On the Interpolation Formula Based upon the Asymptotic Solution to the Kinetic Model Equation

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1. Introduction

When the Knudsen number Kn, the ratio of the mean free path for gas molecules to a reference length of the object, has a finite value, slips of flow velocity, density and temperature affect the macroscopic flow outside of the Knudsen layer. For $Kn \ll 1$ or for $Kn \gg 1$, an asymptotic solution in the power series of Kn or a solution for the free molecular flow is adequate to the problem. The effects of the slips, however, are significant in the transition regime ($0.1 \lesssim Kn \lesssim 10$), where the both solutions become invalid; the asymptotic solution may diverge as Kn increases. Accordingly, the solution in this regime were usually obtained numerically [1]–[5].

Another approach to the problem is to obtain an interpolation formula being valid in this regime, using the asymptotic solutions. Sherman's formula [6] for a property $Q(Kn)$ is given by

$$Q(Kn)/Q_f = (1 + Q_f/Q_c)^{-1}$$

where the suffixs c and f denote the values for continum and free molecular flow limits, respectively. When a good approximation Q^* for $Q(Kn \ll 1)$ is obtained, Sherman's formular yields

$$Q(Kn)/Q_f = Q^*/Q_f - Q^{*2}/(Q_f^2 + Q_f Q^*) \approx Q^*/Q_f + O(Kn^2/Q_f^2).$$

Thus, the Scherman's formula dose not correspond to the accuracy of the asymptotic solution.

In the present paper, we aim to obtain an interpolation formula which is valid in the whole range of the Knudsen number and its accuracy is corresponding to the accuracy of the asymptotic solution. As an example, we consider the cylindrical Couette flow problem to which the numercial solutions were obtained [1], [4].

2. Asymptotic Solution for $Kn \ll 1$

We consider a cylindrical Couette flow where the inner cylinder (radius a) is rotating with the circular velocity v_0 while the outer cylinder (radius b) stands still. The linearized Boltzmann equation in the cylindrical coordinates is given by

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$$c_r \frac{\partial \phi}{\partial r} + \frac{c_\theta^2}{r} \frac{\partial \phi}{\partial r_r} - \frac{c_r c_\theta}{r} \frac{\partial \phi}{\partial c_\theta} = J; \qquad J = \sum \lambda_i \phi_i, \tag{1}$$

where $\phi = \phi(r, c_r, c_\theta, c_z)$ is the linearized velocity distribution function, c_r the r component of the particle velocity c, c_θ the θ component, c_z the z component, λ_i an eigenvalue of the Boltzmann equation corresponding to an eigenfunction ϕ_i. A collision model that retains the correct thirteen moments of the Boltzmann equation is given by

$$J = \frac{\beta}{\varepsilon} [N + 2cv + t(c^2 - 3/2) + (1 - 1/\beta) p_{ij}(c_i c_j - c^2 \delta_{ij}/3) + (4/5)(1 - Pr/\beta) q_i c_i (c^2 - 5/2) - \phi]. \tag{2}$$

By use of the eigenvalue $-Pr(p/\mu)$ corresponding to the eigenfunction $\phi_i = c(c^2 - 5/2)$, Eq. (2) yields

$$J = Pr(p/\mu)[N + 2cv + t(c^2 - 3/2) + (1 - 1/Pr) p_{ij}(c_i c_j - c^2 \delta_{ij}/3) - \phi], \tag{2'}$$

where μ is the viscosity, p the pressure and Pr the Prandtl number.

The perturbed values of density N, temperature t, velocity v, and the shear force p_{ij} are given by

$$\begin{bmatrix} N \\ v \\ t \\ p_{ij} \end{bmatrix} = \pi^{-3/2} \iiint_{-\infty}^{\infty} \begin{bmatrix} 1 \\ c \\ (2/3)(c^2 - 3/2) \\ 2(c_i c_j - c^2 \delta_{ij}/3) \end{bmatrix} \phi e^{-c^2} dc, \tag{3}$$

where δ_{ij} is the Kronecker's delta.

