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Theoretical Study of Strong Interaction of Vortex Filaments

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Summary: A brief report is made on the recent analysis on the recombination of two vortex filaments in a viscous incompressible fluid. The analysis is confined to a local flow field where this interaction occurs, and is based on the vorticity equation with the viscous term along with several assumptions, including fluid impulse conservation and spatial symmetry of the flow field. The flow field is expanded into polynomials of coordinates and the coefficient, which are functions of time, are obtained by substituting the expansions into the vorticity equation. It is proved that this process is completed within a short time $t^* = 0(\sigma^2/\Gamma)$, where σ and Γ are the size and the circulation of the vortex filaments respectively, and that the viscous effect is necessary for occurrence of the process.

§I. INTRODUCTION

Vortex dynamics is a powerful tool for theoretical investigation of flows in which the vorticity is confined within a narrow region and the viscosity plays no essential role. The attractiveness of the vortex approach lies in the fact that vortex motions are governed by simple equations derived from the Navier-Stokes equation. Two cases have been investigated extensively; one is an ensemble of two-dimensional point vortices whose motions are governed by the Hamiltonian dynamics [*1*], and another is a single or several vortex filaments in three-dimensional motion which is determined uniquely by Biot-Savart's law. The motion of a single vortex filament is analysed by Hama [2], Betchov [*3*], Hasimoto [*4*] and Takaki [*5*], and interactions of two filaments are treated by Crow [*6*], Moffatt [*7*], Hopfinger *et al.* [*8*] and Takaki & Hussain [9]. The short wave instability, investigated by Widnall *et al.* [*10*] and Saffman [*11*], is also a phenomenon caused by the interaction between vortex filaments.

In all these cases, the distance between neighboring vortices is assumed larger than the vortex core size, and the potential flow theory is applied. On the other hand, there are certain types of interactions in which vortices approach each other and undergo strong interactions. A typical example of the strong interaction in the two-dimensional flow is the vortex merger, which has been investigated experimentally (Winant & Browand [*12*], Brown & Roshko [*13*], Hussain & Zaman [*14*]) and theoretically (Acton [*15*], Corcos & Sherman [*17*], Ashurst [*18*], Takaki & Kovasznay [*19*], Aref & Siggia [*20*]; Takaki & Tanaka [*21*]). Theoretical study of vortex merger in the two-dimensional case can be made by the use of the inviscid equation, because the vorticity fields is approximated well by a cloud of point vortices and the viscous effect is not strong compared with the mutual velocity induction.

The strong interaction in the three-dimentional case is the recombination of two filaments observed by several authors (Hama [22]; Crow [6], Kambe & Takao [23], Oshima & Asaka [24], Fohl & Turner [25]), where local elements of two filaments approach each other, are cut and then connected after being switched. Except for the MHD flow (Yeh & Axford [26]) this mechanism has not been studied theoretically so far and remains a big challenge for theoretists as Saffman & Baker [26] points out. This phenomenon is quite complicated, because the fluid motion varies rapidly in essentially three-dimentional manner. What is unknown even qualitatively is the behaviour of flow field in the localized region where two vortex filaments get contact to each other and the vorticity lines are cut and reconnected.

The purpose of the present analysis is to cope with this phenomenon by a simple model of time-dependent flow field which simulates the real flow within this region. Proposed model flow must satisfy the Navier-Stokes equation or the vorticity equation with the viscous term, because the viscosity plays an essential role in the mechanism. The full paper is now being prepared for a regular journal, and an outline is given here.

The present analysis is motivated not only by the interest in vortex dynamics but also by our claims that this process is one of the main aspects of development of three-dimensional turbulence from two-dimensional, less chaotic flows and that the mixing and noise production in the near field of a jet are considered to come primarily from the breakdown process of the initial toroidal structures through the recombination process (Hussain [27]).

§II. Basic Assumptions

Consider that two vortex filaments with opposite sense of rotation approach each other. Then, parts of the filaments close to each other will deform strongly and are pushed upwards as shown in Figure 1(a). After a recombination process the new filaments are arranged as shown in Figure 1(b), and the adjacent parts move downwards. It is reasonable to assume that this motion is confined to a local part of the filaments and that the distant parts remain unaffected for a short time during the interaction. This local fluid region is called an interaction region.

