structure	 Saddle (stable & unstable Manifold)	 Center manifold	 Complex saddle

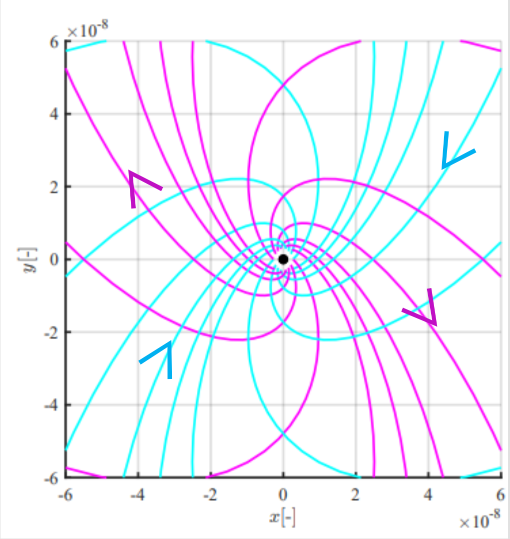
Types of invariant manifolds

Eigenvalue	$\pm\lambda$	$\pm i\omega$	$\pm(\lambda \pm i\omega)$
Dynamical structure	 Saddle (stable & unstable Manifold)	 Center manifold	 Complex saddle

Types of invariant manifolds

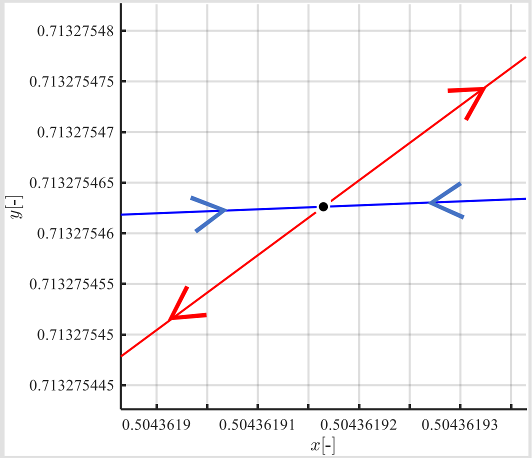
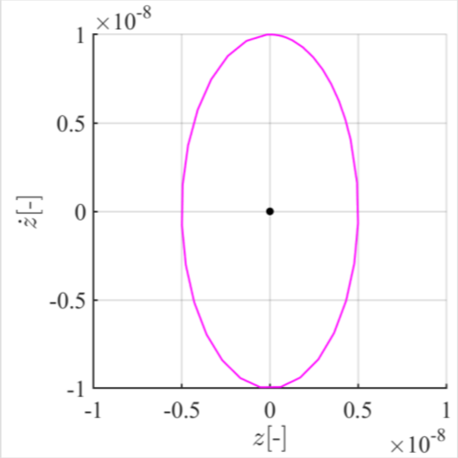
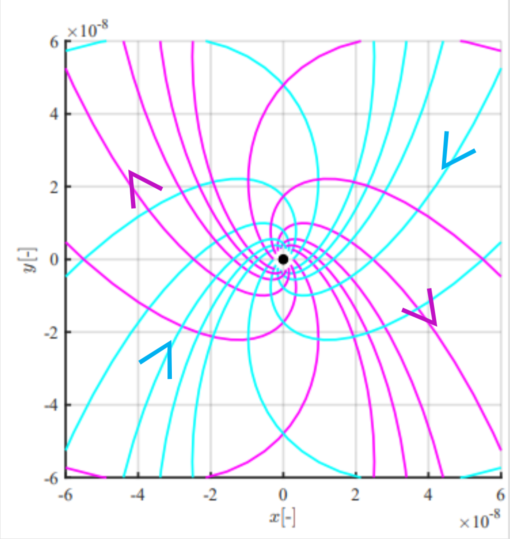
<table>
<tr><th>Eigenvalue</th><th>$\pm\lambda$</th><th>$\pm i\omega$</th><th>$\pm(\lambda \pm i\omega)$</th></tr>
<tr><td>Dynamical structure</td>
<td>

Saddle
(stable & unstable
Manifold)</td>
<td>

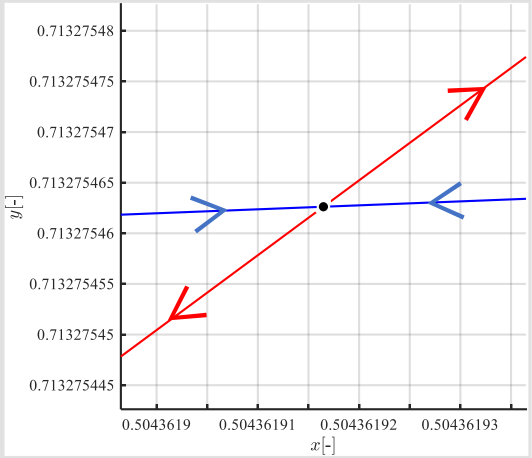
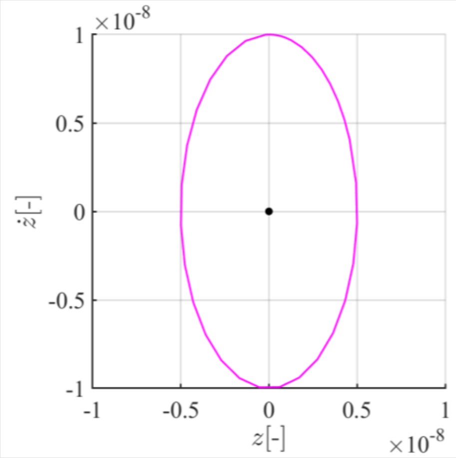
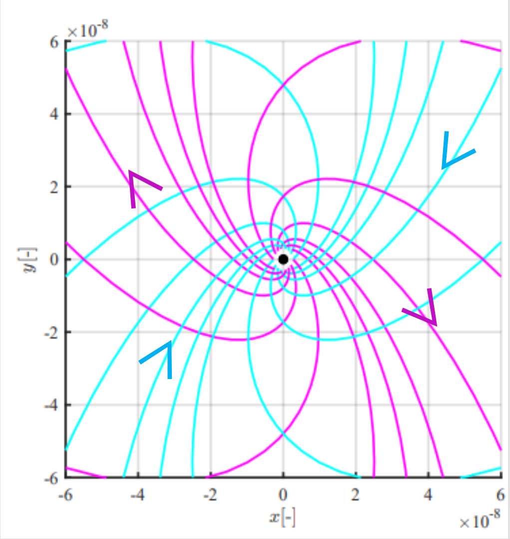
Center manifold</td>
<td>

Complex saddle</td></tr>
</table>

Types of invariant manifolds

Eigenvalue	$\pm\lambda$	$\pm i\omega$	$\pm(\lambda \pm i\omega)$
Dynamical structure	 Saddle (stable & unstable Manifold)	 Center manifold	 Complex saddle

Types of invariant manifolds

Eigenvalue	$\pm\lambda$	$\pm i\omega$	$\pm(\lambda \pm i\omega)$
Dynamical structure	 Saddle (stable & unstable Manifold)	 Center manifold	 Complex saddle

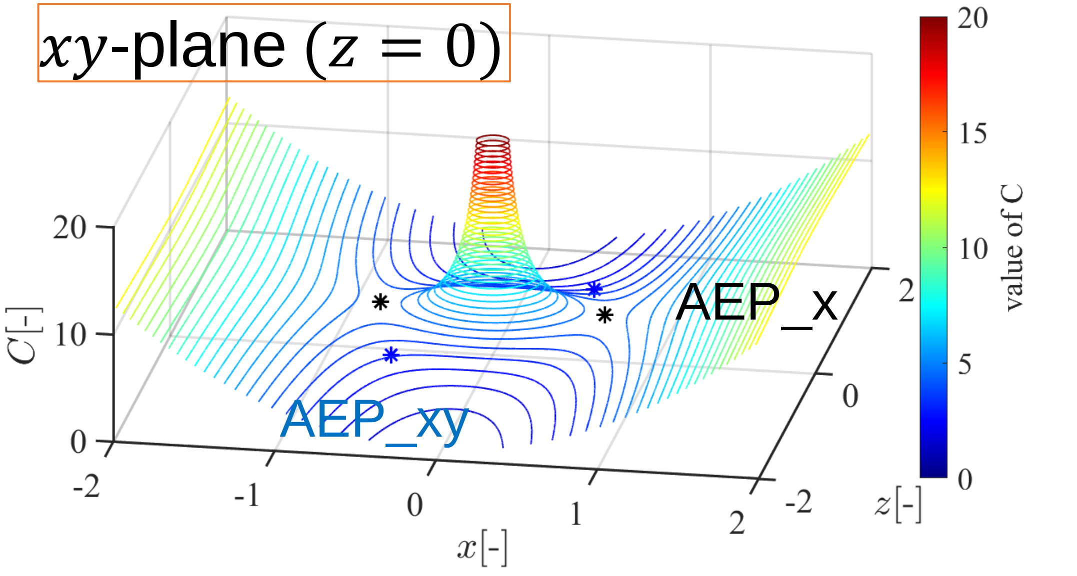
Do not exist
in natural dynamics

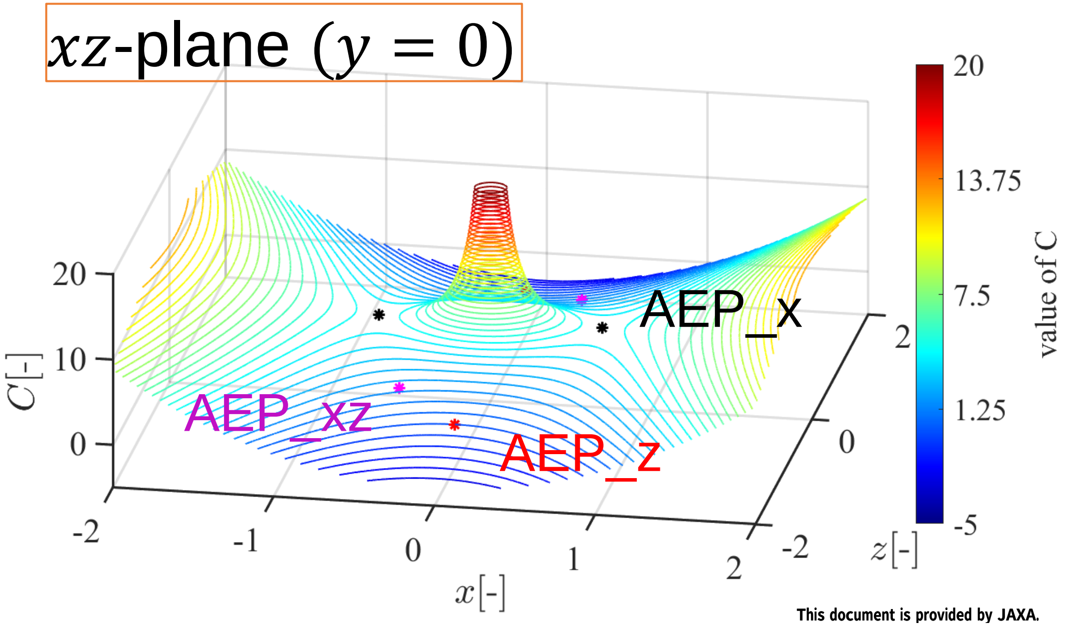
Equilibrium points in Hill3BP ($\boldsymbol{Q} = \mathbf{0}, \boldsymbol{R} = \boldsymbol{I}$)

Artificial Equilibrium Point
for minimum energy problem in Hill3BP (AEP)

Natural equilibrium points

	place	coordinates[x, y, z]
AEP_x	x-axis	$\left[\pm\left(\frac{1}{3}\right)^{\frac{1}{3}}, 0, 0\right]$
AEP_z	z-axis	$\left[0, 0, \pm 2^{\frac{1}{3}}\right]$
AEP_xy	xy-plane	$\left[\pm\frac{1}{\sqrt{3}}\left(\frac{2}{3}\right)^{\frac{1}{3}}, \pm\sqrt{\frac{2}{3}}\left(\frac{2}{3}\right)^{\frac{1}{3}}, 0\right]$
AEP_xz	xz-plane	$\left[\pm\frac{1}{\sqrt{6}}, 0, \pm\sqrt{\frac{5}{6}}\right]$