Let introduce new variables ψ_k's as follows:

$$\psi = (1/\pi) \iint_{-\infty}^{\infty} H_k(c_\theta) \phi \exp(-c_\theta^2 - c_z^2) dc_\theta dc_z,$$

where $H_k(c_\theta)$ is the kth order Hermite polynomial, Multiplying Eq. (1) by H_k and integrating with respect to c_θ and c_z, we have

$$c_r \frac{\partial \psi_k}{\partial r} + \frac{1}{r} \frac{\partial}{\partial c_r} \left[\psi_{k+2} + \frac{2k-1}{2} \psi_k + \frac{k(k-1)}{4} \psi_{k-2} \right] - \frac{c_r}{r} (2\psi_{k+2} + k\psi_k) = (p/\mu)(v_\theta \delta_{1k} - \psi_k), \tag{4}$$

where v_θ is the circular velocity of the flow. In deriving Eq. (4), we assumed that the Mach number of the flow is sufficiently small and the temperature of the inner cylinder is same as the temperature of the outer cylinder, i.e., $N = t = 0$.

If we assume the diffuse reflection at the wall, the boundary conditions for solving Eq. (4) are given by

$$\psi_1(a, c_r > 0) = v_0, \qquad \psi_k(a, c_r > 0) = 0 \ (k \neq 1), \qquad \psi_k(b, c_r < 0) = 0. \tag{5}$$

A. Solution in the Knudsen Layer

We introduce a length scale d, $d=b-a$, and define a new variable x by $r=a+\tilde{K}nx$; $\tilde{K}n=(\mu/\rho d)(\tilde{R}T)^{1/2}$, where ρ is the density of gas and $\tilde{R}$ the gas constant. Expanding ψ_k in the power series of $\tilde{K}n$ as

$$\psi_k=\sum_{i=0}^{\infty}\psi_k^{(i)}\tilde{K}n_i,$$

and substituting into Eq. (4), we have a set of differential equations of $\psi_k^{(i)}$'s,

$$c_r\frac{\partial\psi_k^{(i)}}{\partial x}+\sum_{m=0}^{i-1}\left(\frac{x^{m-1}}{r_0^m}\right)\Delta_m(\psi_{k1}^{(m)},\ \psi_{k2}^{(m)},\ \cdots)=v_\theta^{(i)}\delta_{1k}-\psi_k^{(i)} \tag{6}$$

where $r_0=a/d$. The concrete form of Δ_m can be easily found from Eq. (4). Equation (6) and the boundary conditions (5) with the restriction that $\psi_k^{(i)}$ is finite at $x\to\infty$ yield

$$\psi_1^{(0)}=v^{(0)}=v_0,\qquad \psi_3^{(0)}=\psi_3^{(1)}=\psi_5^{(0)}=\psi_5^{(1)}=\psi_5^{(2)}=0.$$

Knowing the solution for $\psi_1^{(0)}$, Eq. (6) for i is solved one by one.

Equation (6) for $i=1$ is solved as follows: Let expand $\psi_1^{(1)}$ by use of the half-range Hermite polynomials, $\tilde{H}_k(\eta)$ [7],

$$\psi_1^{(1)}(c_r\gtrless 0)=\sum_{i=1}^{n}a_i^{\pm}(x)\tilde{H}_i(\eta);\quad \eta=|c_r|. \tag{7}$$

The orthogonal relation of the half-range Hermite polynomials is given by

$$\int_{-\infty}^{\infty}\tilde{H}_m\tilde{H}_n\exp(-\eta^2)d\eta=\delta_{mn}.$$

Substituting the expression (7) into Eq. (6), multiplying it by $H_k(\eta)$ and integrating with respect to η from 0 to ∞, we have a set of ordinary differential equations of the coefficients $a_i^{\pm}$'s,

$$\frac{dX}{dx}=\Gamma X;\qquad X=(a_1^+,\ a_2^+,\ \cdots,\ a_n^+,\ a_1^-,\ a_2^-,\ \cdots,\ a_n^-)^t, \tag{8}$$

where Γ is the square matrix of order $2n$ and the superscript t denotes the transpose of a vector or a tensor. (In details, see reference 7). A general solution of Eq. (8) is then obtained as

$$\begin{aligned}X^{(1)}&=\sum_{i=1}^{n-1}p_i^n u_i^n\exp(-\lambda_i x)+\sum_{i=1}^{n-1}p_i^p u_i^p\exp(\lambda_i x)+X_f^{(1)};\\ X_f^{(1)}&=(v^{(1)}-\sigma^{(1)}x)X_1+\sigma^{(1)}X_2.\end{aligned} \tag{9}$$

Here, X_f is the fluid dynamic solution corresponding to the macroscopic equation, i.e., Navier-Stokes equation. The vectors X_1 and X_2 are, respectively, the vectorial forms of 1 and c_r.

