The Institute of Space and Astronautical Science
Report No. 605, September 1983

Synthesis of Insensitive Controllers in Linear Quadratic Control Problems

By

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(June 1, 1983)

Summary: Usually, the design objectives of control systems are not only to achieve specified performances at the nominal operational points but also to guarantee admissible performances over the range of parameter variations which inevitablly exist in the actual systems. In this paper we consider the insensitive controller which guarantees the robustness explicitly in terms of parameter variations while preserving the performance in terms of cost.

There have been reported many design methods as to this subject. Some investigated the techniques which augmented the system structures such as the adaptive systems or others. But here we consider the insensitive controller design that keeps its structure as usual regulators from the view point of reliability. For this purpose we evaluate the "Additive Term Design" methods in which the proper additional terms are introduced in the covariance propagation equations or the cost equations in order to assess the deterioration of the performance index (cost).

Reviewing the previous works we clarify some properties of these "Additive Term Designs". And devising a few of new approaches we show that these techniques can improve the cost surfaces and the stability and the stability margins and the sensitivity with respect to the equivalent open loop systems.

To show the practical applicability, these techniques are applied to three numerical examples. One is the longitudinal autopilot of aircraft and the others are the estimation problem of radar tracking and the attitude control system of a large flexible booster. These reveal that the "Additive Term Design" techniques are very effective to those systems and that particularly the M.C.V. ("Maximum Cost Variation") method improves the cost surfaces satisfactorily. Since this robustness realization requires only a slight cost increase at the nominal operational point, these techniques are expected to be applied to a large number of systems with satisfactory improvement.

CONTENTS

Chapter 1. Introduction 2
Chapter 2. Previous Works 6
2–1 Output Regulation and Structually Stable System 6
2–2 Sensitivity Problems and Its Reduction 7
2–3 Recent Developments on Robustness 8
2–4 Existing Methods 10
2–4–1 Augmented System Design 10
2–4–2 Parameter Adjustment 13
2–4–3 Additive Term Designs 14
2–4–4 Other Works 16

Chapter 3. New Design Methods and Discussions 17
3–1 A New Approach to "Guaranteed Cost Control" (G.C.C.)
—"Maximum Cost Variation" (M.C.V.) 17
3–2 Cost and Stability Improvement of Additive Term Designs 24
3–2–1 Cost Improvement 24
3–2–2 Stability Improvement 30
3–2–3 Other Cost Properties and Their Evaluations.................. 35
3–3 Alternative Design Methods of "Additive Term Designs" 41
Chapter 4. Applications to Practical Systems 43
4–1 Systems Containing Control Matrix Uncertainty and Other Uncertainty Sources 43
4–2 Discrete Systems 44
4–3 Dynamically Compensated Systems (Output Feedback Systems) 46
4–4 Computation Algorithm of Optimal Output Feedback Systems with Additive Terms 48
Chapter 5. Numerical Examples 51
5–1 Longitudinal Autopilot 51
5–2 Nonlinear Filtering 77
5–3 The Attitude Control System of a Large Flexible Booster......... 93
Chapter 6. Concluding Remarks 109
Acknowledgement 110
Appendix 1. The Equations of Attitude Motion of a Flexible Booster 110
Appendix 2. The Model Simplifying Technique 111
Appendix 3. Nomenclature 112
References 113

1. Introduction

For a few decades, modern control theory has been applied to many control systems. Particularly, linear quadratic (L.Q.) or linear quadratic Gaussian (L.Q.G.) designs have been utilized in order to minimize the performance index (cost) associated with state and control variables. The reasons why many designers used these methods are first of all their lowest cost, but the other important reasons are their larger stability margins than other design methods, as they are fairly important properties in system designs. L.Q. controllers guarantee at least a half of gain tolerance and 60 degrees phase margin automatically at the point of nominal parameters. But in many cases, state feedback on which such properties are based is impossible and obtaining wider class of stability margins is not a main goal of L.Q. design, because they are not evaluated explicitly in such designs. Therefore one cannot always want the L.Q. controllers to have such properties, and only expect them indirectly. The situation like this is common with the cases designed by classical methods. Finally and essentially, even when this L.Q. design is employed, design procedures of controllers are reduced to trade-off between obtaining specified performances at the point of nominal parameters and making their performances insensitive to parameter variations. In this paper, the author attempted to circumvent these situations through the explicit evaluations of insensitivity in L.Q. designs.

In view of practical control systems, there exist many uncertain sources in their own models. In some cases, these are produced by nonlinearities in plants which cannot be described easily or exactly and are approximated as first order into linear models; such as actuator nonlinearities. In other cases, the causes are due to the lack of exact parameter informations when many identification tests cannot be executed; such as rocket or spacecraft dynamics, and sometimes aging effects or the circumstances under which the systems are driven; such as aircraft dynamics. In these cases, the most significant objective of system designs is of course to keep admissible or specified performances retained over a wide range of parameter variations. The problem of assuring large stability margins is sometimes called as "robustness" property. The term "robustness" indicates the total properties associated with the insensitive controllers which have the large stability margins or stable regions of parameters. This concept is defined not locally but constructed as to the large parameter variations. Recently, "robustness" property is well developed and considered by Doyle [*28*] and other researchers. In this paper, the author often dares to term these properties roughly "insensitive" properties involving the preservation of performances or costs. Therefore design purposes in our cases are as follows.

1) reducing the performance indices (costs) with nominal parameters as small as possible.
2) and also keeping the performances retained at admissible or specified levels over a preassigned range of parameters.
   (This means robust or insensitive property.)

While the former leads to the optimization of design parameters with respect to the nominal cost, the latter does not imply that but suggests the another sense of total optimization evaluating the sensitivity. Namely the design is near optimal or suboptimal one concerning the nominal cost with insentitive properties. The meaningful quantity of performances is not defined so easily. In this paper, this quantitative measure is taken as the same type of performance indices (costs) as in the usual L.Q. or L.Q.G. problems. The second purpose above is equivalent to the following.

3) reducing cost increase due to the system variations as small as possible or minimize the highest cost value over a specified range of parameters.

In this paper, our main discussions are in the analyses of the new design method "Maximum Cost Variation" (M.C.V.). These considerations are made via the elucidation of desensitizing mechanism by the general form of "Additive Term Designs", which introduce some additional terms into L.Q. designs. Moreover through these the sufficient condition of cost surface improvement and the expanding mechanism of stability margins are clarified. And particularly for M.C.V, that condition of uniform cost surface improvement and the considerably large stability margins are made clear.

In this subject, many design procedures have been proposed and developed by some investigators. Harvey in [*1*] has surveyed many existing techniques and some new concepts proposed there and compared with one another from the view point of cost, when state feedback is possible. According to his results, "Uncertainty Weighting" method (to be shown later) is most effective and preferable. But historically speaking, the first attempt to this problem is performed as the output regulation problem: Even when the disturbances and the changes of reference levels affect the system through the uncertain mechanisms, the outputs of the system behave invariantly if the internal stability is assured; This can be achieved by the augmented systems which contain the same internal models as in disturbances and reference level dynamics. This property is termed as "Internal Model Principle" and such output regulation is called "Structually Stable" property if they are retained even in the existence of uncertainties. (Davison [*15–17*] and others) But this holds only in the case when internal stability is assured in spite of such uncertainties. Consequently, one cannot expect this property because the stability of the total system with uncertainties is hardly clear. The concept of sensitivity is considered by Cruz and Pearkins [*18*] first. Their concepts are the extensions of single-input and single-output system sensitivity, and this is the transfer function between the equivalent open loop and the closed loop system with same parameter variations in multivariable systems. This sensitivity relates to "Return Difference Matrix" and is utilized to recent investigations and robust properties. Kreindler also investigated this and showed that the sensitivity is reduced under unity compared with the equivalent open loop system if the optimal L.Q. control laws are employed. (Kreindler [*24*], Cruz [*27*]) Bhatacharyya [*37, 38*] and others [*39, 41*] considered the idea of "Disturbance Eliminating". This concept is constructed under the assumption that interaction mechanisms between plants and other dynamics are exactly known. Therefore the uncertainty considered there is confined to the exogeneous dynamics and the mathematical constraint is very strict one. Hence, this idea will not be useful in highly uncertain systems.

Generally, there are two types of design methods which have ever been considered as the procedures for obtaining the insensitive controllers. One type is as it were "Augmented System Design" methods by which the total system structures are changed to higher order systems. The other type is contrary to this, by which the structures are retained as nominally designed ones such as L.Q. systems. "Parameter Adjustment" and "Additive Term Design" belong to this type. In "Augmented System Design", Landau [*42*] surveyed so to speak adaptive systems. And Kreindler in [*1*] also considered "Sensitivity Vector Augmentation" design which defines first order variations of states as augmented system variables and constructs the compensators for them. The basic idea of this is one of estimating unknown disturbing states. This leads to "Mismatch Estimation" of Kleinmann in [*1*], and "Orthogonal Filter" of Skelton [*52, 53*] and other real time parameter identification techniques. "Parameter Adjustment" is a very direct method to insensitive purposes. Salmon [*54*] devised "Mini-Max Design" algorithm and he applied it to insensitive controller designs. Kleinmann also considered "Maximum Difficulty Design" which utilized the same idea as Salmon's. "Additive Term Designs" are curious methods which contain various types of concepts. These concepts evaluate the cost increase over a range of parameters by using corresponding types of additive terms. Therefore each method of these is characterized and classified according to the respective philosophy of evaluating the cost variations. Kleinmann [*58*, *59*] reported "State (or Control) Dependent Noise" concept which models system variations as persistently changing ones or random processes. "Uncertainty Weighting" method has the simplest form of the additive terms which is introduced in Harvey's [*1*]. Peng [*63*] showed the "Guaranteed Cost Control" (G.C.C.) concept and Vinkler and others [*65*] reported the applicability of this to practical systems. Another form of additive terms in G.C.C. is found in Jain's [*66*] which leads to M.C.V. design in this paper. Moreover, many researchers have considered these robust designs by practical or classical techniques.

Among the various kinds of design procedures, the author focused his attention on "Additive Term Designs" in this paper, because of the simplest structure of resulted control systems. Even if the highly insensitive properties are made by the use of complex higher order controller structures, the utilization of them cannot be always accepted. And from the practical aspects, even the simplest controller will have the insensitive properties. Hence the objective of this paper is confined to the problem; What type of additive terms is effective to designing insensitive controllers? The outline of this paper is the following.