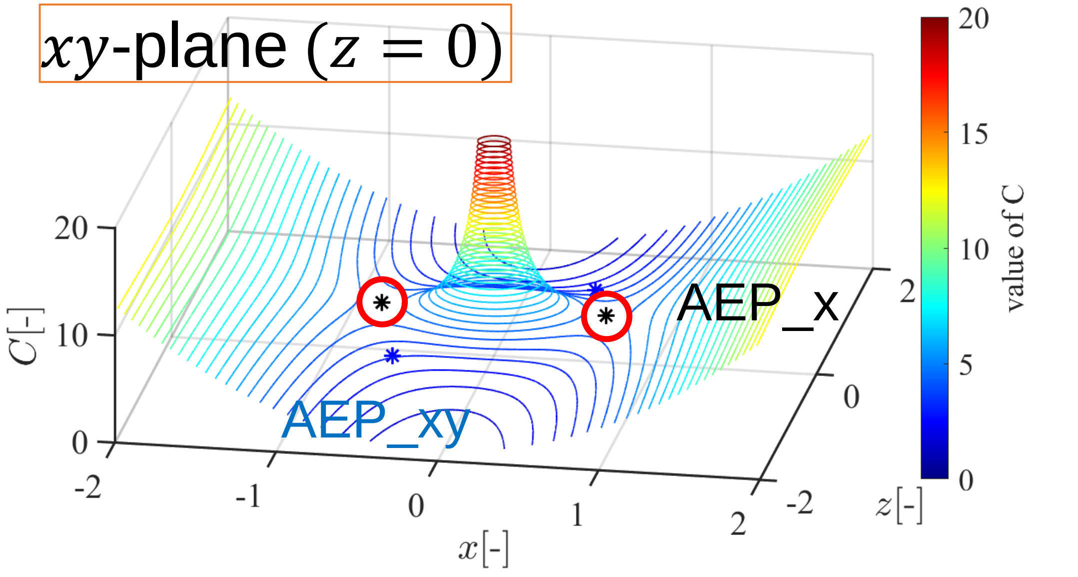


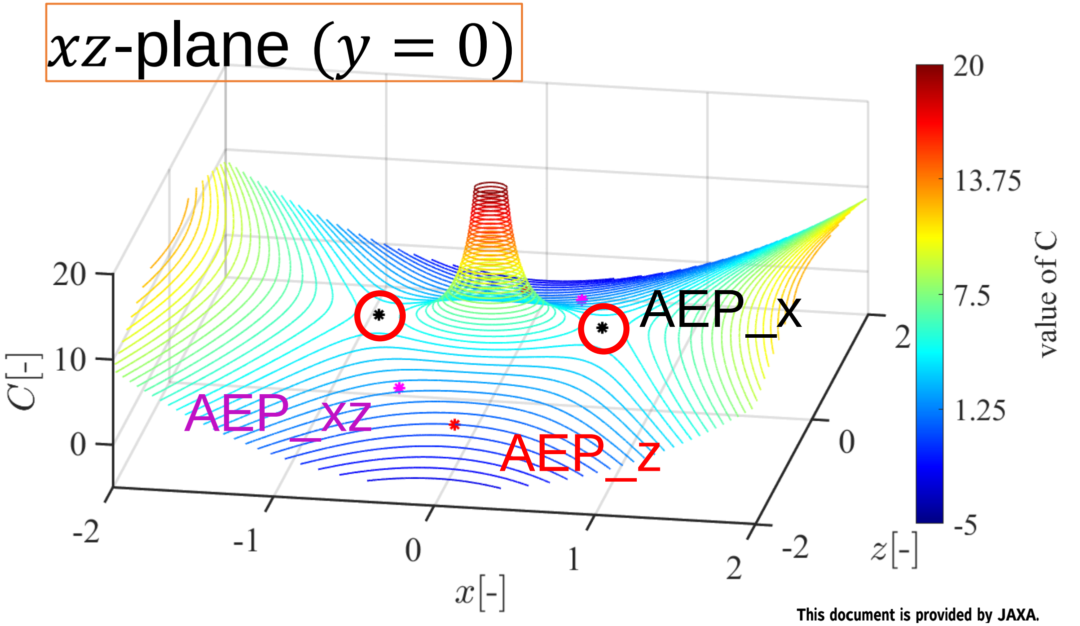
Equilibrium points in Hill3BP ($\boldsymbol{Q} = \boldsymbol{0}$, $\boldsymbol{R} = \boldsymbol{I}$)

Artificial Equilibrium Point
for minimum energy problem in Hill3BP (AEP)

Natural equilibrium points

	place	coordinates[x, y, z]
AEP_x	x-axis	$\left[\pm\left(\frac{1}{3}\right)^{\frac{1}{3}}, 0, 0\right]$
AEP_z	z-axis	$\left[0, 0, \pm 2^{\frac{1}{3}}\right]$
AEP_xy	xy-plane	$\left[\pm\frac{1}{\sqrt{3}}\left(\frac{2}{3}\right)^{\frac{1}{3}}, \pm\sqrt{\frac{2}{3}}\left(\frac{2}{3}\right)^{\frac{1}{3}}, 0\right]$
AEP_xz	xz-plane	$\left[\pm\frac{1}{\sqrt{6}}, 0, \pm\sqrt{\frac{5}{6}}\right]$





Equilibrium points in Hill3BP ($\boldsymbol{Q} = \boldsymbol{0}$, $\boldsymbol{R} = \boldsymbol{I}$)

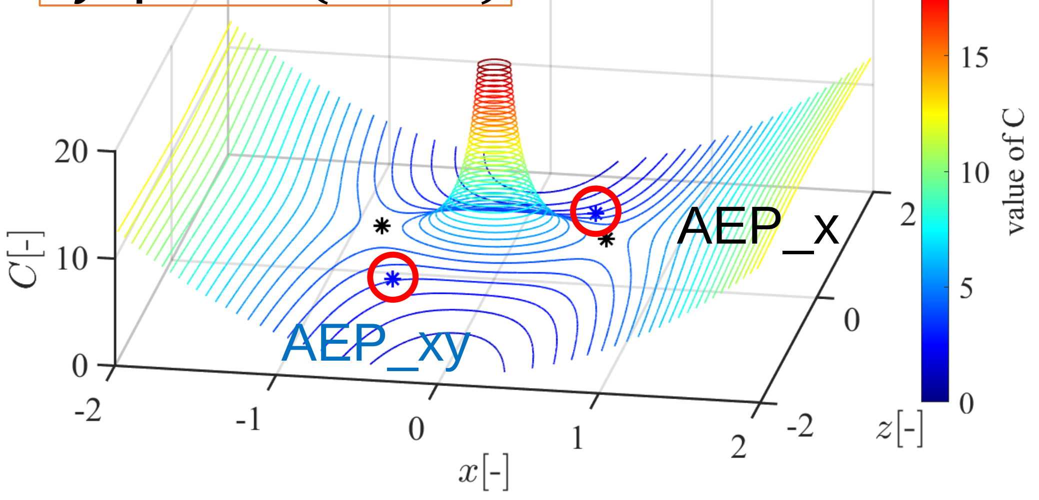
Artificial Equilibrium Point
for minimum energy problem in Hill3BP (AEP)

	place	coordinates[x, y, z]
AEP_x	x-axis	$\left[\pm\left(\frac{1}{3}\right)^{\frac{1}{3}}, 0, 0\right]$
AEP_z	z-axis	$\left[0, 0, \pm 2^{\frac{1}{3}}\right]$
AEP_xy	xy-plane	$\left[\pm\frac{1}{\sqrt{3}}\left(\frac{2}{3}\right)^{\frac{1}{3}}, \pm\sqrt{\frac{2}{3}}\left(\frac{2}{3}\right)^{\frac{1}{3}}, 0\right]$
AEP_xz	xz-plane	$\left[\pm\frac{1}{\sqrt{6}}, 0, \pm\sqrt{\frac{5}{6}}\right]$

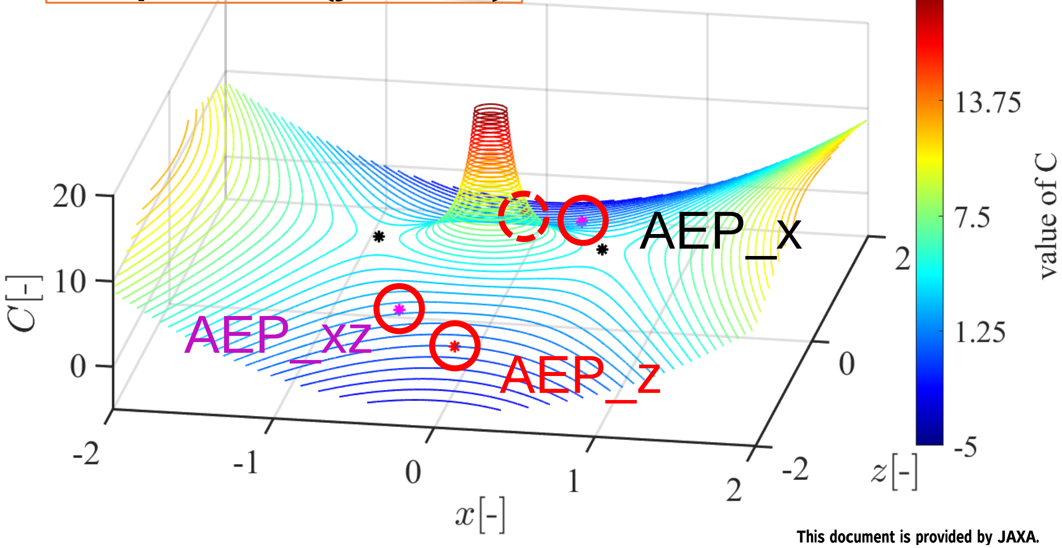
Natural equilibrium points: AEP_x

New equilibrium point with optimal control inputs: AEP_z, AEP_xy, AEP_xz

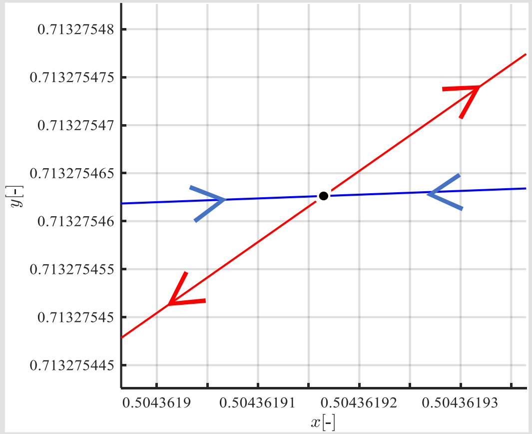
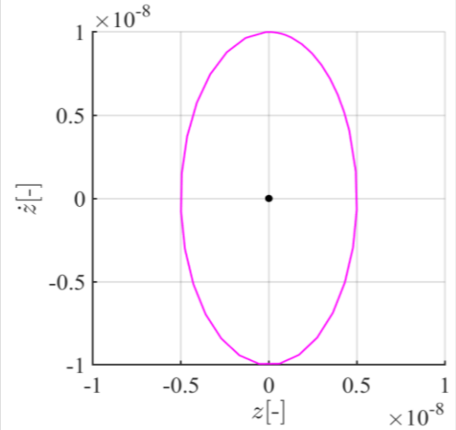
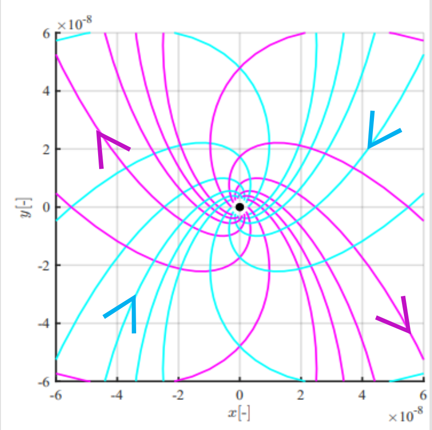
xy-plane ($z = 0$)