In the vectorial forms, the boundary conditions (5) at $x=a$ are reduced to $X^{+}(a)=0$, where the superscript + implies the upper half of the vector X. The restriction that the value of X must be finite at $x\to\infty$ requires $p_i^p=0$. Then, we have

$$\sum_{i=1}^{n-1} p_i^n u_i^{n+} + X_f^{(1)}(a) = 0,$$

from which the parameters p_i^n's and $v^{(1)}$ are determined. Thus, the general solution is expressed as

$$X^{(1)} = UQp^{(1)} + x_f^{(1)}\,;\quad U=(u_1, u_2, \cdots, u_{n-1}, X_1, X_2),$$
$$p^{(1)} = (p_1^n, p_2^n, \cdots, p_{n-1}^n)^t,$$

and the matrix $Q=\{Q\}$ is given by

$$Q_{ij} = \exp(-\lambda_i x)\delta_{ij}(1\leqq i\leqq n-1), \qquad Q_{ij} = \exp(\lambda_i x)\delta_{ij}(n\leqq i\leqq 2n-1),$$
$$Q_{2n-1\,j} = \delta_{2n-1\,j} + x\delta_{2n\,j}, \qquad Q_{2n\,j} = \delta_{2n\,j}.$$

The slip velocity $v^{(1)}$ is given by

$$v^{(1)} = c_f \sigma^{(1)}. \tag{10}$$

Equation (6) for $i=2$ is an inhomogeneous equation including $\psi_1^{(1)}$ obtained above;

$$\frac{dX^{(2)}}{dx} + \frac{1}{r_0}\Delta_1(X^{(1)}) = \Gamma X^{(2)}. \tag{11}$$

A general solution of Eq. (11) is given as

$$X^{(2)} = UQp^{(2)} + \frac{1}{r_0}UQ\int Q^{-1}U^{-1}\Delta_1(X^{(1)})dx + X_f^{(2)}\,; \tag{12}$$

$$X_f^{(2)} = \left\{v^{(2)} + \frac{x}{r_0}v^{(1)} + \left(\frac{1}{2}\right)\left(\frac{x}{r_0}\right)^2\sigma^{(1)} - \frac{x}{r_0}\sigma^{(2)}\right\}X_1$$
$$+\frac{1}{r_0^2}\left(\sigma^{(2)} - \frac{2x}{r_0}\sigma^{(1)}\right)X_2 + \frac{3}{r_0^3}\sigma^{(1)}X_3,$$

$$v^{(2)} = c_f\sigma^{(2)} + (-3A_t/r_0)\sigma^{(1)} + (P_k^{(2)}/r_0)\sigma^{(1)}, \tag{13}$$

where X_3 is the vectorial form of $(c_r^2-1/2)$. The parameters in $p^{(2)}$ and $v^{(2)}$ are obtained from the boundary conditions (5) and the condition $p_i^p=0$. The third order solution for $i=3$ is obtained in a similar manner as the case for $i=2$ and yields

$$v^{(3)} = c_f\sigma^{(3)} + \left(\frac{-3A_t}{r_0}\right)\sigma^{(2)} + \left(\frac{-12B_t}{r_0^2}\right)\sigma^{(1)} + \left(\frac{r_0P_{k1}^{(3)} + P_{k2}^{(3)}}{r_0^2}\right)\sigma^{(1)}. \tag{15}$$

The coefficients in $v^{(i)}$ were obtained as follows:

$$c_f = Q_1 = -1.0162,\quad A_t = -Q_2 - 1/2 = 0.7763,\quad B_t = -Q_3 - (3/2)Q_1 = 0.2964,$$
$$P_k^{(2)} = 0.7488,\qquad P_{k1}^{(3)} = 0.5430,\qquad P_{k2}^{(3)} = 0.9054,$$

where Q_i's are the notation given by Cercignani [8] and $P_k^{(i)}$'s are the contributions of the inhomogeneous terms to the slip velocity. Other higher order solutions may be obtained in the similar manner mentioned here.

