The Third Order Compact Schemes for Fluid Dynamic Problems

By

A. I. Tolstykh*

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1. The algorithms for viscous flows computations should meet higher requirements to such kinds of their properties as stability, the absence of "scheme" oscillations etc. The first of these properties makes preferable implict schemes, the second one leads to the odd order of approximation. Computational practice has shown that from this point of view the "three-point" ("compact") third order schemes proposed and investigated by the author [1,2] are efficient. Those are actually high accuracy schemes when supplemented with grid points condensing in the computational regions with strong gradients [2]. The discription of the third order compact schemes and of some of their applications to fluid dynamic problems is given below.

2. Consider simple equation

$$u_t+\varphi(n)_x\neq f, \tag{1}$$

and the three-point operators

$$A_{\pm}f_j=(5f_{j\mp1}+8f_j-f_{j\pm1})/12, \qquad \Delta_{\mp}f_j=\pm f_{j\mp}f_{j\mp1}, \tag{2}$$

defined on the mesch ω_h ($x_n=nh$, $h=\text{const}$). Using symbolic representation

$$A_{\pm}f_j=\left(E\pm\frac{h}{2}D_x+\frac{h^2}{6}D_x^2+O(h^3)\right)f, \qquad \Delta_{\mp}f_j=\left(E\pm\frac{h}{2}D_x+\frac{h^2}{6}D_x^2+O(h^3)\right)D_xf,$$

where D_x is the operation of the first derivative in respect to x in the grid point $x=x_j$, one can easily see that for sufficiently smooth functions u, φ and f the following relation holds:

$$A_{\mp}^{-1}\Delta\pm f_j/h=f_x|_{x=xj}+O(h^3).$$

Below it is covenient to present the operators $A_{\mp}$, and as the sums of self-adjoint and skew-symmetric operators:

$$A_{\pm}=A(S)=A_0-0.25S\Delta_0, \qquad \Delta_{\pm}=0.5(\Delta_0-S\Delta_2); \tag{3}$$

* Computing Center of USSU Acamemy of Sciences.

here $\Delta_0 f = f_{j+1} - f_{j-1}$, $\Delta_2 f = f_{j+1} - 2f_j + f_{j-1}$, $A_0 f = (f_{j-1} + 4f_j + f_{j+1})/6$ and $S = \mp 1$ is the parameter which defines characteristic "orientation" of one-sided difference $\Delta\pm$. Using (3) for approximation $\varphi(u)_x$ one can construct the difference scheme which after multiplying by
A(S) can be written in the form

$$A(S_j)\langle u_t\rangle_j^{m+1} + \Delta(S_j)\varphi(u_j^{m+1})/h = A(S_j)f_j^{m+1}. \quad (4)$$

Here $\langle u_t\rangle^{m+1}$ is some discretization of the u_t on the mesh $\omega_\tau(t_m = m\tau)$ with the order k and $S_j = \text{sign}\ \varphi_u(u_j^m)$. The truncation error of the scheme (4) is $0\ (h^3 + \tau^k)$. One can also write instead of (4) the scheme with weight factors corresponding to levels t=const. Moreover the operator $A^{-1}(S)\ \Delta(S)$ can be used in many other "traditional" schemes for approximation $\varphi(u)_x$ derivative. Bellow we shall denote A (S) and Δ (S) operators by A and Δ having in mind that for each grid point $S_j = \text{sign}\ \varphi_u u\ (u_j^m)$.

3. Let us consider some properties of the scheme (4) with the operators (3).

Stability. Let $\varphi(u) = a\ u$, a=const ("frozen coefficient" approximation). Then $A^{-1}\Delta > 0$ [3] and the corresponding schemes are conditional or absolute stable. In particular, one can easily verify by the Fourier method that two-level scheme (4) with $\langle u_t\rangle^{m+1} = (u^{m+1} - u^m)/\tau$ is absolutely stable.