In this paper, at first many previous works considering the insensitive controller designs are introduced, which are concerned with later discussions. Particularly, the concept of robustness and some "Additive Term Design" methods are stated in detail. (Chapter 2)

Next, some new design methods of these are investigated, where the "Maximum Cost Variation" (M.C.V.) concept that assures uniform cost improvement as G.C.C. is shown and derived. Some discussions on cost and stability improvement under "Additive Term Designs" that have not ever been studied in detail appear next, and the comparisons of these with the existing designs are shown, where a few of new useful properties and mathematical studies concerning "additive term designs" are established; such as the condition of insensitivity achievement and considerably large stability margins of M.C.V. and other properties. (Chapter 3)

While the discussions are made for system matrix uncertainties till then, we next proceed to the systems which have the other types of uncertain mechanisms such as control matrix uncertainties, and the application of this method to the discrete systems and the dynamically compensated systems will appear before computation algorithms. Some computation algorithms of robust output feedback systems with M.C.V. type additive terms which are utilized in later examples and practically compensated systems are shown next. (Chapter 4)

Finally the author presents three numerical examples with satisfactory insensitive properties. First example deals with the longitudinal autopilot of aircraft with uncertainties of static stability stiffness and the damping derivatives. The example has the same model as that of our main discussions in this paper, where the M.C.V. concept reduces the cost variation considerably with only a little increase of the nominal cost. Second example is the application to the estimation problem, where the positions and velocities of a re-entry vehicle with unknown aerodynamic coefficients are estimated by nonlinear filtering procedures. The robust design methods are utilized to covariance propagation equations with highly improved results. The last example deals with the attitude control system of a large flexible booster. The system is a large scale one, in which 23 state variables and the dynamic compensator of several degrees is considered. And model simplifying techniques are used for designing compensators and the M.C.V. type robust output feedback technique is applied to the total optimization of this discrete system. The results show that a better insensitivity is achieved by some existing methods than M.C.V, because of ambiguous or aprroximated modeling of system variations due to the transformation between the continuous and the discrete systems. But the designed systems are very simple ones with sufficient robustness. (Chapter 5)

2. Previous Works

2–1. *Output Regulation and Structually Stable System*

As many controller designs are applied to multivariable systems, system designers have been required to improve the steady state performances of control systems. These properties are similar to the type problem in classical single-input single-output systems, such as the type 1 servo compensator. These servo compensators demand the system to have P. I. type feedback if the reference level changes dependent on time. When the disturbances or external system outputs affect the system to be controlled, these type problems must be solved. At first, Young [*9*], Johnson [*10*, *11*], Pearson [*13*] and others [*12*] considered this type problem and derived the integration feedback servo systems in multivariable systems as in classical cases. And Davison [*2*, *3*, 4] and Wonham [*5*] investigated the feedforward control for obtaining the output regulation as to the systems with the disturbances and the reference level changes. Of these two types of regulators, the feedforward controllers are sensitive to the system parameters, that is; if the system parameters change, the output error takes place with finite or infinite quantity. But the integration feedback system is so to say robust, where the output is well regulated for a certain amount of parameter variations. After these researches, Davison [*14*], Wonham [*6*] and others established the "Internal Model Principle", which means the servo control systems must have some rational numbers of copied dynamic models under which the disturbances and the reference levels behave, through the study on the role of transmission zero [7]. Further they made discussions on servo compensators in which any subsystem or internal system is stable, and such controllers are termed as "Internally Stable Output Regulators". After the detailed considerations, regardless of the interaction mechanisms between the plant and the exogeneous models, the output regulation is expected to be achieved. In fact, Davison [*15*, *16*, *17*] established the concept of "Structually Stable Output Regulation" which keeps the output regulation properties retained no matter what interaction mechanisms exist, assuming that the dynamics of external models are known exactly and internal stability is guaranteed in spite of interaction mechanisms. He classifies the compensator mechanisms into two groups; stability compensators and servo compensators. Though these concepts will be of use to many control systems, in highly uncertain systems the properties assumed above cannot be expected, particularly as to the internal stability. Hence, the robust controller designs have become to be considered from these output regulation problems to the sensitivity or other ones.

2–2. *Sensitivity Problems and Its Reduction*

It is very ambiguous to define the sensitivity of control systems. Particularly in multivariable systems, there is no definite selection of sensitivity quantity. The most primitive idea is cost (performance index) sensitivity versus parameter variations. This concept is accepted by some investigators, and is to be treated in detail in this paper. The sensitivity idea, which relates to both the extension of single-input single-output system properties and the return difference concept, was established by Cruz and Pearkins [*18*]. The motivation of their concept is demonstrated in Fig. 1. Consider the equivalent open loop system to the designed closed loop system. Then the output deviations of the open loop system and the closed system are related with each other as follows:

$$(Y_c(s)-Y_{cn}(s))=S(s)\cdot(Y_o(s)-Y_{on}(s)),\qquad S(s)=(I+P(s)\cdot F(s))^{-1}. \qquad (1)$$

where subscripts c and o denote closed and open loop respectively. They defined the sensitivity as this $S(s)$ matrix.

They also considered the insensitivity conditions that the time-integration of the output deviation norm is reduced in closed systems comparing with nominally equivalent open loop systems. That is

$$\int_0^{t_1} e_c(t)^T\cdot Z\cdot e_c(t)\cdot dt \leq \int_0^{t_1} e_o(t)^T\cdot Z\cdot e_o(t)\cdot dt \quad \text{for } {}^\forall t_1>0,\quad {}^\forall Z>0,$$
$$\text{where } e_c(t)=y_c(t)-y_{cn}(t),\quad e_o(t)=y_o(t)-y_{on}(t). \qquad (2)$$

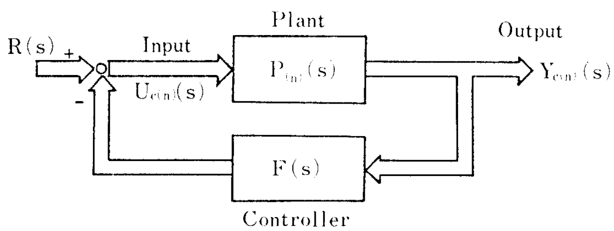
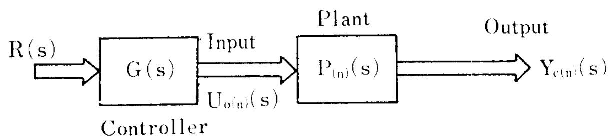


Fig. 1–a. Closed Loop System.



$$;G(s)=(I+F(s)\cdot P_n(s))^{-1}$$
$$=I-F(s)(I+P_n(s)\cdot F(s))^{-1}\cdot P_n(s)$$

Fig. 1-b. Equivaient Open Loop System.

This attempt is also studied by Kreindler, who derived through using Parseval's theorem the sufficient condition for the inequality above to hold as follows:

$$S^T(-j\omega)\cdot Z\cdot S(j\omega)\leq Z \quad \text{or} \quad (I+P(-j\omega)F(-j\omega))^T\cdot Z\cdot(I+P(j\omega)F(j\omega))\geq Z,$$
$$\text{for } {}^\forall\omega>0. \qquad (3)$$

Therefore, this sensitivity is well characterized in the sense of the inequality (2); that is, from equation (3) in order for the inequality (2) to hold, it is sufficient that this sensitivity is under unity.

These discussions are also noted briefly by Kimura [*130*]. And other attempts to define sensitivities of control systems are considered by Porter [*19*].

But there is the most important question left to be answered. That is—Are there any design methods by which the sensitivity is kept under unity? Essentially the answer is in the affirmative. Kreindler [*24*, *25*] and Cruz [*27*] stated in their reports that if the optimal control laws (L. Q.) are employed, then the sensitivity of designed closed loop systems is reduced under unity. (i.e. The inequality (2) holds in such systems.) The conceptual basis of this property is the circle condition of L. Q. regulators, which is discussed in later chapters of this paper. This property is the one of the most important characteristics of L. Q. regulators. However, of course, if the state feedback is not available, these merits can not be expected. And the system compared with such regulators in this discussion is only the equivalent open loop one, therefore it is unable to insist that L. Q. regulators are superior to the controllers designed by other methods.

The discussions on this return difference property is well developed as to the robustness property recently.

2–3. *Recent Developments on Robustness*

Recently many engineers have been studying the robustness property of multivariable control systems utilizing "Multivariable Nyquist" criterion. These researches are made on the basis of classical Nyquist stability criterion, and using the analogy of multivariable transfer functions. These developments are briefly surveyed in Kodama [*33*] and Kimura [*34*].

Doyle [*28*, *29*] considered the multiplicative form of uncertainties as in Fig. 2. And he derived the necessary and sufficient condition of robustness; i.e. To what amount the system stability is retained? The result is

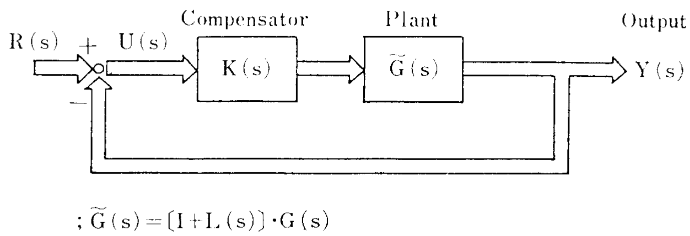


Fig. 2. Uncertain System Model (I).

If $\bar{\sigma}(G\cdot K\cdot(I+G\cdot K)^{-1})<1/1m(\omega)$ or $\underline{\sigma}(I+(G\cdot K)^{-1})>1m(\omega)$ for $\forall\omega>0$, then $\forall L(s)$ such that $\bar{\sigma}(L(j\omega))\leq 1m(\omega)$ is permitted for the system to remain stable, where $\bar{\sigma}(*)$ and $\underline{\sigma}(*)$ denotes the maximum and minimum spectrum radius respectively. (4)

This guarantees that any linear elements $L(j\omega)$ which do not violate the inequality above are permitted for the system to remain stable. He continued the discussions to the design procedures qualitatively, evaluating both robustness and insensitivity defined by Cruz and Pearkins. The requirement for obtaining robustness is distinguished from that of sensitivity reduction, and he proposed the trade-off design technique "σ-plot", because the robustness requirement and the insensitivity requirement are qualitatively expressed as follows:

$\underline{\sigma}(G\cdot K)>P(\omega)/(1-1m(\omega))$ for insensitivity, where $P(\omega)$ is a certain large positive function, and $\bar{\sigma}(G\cdot K)<1/1m(\omega)$ for robustness. (5)

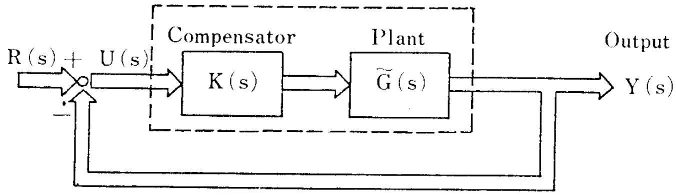
He recommended the procedures that both of two inequalities above should hold over a sufficiently wide range of frequency. Finally he investigated the robustness of L. Q. regulators and displayed the same results of Athans [*32*] and others; i.e. a half value of gain tolerance and at least 60 degrees phase margin.