Now, we consider a change of the fluid impulse between the initial and the final states. The impulse is defined by

$$\boldsymbol{P}=\frac{1}{2}\rho\int \boldsymbol{r}\times\boldsymbol{\omega}\, dV, \tag{1}$$

where ρ is the fluid density, $\boldsymbol{r}$ is the position vector, ω is the vorticity, and the volume integral is taken over whole fluid region (Batchelor [28]). From the above assumption the change in P comes only from the integral in the interaction region. Contributions from the interaction regions give an upward- and a downward-directed impulses for the initial and the final states, respectively, hence violating impulse conservation. To compensate this difference, we need to superpose another fluid motion, i.e. a recoil

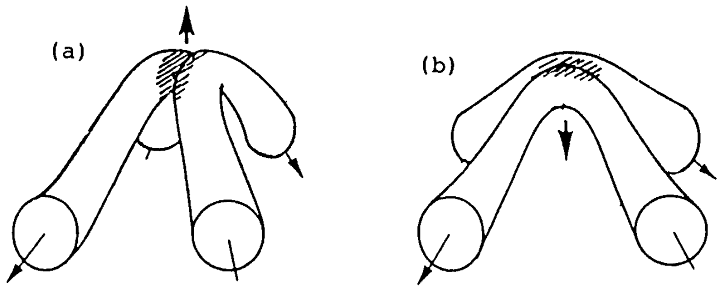

Fig. 1. Vortex configurations before (a) and after (b) the recombination process.

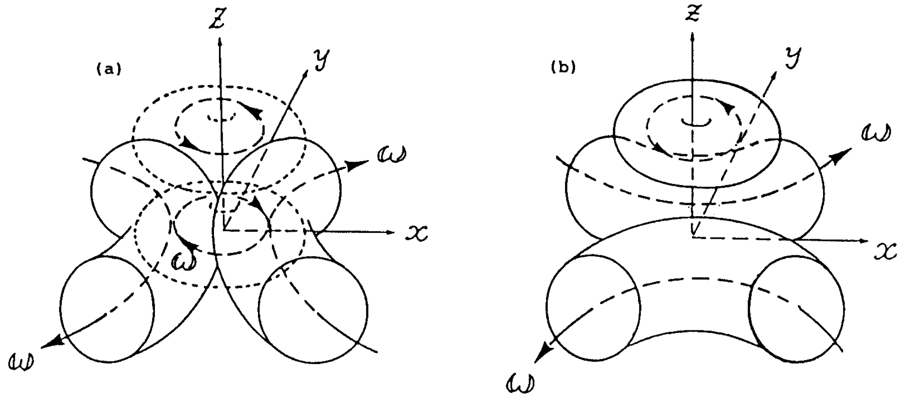

Fig. 2. Assumed mechanism of the process.

motion of a small-size fluid element ejected upwards.

This complicated process can be looked upon as superposing a pair of colinear, small ring vortices on the initial vortex filaments as shown in Fig. 2. These rings have equal size and opposite sense of rotation and of equal circulations, which are also equal to that of either of the initial filaments. Their center is at a position above that of the interaction region, so that the lower ring overlaps with the interaction region and produces a new vortex configuration, while the upper ring plays a role of recoil (see Fig. 2(b)).

This process is expressed quantitatively by asymptotic conditions of the velocity (v) and the vorticity (ω) fields. Let these fields in the initial and the final state and the ring pair be denoted by v_i, ω_i, v_f, ω_f, v_r, ω_r, respectively. The origin of t is chosen so that the process occurs at $t \cong 0$. Since the process is a very rapid one, instants before and after the process, to which the asymptotic conditions of the intermediate flow functions should refer, correspond to $t=0-$ and $0+$ on the time scales of the initial and the final states, respectively. Then, the asymptotic conditions are

$$\boldsymbol{u}_r(-\infty)=0 \quad \text{and} \quad \boldsymbol{u}_i(0-)+\boldsymbol{u}_r(t)\rightarrow\boldsymbol{u}_f(0+) \quad \text{for } t\rightarrow\infty, \tag{2a}$$

$$\boldsymbol{\omega}_r(-\infty)=0 \quad \text{and} \quad \boldsymbol{\omega}_i(0-)+\boldsymbol{\omega}_r(t)\rightarrow\boldsymbol{\omega}_f(0+) \quad \text{for } t\rightarrow\infty. \tag{2b}$$

The coordinate axes are introduced as shown in Fig. 2. It is assumed that the two filaments have a configuration with two planes of inversion symmetry: the xz- and yz-planes. The initial and final states differ in dependence on x and y, because the vortex filaments are aligned nearly with the y- and the x-axis and the velocity vectors are nearly confined in the xz- and yz-plane in these states, respectively. Next, the velocity and the vorticity fields are expanded by polynomials of coordinates up to $O(r^2)$ and $O(r^1)$, respectively, where $O(r^m)$ denotes the m-th order polynomial.

Now, since the x-component of the velocity u is an odd function of x and an even function of y and is independent on y in the initial state (see Fig. 2), only polynimials up to $O(r^2)$ allowed for u in the initial state are x and xz. The y-component v vanishes in the initial state. The z-component w is an even function of both x and y, and allowed polynomials are $1, z, x^2, y^2$. Polynomials for the vorticity and also the flow fields in the final state are obtained in the same way. The quantities $\Delta\omega_x$ and $\Delta\omega_y$ are expanded independently.