Fulfillment of conservation laws. The schemes of the form (4) can be derived from approximation of the conservation law $\oint u dx - \phi(u)dt = 0$ for each cell $x_j \le x \le x_{j+1}$, $t_m \le t \le t_{m+1}$. It follows from the fact that A (S) defines numerical integration formula

$$A(S)f_j = Sh\int_{x_j}^{x_{j+S}} f(x)dx$$

and that $\langle u_t\rangle^{m+1}$ may correspond to some integration formula for $t_p \le t \le t_{m+1}$, $p \le m$. The "flows" across the cell boundaries are annihilated after summing difference equations at each grid point if S_j doesn't change its sign. The computation practice has shown that the absence of annihilation which appears at some grid points where S_j changes its sign hasn't any significant effect on the solutions obtained. Nevertheless one can prserve this property by slight modification of the operators $S\Delta_0$ and $S\Delta_2$: one should use instead of them $\Delta_0 S$ and $\Delta(-1)\ (T_{1/2}S)\ \Delta(1)$ where $T_{1/2}f = f_{j+1/2}$.

Dispersion and dissipation. Let us estimate these properties for eq. (1) with $\varphi(u) = au$, a=const. In order to eliminate their dependence on the way of discretization of the u_t derivative consider equation $u_t + a\ A^{-1}\ h^{-1}\Delta u = 0$. Substituting function v(t) exp (−ik h_j) and comparing the obtained result with exact solution one can derive the following expressions for scheme phase velocity a_s and garmonics attenuation coefficient λ per step τ:

$$\begin{aligned} &a_S/a = W_1(\alpha)/\alpha, \quad \lambda = \exp(-aW_0(\alpha)\tau/h), \quad \alpha = kh, \\ &W_0(\alpha) = 96\,D\sin^4\alpha/2, \quad W_1(\alpha) = 24\,D\sin\alpha\,(7 - \cos\alpha), \\ &D = ((8 + 4\cos\alpha)^2 + 36\sin^2\alpha)^{-2}. \end{aligned} \quad (5)$$

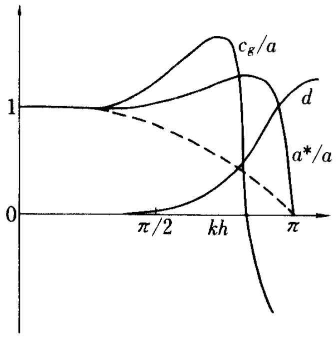

Fig. 1.

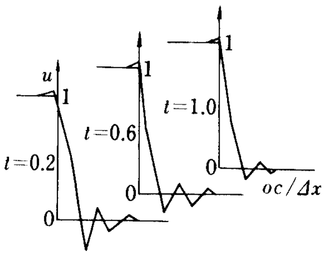

Fig. 2.

It follows from (5) that $a_s/a=1-0\ (\alpha^4)$, $\lambda=1-0\ (\alpha^4)$. Parameters a_S/a and $d=1-\lambda$ are presented in Fig. 1 as functions of $\alpha=kh$. The scheme group velocity Cg and dispersion curve for central difference approximation of u_x are also presented in Fig. 1. As can be seen from the Figure, the phase errors are small for long and medium waves (relative to h); in the regions of kh variation where the errors are significant and the group velocity is negative, the dissipation is pronounced; this leads to damping undesirable high frequency oscillations and improving monotonicity property of difference solutions. As an illustration, numerical solution of the equation $u_t+0.5(u^2)_x=0$ in the case of compact scheme with truncation error $0\ (\tau^3+h^3)$ is presented in Fig. 2.

The presence of damping ocsillations ''in front'' of discontinuity can be explained by unequality $Cg/a> 1$ which is seen in Fig. 1 for the corresponding range of wave lengths.

In order to preserve favourable dispersion and dissipation properties in case of unsteady problems it is desirable to use high order approximation of u_t. There are many ways of doing so (see e.g. [*4*]). However in the following description we shall concentrate on the steady problems for which simple discretions of u_t is sufficed.