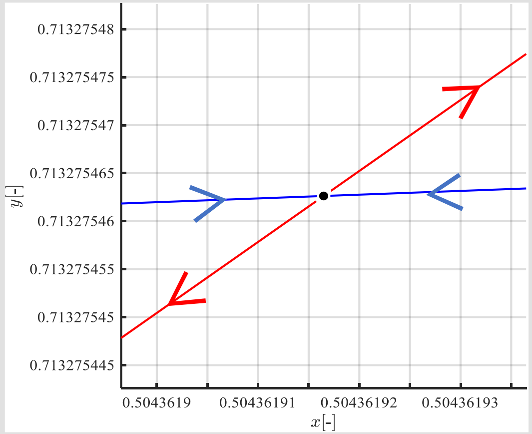
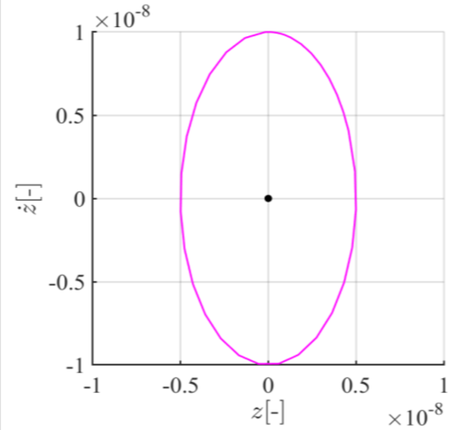
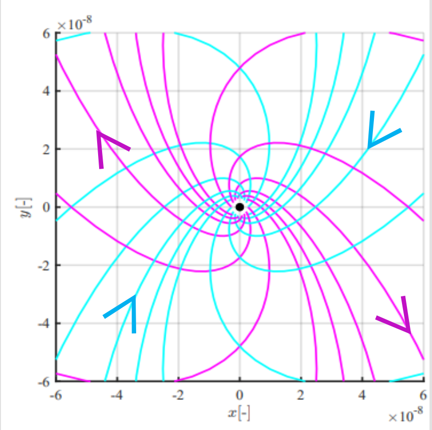
xz-plane ($y = 0$)



The stability of equilibrium points in Hill3BP

Equilibrium point	The number of set of eigenvalue		
	$\pm\lambda$	$\pm i\omega$	$\pm(\lambda \pm i\omega)$
Natural	1	2	0
AEP_x	1(double)	2(double)	0
AEP_z	1	1	2
AEP_xy	1	3	1
AEP_xz	0	2	2
Dynamical structure	 Saddle	 Center	 Complex saddle

The stability of equilibrium points in Hill3BP

Equilibrium point	The number of set of eigenvalue		
	$\pm\lambda$	$\pm i\omega$	$\pm(\lambda \pm i\omega)$
Natural	1	2	0
AEP_x	1(double)	2(double)	0
AEP_z	1	1	2
AEP_xy	1	3	1
AEP_xz	0	2	2
Dynamical structure	 Saddle	 Center	 Complex saddle

Generalized eigenvector (GE)

$$\dot{\boldsymbol{x}} = \boldsymbol{A}|_{\boldsymbol{x}_0}\boldsymbol{x}$$

$$\rightarrow \boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

Generalized eigenvector (GE)

$$\dot{\boldsymbol{x}} = \boldsymbol{A}|_{\boldsymbol{x}_0}\boldsymbol{x}$$

$$\rightarrow \boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}$$

$\boldsymbol{J}$: Jordan normal form

Generalized eigenvector (GE)

Diagonal matrix of eigenvalues

$$\boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

Jordan normal form

$$\boldsymbol{J} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

Generalized eigenvector (GE)

Diagonal matrix of eigenvalues

$$\boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

Eigenvector

$$\boldsymbol{E} = [\boldsymbol{v}_1 \quad \boldsymbol{v}_1 \quad \boldsymbol{v}_2]$$

Jordan normal form

$$\boldsymbol{J} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

Generalized eigenvector (GE)

$$\boldsymbol{V} = [\boldsymbol{v}_1 \quad \boldsymbol{v}_{1\_GE} \quad v_2]$$

Generalized eigenvector (GE)

$$\dot{\boldsymbol{x}} = \boldsymbol{A}|_{\boldsymbol{x}_0}\boldsymbol{x}$$

$$\rightarrow \boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}$$

$\boldsymbol{J}$: Jordan normal form

$$\boldsymbol{x}(t) = e^{\boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}t}\boldsymbol{x}(0)$$

$$\boldsymbol{J} = \boldsymbol{D} + \boldsymbol{N}$$

$$\boldsymbol{D} = diag(\lambda_1, \lambda_2, \cdots, \lambda_{12})\ (\lambda_{2i-1} = \lambda_{2i})$$

$$\boldsymbol{N} = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 0 & & \mathbf{0} & \\ & & 0 & 1 & & \\ & & & 0 & 0 & \\ & \mathbf{0} & & & 0 & \ddots \\ & & & & & \ddots \end{bmatrix}$$

Generalized eigenvector (GE)

$$\dot{\boldsymbol{x}} = \boldsymbol{A}|_{\boldsymbol{x}_0}\boldsymbol{x}$$
$$\rightarrow \boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}$$

$\boldsymbol{J}$: Jordan normal form

$$\boldsymbol{x}(t) = e^{\boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}t}\boldsymbol{x}(0)$$

$$\boldsymbol{J} = \boldsymbol{D} + \boldsymbol{N}$$

$$\boldsymbol{D} = diag(\lambda_1, \lambda_2, \cdots, \lambda_{12})\ (\lambda_{2i-1} = \lambda_{2i})$$

$$\boldsymbol{N} = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 0 & & \mathbf{0} & \\ & & 0 & 1 & & \\ & & & 0 & 0 & \\ & \mathbf{0} & & & 0 & \ddots \\ & & & & & \ddots \end{bmatrix}$$

$$\boldsymbol{x}(t) = e^{\boldsymbol{V}(\boldsymbol{D}+\boldsymbol{N})\boldsymbol{V}^{-1}t}\boldsymbol{x}(0)$$
$$= e^{\boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1}t}e^{\boldsymbol{V}\boldsymbol{N}\boldsymbol{V}^{-1}t}\boldsymbol{x}(0)$$

Generalized eigenvector (GE)

Because $e^{\boldsymbol{A}t} = I + At + \frac{1}{2!}A^2t^2 + \cdots$

$$\Rightarrow \begin{cases} e^{\boldsymbol{VDV}^{-1}t} = \boldsymbol{V}\boldsymbol{e}^{\boldsymbol{D}t}\boldsymbol{V}^{-1} \\ e^{\boldsymbol{VNV}^{-1}t} = \boldsymbol{V}(\boldsymbol{I} + \boldsymbol{N}t)\boldsymbol{V}^{-1}\ (\because \boldsymbol{N}^2 = 0) \end{cases}$$

Generalized eigenvector (GE)

Because $e^{\boldsymbol{A}t} = I + At + \frac{1}{2!}A^2t^2 + \cdots$

$$\Rightarrow \begin{cases} e^{\boldsymbol{VDV}^{-1}t} = \boldsymbol{V}\boldsymbol{e}^{\boldsymbol{D}t}\boldsymbol{V}^{-1} \\ e^{\boldsymbol{VNV}^{-1}t} = \boldsymbol{V}(\boldsymbol{I} + \boldsymbol{N}t)\boldsymbol{V}^{-1} \ (\because \boldsymbol{N}^2 = 0) \end{cases}$$

$$\Rightarrow \boldsymbol{x}(t) = e^{\boldsymbol{VDV}^{-1}t}e^{\boldsymbol{VNV}^{-1}t}\boldsymbol{x}(0)$$

$$= \boldsymbol{V}\boldsymbol{e}^{\boldsymbol{D}t}(\boldsymbol{I} + \boldsymbol{N}t)\boldsymbol{V}^{-1}\boldsymbol{x}(0)$$

Generalized eigenvector (GE)

Because $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots$

$$\Rightarrow \begin{cases} e^{\boldsymbol{VDV}^{-1}t} = \boldsymbol{V}\boldsymbol{e}^{\boldsymbol{D}t}\boldsymbol{V}^{-1} \\ e^{\boldsymbol{VNV}^{-1}t} = \boldsymbol{V}(\boldsymbol{I} + \boldsymbol{N}t)\boldsymbol{V}^{-1} \ (\because \boldsymbol{N}^2 = 0) \end{cases}$$

$$\Rightarrow \boldsymbol{x}(t) = e^{\boldsymbol{VDV}^{-1}t}e^{\boldsymbol{VNV}^{-1}t}\boldsymbol{x}(0)$$

$$= \boldsymbol{V}\boldsymbol{e}^{\boldsymbol{D}t}(\boldsymbol{I} + \boldsymbol{N}t)\boldsymbol{V}^{-1}\boldsymbol{x}(0)$$

Usual solution

$$\boldsymbol{x}(t) = \boldsymbol{E}\boldsymbol{e}^{\boldsymbol{D}t}\boldsymbol{E}^{-1}\boldsymbol{x}(0)$$

Generalized eigenvector (GE)

Because $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots$

v_{2i-1} : eigenvector
v_{2i} : general eigenvector

$$\Rightarrow \begin{cases} e^{VDV^{-1}t} = \boldsymbol{V}\boldsymbol{e}^{\boldsymbol{D}t}\boldsymbol{V}^{-1} \\ e^{VNV^{-1}t} = \boldsymbol{V}(\boldsymbol{I} + \boldsymbol{N}t)\boldsymbol{V}^{-1} \ (\because \boldsymbol{N}^2 = 0) \end{cases}$$

$$\Rightarrow \boldsymbol{x}(t) = e^{VDV^{-1}t}e^{VNV^{-1}t}\boldsymbol{x}(0)$$

$$= \boldsymbol{V}\boldsymbol{e}^{\boldsymbol{D}t}(\boldsymbol{I} + \boldsymbol{N}t)\boldsymbol{V}^{-1}\boldsymbol{x}(0)$$

$$= [e^{\lambda_1 t}\boldsymbol{v}_1 \quad e^{\lambda_1 t}(t\boldsymbol{v}_1 + \boldsymbol{v}_2) \quad e^{\lambda_3 t}\boldsymbol{v}_3 \quad \ldots]\boldsymbol{V}^{-1}\boldsymbol{x}(0)$$

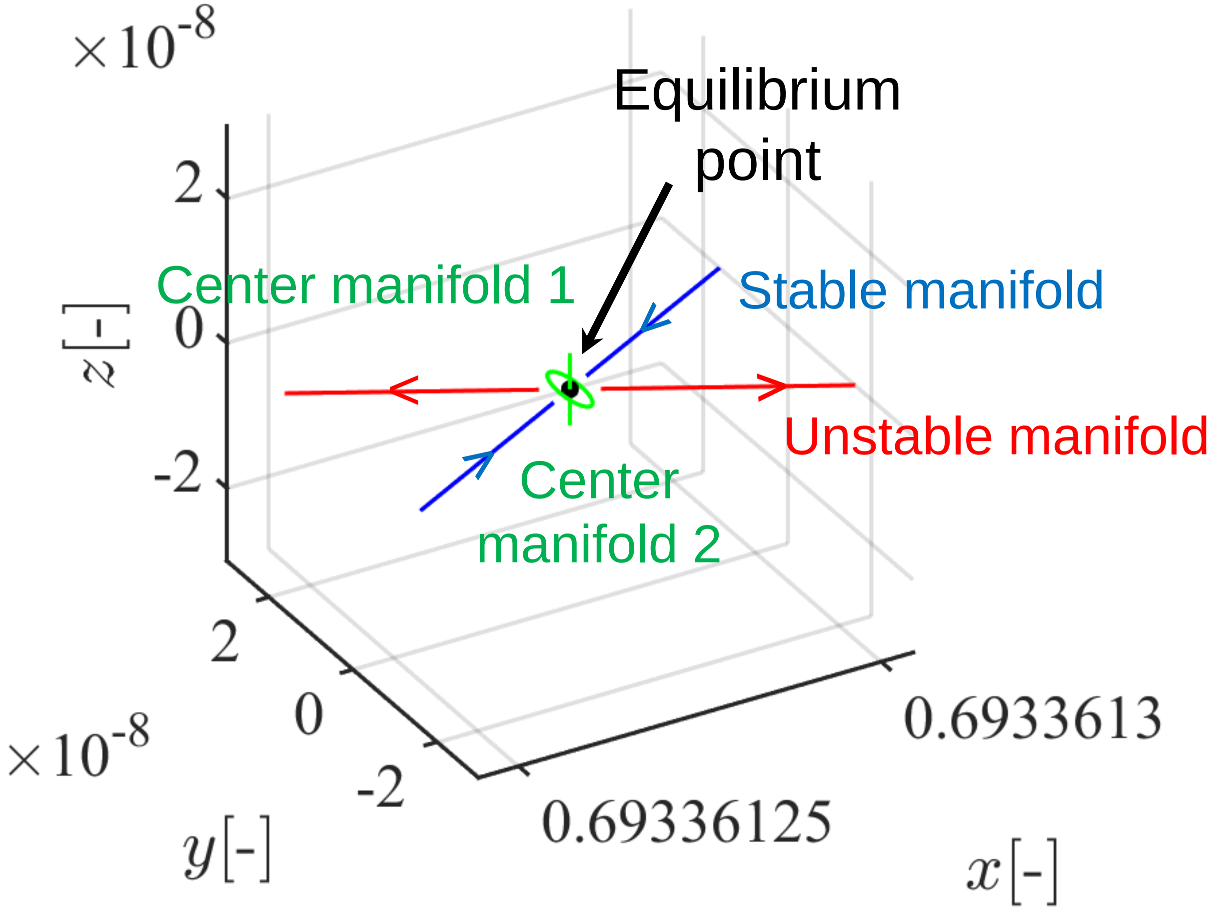
$$= \sum_{i=1}^{6}\{\underbrace{C_{2i-1}e^{\lambda_{2i-1}t}\boldsymbol{v}_{2i-1}}_{\text{natural manifold}} + \underbrace{C_{2i}e^{\lambda_{2i}t}(t\boldsymbol{v}_{2i-1} + \boldsymbol{v}_{2i})}_{\text{non-natural manifold}}\} \quad (e^{\lambda_{2i-1}t} = e^{\lambda_{2i}t})$$