Another consideration is held by Lethomaki [*30*] and others. They noted the correct utilization of "Multivariable Nyquist" criterion which was established by Rosenblock. Their treatment is slightly different from Doyle's; The uncertainties are modeled as in Fig. 3. While Doyle separated the robust properties and the insensitive ones, Lethomaki discussed these by using a single form of inequality. His formulations are highly relating to the stability margins and give us deeper insight into the cross feed perturbations and other kinds of uncertain sources. The basic results are as follows:

If $\alpha_0\leq\underline{\sigma}(I+G)$ and $\alpha_0\leq 1$, then gain factor $\geq \dfrac{1}{1\pm\alpha_0}$, phase shift $\leq\cos^{-1}\left(1-\frac{1}{2}\alpha_0^2\right)$ or $\forall X(s)$ such that $\bar{\sigma}(X(s))<\alpha_0$ contained in

$$L(s)=\begin{bmatrix} I & X(s) \\ O & I \end{bmatrix} \text{ or } \begin{bmatrix} I & O \\ X(s) & I \end{bmatrix} \text{ (cross feed perturbation)}$$

is permitted for the system to remain stable. (6)



; $\tilde{G}(s)=G(s)\cdot L(s)$
(Perturbed System)

Fig. 3. Uncertain System Model (II).

And similar evaluations of L. Q. regulators to Doyle's are made resulting in well known stability margins mentioned above. They insist that from the aspects of stability margins it will be advantageous to design controllers via Lyapunov equations rather than usual Riccati equations.

And some interesting considerations are mentioned in Cruz [*27*] and Sezer [*35*], Barrett [*36*].

2–4. *Existing Design Methods*

2–4–1. *Augmented System Design*

For these several years, there have been developed a number of robust controller designs. As mentioned before, roughly speaking, two types of design methods exist today. One type is the "Augmented System Design" by whose techniques the constructed systems are changed complex to higher order systems, and the other is either "Parameter Adjustment" technique or "Additive Term Design" that maintains the structures as in the cases by the nominal controller (L. Q.) designs.

There appear to exist 5 classical approaches in "Augmented System Designs". These are as follows.

a) Adaptive Systems; Landau
b) Sensitivity Vector Augmentation; Kreindler
c) Multi-Plant Designs; Harvey
d) Mismatch Estimation; Kleinmann
e) Orthogonal Filters; Skelton

Adaptive system designs have spread over a great number of fields of control systems. The most superior characteristics of this approach are their successful asymptotic features. These features are supported by certain sufficient conditions as to stability, which are assured by absolutely nonlinear evaluations. Landau [*42*] reviews some types of M. R. A. C. S. (Model Reference Adaptive Control Systems), and Kreisselmeier [*47*] devised the adaptive observers that have more feasible structures in the control systems. But generally these adaptive systems are fairly inferior to nominal controllers because of their slower damping rates of convergence. Ljung [*45*, *46*] also studied the real time parameter identification method—extended Kalman filtering. In practical systems, there hardly exist the cases in which any informations of system structures and their variations can not be obtained. System engineers are usually informed of the tendency of the overall system parameter variations and of some exact parameters. Therefore these adaptive systems are not required and too complex to be accepted in usual control systems.

The second method "Sensitivity Vector Augmentation" method is the technique where the sensitivity vector states are assumed to behave according to certain dynamical models. The design purposes are to suppress the magnitudes of these sensitivity vectors, and thus the total system performances appear to be retained more or less. These concepts are exploited by Kreindler in [*1*], O'Reilly [*50*], Fleming [*51*] and other researchers. Along Kreindler's the discussions proceed as follows. Consider the system model with a paramerter vector p,

$$\dot{x}=F(p)\cdot x+G\cdot u. \tag{7}$$

If we define the sensitivity vector as the partial derivatives of states (or trajectories) by parameters,

$$\sigma=\frac{\partial x}{\partial p}, \tag{8}$$

then first order simplified dynamical models of (7) are formulated as follows:

$$\dot{\sigma}=F\cdot\sigma+\left.\frac{\partial F}{\partial p}\right|_{p=p_0}\cdot x. \tag{9}$$

Consequently, we define the augmented system state variables, and arrange the system as folows:

$$\dot{\tilde{x}}=\begin{bmatrix} F & O \\ \left.\frac{\partial F}{\partial p}\right|_{p=p_0} & F \end{bmatrix}\tilde{x}+\begin{bmatrix} G \\ O \end{bmatrix}u, \qquad \tilde{x}^T=(\tilde{x}^T, \sigma^T). \tag{10}$$

Thus the design model is constructed as eq. (10). But we can clearly recognize that the order of states may be so higher that design computational load will increase considerably. And the resulted system is never promised to have the robustness property, so even though this technique is adopted in designs, no improvement of cost or stability might be realized for the augmented system variables have no physical meanings.

"Multi-Plant" designs are mentioned in Harvey's [*1*]. The motivation of this idea is very primitive one. If we expect the total system to be stable at several points of parameters, then we have only to prepare the expanded system model such as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix}=\begin{bmatrix} F(p_1) & & O \\ & \ddots & \\ O & & F(p_n) \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}+\begin{bmatrix} G & & O \\ & \ddots & \\ O & & G \end{bmatrix}\begin{bmatrix} u \\ \vdots \\ u \end{bmatrix} \tag{11}$$

with the feedback constraints of gain matrix K as

$$\begin{bmatrix} u \\ \vdots \\ u \end{bmatrix}=\begin{bmatrix} K & & O \\ & \ddots & \\ O & & K \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}. \tag{12}$$

If these design procedures are finished successfully, system stability at these specified points is guaranteed naturally. And physical interpretations of cost associated with these expanded models are defined meaningfully. However these computational load is very severe because of the common reasons with "Sensitivity Vector Augmentation" algorithm; i.e. so to say the curse of dimensionality. Namely if a designer wants to stabilize a 10-th order system at (10×10) 2-parameter variated points by this robust design method, we must consider surprisingly large a number of higher order system (1000-th order expanded system), which can be seldom solved. And effective calculation techniques for solving the gain matrix K like the equations above are not established. In fact Harvey reported this difficulty and commented the calculation method in [1], though this technique leads to the usual simple system structures and may be expected to have the meanings as in mini-max design of "Parameter Adjustment" techniques discussed later.

There are more tactical compensation techniques which are deeply concerned with adaptive systems; "Mismatch Estimation" method and its alternative concept "Orthogonal Filter" techniques. Kleinmann considers in Harvey's [1] "Mismatch Estimation" method. The main principle of this formulation is recognized as that the deviations of state trajectories are caused by other some outputs of external dynamical systems about which any information is not available. The solid formulations are as follows:

$$\dot{x}=F\cdot x+D\cdot\xi+G\cdot u,$$
$$\dot{\xi}=-\Upsilon\cdot\xi+\lambda \qquad \text{(external model).} \qquad (13)$$

It is well known to us that this type of augmented systems resembles to real time parameter identification techniques except for the absence of nonlinear features, where extended states are modeled as a first order Marcov process. Because all the causes of model mismatching are assumed to be the existence of exogeneous state variables, hence we must install the state estimators or observers with the control systems, and we must construct the feedback control laws under the higher order augmented systems. There are no justification and rational detemination techniques of the additional system matrices D, Υ and the random inputs λ.

A similar and more tactical design technique "Orthogonal Filter" is investigated by Skelton [52, 53]. He adopted the following type of equations like Kleinmann's above. The difference between Kleinmann's and Skelton's is the structure of the additional system matrices:

$$\dot{x}=A\cdot x+B\cdot u+P_x\cdot s, \quad \dot{s}=D\cdot s, \quad y=C\cdot x+P_y\cdot s,$$

$$D=\frac{2}{\tau}\begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 4 & 0 & \cdot & \cdot & \cdot & \cdot \\ 3 & 0 & 6 & 0 & \cdot & \cdot & \cdot \\ 0 & 8 & 0 & 8 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \qquad (14)$$

In Skelton's formulations, external states are tuned to behave as Chebyshev polynomials over the time interval $[0, \tau]$:

$$s(t)=\begin{bmatrix} 1 \\ \sigma \\ 2\sigma^2-1 \\ 4\sigma^3-3 \\ \vdots \\ \cos(i\cdot\cos^{-1}\sigma) \end{bmatrix}, \quad i=0, 1, \cdots, d-1, \quad \sigma=\frac{2t}{\tau}-1. \qquad (15)$$

These polynomials span the orthogonal basis of d-th order in time domain, if the dimensions of external states are modeled as d-th order. Namely over the time-interval $[0, \tau]$, these introduced variables $s(t)$ express $t^0, t^1, \cdots, t^{d-1}$ time dependent functions. (often called Chebyshev Filter) Skelton commented that this technique will be effective to not only parameter variations but observation spillover problems in truncation, even if the order of external system is quite small. But we should note that theoretically there are some restrictions naturally that only a few of time-dependent functions are assumed to be causal sources, and that these are not almighty but some buffers for output deviations.

2–4–2. *Parameter Adjustment*

While the previous discussions are treating higher order augmented systems, the considerations here are relatively simple ones. As commented before, we designers may well accept the simpler controllers, if the prepared design parameters as in nominal or specified systems are adjusted properly and so that the robustness is accomplished. Analogous procedures have been performed in classical designs, too. In classical approaches, it is a main design work to adjust total loop gain and the mixing ratio of some outputs of sensors and filters. But in multivariable systems, these trial and error approaches become highly complex and cannot be synthesized in detail. Salmon [54] devised the effective mini-max calculation algorithm evaluating control cost;

> To find the gain matrix so as to minimize the maximum cost over the parameter variations.

This is one of the most direct and successfull approaches to obtain robust controllers. In fact the calculation algorithm is finished with finite cost J, then the robustness property is guaranteed over the specified range of parameter variations. But the computational load increases so highly at the presence of many uncertain parameters that the adoption of this may be rejected usually. Kleinmann in (1) attempted "Maximum Difficulty Design", and Pearson [55] "Worst Case Design", avoiding this computational difficulty, leading to the "Off-Set Design" noted later in this paper.

Yahagi [57] attempted the different "Parameter Adjustment" approach, who evaluated cost sensitivity explicitly and designed with additional weighting of these cost sensitivity matrices in output feedback systems. The evaluation of cost sensitivity is very interesting and will be utilized later. His formulations are as follows.