Since the center (denoted by $z=z_0$) of the interaction region moves in the z-direction, say with a velocity $w_0(t)$, during the process, the variable z in the polynomial expansion should be replaced by $z-z_0$, where

$$z_0(t)=\int w_0(t)dt. \tag{3}$$

The origin of the z-axis is chosen so that $z_0(0)=0$. The velocity $w_0(t)$ constitutes the $O(r^0)$ terms for w_i and w_f.

Now, the flow fields with polynomial expansion are obtained as follows:

$$\left.\begin{aligned} &u_i=-m_i x+k_i x(z-z_0),\\ &v_i=0,\\ &w_i=w_0(t)+m_i(z-z_0)-\frac{k_i}{2}(z-z_0)^2-\frac{l_i}{2}x^2. \end{aligned}\right\} \tag{5a}$$

$$\left.\begin{aligned} &\omega_{ix}=0,\\ &\omega_{iy}=(k_i+l_i)x+n_i x(z-z_0),\\ &\omega_{iz}=0. \end{aligned}\right\} \tag{5a}$$

$$\Delta\omega_{ix}=g_{ix}y, \qquad \Delta\omega_{iy}=g_{iy}x. \tag{5b}$$

In the same way for the final state we have

$$\left.\begin{aligned} &u_f=0,\\ &v_f=m_f y-k_f y(z-z_0),\\ &w_f=w_0(t)-m_f(z-z_0)+\frac{k_f}{2}(z-z_0)^2+\frac{l_f}{z}y^2. \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} &\omega_{fx}=(k_f+l_f)y+n_f y(z-z_0),\\ &\omega_{fy}=0,\\ &\omega_{fz}=0. \end{aligned}\right\} \tag{7a}$$

$$\Delta\omega_{fx}=g_{fx}y, \qquad \Delta\omega_{fy}=g_{fy}x. \tag{7b}$$

In these expansions the coefficients m_i, m_f, k_i, k_f, l_i, l_f, n_i, n_f, g_i, g_f, g_{fx}, g_{fy}, are functions of time.

The term $-m_i x$ or $m_f y$ is a flow pressing or separating the two filaments before or after the interaction. They are relatively weak, because these flow fields are contributed by far parts of filaments. Terms with coefficients k_i, k_f, l_i, l_f, n_i, n_f, come from vorticity located close-by, and are relatively strong.

Order estimations of these coefficients are made in terms of the core size, σ, the circulation Γ and a curvature L of the filament, as follows:

$$\left.\begin{aligned} &k_i,\, k_f,\, l_i,\, l_f = O(\Gamma/\sigma^3); \quad n_i,\, n_f = O(\Gamma/\sigma^4); \\ &w_0(t) = O(\Gamma/\sigma), \qquad m_i,\, m_f = O(\Gamma/L^2). \end{aligned}\right\} \tag{8}$$

The equation which governs variations of these coefficients is derived from the vorticity equation.

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{\omega} = 0, \tag{9}$$

where ν is the kinematic viscosity. Substitution of (4), (5a, b) into this equation yields equations for k_i+l_i and k_f+l_f, which prove to suggest slow variation of these quantities.

The pair of vortex rings superposed on the initial state has the same symmetry as the initial and final states with respect to the x and y axes. It is assumed also to have an axisymmetry around the z-axis and an inversion symmetry with respect to the z coordinate, the center of the symmetry being at $z=O(\sigma)>0$. Therefore, the polynomial expansion beings with a term of $O(z^0)$.

In the same way as in the initial and final states polynomial expansions up to $0(r^2)$ are written as

$$\left.\begin{aligned} &u_r = (c_1 - c_2 z)x, \\ &v_r = (c_1 - c_2 z)y, \\ &w_r = -c_0 - 2c_1 z + c_2 z^2 + c_2'(x^2 + y^2)\cdot \end{aligned}\right\} \tag{10}$$

$$\omega_{rx} = (c_2 + c_2' - c_3 z)y, \qquad \omega_{ry} = -(c_2 + c_2' - c_3 z)x, \tag{11a}$$

$$\Delta\omega_{rx} = g_{rx} y, \qquad \Delta\omega_{ry} = -g_{ry} x, \tag{11b}$$

where the coefficients in these expressions are functions of time. Since the vortex rings appear at the beginning of the process and yield the final state, these coefficients must vanish for $t=-\infty$ and asymptote to some finite values for $t=\infty$.