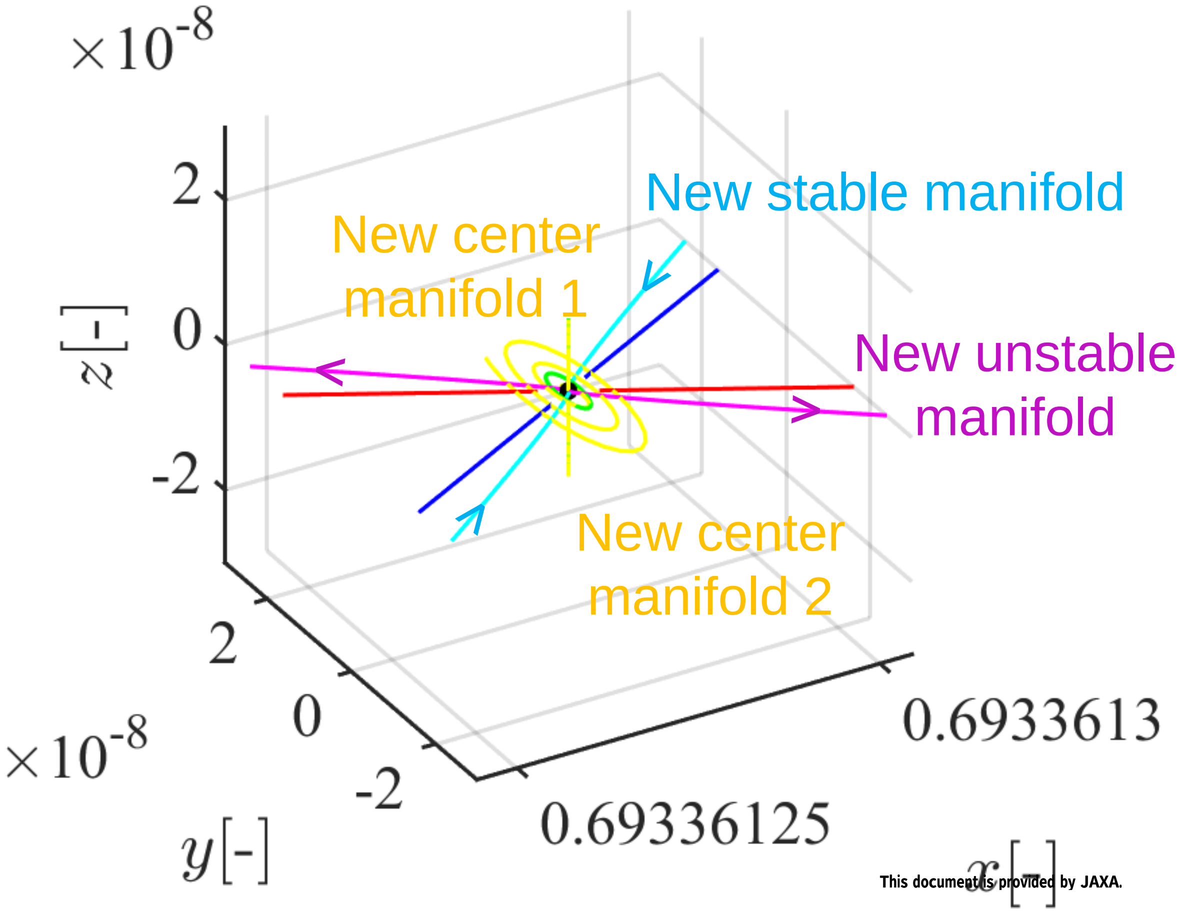
Generalized eigenvector (GE)

Difference of the solution using eigenvector and GE

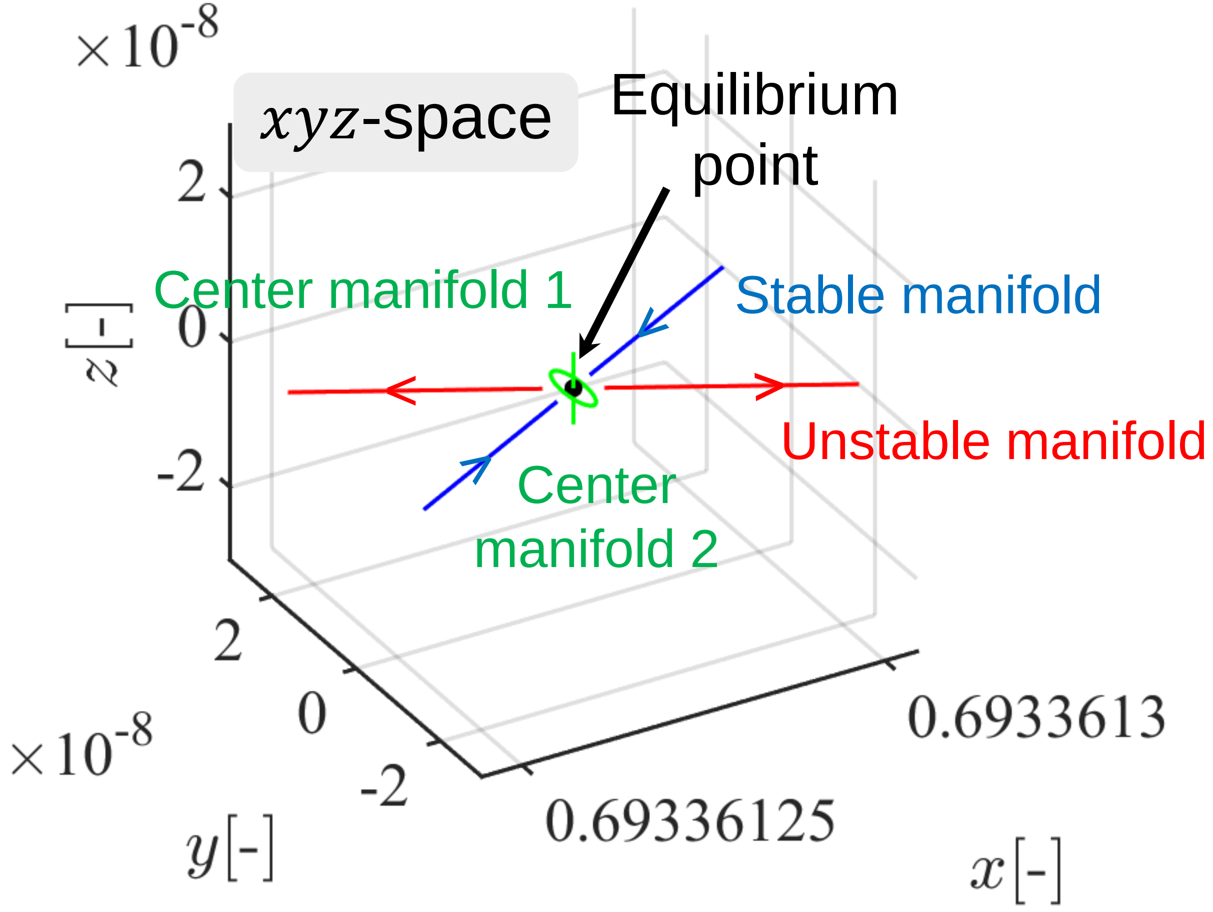
$$\text{Using Eigenvector } (\boldsymbol{E}) \rightarrow \boldsymbol{x}(t) = \boldsymbol{E}\boldsymbol{e}^{\boldsymbol{D}t}\boldsymbol{E}^{-1}\boldsymbol{x}(0)$$

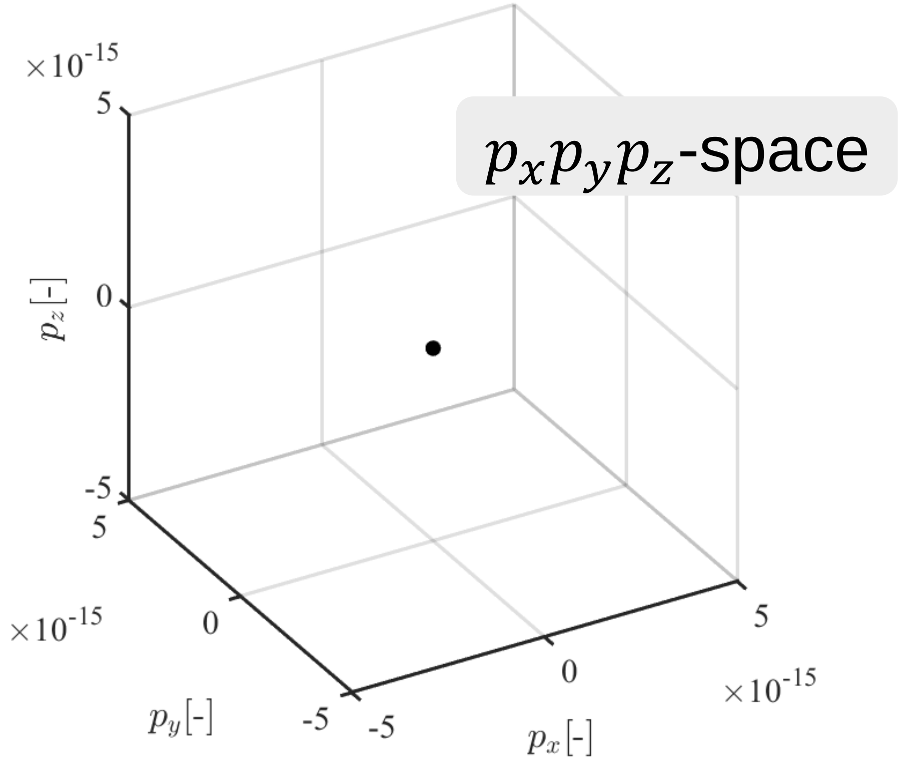
$$\text{Using GE } (\boldsymbol{V}) \rightarrow \boldsymbol{x}(t) = \boldsymbol{V}\boldsymbol{e}^{\boldsymbol{D}t}(\boldsymbol{I} + \boldsymbol{N}t)\boldsymbol{V}^{-1}\boldsymbol{x}(0)$$