Under the usual problem;

$$\dot{x}=A\cdot x+B\cdot u,\ u=K\cdot y,\ y=C\cdot x,\ D=A+B\cdot K\cdot C,$$

$$J_1=\int_0^\infty (x^T\cdot Q\cdot x+u^T\cdot R\cdot u)\cdot dt \text{ to be minimized, } \frac{\partial J_1}{\partial A}=2\cdot S\cdot P, \text{ where}$$

$$P=\int_0^\infty \exp(D\cdot t)\cdot x_0\cdot x_0^T\cdot \exp(D^T\cdot t)\cdot dt \text{ and } D^T\cdot S+S\cdot D+Q+C^T\cdot K^T R\cdot K\cdot C=0,$$

the modified problem is formulated as

$$J_2 = J_1 + \text{tr.}\left(\left(\frac{\partial J_1}{\partial A}\right)^T \cdot L \cdot \frac{\partial J_1}{\partial A}\right) \; (L>0) \text{ to be minimized.} \tag{16}$$

Notwithstanding these quantitative approaches, the computational load is still high, because the effective algorithm is not established. He investigated the numerical technique like gradient method and showed some examples through complex calculations.

2-4-3. *Additive Term Designs*

Apart from the discussions before, quite different approaches are reported by many authors. In this paper, these are termed as "Additive Term Design" methods after the structures of resulted Riccati type equations. There are three types of existing design methods which belong to this "Additive Term Design" group. One is the "Additive Noise" or "State or Control Dependent Noise" concept considered by Kleinmann [*58*, *59*], McLane [*60*, *61*] and others. And the second is "Uncertainty Weighting" method in Harvey's [*1*], and the third is "Guaranteed Cost Control" technique established by Peng [*63*], Vinkler [*65*] and Jain [*66*].

Generally, if the optimal feedback controllers are designed, we must solve so to speak the Riccati type equations which express the value of the performance index (cost) associated with the covariance propagation equations. This titled "Additive Term Design" methods use the modified versions of ordinary cost equations or covariance equations, in which a few of additional terms are introduced against each philosophy. The most advantageous characteristics of these are feasible amount of computational load and the theoretical justification of treatment.

Kleinmann [*58*, *59*] and McLane [*60*, *61*] devised the "State or Control Dependent Noise" concept. These are motivated by the equivalent assumption of parameter variations to random processes dependent on states or control inputs as follows:

$$\dot{x} = A\cdot x + B\cdot u + \delta A\cdot x + \delta B\cdot u \longleftrightarrow dx = A\cdot x\cdot dt + B\cdot u\cdot dt + \delta A\cdot x\cdot d\xi + \delta B\cdot u\cdot d\eta,$$

where ξ, η are Wiener Processes with following properties:

$$E((\xi(t)-\xi(t_0))(\xi(s)-\xi(t_0))^T) = \int_{t_0}^{\min(t,s)} \sigma_\xi^2(\tau)\cdot d\tau,$$
$$E((\xi(t)-\xi(t_0))(\eta(t)-\eta(t_0))^T) = 0,$$
$$E((\eta(t)-\eta(t_0))(\eta(s)-\eta(t_0))^T) = \int_{t_0}^{\min(t,s)} \sigma_\eta^2(\tau)\cdot d\tau. \tag{17}$$

Of course, correctly speaking, these are somewhat strange models, because the parameter variations are not processes but statistical quantity. But they adopted the straightforward interpretations of ergodic characteristics and derived following covariance propagation equations through the use of Ito's calculus on stochastic differential equations:

$$\dot{P} = \tilde{A}\cdot P + P\cdot\tilde{A} + \sigma_\xi^2\cdot\delta A\cdot P\cdot\delta A^T + \sigma_\eta^2\cdot\delta B\cdot K\cdot P\cdot K^T\cdot\delta B^T, \; P(t_0) = P_0,$$
$$\text{where } u = -K\cdot x, \; \tilde{A} = A - B\cdot K. \tag{18}$$

And they reduced the cost equations and feedback laws of optimal control problems as follows.

Under the problem of

$$J=\int_{t_0}^{t_f}(x^T\cdot Q\cdot x+u^T\cdot R\cdot u)\cdot dt \text{ to be minimized,}$$
$$-\dot{S}=\tilde{A}^T\cdot S+S\cdot\tilde{A}+\sigma_\xi^2\delta A^T\cdot S\cdot\delta A+\sigma_\eta^2\cdot K^T\cdot\delta B^T\cdot S\cdot\delta B\cdot K+Q+K^T\cdot R\cdot K,$$
$$S(t_f)=0,\ K=(R+\sigma_\eta^2\delta B^T\cdot S\cdot\delta B)^{-1}\cdot B^T\cdot S. \tag{19}$$

By this method, it is clearly interpreted that high gain feedback laws are required at the presence of state matrix ambiguities, and that low feedback gains are needed at the presence of control matrix uncertainties. Kleinmann proceeded next to the application of these to the aircraft control problems [*59*]. And McLane considered the formulations of output feedback control problems with these dependent noises [*61*].

There was reported in Harvey's [*1*] that the very intuitive realization of additive terms is "Uncertainty Weighting". This design method is fairly simple one and from the point of computational load this has the same complexity as the nominal L. Q. ones. The fundamental basis of this concept is as follows. Consider the original system and first order variated system:

$$\dot{x}=A\cdot x+B\cdot u+\delta A\cdot x,$$
$$\delta\dot{x}=A\cdot\delta x+\delta A\cdot x. \tag{20}$$

If we expect the trajectories preserved, it is intuitively effective to evaluate or weight the external forcing vector δAx. This idea leads to the modification of the performance index as

$$J=\int_{t_0}^{t_f}(x^T\cdot Q\cdot x+u^T\cdot R\cdot u)dt+\int_{t_0}^{t_f}x^T\cdot\delta A^T\cdot M\cdot\delta A\cdot x\cdot dt$$

to be minimized, where M is a certain positive definite matrix. (21)

And the resulted cost equations to be solved are derived easily like the nominal formulations:

$$-\dot{S}=\tilde{A}^T\cdot S+S\cdot\tilde{A}+Q+K^T\cdot R\cdot K+\delta A^T\cdot M\cdot\delta A,$$
$$S(t_f)=0,\ K=R^{-1}\cdot B^T\cdot S. \tag{22}$$

The inspection of the additive term above reveals the simplest feature of not containing cost matrix S to be solved, like the same treatment as state weighting matrix Q. Moreover Harvey reported that this method is superior to any other design techniques contained in his literature. It will be noted later in this paper that this concept is only a simplified version of "State or Control Dependent Noise" or next "Guaranteed Cost Control" techniques. But whether robustness property is achieved or not if we employ this method is only a qualitative analogy.

The last concept "Guaranteed Cost Control" was established by Peng [*63*] and others. This idea has two aspects of characteristics, one of which is the mini-max design feature and the other is Lyapunov's sufficient stability criterion. By the first characteristics, this evaluates the upper limit of maximum cost, and minimizes that cost. As to the stability, when the Lyapunov's criterion is used, this design guarantees the stability over the wide range of specified parameter variations. The discussion proceeds as follows. Consider the cost equation of the variated system.

Under the same problem as (19)

$$-\dot{S}=\tilde{A}^T\cdot S+S\cdot\tilde{A}+Q+K^T\cdot R\cdot K+\delta A^T\cdot S+S\cdot\delta A, \qquad S(t_f)=0. \tag{23}$$

If we adopt the following values instead of the last two terms above,

$$|\delta A^T\cdot S+S\cdot\delta A|=T^T\cdot|\Lambda|\cdot T,$$
where T is a transformation matrix of $(\delta A^T\cdot S+S\cdot\delta A)$ into a real diagonal matrix Λ, (24)

then at any time the calculated cost is proved to be higher than the true cost value. Regarding this equation as the definition of cost along the trajectories, it will be a best choice of feedback laws K to minimize the cost tr. $[S]$ at initial time:

$$-\dot{S}=\tilde{A}^T\cdot S+S\cdot\tilde{A}+Q+K^T\cdot R\cdot K+\delta A^T\cdot S+S\cdot\delta A, \qquad S(t_f)=0,$$
$$K=R^{-1}\cdot B^T\cdot S(t_0). \tag{25}$$

If we confine ourselves to time-invariant and infinite-time cases, the stability is guaranteed like followings. In case that parameters are variated, rewriting the cost equation as

$$\begin{aligned}O=&(\tilde{A}+\delta A)^T\cdot S+S\cdot(\tilde{A}+\delta A)+Q+K^T\cdot R\cdot K\\&+(|\delta A^T\cdot S+S\cdot\delta A|-(\delta A^T\cdot S+S\cdot\delta A))\end{aligned} \tag{26}$$

and applying the Laypunov's stability criterion, we easily find out that over a specified range of parameter variations the stability properties are assured sufficiently. According to the discussions made here, this concept appears to be an applicable design method. But in practical installations, we should note in our mind that this might not improve the cost and require the high gain feedback laws unnecessarily. There would be the consideration of cost improvement in this paper; i.e. insensitivity realization in robust sense. Vinkler [*65*] attempted to avoid high gain feedback laws which may be required by "Guaranteed Cost Control" and proposed the multistep guaranteed cost control design. And Jain [*66*] considered the applicability of this to estimation problems named "Guaranteed Error Estimation" with another form of upper limit evaluations, whose form is considered as M.C.V. concept later in this paper.

2–4–4. *Other Works*

There have been approached to insensitive controller designs other than the methods mentioned here. Alike the structually stable output regulation, Bhattacharyya [*37 38*], and Mita [*39, 40*], Furuta [*41*] and others intended to eliminate the affection caused by exogeneous disturbances or external unknown dynamics.

This is usually termed "Disturbance Elimination" or "Disturbance Localization". While they considered these "Unknown Input Observers", Kimura [*130*] studied this type of controllers. And other classical approaches to this item are shown by Horowitz [*70*]–[*79*]. And other researches are presented by Sugano [*82*], Barnet [*69*], Krogh [*84*] and so many authors [*67*, *68*] and [*80*, *81*]. Any way, we would like to discuss our main body considerations of this paper in next chapter based on the previous works studied in this chapter.

3. New Design Methods and Discussions

Here we derive some new design techniques and discuss cost and stability improvement under "Additive Term Designs." Particularly the "Maximum Cost Variation" (M.C.V.) concept which assures uniform improvement of cost is formulated, where the additive terms that have the monotonous features are utilized. Some useful properties that have not ever been studied in detail are established; such as the condition of cost improvement and considerably large stability margins of M.C.V. and others.

3-1. *A New Approach to "Guaranteed Cost Control" (G.C.C.)—"Maximum Cost Variation" (M.C.V.)*

The concept of "Guaranteed Cost Control" (G.C.C.) presented by Peng was reviewed in the previous chapter. Here we derive the new type of G.C.C. method, which is a little different one from his; i.e. with monotonous characters and leading to uniformly insensitive controllers, named "Maximum Cost Variation" (M.C.V.). Now we formulate this through the covariance equations instead of cost equations.