Difference equations. The set of difference equations has tridiagonal matrix which, due to orientation of one-sided differences, has ''strengthened'' diagonal in comparison with the centered difference approximations. It can be shown that this set is well-defined and the duble sweeps (factorisation) technique of its solution is stable.

Symmeritrization of the operators. The forth order schemes of type [*5–6*] can be obtained by using the operators $A(s)=[A(1)+A(-1)]/2$ and $\Delta_0=[\Delta(1)+\Delta(-1)]/2$ instead of A (S) and Δ (S). However, they lead to strong spurious oscillations in cases of discontinuous or high gradients solutions; ''smoothing'' is needed for their elimination.

4. The operators a and Δ can be used for approximating the hyperbolic sets of equations and the gas dynamic equations in particular. For the set of type (1) with u, φ and f being p-component vectors let us consider corresponding linearized set $u_t+Qu_x=0$, where Jacobian matrix $Q=\varphi_u(u)$ has real eigen values $\lambda_i(i=1,p)$. Let matrix Q be constant. Then the set can be reduced to the form $W_t^{(k)}+\lambda_k W_x^{(k)}=0$, $k=1,p$, after introducing transformation $u=SW$ such that $S^{-1}QS=\mathrm{diag}(\lambda_i)$, $W=(W^{(1)}, W^{(2)}\ldots W^{(p)})^T$.

For each independent equation, one can obtain the scheme of type (4) and after that to return to the variable u. Then the operators $A=A_0-0.25\ M\Delta_0$, $\Delta=0.5\ (\Delta_0-M\Delta_2)$ will be used in the vector analogue of eq. (5) instead of the operators (2) with $M=SDS^{-1}$, $D=\mathrm{diag}\ (\mathrm{sign}\ \lambda_i)$. In case of variable matrix Q a slight modification of these operators is needed to preserve the third order approximation. Finally, two-level scheme with the truncation error $0(\sigma-0.5)\tau+h^3)$ can be presented in the form

$$(B_x+\sigma\tau C_x Q)(u^{m\pm1}-u^m)/\tau+C_x\varphi(u^m)=B_x f, \quad O\leq\sigma\leq 1, \tag{6}$$

where $B_x=A_0-0.25\ \Delta_0\ {}_0M$, $C_x=0.5(\Delta_0-\Delta(-1)\ (T_{1/2}M)\ \Delta(1))/h$. Let H be the Hilbert space of finitely supported grid functions with the inner product $(u, v)=\Sigma_{j=-\infty}^{\infty} u_j\cdot v_j$, where $u_j\cdot v_j$ is inner product of the vectors u_j and v_j corresponding to the node $x_j=jh$. Then the operator unequalities $B_x\geq E/3$, $C_xQ\geq 0$, $B_xC_xQ\geq 0$, $B^{-1}C_xQ\geq 0$ can be proved [*3*] for the constant symmetric matrices. It is easy to show on the basis of these unequalities that scheme (6) is absolutely stable in the norm $||\cdot||_H$ for $\alpha\geq=0.5$. The vector variant of the double sweep technique may be used for solving difference equations (6) with block-tridiagonal matrix.

5. In case of n space variables. $x_1, x_2 \ldots x_n$ the operators B_{xj} and $C_{xj}(j=1,n)$ which are similar to B_x and C_x from (6) can be utilized for approximation of the hyperbolic set

$$u_t+\sum_{q=1}^{n}(\varphi_q(u))_x=f, \tag{7}$$

with Jacobian matrices $Q_j=(\varphi_j)_u$having real eigen values. One of the possible algorithms can be presented in the following way. Let $y_q={}_{xq}^{-1}\ C_{xq}\varphi_q(u^m)$, $q=1,n$; for determination of y_q it is sufficient to solve n one-dimensional sets $B_{xq}y_q=C_{xq}\varphi_q^m$. Now the difference analogue of (7) can be given in the form

$$(E+\sigma\tau B_{xq}^{-1}C_{xq}Q_q)(u^{m+1}-u^m)/\tau+\sum_{q=1}^{n}Y_q=\sigma f^{m+1}+(1-\sigma)f^m. \tag{8}$$