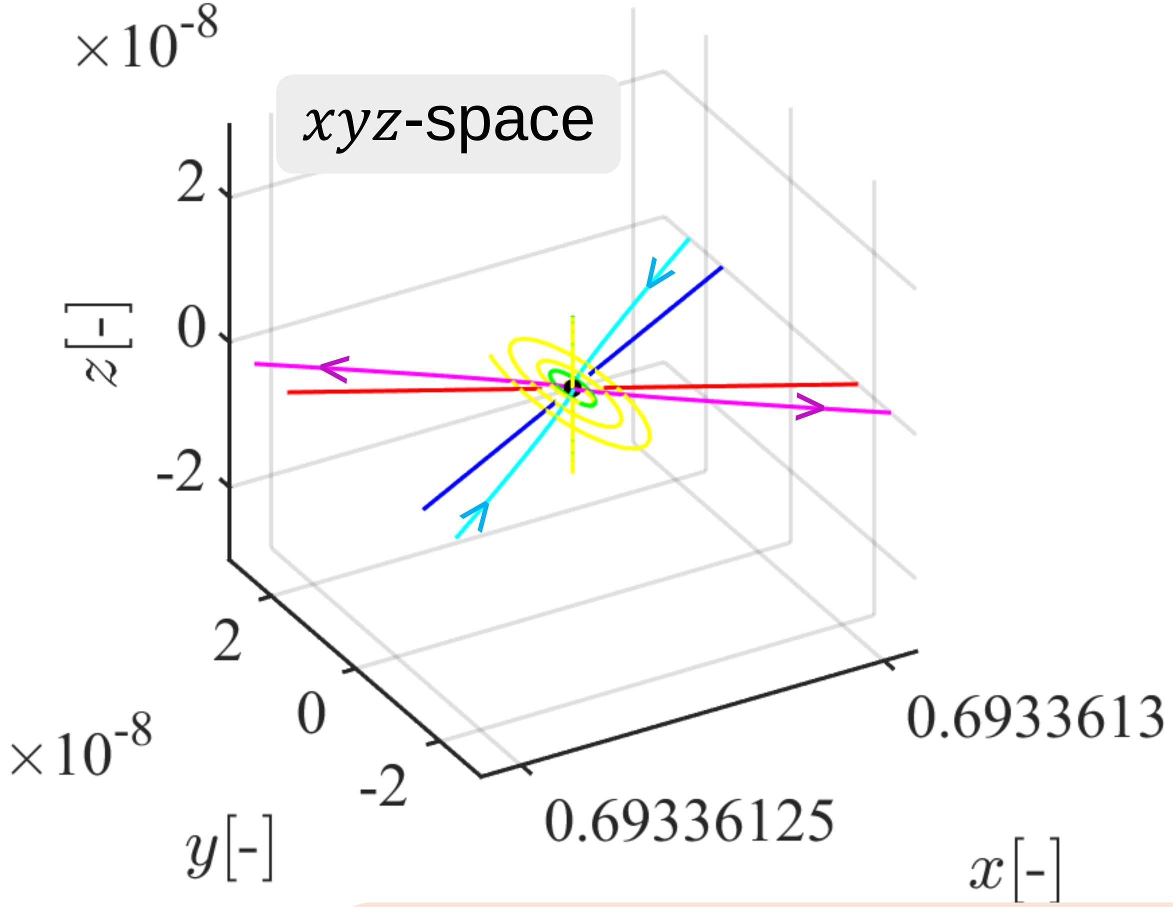
12-dim invariant manifold around AEP_x using eigenvector

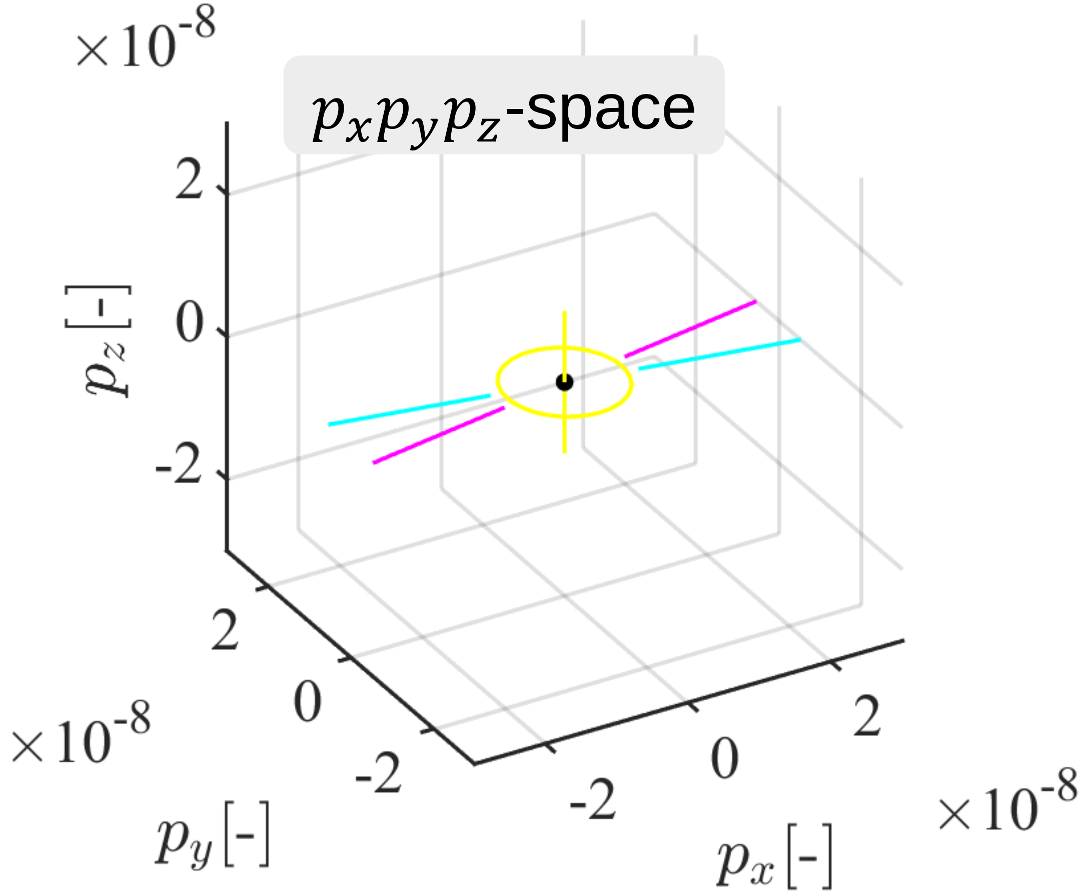




- Invariant manifolds on $p_x p_y p_z$-space stay at the origin
- Invariant manifolds with no control inputs
- Invariant manifolds have (x, y, p_x, p_y) elements, or (z, p_z) elements

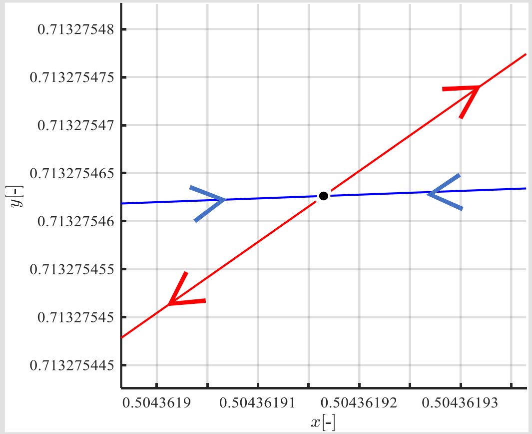
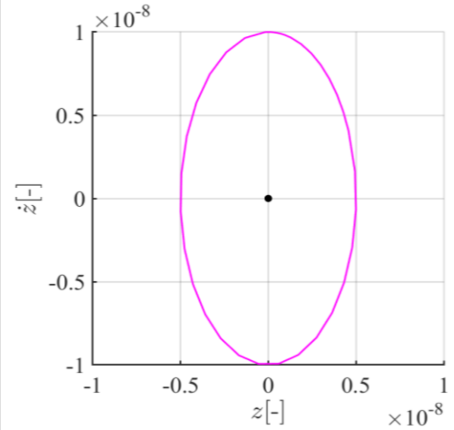
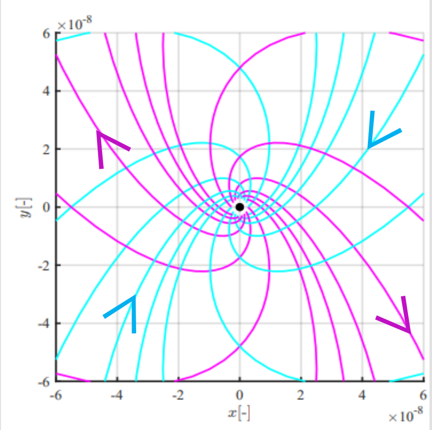
12-dim invariant manifold around AEP_x using GE



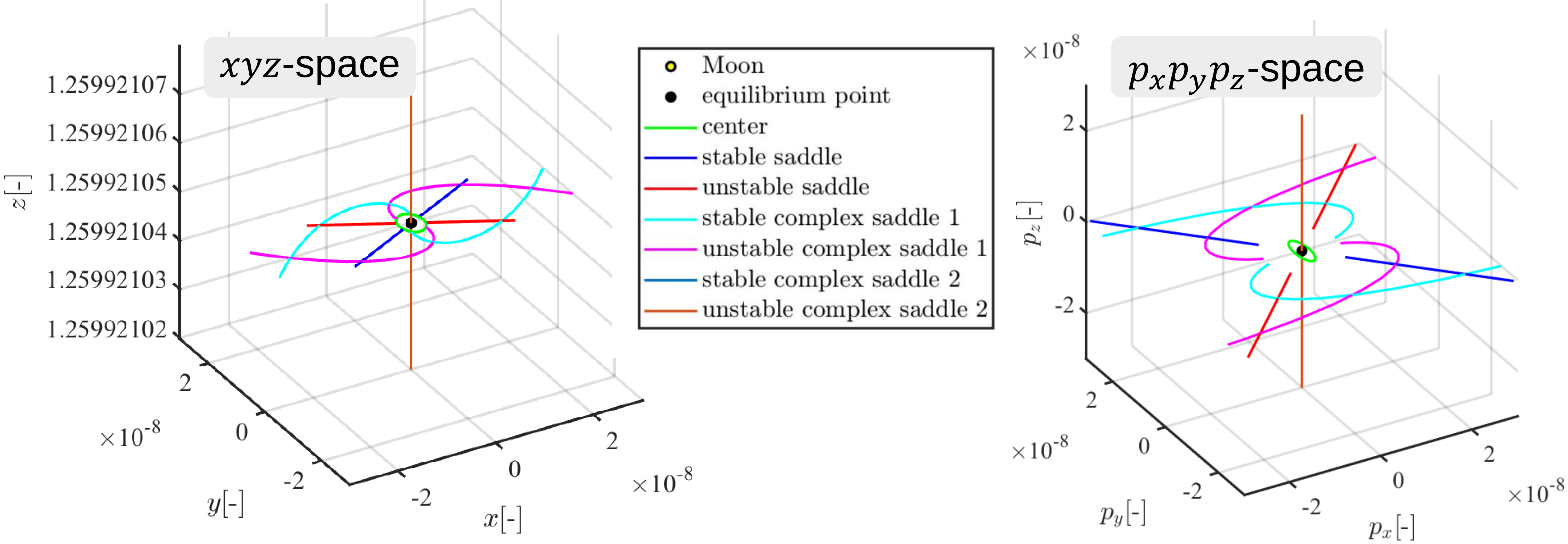


- There are invariant manifolds on $p_x p_y p_z$-space
- Invariant manifolds with optimal control inputs
- Invariant manifolds have (x, y, p_x, p_y) elements, or (z, p_z) elements

The stability of equilibrium points in Hill3BP

Equilibrium point	The number of set of eigenvalue		
	$\pm\lambda$	$\pm i\omega$	$\pm(\lambda \pm i\omega)$
Natural	1	2	0
AEP_x	1(double)	2(double)	0
AEP_z	1	1	2
AEP_xy	1	3	1
AEP_xz	0	2	2
Dynamical structure	 Saddle	 Center	 Complex saddle

12-dim invariant manifold around AEP_z



- Invariant manifolds have (x, y, p_x, p_y) elements, or (z, p_z) elements
- Invariant manifolds with optimal control inputs