A) Derivation

Let's consider the nominal system

$$\dot{x}=A\cdot x+B\cdot u \tag{27}$$

and uncertainly variated system:

$$\dot{x}=A\cdot x+B\cdot u+f(q, t, x, u). \tag{28}$$

Where $f(*)$ is a nonlinear vector function of x, u, t specified by the parameter vector $q\in\Omega$.

Assumption 1.

Ω is a compact and convex set in R.

Assumption 2.

$$f(0, t, x, u)=0$$

and $f(*)$ is a C^1-class vector function with repect to x and u on Ω.

Assuming the adoption of usual type of feedback control law K, we can obtain as it were covariance propagation equations as follows:

$$\dot{P}=\tilde{A}\cdot P+P\cdot\tilde{A}^T+f\cdot x^T+x\cdot f^T,\quad P(t_0)=P_0,$$
$$\text{where } u=-K\cdot x,\quad \tilde{A}=A-B\cdot K,\quad P=x\cdot x^T. \tag{29}$$

Now we set our attention on the minimization of following performance index (cost):

$$J=\int_{t_0}^{t_f}(x^T\cdot Q\cdot x+u^T\cdot R\cdot u)dt=\int_{t_0}^{t_f}\text{tr.}\,((Q+K^T\cdot R\cdot K)\cdot P)dt. \tag{30}$$

From this equation, we can easily find out that the larger the covariance matrix becomes the higher the cost increases. Consequently, in order to evaluate cost deterioration over the range of parameters' set, we seek for the covariance propagation equations expressing maximum covariance. (Here the readers should note that the terms "maximum" or "highest" values etc. do not always imply the actual maximum one. In this paper these are usually used to denote the upper bound values, which might not be identical to the actual maximum one but are over it.) For this purpose, we assume followings.

Assumption 3.

There exists a monotone increasing positive semi-definite matrix function $h(q, P)$ of q and P, where q denotes the properly defined norm of parameter vector, and is assumed to satisfy the following inequality:

$$h(0, P)=0 \quad\text{and}\quad f\cdot x^T+x\cdot f^T\leq h(q, P) \quad\text{and}\quad h(q, P)\geq 0. \tag{31}$$

Thereby considering the modified covariance propagation equation as follows,

$$\dot{\tilde{P}}=\tilde{A}\cdot\tilde{P}+\tilde{P}\cdot\tilde{A}+h(q, \tilde{P}),\qquad \tilde{P}(t_0)=P(t_0)=P_0, \tag{32}$$

we can reach to the following result, considering the monotonous feature of $h(q, P)$ with respect to P.

Theorem 1.

Under the assumptions made above, as to any specified parameter q,

$$\tilde{P}(t)\geq P(t)\quad\text{at}\quad \forall t\in[t_0, \infty). \tag{33}$$

Proof.

If we define the following successive series P^i

$$\dot{P}^{i+1}=\tilde{A}P^{i+1}+P^{i+1}\tilde{A}^T+h(q, P^i)\qquad(i=0, 1, \cdots)$$

and taking $P^0(t)=P(t)$, then the results above are easily verified, providing $\tilde{P}(t)$ exists and $\tilde{A}$ is stable.

Thus we have the G.C.C. system model eq. (32) which evaluates the highest cost over the parameter variations from eq. (30).

B) Control Problem and the Improvement of Cost Deterioration Limit

From this point, we are ready to set up the G.C.C. problem with the performance index like eq. (30),

$$J=\int_{t_0}^{t_f} \text{tr.}\,[(Q+K^T\cdot R\cdot K)\tilde{P}]dt. \tag{34}$$

Applying the matrix type calculus of variations regarding eq. (32) as the virtual system model, we can get the optimal (; in the true sense suboptimal) control laws as follows. (Here the design range q in eq. (32) is selected as the boundary norm of parameters.) Defining the Hamiltonian as

$$H=\text{tr.}\,[(Q+K^T\cdot R\cdot K)\cdot\tilde{P}]+\text{tr.}\,[S\cdot\dot{\tilde{P}}];\quad \text{Hamiltonian} \tag{35}$$

and differentiating this by K, $\tilde{P}$ and S, we have the following constraint equations:

$$\begin{aligned} &O=R\cdot K\cdot\tilde{P}-B^T\cdot S\cdot\tilde{P}+\frac{\partial\,\text{tr.}\,[S\cdot h]}{\partial K},\\ &-\dot{S}=\tilde{A}^T\cdot S+S\cdot\tilde{A}+Q+K^TR\cdot K+\frac{\partial\,\text{tr.}\,[S\cdot h]}{\partial\tilde{P}},\qquad S(t_f)=0,\\ &\dot{\tilde{P}}=\tilde{A}\cdot\tilde{P}+\tilde{P}\cdot\tilde{A}^T+h(q,\tilde{P}),\qquad \tilde{P}(t_0)=P_0. \end{aligned} \tag{36}$$

In this case, the evaluated maximum cost (in upper bound sense) which should be minimized through the procedures above is calculated as follows:

$$\begin{aligned} &\frac{d}{dt}\,\text{tr.}\,[S\cdot\tilde{P}]=-\text{tr.}\,[(Q+K^T\cdot R\cdot K)\cdot\tilde{P}]+\text{tr.}\left[S\cdot h-\frac{\partial\,\text{tr.}\,[S\cdot h]}{\partial\tilde{P}}\,\tilde{P}\right],\\ &J=\text{tr.}\,[S(t_0)\cdot P(t_0)]+\int_{t_0}^{t_f}\text{tr.}\left[S\cdot h-\frac{\partial\,\text{tr.}\,[S\cdot h]}{\partial\tilde{P}}\cdot\tilde{P}\right]dt. \end{aligned} \tag{37}$$

As is easily recognized, if we attempt to consider the infinite-time problem, the second term must be a finite value, which requires that the integrant may not be positive or negative definite value. (This is reviewed again later in this chapter.)

Assumption 4.

The additive term h is selected such that the integration of eq. (37) exists with finite value.

Now returning to the control problem, we define the cost deterioration limit in the large sense within the norm q_0 concerning a certain gain matrix K as follows.

Definition 1.

$$\begin{aligned} &\delta J_{\text{limit}}(q_0,\,K)=\int_{t_0}^{t_f}\text{tr.}\,((Q+K^TRK)\delta P)dt,\\ &\text{where } \delta P=\tilde{P}(q=q_0)-P(q=0), \end{aligned} \tag{38}$$

and where $\tilde{P}$ and P are defined in eq. (32) and eq. (29) respectively.

Then the following result is clearly true.

Theorem 2.

The cost deterioration limit defined above is uniformly reduced within the specified range of parameter variations, q_0, by the control laws K_2 in eq. (36), comparing the case by the nominal optimal control laws K_1. Where the term "uniformly" indicates

$$\delta J_{\text{limit}}(q, K_2) \leq \delta J_{\text{limit}}(q_0, K_2) \leq \delta J_{\text{limit}}(q_0, K_1); \ |q| \leq q_0. \tag{39}$$

C) Candidates of Additive Terms and Their Features

Through the discussions before, we have the question,

Is the additive term written explicitly which satisfies Assumption 3?

and the questions on stability, stability margins, cost improvements. The author shows these characteristics in next sections in detail, but would like to answer the question above.

In linear time-invariant systems and when the variation is linear and the uncertainty is only in the system matrix that is characterized by the single parameter q, equating f as $q\delta Ax$, we can rewrite Assumption 3 as

$$q \cdot (\delta A \cdot P + P \cdot \delta A^T) \leq h(|q|, P), \qquad h \geq 0. \tag{40}$$

The cases where the other types of variations than system matrices exist are treated later in this paper. In view of an easily recognized inequality

$$\left(\sqrt{\alpha} \cdot I \pm \frac{1}{\sqrt{\alpha}} \cdot \delta A\right) \cdot P \cdot \left(\sqrt{\alpha} \cdot I \pm \frac{1}{\sqrt{\alpha}} \cdot \delta A\right)^T \geq 0, \qquad \alpha > 0, \tag{41}$$

we can accept the following inequality

$$q \cdot (\delta A \cdot P + P \cdot \delta A^T) \leq |q| \cdot \left(\alpha \cdot P + \frac{1}{\alpha} \cdot \delta A \cdot P \cdot \delta A^T\right), \qquad \alpha > 0, \tag{42}$$

where α is an adjustable parameter. Or, using the spectrum radius of δA denoted by σ,

$$q \cdot (\delta A \cdot P + P \cdot \delta A^T) \leq |q| \cdot \left(\alpha + \frac{1}{\alpha} \cdot \sigma^2\right) \cdot P, \tag{43}$$

otherwise utilizing the minimized value of the right hand side of the inequality above,

$$q \cdot (\delta A \cdot P + P \cdot \delta A^T) \leq |q| \cdot 2 \cdot \sigma \cdot P. \tag{44}$$

If δA has full rank, minimum spectrum radius $\sigma_{\min}$ is not zero. Therefore,

$$q \cdot (\delta A \cdot P + P \cdot \delta A^T) \leq \frac{2 \cdot \sigma}{\sigma_{\min}^2} \cdot |q| \cdot \delta A \cdot P \cdot \delta A^T \tag{45}$$

is a candidate form of $h(*)$ in eq. (40). These are the answers for the question.

The readers may well devise another type of inequality such as

$$\left(\sqrt{\alpha} \cdot P \pm \frac{1}{\sqrt{\alpha}} \cdot \delta A\right) \cdot \left(\sqrt{\alpha} \cdot P \pm \frac{1}{\sqrt{\alpha}} \cdot \delta A\right)^T > 0. \tag{46}$$

In fact this inequality holds itself clearly. But if this form is applied to, the result is

$$q\cdot(\delta A\cdot P+P\cdot\delta A^T)\leq|q|\cdot\left(\alpha P^2+\frac{1}{\alpha}\cdot\delta A\cdot\delta A^T\right). \tag{47}$$

In this form of $h(*)$, the monotonous feature with respect to q holds, too, but this property with respect to P might not be maintained except for the special cases. And similar forms as in G.C.C. of Peng may be a candidate:

$$q(\delta A\cdot P+P\cdot\delta A^T)\leq|q|\cdot T^T|\Lambda|\cdot T \text{ as in eq. (24).} \tag{48}$$

But this form also may lose the monotonous character as to P.

By the two types of inequalities (47), (48) the cost deteriolation in the large sense will be improved between the nominal point and the specified variated point. But in these cases the uniform improvement of cost cannot be promised in sufficient sense, because in these cases $\tilde{P}$ cannot be always the monotone function of q. This is the difference between M.C.V. and usual G.C.C.