The scheme (8) is equivalent (7) to the order of $0(\sigma-0.5)\tau+\tau^2+\Sigma_{j=1}^{n} h_j^3)$. It can be shown that in case of commute operators corresponding to different x_q the scheme is absolutely stable for $n\leq 2$ and $\sigma\geq 0.5$ in the norm $||\cdot||_H$ generalized for multidimensional problems. For $n>2$, the algorithm (8) is only conditionly stable as in case of any factorized scheme of this type. However, there is some possibility to make (8) absolutely stable by introducing internal operations. The possibility is not considered here.

6. The schemes (6), (8) can be used for approximation eq. (1), (7) with added difusive terms (e.g Navier Stokes equations); the proper discretization of the second derivatives is only needed for that. Consider, for simplicity the scalar equation with diffusive term:

$$u_t+(\varphi(u)-\mu q)_x=0, \qquad u_x=q, \tag{9}$$

One can introduce a joint operators $\tilde{A}=A^*$, $\tilde{\Delta}=-\Delta^*$ (in fact, the operators corresponding to $S_j=-\mathrm{sign}\ \varphi_u(U)^m)$. Now the scheme with truncation error of the order $0(h^3)$ can be given in the form

$$\left(A+\frac{\tau}{h}\Delta\varphi'(u^m)\right)\frac{u^{m+1}-u^m}{\tau}-\frac{\Delta_\mu q^{m+1}}{h}+\frac{\Delta\varphi(u)^m}{h}=0, \qquad \tilde{A}q^{m+1}\frac{\tilde{\Delta}u^{m+1}}{h}. \tag{10}$$

The set of the difference equations (10) has blook-tridiagmal matrix with the block dimension 2×2; it can be easily inverted. However, inversion of 2p×2p matrices is needed in case of p-component vectors. There are several ways of overcoming this difficulty; they are based on internal iterations at each time step. The following simple way may be also useful. One can change the operator $\tilde{A}$ is the second eq. (10) by the unit operator. In fact, it means that three-point approximation $\Delta\mu\tilde{\Delta}u^{m+1}/h^2$ of the derivative $(\mu U_x)_x$ is used instead of the term $\Delta\mu q^{m+1}/h^2$ of the derivative $(\mu U^x)_x$ is used instead of the term $\Delta\mu q^{m+1}/h$. The truncation error of such a scheme may be presented as $0(h^3+\mu h)$. The terms with the order $0(\mu h)$ are insignificant on the regions beyond ones with high gradients if μ is small. In the last ones the scheme has formally the first order; however, its truncation error is not "scheme viscosity". Numerical experiments has shown that with proper stretching the high gradients regions the introduction of such approximation doesn't significantly influence the solutions obtained.

7. Different variants of the above considered schemes were used for numerical solutions of compressible and incompressible gas flow equations. The main advantage of these schemes consists in obtaining sufficiently exact solutions on moderate meshes without pronounced spurious oscillations. Some examples of computations for the steady compressible flows are presented below.

Two kinds of algorithms were used: the first one has the line relaxation type and was designed chiefly for "unidirectional" flows without separation and the second one was factorized scheme (8) with approximations of the viscous terms to the order $0(\mu h^k)$, $k\geq 2$.

In the first case operators A and Δ were used only for normal to the wall coordinate after performing the transformation which automatically condenses the grid points in normal direction. The derivatives along the "marching" coordinate x were approximated to the second order. The iterative algorithm had the following properties. The pressure and velocity components corresponding to the "forward" grid points relative to the line of sweeps were taken from the previous iteration. Each of the Navier-Stokes equatians was considered independently of the others. This permitted to perform scalar sweeps for solving difference equations corresponding to the x=const line.