B. Fluid Dynamic Solution in the Scale of r

A fluid dynamic solution to the Boltzmann equation can be obtained by substituting approximated expressions for the pressure tensors and heat fluxes, i.e., Equilibrium, Chapman-Enskog, Burnett, and the third order approximate solutions and so on. For the linearized model equation, these solutions can be easily found one by one by substituting the expansion form,

$$\psi_k=\sum_{i=0}^{\infty} \Psi_k^{(i)}\tilde{K}n,$$

into Eq. (4) where $\Psi_k^{(i)}$ is the ith order approximate solution of the distribution function. These are given as

$$\begin{aligned}
&\Psi_1^{(0)}=V^{(0)}, \qquad \Psi_1^{(1)}=V^{(1)}-c_r\sigma_0^{(1)},\\
&\Psi_1^{(2)}=V^{(2)}-c_r\sigma_0^{(2)}+(c_r^2-1/2)r\frac{\partial}{\partial r}\left(\frac{\sigma_0^{(1)}}{r}\right)+\frac{1}{2r^2}\frac{\partial}{\partial r}(r^2\sigma_0^{(1)}),\\
&\Psi_1^{(3)}=V^{(3)}-c_r\sigma_0^{(3)}+(c_r^2-1/2)r\frac{\partial}{\partial r}\left(\frac{\sigma_0^{(2)}}{r}\right)-c_r(c_r^3-3/2)r\frac{\partial^2}{\partial r^2}\left(\frac{\sigma_0^{(1)}}{r}\right).
\end{aligned} \tag{16}$$

Since, for the shear flow problem, the conservation equations of mass and energy are automatically satisfied, i.e., the radial velocity $v_r=0$ and T=const., we only need to consider the momentum equation of the circular velocity v_θ,

$$\frac{d}{dr}\left[r^2\int_{-\infty}^{\infty} c_r\Psi \exp(-c_r^2)dc_r\right]=0. \tag{17}$$

Substituting the asymptotic solution (16) into Eq. (17), it gives

$$v_\theta=v(0)(r/r_0)-(\sigma_0/2r_0)(r/r_0-r_0/r), \tag{18}$$

where $v(0)$ is the macroscopic flow velocity outside of the Knudsen layer at $r=a$. If Eq. (18) is consistent with the solution in the Knudsen layer, we have

$$v(0)=\sum_{i=0}^{\infty} V^{(i)}\tilde{K}n^i=\sum_{i=0}^{\infty} v^{(i)}\tilde{K}n^i, \qquad \sigma_0=\sum_{i=0}^{\infty}\sigma^{(i)}\tilde{K}n^i. \tag{19}$$

C. Asymptotic Solution in the Power Series of Kn

The solution near the outer cylinder is obtained, changing a, x, c_r, and σ_0 by b, $-x$, $-c_r$, and $(a/b)^2\sigma_0$, respectively. Combining the solution in the Knudson layer adjacent to the inner cylinder and that of outer cylinder through the solution (18) with Eq. (19), we obtain

$$v^{(i)}(b)=v^{(i)}(a)(b/a)-(\sigma^{(i)}/2r_0)(b/a-a/b) \qquad (i=1,2,3,\cdots). \tag{20}$$

Equation (20) with Eqs. (10), (13) and (15) yields

$$\tilde{\sigma}^{(1)}=v_0RA,\ \tilde{\sigma}^{(2)}=AB\tilde{\sigma}^{(1)},\ \tilde{\sigma}^{(3)}=A[B\tilde{\sigma}^{(2)}+C\tilde{\sigma}^{(1)}+C(-P_k^{(2)}/3A_t)\tilde{\sigma}^{(1)}],$$
$$\tilde{\sigma}^{(4)}=A[B\tilde{\sigma}^{(3)}+C\tilde{\sigma}^{(2)}+D\tilde{\sigma}^{(1)}+C(-p_{k1}^{(3)}/3A_t)\tilde{\sigma}^{(1)}+D(-P_{k2}^{(3)}/12B_t)\tilde{\sigma}^{(1)}], \tag{21}$$

where

$$\tilde{\sigma}^{(1)}=\sigma^{(i)}/r_0^2,\qquad A=(2R/r_0)/(R^2-1),\qquad B=(c_f/R^2)/(R^3+1),$$
$$C=(-3A_t/r_0)(R^4+1)/R^3,\quad D=(-12B_t/r_0^2)(R^5+1)/R^4,\quad R=b/a,$$

The nth order approxixmation for the shear force at the inner cylinder and the flow velocity are given by

$$\tau=\sum_{i=0}^{n}\tilde{K}n^{(i)}\tilde{\sigma}^{(i)}r_0^2,\qquad v_{\theta n}(r)=X_1\cdot\left(\sum_{i=0}^{n}\tilde{K}n^iX^{(i)}\right)_{x\to r}, \tag{22}$$

where $\cdot$ implies the inner product of the vectors.