And the author notes that the former three types of inequalities eq. (42), (44), (45) satisfy Assumption 4, because the integrant of eq. (37) vanishes. In view of these three types, the relationships between existing "Additive Term Designs" and these will be clarified. In fact the third form is the same one as that of the "State Dependent Noise" concept by Kleinmann, which shows that if the variation δA has full rank, M.C.V. design is almost equivalent to "State Dependent Noise" design. And as to the second form following interpretations can be made; If this form is taken in designs, the open loop system which should be controlled is modeled as that having its pole locations shifted in the right direction a little. Thus it can be said that this method is almost equivalent to the worst case designs as it were. The first form (42) is the mixed type of the second and the third, i.e. the character is the middle one between the "State Dependent Noise" and the worst case designs depending on the selection of α. And the fourth type (47) which cannot guarantee the uniform insensitivity realization reveals that this form relates deeply with "Uncertainty Weighting" method in view of the second term of it. In this paper, while the first form design (42) is termed simply as the M.C.V. design, the fourth form design (47) is called "Uncertainty Weighting type M.C.V." (U.W. type M.C.V.). And the other forms are not considered as M.C.V. designs. (The M.C.V. form stated here is shown in Jain's [66]. Though his form happens to resemble this, there's no consideration like the monotonous character or the relation to the alternative forms here.)

Thus we have prepared the solid expressions of the additive terms, but the readers may ask what value of the design range q should be specified. Simply speaking, this range can be taken as the boundary norm of the parameter variations of which the designers are informed a priori. But the value taken in this way may produce considerably so high nominal cost or gain that we cannot adopt these results practically. (These situations are shown in numerical examples later.) Hence this range q should be searched from practical points of view together with an adjustable parameter α.

D) Some Foundations of M.C.V.

Finally, we show the existence condition of the M.C.V. surface and commented the monotonous feature. For simplicity, we restrict our discussions to the time-invariant infinite time problems associated with the first type of M.C.V. additive term. In this case, the cost equation corresponding to the M.C.V. type covariance propagation is written as

$$O=\tilde{A}^T\cdot S+S\cdot\tilde{A}+Q+K^T\cdot R\cdot K+|q|\cdot\left(\alpha\cdot S+\frac{1}{\alpha}\cdot\delta A^T\cdot S\cdot\delta A\right). \tag{49}$$

This defines the M.C.V. cost value. For this equation is linear one with respect to cost S, therefore the necessay and sufficient condition for the existence of soluiton S is readily derived.

Theorem 3.

The necessary and sufficient condition that the unique solution of eq. (49) exists is

$$\lambda_i\cdot\left(\tilde{A}^T\otimes I+I\otimes\tilde{A}^T+|q|\cdot\alpha\cdot I\otimes I+\frac{|q|}{\alpha}\cdot\delta A^T\otimes\delta A^T\right)\neq 0 \quad (i=1,2,\cdots,n^2),$$

where $\lambda_i(*)$ denotes the eigenvalues. (50)

If the assumption in above theorem holds, M.C.V. surface is well defined and the monotonous character is proved to exist as follows. Let the two solutions for distinct parameter values q_1 and q_2,

$$O=S_1\cdot\tilde{A}+\tilde{A}^T\cdot S_1+Q+K^T\cdot R\cdot K+q_1\cdot\left(\alpha\cdot S_1+\frac{1}{\alpha}\cdot\delta A^T\cdot S_1\cdot\delta A\right),$$

$$O=S_2\cdot\tilde{A}+\tilde{A}^T\cdot S_2+Q+K^T\cdot R\cdot K+q_2\cdot\left(\alpha\cdot S_2+\frac{1}{\alpha}\cdot\delta A^T\cdot S_2\cdot\delta A\right),$$

where $q_1\geq q_2$, (51)

and subtracting each other, and note that the following equation for cost variation holds.

Lemma 1.

The solution of

$$O=S\cdot\tilde{A}+\tilde{A}^T\cdot S+M+(\alpha\cdot S+\beta\cdot\delta A^T\cdot S\cdot\delta A),\quad \alpha,\ \beta>0,\quad M>0 \tag{52}$$

where $\tilde{A}$ is stable, satisfies

$$S\geq S^*\ (\geq 0), \tag{53}$$

where S^* is the solution of

$$O=S^*\cdot\tilde{A}+\tilde{A}^TS^*+M. \tag{54}$$

Using this lemma and the equation above, through the monotonous feature of the additive term with respect to q and S, it is clear that M.C.V. cost surface is monotonous:

$$S_2 \geq S_1. \tag{55}$$

E) *An Example*

We would like to present a very simple and comprehensive example. While the cost is improved in the upper limit sense, it is commented here that the cost surface may be improved actually by the M.C.V. design, whose mechanism is deeply investigated in next section.

Example 1.

We consider the scalar system and its variated one

$$\dot{x}=x+u, \quad u=-k\cdot x \quad \text{and} \quad \dot{x}=x+u+q(\delta ax+\delta bu) \tag{56}$$

with the following performance index to be minimized:

$$J=\int_0^\infty (x^2+u^2)dt. \tag{57}$$

The usual regulator is solved by the L.Q. method and this results in

$$k=2.414. \tag{58}$$

The corresponding stability properties are

$$\text{gain factor} \geq 0.414, \text{ phase shift} \leq 65.5 \text{ (degrees)} \tag{59}$$

and the stable regions are as in Fig. 4.

The M.C.V. design for this problem is solved for the following uncertainties, taking the design range q as 1,

$$\delta a=\delta b=0.333 \tag{60}$$

and this results in the following gain and stability properties. (Control coefficient variation δb is treated as δbk system coefficient variation, though the exact treatment of this variation is omitted here which is shown in chapter 4.)

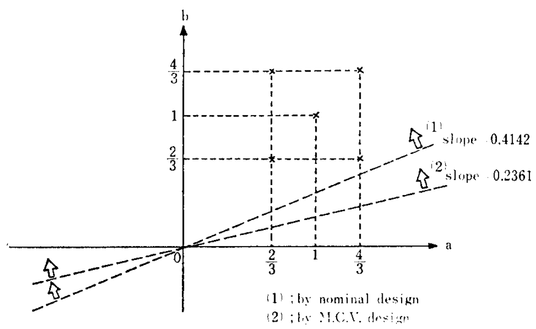


Fig. 4. Robust Design Example—Stable Region.

$$k=4.236, \tag{61}$$

$$\text{gain factor} \geq 0.236, \text{ phase shift} \leq 76.4 \text{ (degrees)} \tag{62}$$

and the corresponding stable region is illustrated also in Fig. 4. The cost values at some variated points are compared with each other as followings.

	Cost of the Robust Controller	Cost of the Nominal Controller
$a=b=1$,	2.927	2.412
$a=\frac{4}{3}$, $b=\frac{2}{3}$,	6.354	12.345
$a=\frac{2}{3}$, $b=\frac{4}{3}$,	1.901	2.797
$a=\frac{2}{3}$, $b=\frac{2}{3}$,	4.391	3.618
$a=\frac{4}{3}$, $b=\frac{4}{3}$,	2.195	1.811

It is clear that the cost surface and the stability and stability margins are improved at some points by slight amount of the cost at the origin. The readers may well ask if these properties are improved really, and at what regions these improvements are achieved. In next section the author discusses these problems in detail.

3–2. *Cost and Stability Improvement of Additive Term Designs*

From the example above, we can expect the cost surface and stability and stability margins are improved actually by M.C.V. design. In order to make sure of these properties, we would like to examine and analyze some improvement mechanisms of these generally. But in practical systems it is almost impossible to realize state feedback control laws because the number of sensors is limited. Hence the cost and stability margin improvements should be analyzed in each practical case. But even when the optimal output feedback laws are taken, the corresponding cost equation is still considerably complex, so we cannot discuss these subjects except for the ideal case, in which state feedback is possible. Therefore hereafter we will limit our discussions to the comparison between ideal optimal systems and the suboptimal robust ones designed by "Additive Term Designs", assuming that state feedback is possible in both systems. (And implicitly the systems are assumed to be time-invariant and to be considered in infinite-time problems without any comment.) And from now on in our discussions, the uncertain sources are assumed to be only in system matrices, which is in practical systems not restrictive, for the uncertainty of control matrices can be modeled as the actuator dynamics in system matrices.

3–2–1. *Cost Improvement*

A) Sufficient Condition 1—Improvement Mechanisms

Now, we deeply consider the problem whether the true cost surface is improved or not by "Additive Term Designs" and particularly by the M.C.V. technique.

Let's consider the nominal system design and the "Additive Term Design" (insensitive controller design) as

$$0=S_{10}\tilde{A}_1+\tilde{A}_1^T S_{10}+Q+K_1^T RK_1, \quad K_1=R^{-1}B^T S_{10}, \quad \tilde{A}_1=A-BK_1$$
$$0=S_{20}\tilde{A}_2+\tilde{A}_2 S_{20}+Q+K_2^T RK_2+h(q_0, S_{20}),$$
$$K_2=R^{-1}B^T S_{20}, \quad \tilde{A}_2=A-BK_2. \tag{63}$$

Thus we have the ideal optimal control law K_1 and the suboptimal robust control law K_2 of M.C.V. at the design range q_0. After this point, we define S_1 surfaces as overall true cost surfaces and S_2 surfaces as design cost surfaces by which sometimes the M.C.V. surfaces are meant. And we temporarily define $\tilde{S}_1$ surfaces as true cost surfaces using the insensitively designed control law K_2. Then the two types of true surfaces corresponding to the equations above are determined by following usual cost equations:

$$0=S_1(q)\tilde{A}_1+\tilde{A}_1^T S_1(q)+Q+K_1^T RK_1+S_1(q)\delta A(q)+\delta A(q)^T S_1(q),$$
$$0=\tilde{S}_1(q)\tilde{A}_2+\tilde{A}_2^T\tilde{S}_1(q)+Q+K_2^T RK_2+\tilde{S}_1(q)\delta A(q)+\delta A(q)^T\tilde{S}_1(q). \tag{64}$$

Next, we define cost variation $\delta S(q)$ as $(S_1(q)-\tilde{S}_1(q))$ and write the cost equation of δS:

$$0=\delta S(q)(\tilde{A}_2+\delta A(q))+(\tilde{A}_2+\delta A(q))^T\delta S(q)+K_1^T RK-K_2^T RK_2$$
$$+S_1(q)B(K_2-K_1)+(K_2-K_1)^T B^T S_1(q). \tag{65}$$

If the two systems are stable at these parameter variated points, the sign of $\delta S(q)$ cost depends only on that of the following matrix function by Lyapunov's criterion:

$$f(q)=K_1^T RK_1-K_2^T RK_2+S_1(q)B(K_2-K_1)+(K_2-K_1)^T B^T S_1(q). \tag{66}$$

At the nominal parameter point without any variations, $f(q)$ is reduced to a simple form:

$$f(0)=-(S_{20}-S_{10})BR^{-1}B^T(S_{20}-S_{10})<0. \tag{67}$$

This inequality directly indicates

$$\delta S(0)=S_1(0)-\tilde{S}_1(0)<0, \quad \text{or} \quad S_1(0)<\tilde{S}_1(0). \tag{68}$$