Relationship with stabilization by LQR optimal control

Riccati equation :

$$\boldsymbol{PA} + \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{0}$$

↓ $\boldsymbol{Q} = constant * \boldsymbol{I}$

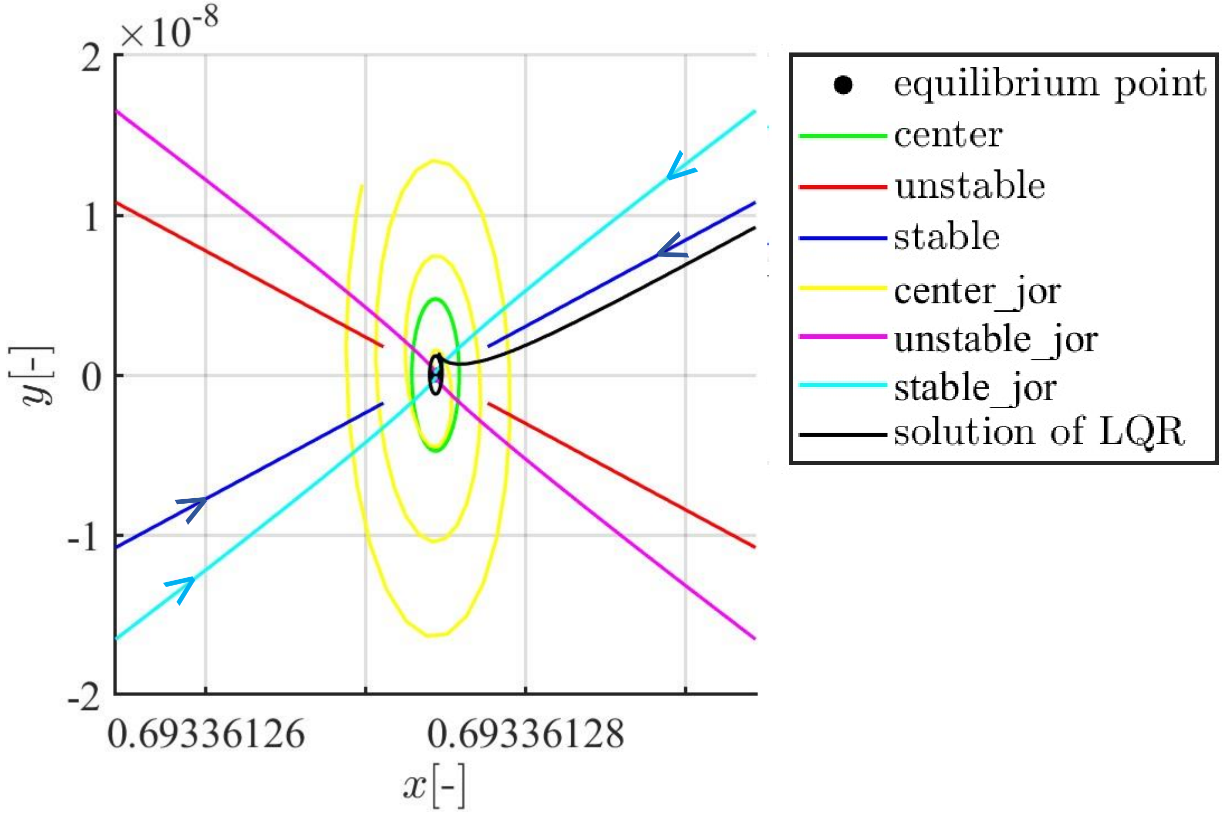
LQR optimal control

- $\boldsymbol{Q} = \boldsymbol{0}$ → cannot stabilize
- $\boldsymbol{Q} > \boldsymbol{0}$ → can stabilize

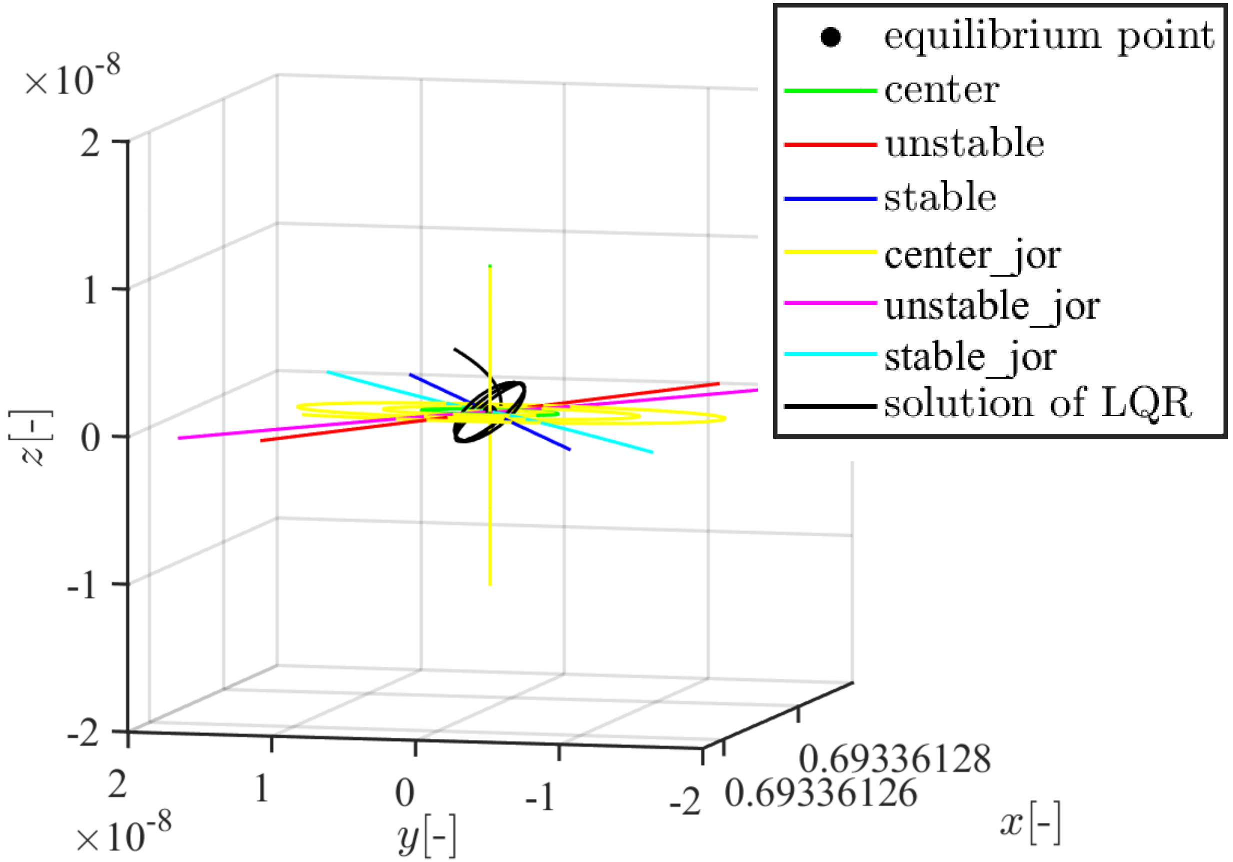
Relationship with stabilization by LQR optimal control

LQR optimal control
- $\boldsymbol{Q} = 0 \rightarrow$ cannot stabilize
- $\boldsymbol{Q} > 0 \rightarrow$ can stabilize

Manifold around AEP_x $(\boldsymbol{Q} = 10^{-16}\boldsymbol{I})$



initial : arbitrary point on the xy-plane

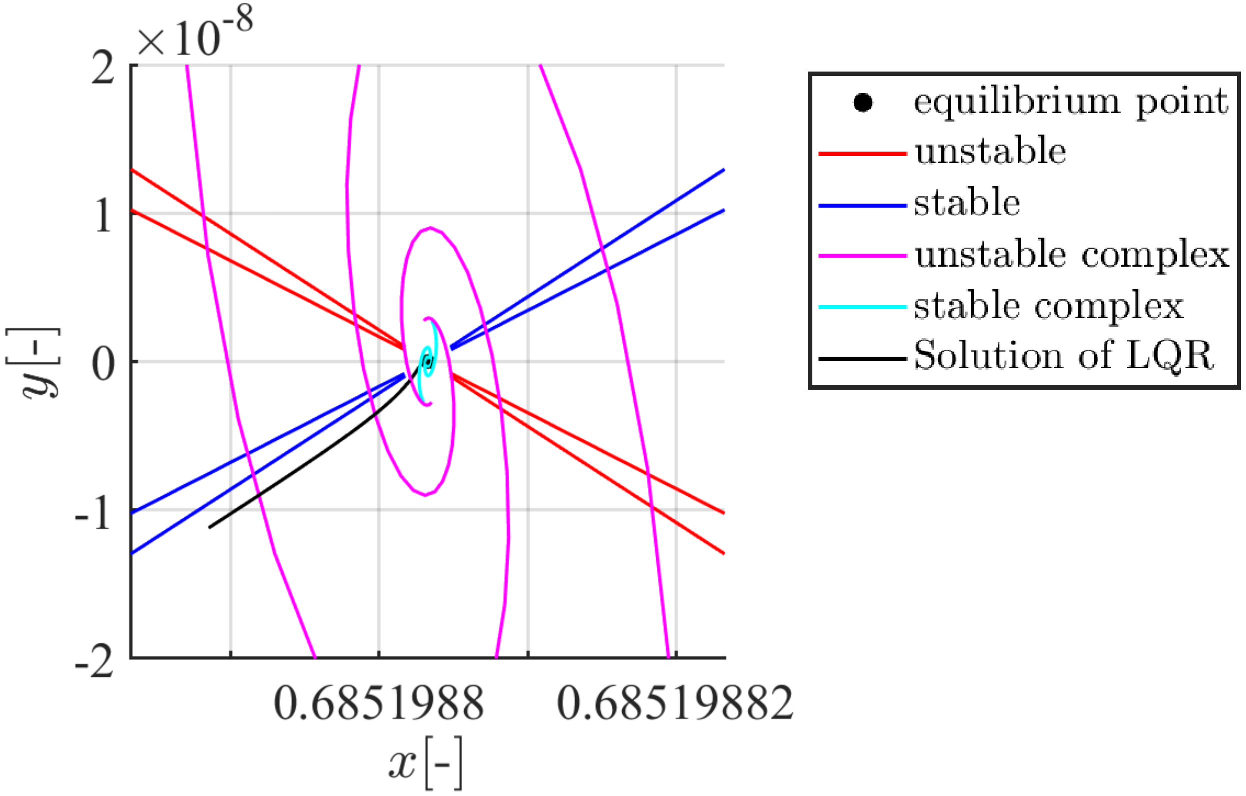


initial : arbitrary point on the xyz-space

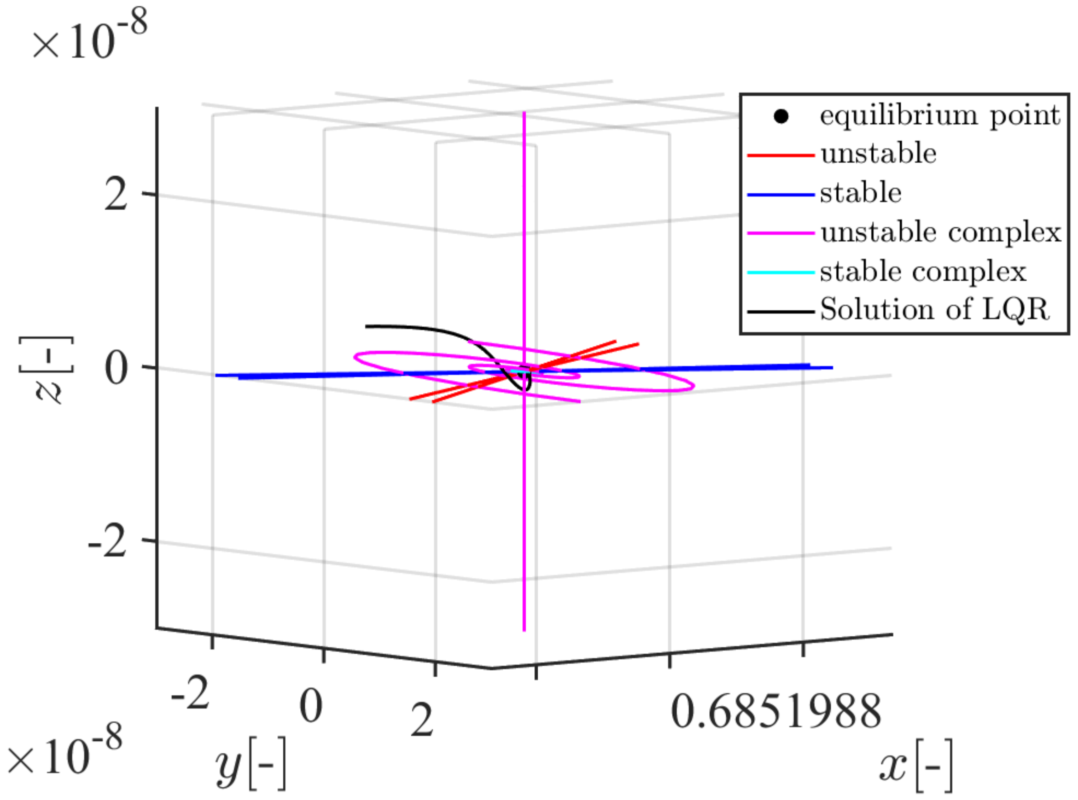
Relationship with stabilization by LQR optimal control