3. Interpolation Formula

Equation (21) shows that the shear force in the power of $\tilde{K}n$ can be obtained one by one from $\tilde{\sigma}^{(1)}$. The nth approximation, however, diverges to $+\infty$ or $-\infty$ as Kn tend to infinity according to the sign of the highest order. Summing up Eq. (21) formally from $i=1$ to ∞, we have

$$\tau/r_0^2=\sum_{i=1}^{\infty}\tilde{\sigma}^{(i)}\tilde{K}n^i$$
$$=\frac{\tilde{\sigma}^{(1)}\tilde{K}n\left[1+AC\sum_{i=1}^{\infty}(-P_{k1}^{(i)}/3A_t)\tilde{K}n^i+AD\sum_{i=3}^{\infty}(-P_{k2}^{(i)}/12B_t)\tilde{K}n^i+\cdots\right]}{1-A(B\tilde{K}n+C\tilde{K}n^2+D\tilde{K}n^3+\cdots)} \tag{23}$$

If the righthand-side of Eq. (23) converges for any values of the Knudsen number, Eq. (23) gives the exact value of the shear force. If the approximation is truncated at n, the numerator has the order of $O(\tilde{K}n^n)$, while the denominator has the order of $O(\tilde{K}n^{n-1})$. Equation (23), then, diverges as $\tilde{K}n$ approaches to infinity.

Here, we transform Eq. (23) as follows:

$$\frac{\tau}{r_0^2}=\left(\frac{\tilde{\sigma}^{(1)}\tilde{K}n}{1+\alpha_1\tilde{K}n}\right)\left[1+\frac{\alpha_2\tilde{K}n^2}{1+a_3\tilde{K}n+\omega\tilde{K}n^2}+O(\tilde{K}n^4)\right], \tag{24}$$

where

$$\alpha_1=AB,\qquad \alpha_2=(1-P_k^{(2)}/3A_t)AC,$$
$$\alpha_3=(D/C)[1-(P_{k1}^{(3)}+P_{k2}^{(3)})/12B_t-ABC/D]/(1-P_k^{(2)}/2A_t).$$

If we choose the value of ω so that $\tau(Kn\to\infty)$ may give $\tau_f=v_0/\sqrt{\pi}$, Eq. (24) yields an interpolation foumula which gives a solution for $\tilde{K}n\ll 1$ including correct terms up to the order of $O(\tilde{K}n^3)$ and gives the exact value for the free molecular flow limit.

In general, Eq. (23) can be expressed as

$$\tau/r_0^2=\left(\frac{\tilde{\sigma}^{(1)}Kn}{1+\alpha_1 Kn}\right)\left[1+\frac{\alpha_2 Kn^2}{1+\alpha_3 Kn+\alpha_4 Kn^2}+\frac{\alpha_5 Kn^5}{1+\alpha_6 Kn+\cdots+\alpha_{10}Kn^5}+\cdots\right]. \tag{25}$$

If we obtain asymptotic solutions in the power series of $\tilde{K}n$ one by one, the values of α_i's can be determined accordingly; the highest coefficient should be determined as in Eq. (24). The interpolation formula is then corresponding to the accuracy of the asymptotic solution. The interpolation formula obtained from the nth order approximation [Eq. (22)] coincides with the one that is obtained from Eq. (25) with nth order approximation.

4. Results and Discussion

The results for $n=3$ are shown in Table I and Figs. 1–3 where the numerical soluations by Cercignani and the results by Sherman's formula are also presented. For the case, $r_0 \leqq 1$, the obtained formula gives fairly good results and indicates the degree of improvement of the present formula to the Sherman's one. For the case, $r_0=2$, present formular shows a better agreement than the Sherman's with the numerical results but the agreement is not so good. This is, obviously, atributed to the insufficiency of the approximation of the asymptotic solution.

Table I. Stress constant τ/τ_f

$1/Kn$	$R=1.235$		$R=2.0$		$R=3.0$	
	Eq. (24)	Ref. 1	Eq. (24)	Ref. 1	Eq. (24)	Ref. 1
0.001	0.9998	0.9997	0.9998		0.9999	
0.4	0.8939	0.8878	0.9372		0.9445	
1.0	0.7442	0.7441	0.8618	0.8863	0.8813	0.9380
2.0	0.5739	0.5768	0.7588	0.7790	0.8071	0.8727
3.0	0.4653	0.4679	0.6727	0.6876	0.7517	0.8101
4.0	0.3909	0.3926	0.6004	0.6118	0.7048	0.7526
5.0	0.3369	0.3377	0.5398	0.5490	0.6622	0.7005
7.0	0.2638	0.2631	0.4461	0.4527	0.5865	0.6116
9.0	0.2167	0.2149	0.3784	0.3830	0.5225	0.5399