This property is quite natural one, for $\tilde{S}_1$ surface is not optimal at the nominal parameter point but only suboptimal. And through the slightly different manipulations the alternative forms of $f(q)$ are written as

$$f(q)=-(S_{20}-S_1(q))BR^{-1}B^T(S_{20}-S_1(q))+(S_1(q)-S_{10})BR^{-1}B^T(S_1(q)-S_{10}), \quad \text{or}$$
$$-(S_{20}-\tilde{S}_1(q))BR^{-1}B^T(S_{20}-\tilde{S}_1(q))+(\tilde{S}_1(q)-S_{10})BR^{-1}B^T(\tilde{S}_1(q)-S_{10}). \tag{69}$$

These two forms appear to be not compatible. But by the assumption that the two systems remain stable at the points considered, both two forms of matrix functions reflect the sign of $\delta S(q)$ cost. Thereby we easily observe that if the designs are performed treating additive terms as $S\delta A(q)+\delta A(q)^T S=h(q, S)$, at $q=q_0$; i.e. $\tilde{S}_1(q_0)=S_{20}$,

$$\tilde{S}_1(q_0)<S_1(q_0). \tag{70}$$

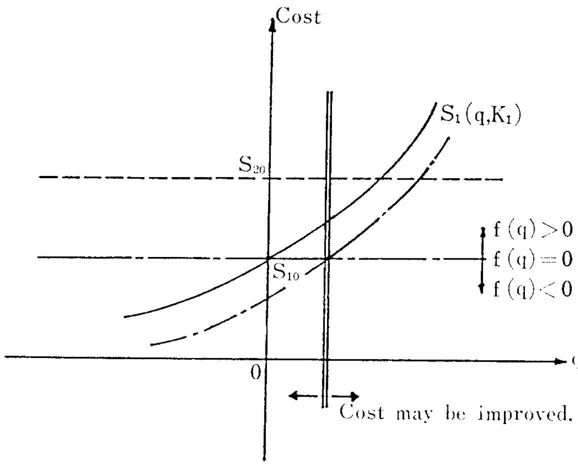


Fig. 5. Intuitive Interpretation of Cost Improvement.

This relation is also quite reasonable, because in this case the design is optimally made at $q=q_0$ naturally.

This rewritten forms of $f(q)$, (69), indicates qualitative interpretations as in Fig. 5. Namely if S_1 surface (nominal surface) is monotonously increasing, then over the region in the direction of higher cost values, this cost surface can be improved. Apparently at a glance, if only the cost matrix S_{20} which is used for control laws is higher, then at any rate cost surface appeares to be improved. In scalar systems and some special systems, this analogy is right, but in many systems it is not so simple as expected to satify the positivity condition of $f(q)$ above. Hence for the purpose of obtaining more informations from the inequality (69), next we would like to proceed to the qualitaive discussions on these mechanisms. In order that the positivity of $f(q)$ in eq. (69) is realized, the following relations are required to hold:

$$\Gamma[(S_{20}-S_1(q))BR^{-1}B^T(S_{20}-S_1(q))]\subseteq\Gamma[(S_1(q)-S_{10})BR^{-1}B^T(S_1(q)-S_{10}))], \quad \text{or}$$

$$D[(S_{20}-S_1(q))BR^{-1}B^T(S_{20}-S_1(q))]\leq D[(S_1(q)-S_{10})BR^{-1}B^T(S_1(q)-S_{10}))],$$

where $\Gamma(H)$ denotes the space in which the matrix produces a non zero image, and $D(H)=\text{dim.}(\Gamma(H))$. (71)

Though these relations are not equivalent to the inequality (69) of $f(q)$ and these interpretations should be made carefully, roughly speaking it is required for cost improvement by "Additive Term Designs" that the "spanned space" of cost matrices would rather be confined ones. More qualitatively speaking, while the design cost matrix S_{20} should evaluate and reflect the true cost variations, but it should not overestimate the cost variations. Particularly if $(S_{20}-S_1(q))$, $(S_1(q)-S_{10})$ and $BR^{-1}B^T$ are positive definite, which is satisfied in scalar systems or some special cases, then this condition is reduced to (see Kodama [*133*])

$$S_1(q)>\tfrac{1}{2}(S_{10}+S_{20}). \qquad (72)$$

This inequality indicates, to tell much more roughly, cost improvement can be achieved over the region denoted above.

B) Sufficient Condition 2—Requirements for Additive Terms

Now next, we discuss the sufficient condition such that the design surface $S_2(q, K_2)$ is below the nominal surface $S_1(q, K_1)$ at the specified design point $q=q_0$. If the design is made by G.C.C. or M.C.V. methods, this condition, if satisfied, absolutely guarantees the cost improvement clearly. Like the manners taken above, consider the nominal surface S_1 and the design cost surface S_2

$$\begin{aligned} 0&=S_1\tilde{A}_1+\tilde{A}_1^T S_1+Q+K_1^T RK_1+S_1\delta A(q)+\delta A(q)^T S_1,\\ 0&=S_2\tilde{A}_2+\tilde{A}_2^T S_2+Q+K_2^T RK_2+h(q, S_2). \end{aligned} \tag{73}$$

Again like the previous procedures, let $\delta S(q)$ be defined as $(S_1(q)-S_2(q))$. Then the equation for $\delta S(q)$ to satisfy is as follows:

$$\begin{aligned} 0=&\delta S(\tilde{A}_1+\delta A(q))+(\tilde{A}_1+\delta A(q))^T\delta S+K_1^T RK_1-K_2^T RK_2-S_2B(K_1-K_2)\\ &-(K_1-K_2)^T B^T S_2-h(q, S_2)+S_2\delta A(q)+\delta A(q)^T S_2. \end{aligned} \tag{74}$$

Consequently the sign of $\delta S(q)$ depends on the following matrix function $f(q)$:

$$\begin{aligned} f(q)=&K_1^T RK_1-K_2^T RK_2-S_2B(K_1-K_2)-(K_1-K_2)^T B^T S_2\\ &-h(q, S_2)+S_2\delta A(q)+\delta A(q)^T S_2\\ =&-(S_{20}-S_{10})BR^{-1}B^T(S_{20}-S_{10})+(S_2(q)-S_{10})BR^{-1}B^T(S_{20}-S_{10})\\ &+(S_{20}-S_{10})BR^{-1}B^T(S_2(q)-S_{10})-h(q, S_2)+(S_2\delta A(q)+\delta A(q)^T S_2). \end{aligned} \tag{75}$$

At the point $q=q_0$ (i.e. the designed point) $S_2(q_0)=S_{20}$, so

$$f(q_0)=(S_{20}-S_{10})BR^{-1}B^T(S_{20}-S_{10})+S_{20}\delta A(q_0)+\delta A(q_0)^T S_{20}-h(q_0, S_{20}). \tag{76}$$

Therefore in order for design surface $S_2(q, K_2)$ to be below the nominal true cost surface $S_1(q, K_1)$ at $q=q_0$, it is sufficient for the following inequality to hold:

$$f(q_0)>0, \quad \text{or} \quad h(q_0, S_{20})<(S_{20}-S_{10})BR^{-1}B^T(S_{20}-S_{10})+S_{20}\delta A(q_0)+\delta A(q_0)^T S_{20}. \tag{77}$$

This inequality does inform us of very important information; The additive term h that is used for robust controller designs would rather be bounded one. Well let's examine this condition particularly as for G.C.C. and M.C.V. designs. Reviewing the requirement for G.C.C. or M.C.V. designs, the additive term $h(*)$ should be such that

$$h(q_0, S_{20})>S_{20}\delta A(q_0)+\delta A(q_0)^T S_{20} \text{ and } h(0, —)=0,$$
and for M.C.V. designs h is a monotone increasing function of q, S. (78)

Connecting these conditions, we can easily reach to the following sufficient condition.

Theorem 4.

The sufficient condition for the design cost S_{20} to be below the nominal cost $S_1(q_0)$ at $q=q_0$ is that the additive term h satisfies, if M.C.V. or G.C.C. design is employed,

$$S_{20}\delta A(q_0)+\delta A(q_0)^T S_{20}<h(q_0, S_{20})$$
$$<S_{20}\delta A(q_0)+\delta A(q_0)^T S_{20}+(S_{20}-S_{10})BR^{-1}B^T(S_{20}-S_{10})$$
and also the conditions of eq. (78). (79)

It will be noted again that this condition, if satisfied, absolutely guarantees the cost improvement. This theorem also reveals that there is really possibility of cost improvement by G.C.C. or M.C.V. from the bandwidth of inequality (79), and that in qualitative sense, additive term or designed cost should be modest rather than arbitrary. (In real computations, eq. (79) in this theorem is not appropriate for this judgement. The direct comparison of true cost surfaces is more effective than this criterion naturally.)

C) Useful Properties—S_3 Surface

The upper limit condition of additive term $h(*)$ in eq. (79) has only to be satisfied for $S_2(q_0)$ (S_{20}) the designed cost matrix. But hereafter in our discussions, in order to obtain more deeper insight into the mechanisms, we investigate some characteristics assuming that such condition as for h is satisfied by any positive semi-definite matrix V. That is

Assumption 5.

For $\forall V>0$,

$$V\delta A(q)+\delta A(q)^T V<h(q, V)<V\delta A(q)+\delta A(q)^T V+(V-S_{10})BR^{-1}B^T(V-S_{10}). \quad (80)$$

First of all, as the preparation for introducing a new cost surface, we repeat the sufficiency of cost improvement under the Assumption 5 again. Considering the nominal cost surface and the upper cost surface corresponding to respective designed control laws again,

$$0=S_1\tilde{A}_1+\tilde{A}_1^T S_1+Q+K_1^T RK_1+S_1\delta A(q)+\delta A(q)^T S_1,$$
$$0=S_2\tilde{A}_2+\tilde{A}_2^T S_2+Q+K_2^T RK_2+h(q, S_2) \quad (73)$$

under the assumptions (80) as to additive term $h(*)$. Then the cost variation $\delta S=S_1-S_2$ satisfies the following equations.

$$0=\delta S(q)(\tilde{A}_1+\delta A(q))+(\tilde{A}_1+\delta A(q))^T\delta S+f(q), \quad (74)$$
where $f(q)$ is defined in eq. (75).

Introducing the inequality of the assumption (80), we have

$$f(q)>-(S_{20}-S_2(q))BR^{-1}B^T(S_{20}-S_2(q)). \quad (81)$$

But by the definition of the upper cost surface which is used for designs, $S_2(q_0)=S_{20}$. Therefore

$$f(q_0)>0, \quad \text{and} \quad S_2(q_0)<S_1(q_0). \quad (82)$$

This result guarantees the cost improvement sufficiently under the Assumption 5, though it's somewhat strict one.