LQR optimal control
- $\boldsymbol{Q} = 0 \rightarrow$ cannot stabilize
- $\boldsymbol{Q} > 0 \rightarrow$ can stabilize

Manifold around equilibrium point when $\boldsymbol{Q} = \boldsymbol{I}$

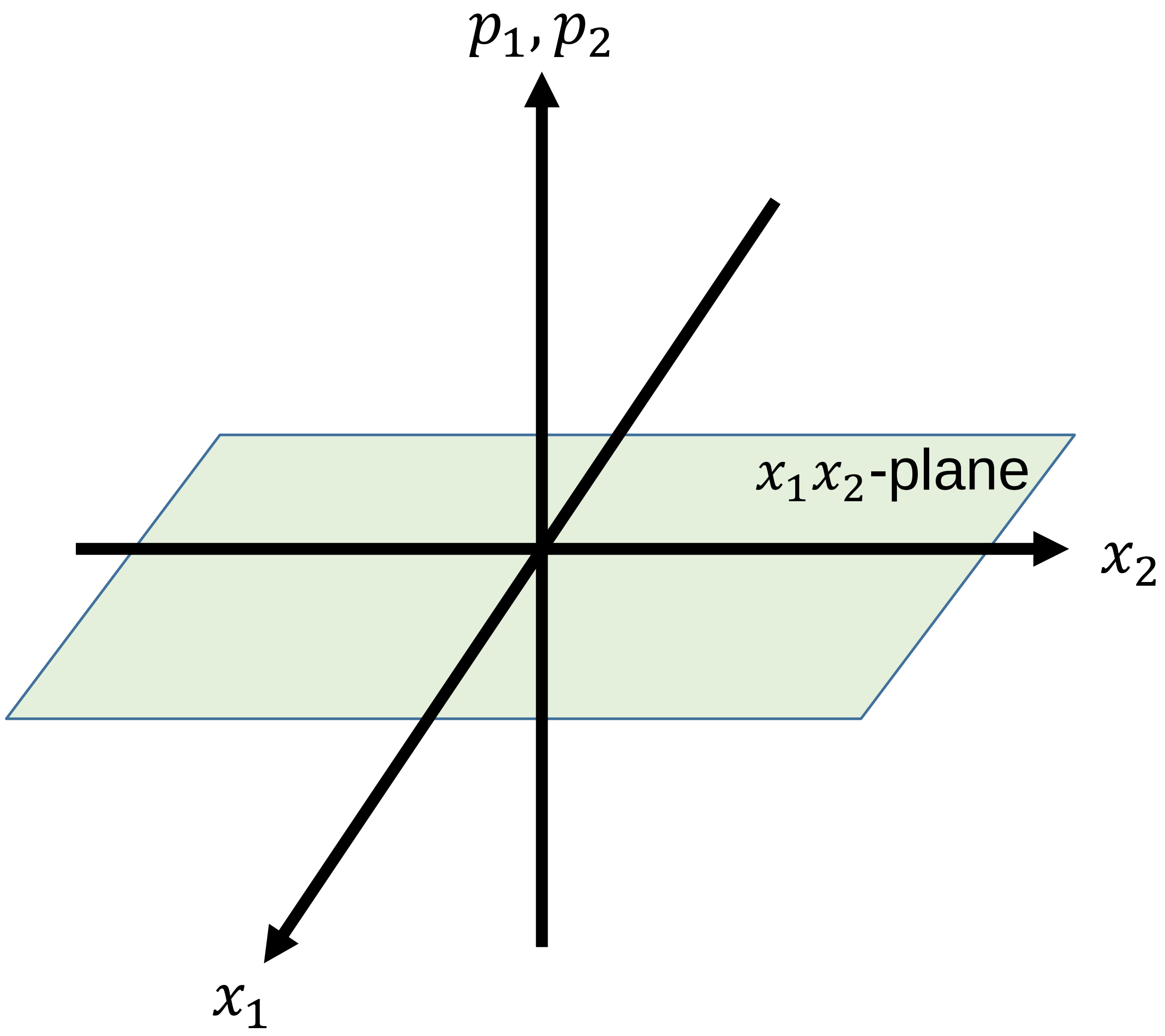


initial : arbitrary point on the xy-plane

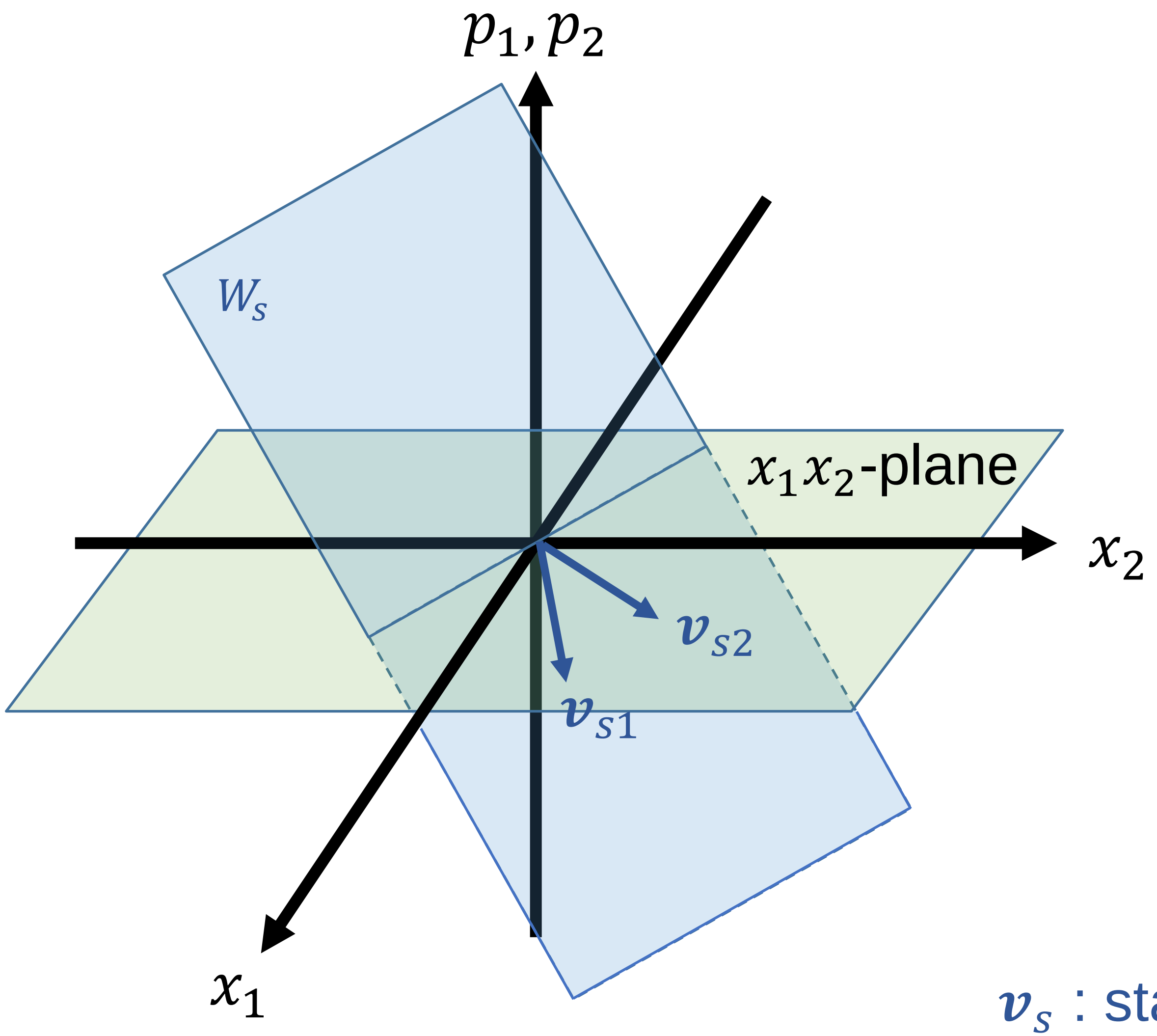


initial : arbitrary point on the xyz-space

Relationship with stabilization by LQR optimal control



Relationship with stabilization by LQR optimal control



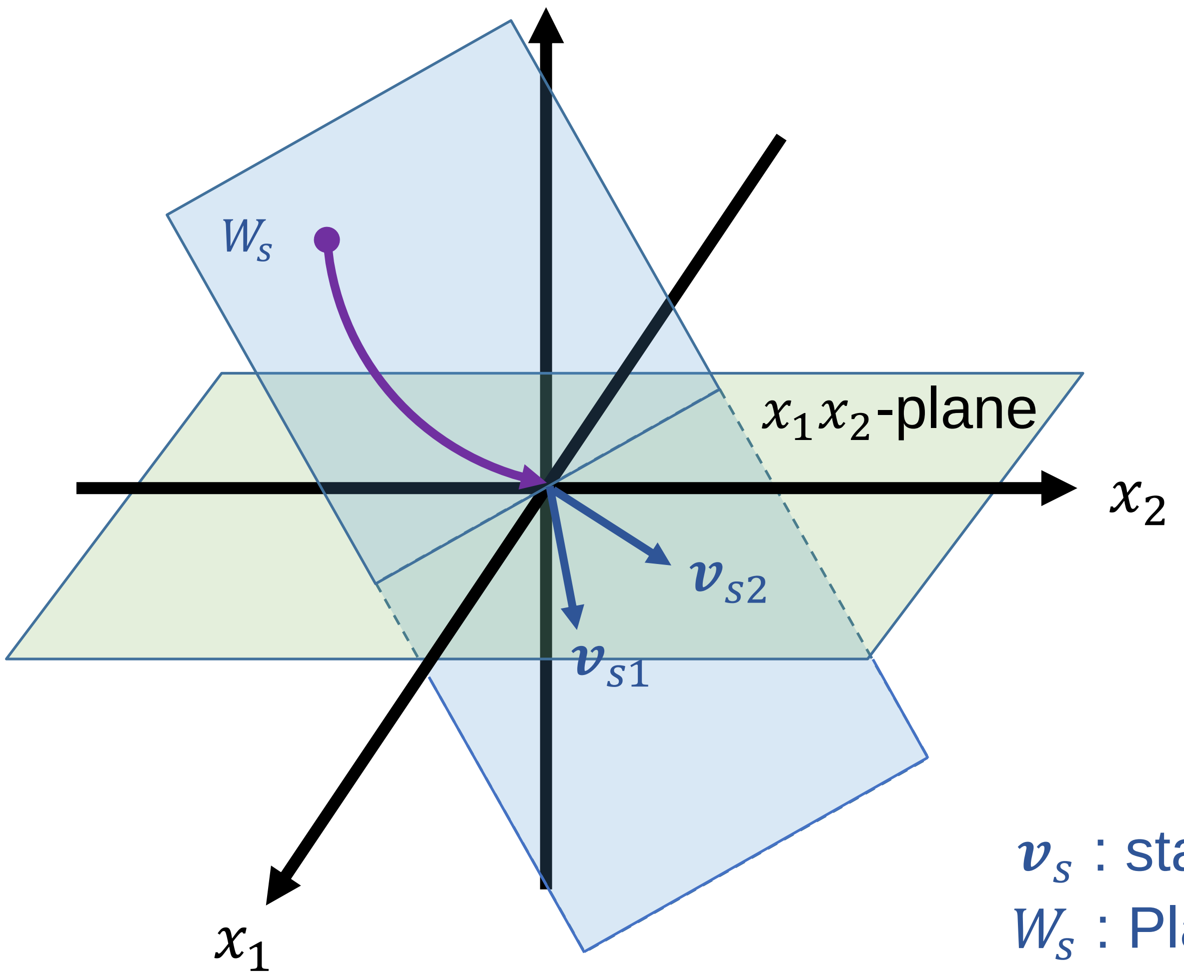
$\boldsymbol{v}_s$: stable eigenvector

W_s : Plane of stable manifold

Relationship with stabilization by LQR optimal control

12-dim dynamical system — Restriction → LQR optimal control $\boldsymbol{p} = \boldsymbol{X}\boldsymbol{x}$