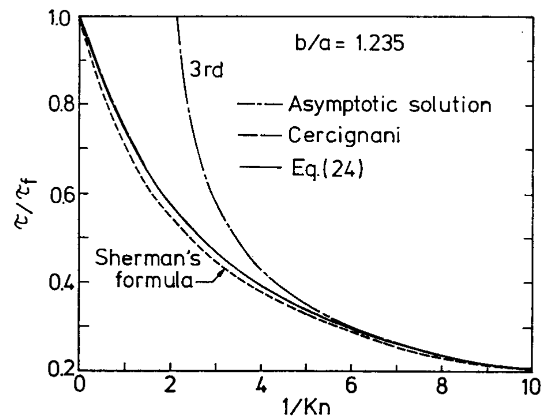

Fig. 1. Stress constant vs inverse Knudsen number; $b/a=1.235$.

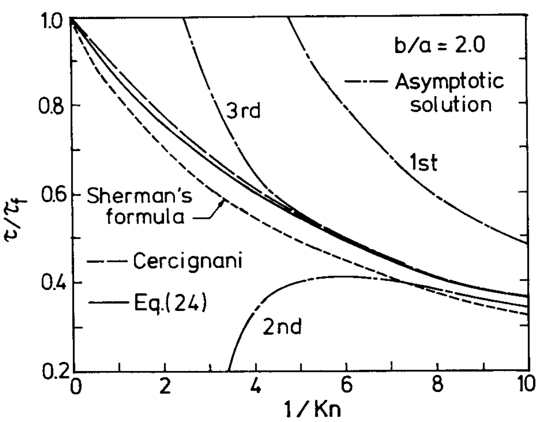

Fig. 2. Stress constant vs inverse Knudsen number; $b/a=2.0$.

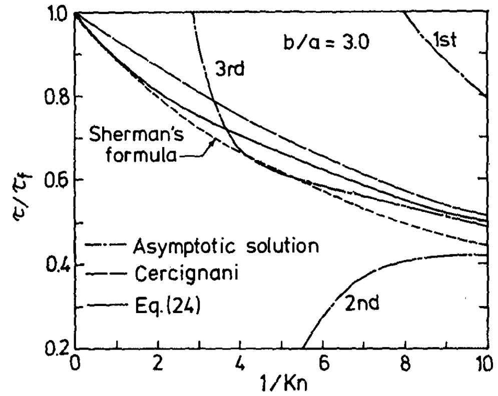

Fig. 3. Stress constant vs invers Knudsen number; $b/a=3.0$.

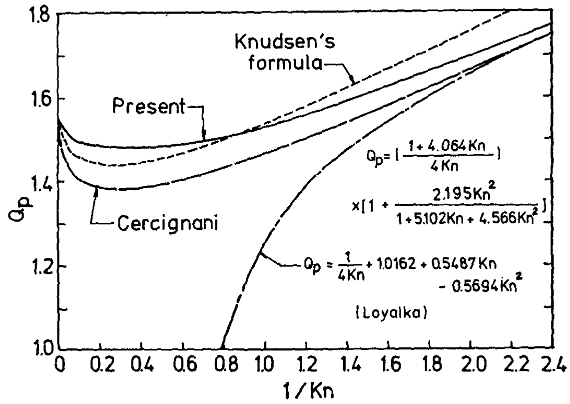

Fig. 4. Mass ffow rate vs inverse Kundsen number.

In Fig. 4 is shown the results of Eq. (24) applied to the Poiseuille flow problem, where the coefficients of the formula were obtained from the asymptotic solution obtained by Loyalka [3]. The derivative of Eq. (24) with respect $1/\tilde{K}n$ gives a finite value at $\tilde{K}n\to\infty$, while the asymptotic solution for $1/\tilde{K}n\ll 1$ [2] gives

$$\{\partial\tau/\partial(1/\tilde{K}n)\}_{\tilde{K}n\to\infty}=O(-\log\tilde{K}n).$$

Thus, the interpolation formula (24) may have a poor convergence for the problems which include singularities at $Kn\to\infty$.

5. Conclusions

The cylindrical Couette flow problem was solved using the half-range Hermite polynomials and the asymptotic solutions in the power series of the Knudsen number were obtained. The nth order interpolation formula was obtained from the nth order asymptotic solution for $\tilde{K}n\ll 1$. The results improved the Sherman's formula to the extent of the accuracy of the asymptotic solution.

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