It is assumed here that the above coefficients are expressed in terms of a single function of time $T(t)$ as follows:

$$\left.\begin{aligned} &c_0(t) = \delta T, \quad c_1(t) = mT, \quad c_2(t) = kT, \quad c_2'(t) = lT, \\ &c_3(t) = nT, \quad g_{rx}(t) = g_x T, \quad g_{ry}(t) = g_y T, \end{aligned}\right\} \tag{12}$$

where T(t) satisfies

$$T(-\infty)=0, \qquad T(\infty)=1. \tag{13}$$

Expansion coefficients in the initial, the intermediate and the final states are related to each other because of the asymptotic conditions (2a, b). After reducing number of independent coefficients by the use of these conditions, the flow field during the whole process of recombination is written up to $O(r^2)$ as

$$\left.\begin{aligned} &u=-m(1-T)x+k(1-T)xz, \\ &v=mTy-kTyz, \\ &w=w_0(0-)-\delta T+m(1-2T)z-\frac{l}{2}(1-T)x^2+\frac{l}{2}Ty^2-\frac{k}{2}(1-2T)z^2. \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} &\omega_x=(k+l)Ty-nTyz, \\ &\omega_y=(k+l)(1-T)x-n(1-T)xz, \\ &\omega_z=0. \end{aligned}\right\} \tag{15a}$$

$$\Delta\omega_x=g_xTy, \quad \Delta\omega_y=g_y(1-T), \quad \Delta\omega_z=0. \tag{15b}$$

The parameters m, k, l, δ and $w_0(0-)$ are treated as given. Parameters g_x and g_y are related to the other parameters, while solving the vorticity equation (9) to obtain $T(t)$, as shown below.

By substituting the final expressions (14) and (15) into the x- and y- components of the vorticity equation (9), we have, up to $O(r)$,

$$x: \quad (k+l)dT/dt+m(k+l)T^2-(w_0(-0)-\delta T)nT+(k+l)Tm(1-T)=\nu g_xT, \tag{16a}$$

$$\begin{aligned} y: \quad -(k+l)dT/dt-m(1-T)(k+l)(1-T)-(w_0(0-)-\delta T)n(1-T) \\ -(k+l)(1-T)mT=\nu g_y(1-T). \end{aligned} \tag{16b}$$

From these equations and condition (13), values of g_x and g_y are determined as follows:

$$\nu g_x=m(k+l)-(w_0(0-)-\delta)n, \qquad \nu g_y=-m(k+l)-w_0(0-)n. \tag{17}$$

Then, two equations (16a, b) are reduced to a same simple form

$$dT/dt-t^{*-1}T+t^{*-1}T^2=0, \tag{18}$$

where

$$t^*=\delta n/(k+l)=O(\Gamma/\sigma^2). \tag{19}$$

This equation does not contain the viscosity explicitly, but, since the parameters in (18) are related to the viscosity through (17), the viscous effect is essential. The two components of the vorticity equation (16a, b) contradict to each other even in the lowest order, i.e. $O(r)$, if the viscous terms are neglected.

Solution of the differential equation (18) satisfying (13) is

$$T = e^{\tau}/(1+e^{\tau}), \qquad \tau = t/t^*. \tag{20}$$

This solution has an expected behavior, because it varies from 0 to 1 during a time of $O(t^*)$.

§III. Discussion

The approximate solution obtained here for the recombination process explains well this rapid fluid motion observed by flow visualization. From the process of solution it is shown that the viscous term in the governing equation is playing an esential role. Recent visualization experiment by Oshima[*28*] seems to suggest a presence of the recoil motion assumed in the above analysis. More experimental investigation of the detailed flow structure at the interacting region is awaited.

Here, some comments are given on the obtained solution. The superposition of the ring pair to the initial state may give us an impression that the kinetic energy is not conserved because the ejected recoil vortex ring carries a definite energy. But, this problem is solved by assuming a growth of the core size after the interaction, which should certainly be expected to occur since the viscosity is essential during the process. Moreover, this quick core growth leads us to a new concept for entrainment of the potential fluid into the vortical region, especially in turbulence. Viscous dissipation in turbulence will occur not only through vortex stretching but also through recombination processes which are expeced to occur continually in the turbulent flows.

The only solution obtained in the past for a recombination process is given by Yeh & Axford [*27*] for a MHD flow, where the magnetic field lines are recombined at the center of a stagnation flow. The fluid is assumed inviscid, but the effect of finite conductivity yields this process. This mechanism is similar to our mechanism in a sense that the dissipative effect is occurring in a relatively narrow region, i.e. the interaction region in our analysis, while the mechanism differs in several points. Their MHD flow is steady and two-dimensional, and the three forces on the fluid, the inertia, the pressure and the Lorentz forces balance with each other. On the other hand, in the recombination of the vorticity lines the inertia, the pressure and the viscous forces balance.

The rapid process of recombination is also considered as an efficient source of the aerodynamic noise. It is suggested here that the jet noise can be predicted in terms of the recombination processes within the turbulent jet shear flow.

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