The Institute of Space and Astronautical Science
Report S. P No. 1, March 1983

Self-Semilar Solutions of Hydrodynamic Equations for Expanding Layer with Boundary Condition Specified by Differential Equation

By

Hiroyuki MURAKAMI*, Moritake TAMBA** and Keishiro NIU***

(February 1, 1983)

Summary: A set of self-similar solutions of the hydrodynamic partially differencial equations are derived for an expanding layer. The equations consist of continuity, momentum and energy equations. Energy is supplied into the layer constantly in time and homogeneously in space. One end surface of the layer is fixed in the space and the motion of the other end is specified by a partially differential equation. By using a similar variable, the equations are transformed to a set of ordinary differential equations. Forms of these ordinary differential equations and hence the self-similar solutions are not unique according to the choice of similar variables. However, the solutions themselves as functions of time and space are shown to be unique.

1. Introduction

Let us investigate the expanding motion of the A layer in Fig. 1. The one end of the A layer is fixed in space at 0. Energy is supplied into the A layer constantly in time and homogeneously in space. The A layer expands and pushes the B layer which does not receive energy from outside, and is accelerated by the pressure of the A layer acting on the boundary between the A and B layers. This situation comes from the light-ion-beam inertial confinement fusion (LIB ICF). The target of LIB consists of three layers of, for example, Pb, Al and DT fuel layers [1]. The ion beam penetrates the Pb (C) layer and deposits its energy in the Al (A) layer. The Al layer expands. Since the Pb layer is heavy, however, the surface a in Fig. 1 can not move. The Al layer pushes the DT layer by the pressure acting on the boundary b. Thus the DT fuel is accerelated to a high velocity.

Now the situation is assumed to be plane and one-dimensional. Of course, the phenomenon is time- and space-dependent. The purpose of this paper is to obtain the self-similar solutions.

* The Metropolitan College of Technology
** The Institute of Physical and Chemical Rerearch
*** Department of Energy Sciences, Tokyo Institute of Technology

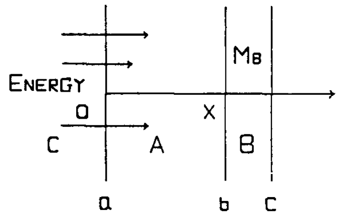

Fig. 1. The profile of the expansion motion of the A layer.

2. Basic Equations

The basic equations of the expanding A layer are written down as follows:
The equation of continuity

$$\frac{\partial \bar{\rho}}{\partial t}+\frac{\partial \bar{\rho}\bar{u}}{\partial x}=0. \qquad (1)$$

The equation of motion

$$\bar{\rho}\frac{\rho \bar{u}}{\partial t}+\bar{\rho}\bar{u}\frac{\partial \bar{u}}{\partial x}=-\frac{\partial \bar{p}}{\partial x}. \qquad (2)$$

The equation of energy

$$\bar{\rho}\frac{\partial \bar{e}_i}{\partial t}+\bar{\rho}\bar{u}\frac{\partial \bar{e}_i}{\partial x}+\bar{p}\frac{\partial \bar{u}}{\partial x}=H_b\bar{\rho}. \qquad (3)$$

The boundary condition between the A and B layers

$$\bar{p}=M_B\left(\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}\right). \qquad (4)$$

Here t denotes the time, x the distance, $\bar{\rho}$ the density, $\bar{u}$ the velocity, $\bar{p}$ the pressure, $\bar{e}_i$ the internal energy, H_b the input power density into the A layer per unit area and M_B the mass of the B layer per unit area. The internal energy $\bar{e}_i$ and the pressure $\bar{p}$ are connected with the temperature $\bar{T}$ or the density $\bar{\rho}$ through $\bar{e}_i=3/2R\bar{T}$, $\bar{p}=\bar{\rho}R\bar{T}$ (R is the gas constant), respectively.

The boundary conditions are as follows; at $x=0$ (at the bounary between the A and C layers) $\bar{u}=0$, and at the boundary between the A and B layers, the velocity of the A layer must coinside with the velocity of the B layer as eq. (4) points out. (The compressibility of the B layer is neglected here.)

3. Transform to Ordinary Differential Equations

Here the following similar variable is chosen as the only one independent variable,

$$\xi=k_1 x^g t^{-h}. \qquad (5)$$

The dependent variables $\overline{T}$, $\bar{u}$ and $\bar{\rho}$ are transformed into $T(\xi)$, $u(\xi)$ and $\rho(\xi)$ as follows;

$$T=k_2 x^A t^D T(\xi), \tag{6}$$

$$u=k_3 x^B t^E u(\xi), \tag{7}$$

$$\rho=k_4 x^C t^F \rho(\xi). \tag{8}$$

where k_1, k_2, k_3, k_4, g, h, A, B, C, D, E and F are constant. By using eqs. (5)–(8), eqs. (1)–(4) are written down as follows:

$$\begin{aligned} &F\xi^G x^{C-gG} t^{F-1+hG}\rho - h\xi^{H+1} x^{C-gH} t^{F-1+hH}\frac{\partial\rho}{\partial\xi} \\ &+k_3(B+C)\xi^I x^{B+C-1-gI} t^{E+F+hI} u\rho \\ &+g k_3 \xi^{J+1} x^{B+C-1-gJ} t^{E+F+hJ}\frac{\partial\rho u}{\partial\xi}=0, \end{aligned} \tag{9}$$

$$\begin{aligned} &E k_3 \xi^K x^{B+C-gK} t^{E+F-1+hK}\rho u - h k_3 \xi^{L+1} x^{B+C-gL} t^{E+F-1+hL}\rho\frac{\partial u}{\partial\xi} \\ &+Bk_3^2 \xi^M x^{2B+C-1-gM} t^{2E+F+hM}\rho u^2 + gk_3^2 \xi^{N+1} x^{2B+C-1-gN} t^{2E+F+hN}\rho u\frac{\partial u}{\partial\xi} \\ &+Rk_2 \xi^O (A+C) x^{A+C-1-gO} t^{D+F+hO}\rho T \\ &+gRk_2 \xi^{P+1} x^{A+C-1-gP} t^{D+F+hP}\frac{\partial\rho T}{\partial\xi}=0, \end{aligned} \tag{10}$$

$$\begin{aligned} &\xi^Q \frac{3}{2} Rk_2 D\, x^{A+C-gQ} t^{D+F-1+hQ}\rho T - \xi^{S+1}\frac{3}{2} Rk_2 h\, x^{A+C-gS} t^{D+F-1+hS}\rho\frac{\partial T}{\partial\xi} \\ &+\xi^U \frac{3}{2} Rk_3k_2 A\, x^{A+B+C-1-gU} t^{D+E+F+hU}\rho u T \\ &+\xi^{V+1}\frac{3}{2} Rk_3k_2 g\, x^{A+B+C-1-gV} t^{D+E+F+hV}\rho u\frac{\partial T}{\partial\xi} \\ &+\xi^W Rk_2k_3 B\, x^{A+B+C-1-gW} t^{D+E+F+hW}\rho u T \\ &+\xi^{X+1} Rk_2k_3 g\, x^{A+B+C-1-gX} t^{D+E+F+hX}\rho T\frac{\partial u}{\partial\xi} - \xi^Y \overline{H} x^{C-gY} t^{F+hY}\rho=0, \end{aligned} \tag{11}$$

$$\begin{aligned} &-\xi^\alpha Rk_2k_4\, x^{A+C-g\alpha} t^{D+F+h\alpha}\rho T + \xi^\beta M_{DT}k_3 E\, x^{B-g\beta} t^{E-1+h\beta} u \\ &-\xi^{\gamma+1} M_{DT}k_3 h\, x^{B-g\gamma} t^{E-1+h\gamma}\frac{\partial u}{\partial\xi} + \xi^\delta M_{DT}k_3^2 B\, x^{2B-1-g\delta} t^{2E+h\delta} u^2 \\ &+\xi^{\varepsilon+1} M_{DT}k_3^2 g\, x^{2B-1-g\varepsilon} t^{2E+h\varepsilon} u\frac{\partial u}{\partial\xi}=0, \end{aligned} \tag{12}$$

where G, H, I, J, K, L, M, N, O, P, Q, R, S, U, V, W, X, Y, g, h, α, β, γ, δ and ε are constant. These constants and A, B, C, D, E, F are determined in order that the powers of t and x if terms in each equation of (9)–(12) are equal with each other. The ratio of g to h is uniquely decided as follows:

$$\frac{g}{h}=\frac{2}{3} \tag{13}$$

The constants A and D, B and E, C and F must satisfy the following equations, respectively:

$$3A+2D=2, \tag{14}$$

$$2E+3B=1, \tag{15}$$

$$3C+2F=-3. \tag{16}$$

A set of self-similar variables to satisfy eqs. (14)–(16) is chosen here as follows:

$$\xi=k_1 x^2 t^{-3}, \tag{17}$$

$$\bar{T}=k_2 t^{2/3} x^{-1/3} T(\xi), \tag{18}$$

$$\bar{u}=k_3 x^{1/2} t^{-1/4} u(\xi), \tag{19}$$

$$\bar{\rho}=k_4 t^{-3/2} \rho(\xi), \tag{20}$$

As is clear from eq. (13), ξ decided by eq. (17) is not unique. The right hand sides of eqs. (17)–(19) can also be multiplied by arbitrary functions of ξ.