Trajectory on stable manifold → Optimal control trajectory



$\boldsymbol{v}_s$: stable eigenvector

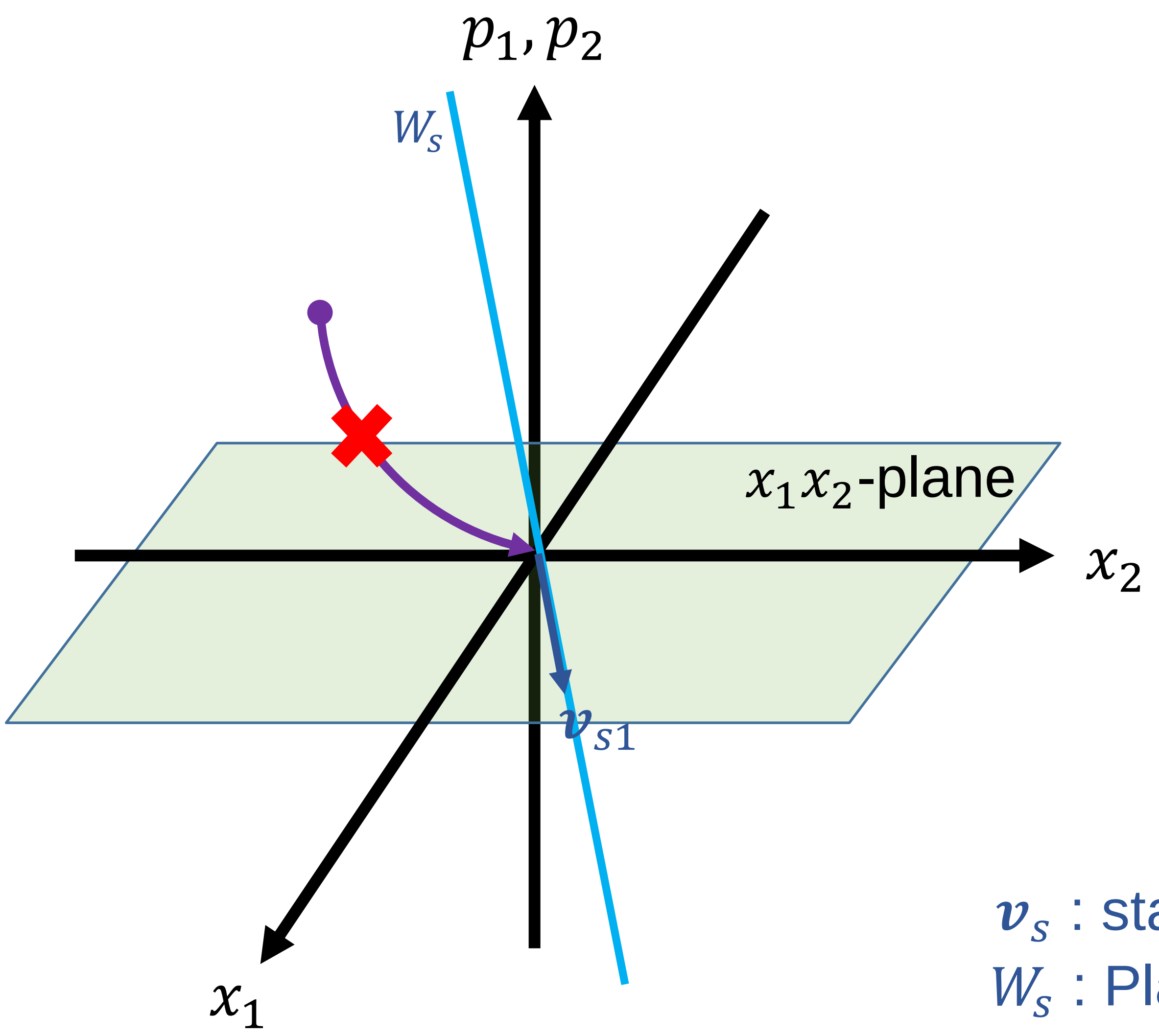
W_s : Plane of stable manifold

Relationship with stabilization by LQR optimal control

12-dim dynamical system
Trajectory on stable manifold

Restriction

LQR optimal control $\boldsymbol{p} = \boldsymbol{X}\boldsymbol{x}$
Optimal control trajectory



$\boldsymbol{v}_s$: stable eigenvector
W_s : Plane of stable manifold

What is center manifold in optimal control theory?

LQR optimal control

- $\boldsymbol{Q} = 0 \rightarrow$ cannot stabilize
- $\boldsymbol{Q} > 0 \rightarrow$ can stabilize

There exists **a stabilizing solution** of Riccati equation. ($\boldsymbol{Q} > \mathbf{0}$)

⟺ There are 6 stable eigenvectors, we can always find a corresponding 12-dim trajectory on stable manifold.

There is **no stabilizing solution** ($\boldsymbol{Q} = \mathbf{0}$)

⟺ There are *6-n* stable eigenvectors, the trajectory converges to the *n*-dim center manifold.

Relationship with stabilization by LQR optimal control

The number of stable manifolds when $\boldsymbol{Q} = \boldsymbol{0}$

Equilibrium point	The number of eigenvector		
	Stable	Unstable	Center
AEP_x (natural)	2 (xy-plane) 0 (z-direction)	2 (xy-plane) 0 (z-direction)	4 (xy-plane) 4 (z-direction)
AEP_z	3 (xy-plane) 2 (z-direction)	3 (xy-plane) 2 (z-direction)	2 (xy-plane) 0 (z-direction)
AEP_xy	3 (xy-plane) 0 (z-direction)	3 (xy-plane) 0 (z-direction)	2 (xy-plane) 4 (z-direction)
AEP_xz	4 (xyz-space)	4 (xyz-space)	4 (xyz-space)

The number of eigenvectors required to stabilize

xy-plane → 4
z-direction → 2
} xyz-space → 6

Relationship with stabilization by LQR optimal control

The number of stable manifolds when $Q = 0$

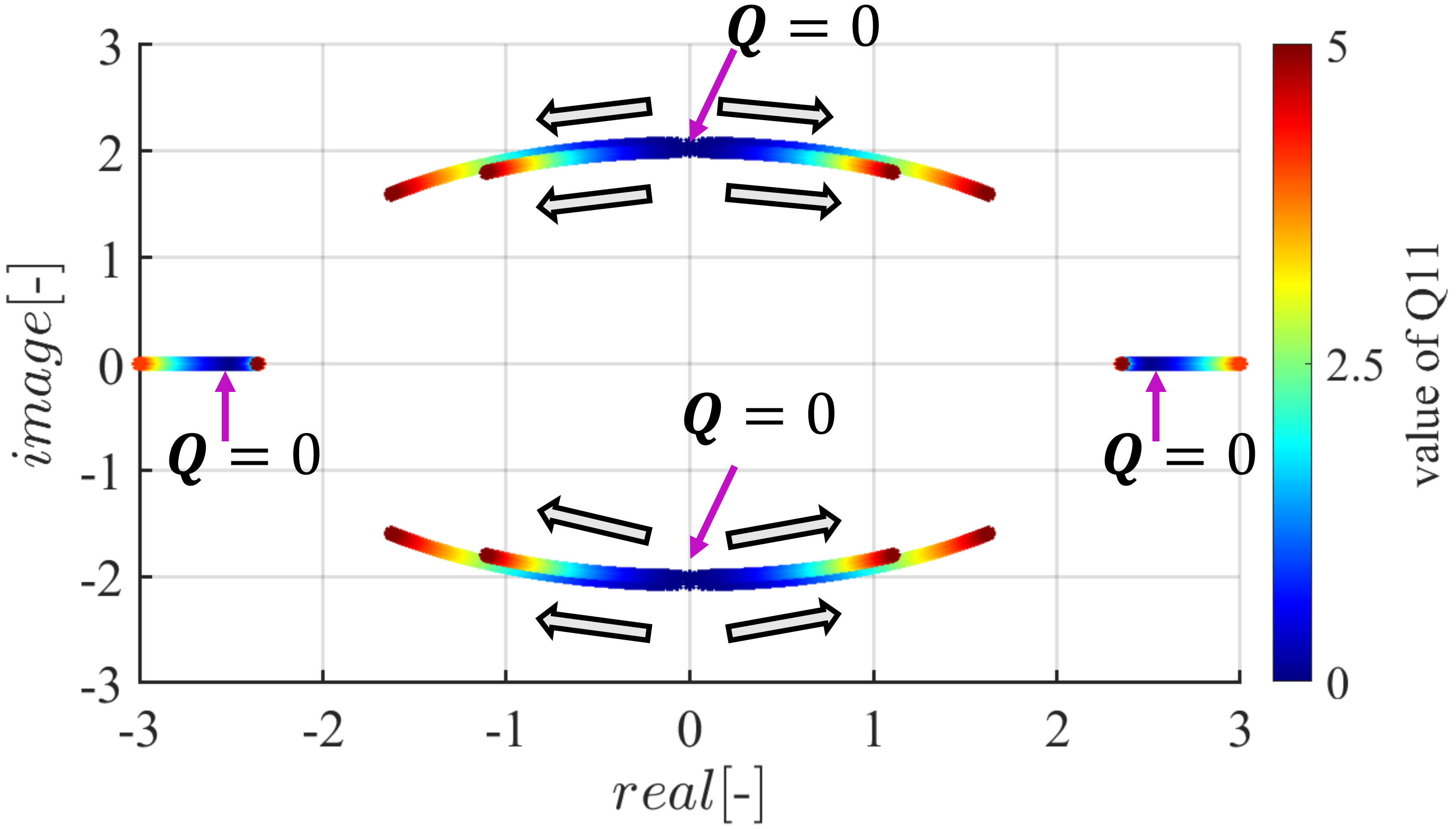
Equilibrium point	The number of eigenvector		
	Stable	Unstable	Center
AEP_x (natural)	2 (xy-plane) 0 (z-direction)	2 (xy-plane) 0 (z-direction)	4 (xy-plane) 4 (z-direction)
AEP_z	3 (xy-plane) 2 (z-direction)	3 (xy-plane) 2 (z-direction)	2 (xy-plane) 0 (z-direction)
AEP_xy	3 (xy-plane) 0 (z-direction)	3 (xy-plane) 0 (z-direction)	2 (xy-plane) 4 (z-direction)
AEP_xz	4 (xyz-space)	4 (xyz-space)	4 (xyz-space)

The number of eigenvectors required to stabilize

xy-plane → 4
z-direction → 2
} xyz-space → 6

Relationship to stabilization by LQR optimal control

For $\boldsymbol{Q} > 0$, we can confirm the 12-dim dynamical system has 6-dim stable manifold from the root locus.



The change of the eigenvalue at the equilibrium point on the x-axis when $\boldsymbol{Q}$ changed.

Conclusion

Conclusion

Investigated the equilibrium point with continuous optimal control inputs and its dynamical structures

✓Equations of motion for the dynamical system with optimal control inputs that minimize the quadratic cost function are derived.

✓Conditions for equilibrium points in Hill3BP were derived.

✓The stability of certain unstable equilibrium points in Hill3BP were investigated and the dynamical structures around them were investigated.

✓Compare the solution of LQR and dynamical structure with optimal control inputs