The following problems were solved supersonic flow past a spherically blunted body in the wide domain of Reynolds number (200–300 iterations were needed for total number N of grid points equal to 10^3) [*1*]; hypersonic low density flow past a finite length flat plate with a velocity sleep and temperature jump as a boundary conditions (400–500 interations for $N=10^3$); base flow (1500–2000 iterations for relatively small Re; quasiperiodic regime for moderate Re). Some results are presented in [2].

The factorized schemes were used for the computation of more complex problems (e.g. the separating flows). For simplicity the symmetrized operator A_0 and Δ_0 were used in some cases instead of A (S) and Δ (S) [*8*]. The high order "smoothing" was needed for the elimination of spurious oscillations in the cases. The utilization of the operators A (S) and Δ (S) leads to more complex computer cod and to more computer

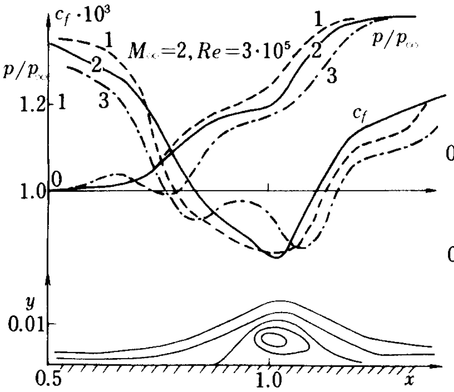

Fig. 3.

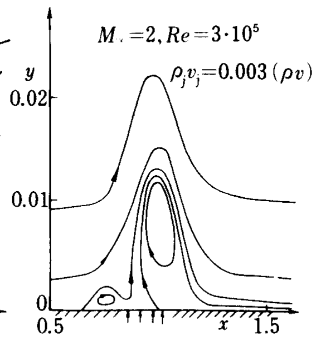

Fig. 4.

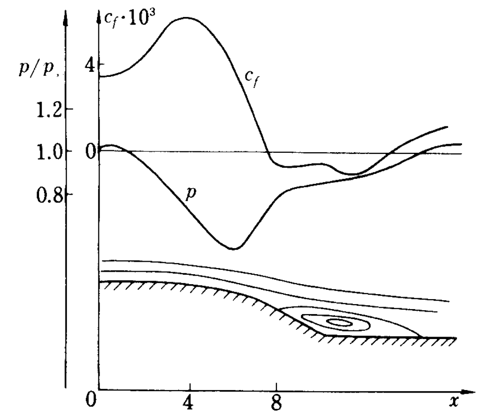

Fig. 5.

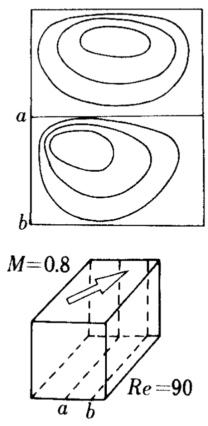

Fig. 6.

time required. However, qualities of the solutions obtained becomes significantly improved; ''smoothing'' isn't needed any more.

The examples of computations of the test problem (the schock wave boundary layer interaction) are shown in Fig. 3–4. The friction coefficient and pressure distribution for the conditions of flow [8] are given in Fig. 4. The curves 1, 2, 3 correspond to the compact schemes of orders $O(h^4)$, $O(h^3)$ and scheme from [8], respectively.

The stream lines pattern in case of injection in the interaotion region (scheme $O(h^3)$) is shown in Fig. 4. The two vortex structure is clearly seen.

Another example of separation flow (the flow about axisymmetric body with curvilinear contour) is shown in Fig. 5.

The scheme (8) was also used for the computations of the channel and nozzle flows (using the Navier-Stokes or Euler equations). High accuracy of the obtained results are deduced from their analysis.

In conclusion, let us mention another problems in which algorithm (8) was effectively used: incompressible fluid flows described by vorticity-stream function equations or by equations for primitive variables (in the last case three-dimensional flows were considered); turbulent fluid or gas flows with different semiempirical mode of turbulence (with the second order moments model in particular); three-dimensional viscous flows. As an example of three-dimensional problem, much number isolines in selected planes of cavern with subsonic flow above it is presented in Fig. 6.

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