To examine the dependences of $\bar{T}$, $\bar{u}$ and $\bar{\rho}$ or t and x at the boundary between the A and C layers (at $\xi=1$), the following equations are obtained by substituting $x=k_1^{-1/2} t^{3/2}$ or $t=k_1^{1/3} x^{2/3}$ into eqs. (18)–(20),

$$\bar{T}\propto t\,T(\xi), \qquad \bar{T}\propto x^{2/3}T(\xi), \tag{21}$$

$$\bar{\rho}\propto t^{-3/2}\rho(\xi), \qquad \bar{\rho}\propto x^{-1}\rho(\xi), \tag{22}$$

$$\bar{u}\propto t^{1/2} u(\xi), \qquad \bar{u}\propto x^{1/3} u(\xi) \tag{23}$$

These dependences of $\bar{T}$, $\bar{u}$ and $\bar{\rho}$ on t and x are kept unchanged for other choice of self-similar variables than those by eqs. (17)–(20). It is concluded that the self-similar solutions are uniquely obtained not only on the boundary but also in the A layer.

By using the self-similar variables given by eqs. (17)–(20), eqs. (1)–(3) are transformed into the following set of the ordinary differential equations.

$$\frac{\partial T}{\partial \xi}=f_8, \tag{24}$$

$$\frac{\partial \rho}{\partial \xi}=\rho f_9, \tag{25}$$

$$\frac{\partial u}{\partial \xi}=f_{12}, \tag{26}$$

where $f_1=gRk_2\xi$, $f_2=3/2(k_3u-h/g\xi^S)$, $f_3=(k_3u-h/g\xi^L)(k_3u-h/g\xi^H)-Rk_2\xi^P T$, $f_4=Rk_2(k_3u-h/g\xi^H)T$, $f_5=k_3u(E\xi^K+Bk_3\xi^M u)$, $f_6=(k_3u-h/g\xi^L)[k_3(B+C)u-F\xi^G]$, $f_7=-Rk_2T(F\xi^G+3/2D\xi^Q)+H_b\xi^Y$, $f_8=[f_4(f_5-f_6)+f_3f_7]/[f_1(f_2f_3-f_4\xi^P)]$, $f_9=[f_1f_8\xi^{P-1}+1/\xi g(f_5-f_6)]/f_3$, $f_{10}=(k_3u-h/g\xi^H)f_9$, $f_{11}=1/g\xi[k_3(B+C)u-F\xi^G]$, $f_{12}=-1/k_3(f_{10}+f_{11})$.

4. Self-Similar Solution

A set of the ordinary differential equations (24)–(26) can be numerically integrated by using the Runge-Kutta-Gill method. Just before the A layer expands, that is, at $t=2.7\times10^{-9}$s the boundary conditions are as follows: at $\xi=0$; $T=5.8\times10^5$K, $\rho=2.79\times10^3$ kg/m^3, u$=0$ m/s, $T(\xi)=1$, $\rho(\xi)=1$, $u(\xi)=1$ and $\xi=1$; $x=5.3\times10^{-5}$. After the A layer expands, that is, $t=7\times10^{-8}$ s, the boundary condition is written down as follows: at $\xi=1$; $x=7\times10^{-3}$ m. The profiles of $T(\xi)$, $\rho(\xi)$ and $u(\xi)$ for ξ are shown in Fig. 2.

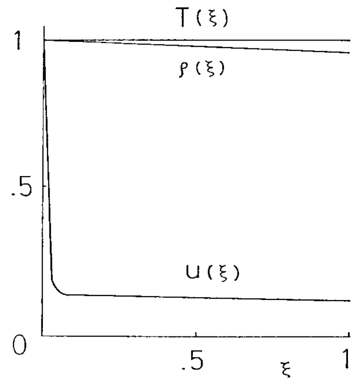

Fig. 2. The profile of $T(\xi)$, $\rho(\xi)$ and $u(\xi)$ for ξ by the numerical calculation.

By using the boundary conditions of $\bar{T}$, $\bar{\rho}$ and $\bar{u}$, we can derive that $k_1=7.00\times10^{-18}$, $k_2=2.13\times10^{14}$, $k_3=0.17\times10^6$ and $k_4=3.91\times10^{-10}$. After all at $\xi=1$ and $t=7\times10^{-8}$ s, we can obtain that $u=1.05\times10^5$ m/s (the Mach number is 7.89), the temperature $T=2.08\times10^4$ K and the density $\rho=20.3$ kg/m^3.

5. Conclusion

A set of self-similar solutions of the hydrodynamic partially differential equations which govern the expansion motion of the layer is derived, substituting the boundary condition which is expressed by a partially differential equation. The self-similar solutions themselves are not unique, but the corresponding solutions in the physical plane are uniquely determined. In general, the charasteristics of time-dependent solutions of partially differential equations can not be comprehensible. However, the behaviors of solutions become much clearer if the equations have the self-similar solutions [2].

References

[1] M. Tamba, N. Nagata, S. Kawata and K. Niu: Laser and Particle Beams 1 (1982) in press.
[2] M. Tamba, H. Murakami and K. Niu: Nagare 1 (1982, in Japanese) 36.