Next let's examine qualitatively the method by which the satisfaction of the inequality in the assumption (80) is judged. Now we define the following new cost surface named as S_3 surface:

$$0=S_3(q)\tilde{A}+\tilde{A}^TS_3(q)+Q+K^TRK+S_3(q)\delta A(q)+\delta A(q)^TS_3(q)$$
$$+(S_3(q)-S_{10})BR^{-1}B^T(S_3(q)-S_{10}), \tag{83}$$

where $\tilde{A}=A-BK$, for some given gain K.

And by similar procedures, we get the relation of the cost variation δS between S_3 cost and S_2 cost at point q:

$$0=\delta S\tilde{A}+\tilde{A}^T\delta S+h(q,\delta S)+S_3\delta A(q)+\delta A(q)^TS_3$$
$$+(S_3-S_{10})BR^{-1}B^T(S_3-S_{10})-h(q,S_3). \tag{84}$$

If the inequality (80) of $h(*)$ additive term holds, using the monotonous feature of this, we have

$$\text{If the eq. (80) is satisfied, then } \delta S>0 \text{ or } S_3(q)>S_2(q). \tag{85}$$

Consequently, the S_3 surface demonstrates the satisfaction of the inequality (80) qualitatively. But the readers should note that this criterion is necessary for that inequality (80) and that is sufficient for cost improvement. Hence the criterion above is only qualitative one rather than quantitative one. But in spite of these ambiguities, this is expected to present us the meaningful informations; such an example is shown in numerical examples later. If the S_3 surface is kept nearly flat under the nominal design, we should recognize that a great amount of cost improvement cannot be provided. And otherwise we may refine the cost surfaces through these "Additive Term Design" techniques M.C.V. or G.C.C.. Schematic illustrations are shown in Fig. 6.

Finally we inquire into the other aspects of S_3 surface, which relate to the boundary nature of cost improvement as mentioned before and lead to the properties to be discussed later. Assume that we minimize the S_3 cost at point q_0 by the control laws as followings:

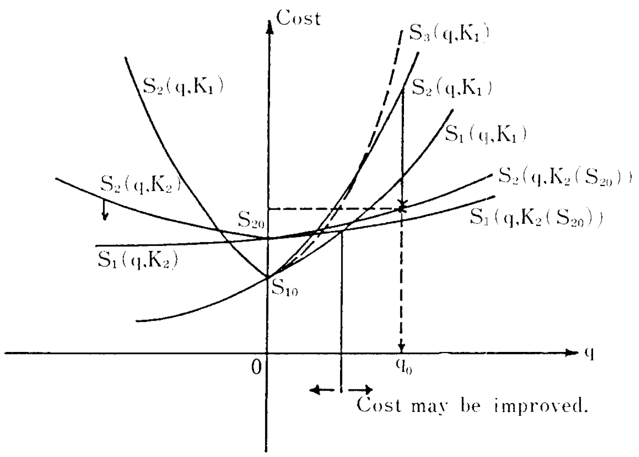


Fig. 6. The Concepts of S_1, S_2, S_3, Cost Surfaces.

$$0=S_3(q_0)(A-BK)+(A-BK)^T S_3(q_0)+Q+K^T RK+S_3(q_0)\delta A(q_0)$$
$$+\delta A(q_0)^T S_3(q_0)+(S_3(q_0)-S_{10})BR^{-1}B^T(S_3(q_0)-S_{10}),$$
$$K=R^{-1}B^T S_3(q_0). \tag{86}$$

And the nominal cost $S_1(q_0)$ is given by

$$0=S_1(q_0)\tilde{A}_1+\tilde{A}_1^T S_1(q_0)+Q+K_1^T RK_1+S_1(q_0)\delta A(q_0)+\delta A(q_0)^T S_1(q_0). \tag{87}$$

Utilizing the relations

$$(S_3(q_0)-S_{10})BR^{-1}B^T(S_3(q_0)-S_{10})=K^T RK+K_1^T RK_1-S_3(q_0)BK_1-K_1^T B^T S_3(q_0),$$
$$S_3(q_0)(A-BK)+(A-BK)^T S_3(q_0)$$
$$=S_3(q_0)\tilde{A}_1+\tilde{A}_1^T S_3(q_0)-2K^T RK+S_3(q_0)BK_1+K_1^T B^T S_3(q_0), \tag{88}$$

we have lastly

$$0=(S_3(q_0)-S_1(q_0))(\tilde{A}_1+\delta A(q_0))+(\tilde{A}_1+\delta A(q_0))^T(S_3(q_0)-S_1(q_0)). \tag{89}$$

Hence providing the unique existence of the solution above, the interesting result

$$S_3(q_0)=S_1(q_0) \tag{90}$$

is obtained. This shows that even if we design by S_3 surface, we can only get the same cost as the nominal one; that is, any cost improvement may not be expected. And easily we recognize that the design cost surface S_2 should be below this S_3 surface, which leads to the previous dicussions, (85), where's qualitative evaluation of costs between S_3 surfaces and S_2 surfaces. We will show the other properties of S_3 surfaces in detail later in this paper.

3–2–2. *Stability Improvement*

In the previous section, we looked into the problem whether the true cost surface is improved or not by "Additive Term Designs", where we discuss it assuming that the system is stable at the points of parameters considered. So the cost improvement property discussed there does not imply the stability improvement because of the assumption made there. In this section, we show that in case "Additive Term Design" techniques are employed, stable regions are expanded through the Lyapunov's sufficient criterion, and that stability margins are also improved roughly and the sensitivity which is defined in Cruz's sense is reduced as in usual L.Q. regulators. And some important results are shown.

A) *Stability and Stable Region*

The cost equations which are constructed in design procedures of "Additive Term Design" techniques are written as follows:

$$0=S\tilde{A}+\tilde{A}^T S+Q+K^T RK+h(q, S), \qquad \tilde{A}=A-BK. \tag{91}$$

Where we assume the additive term $h(*)$ is a positive semi-definite matrix. To speak roughly, this shows the direct insight into stability properties through Lyapunov's criterion. Namely, let the linear system variation $q\delta A$ satisfy the following relation:

$$h(q,\,S)\geq q(S\delta A+\delta A^T S). \tag{40}$$

These class of variations are permitted for the system to remain stable by "Additive Term Designs". Hence the stability over the specified range of parameters is generally expected to be assured. In fact, this perspective is true and justified also for the general variated system eq. (28) by the following manners. Defining the Lyapunov's function V as

$$V=x^T Sx, \tag{92}$$

we attempt to evaluate the time-derivative of V along the trajectories (28):

$$\begin{aligned}\frac{d}{dt}V&=-x^T(Q+K^T RK)x-x^T\frac{\partial\,\mathrm{tr.}\,(Sh)}{\partial\tilde{P}}x+f^T Sx+x^T Sf\\ &\leq-\mathrm{tr}\,[(Q+K^T RK)P]-\mathrm{tr}\left[\frac{\partial\,\mathrm{tr.}\,(Sh)}{\partial\tilde{P}}P-Sh\right].\end{aligned} \tag{93}$$

Consequently for the first term is negative, in order to assure the stability, it is sufficient that the second term is not positive. In view of Assumption 4 and the considerations made in the previous section as to the linear cases, this relation is proved to hold. Hence concerning the M.C.V. design, the following results are obtained.

Theorem 5.

If the M.C.V. method is employed, then the stability is assured over the specified range of parameters.

B) Stability Margins

Thus we obtain the stable regions, eq. (40), assured by "Additive Term Designs" in linear cases and Theorem 5 for M.C.V. Next we pick up the problem of the stability margins. That is—

> To what amount the stability margins are improved by these "Additive Term Designs"?

It is well known to us that the usual L.Q. regulators have a half scale gain tolerance and at least 60 degrees phase margin. We proceed to this subject through the modification of well known circle condition of L.Q. regulators, using the robustness properties that are studied by Doyle [*28*, *29*] and others [*30*].

We begin with the following assumption.

Assumption 6.

The additive term h satisfies

$$h>\mu SBR^{-1}B^T S, \qquad \mu>0. \tag{94}$$

Under the inequality above, introducing the complex variable s (Laplace variable) into the design cost equation,

$$0=-S(sI-A)+(sI+A^T)S+Q-SBR^{-1}B^TS+h(q,\ S) \qquad (59)$$

is obtained. And operating $(sI-A)^{-1}B$ from left and $-B^T(sI+A)^{T-1}$ from right, we have

$$RG(s)+G^*(s)R+(1-\mu)G^*(s)RG(s)>B^T(sI-A)^{*-1}Q(sI-A)^{-1}B\geq 0,$$

where $G(s)=R^{-1}B^TS(sI-A)^{-1}B$; loop transfer function. (96)

Therefore finally we obtain the modified circle condition as follows:

$$\left(\frac{1}{1-\mu}I+G(s)\right)^* R\left(\frac{1}{1-\mu}I+G(s)\right)\geq\frac{1}{(1-\mu)^2}R, \quad \text{or}$$

$$\underline{\sigma}\left(\frac{1}{1-\mu}I+\hat{G}(s)\right)\geq\frac{1}{1-\mu}, \quad \text{where } \hat{G}(s)=R^{1/2}G(s)R^{-1/2}. \qquad (97)$$

This condition implies that the vector locus of the loop transfer function $\hat{G}(s)$ is out of the circle with the radius of $1/(1-\mu)$, whose center lies at $(-1/(1-\mu),\ 0)$. (Without loss of generality, we assume $\mu<1$, because the positive cost solution may not be obtained otherwise.) This implication is schematically illustrated in Fig. 7. Moreover, this condition is manipulated as followings:

$$\langle=\rangle \text{ for } \forall i,\ \left|\frac{1}{1-\mu}I+\lambda_i(\hat{G}(s))\right|\geq\frac{1}{1-\mu}, \quad (i=1,2,\cdots,n). \qquad (98)$$

Operating the mapping

$$z(\lambda)=1+\frac{1}{\lambda}, \qquad (99)$$

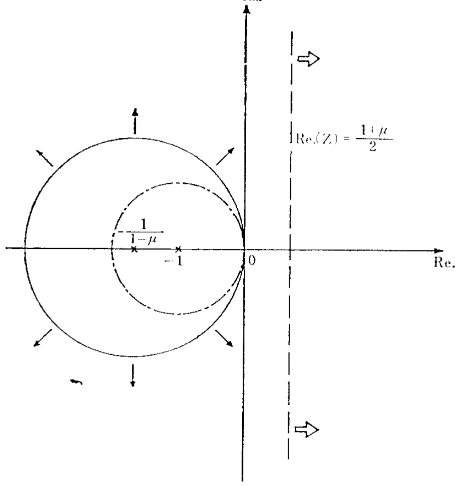


Fig. 7. Modified Circle Condition.

the image of this is

$$\left[z \,|\, \mathrm{Re.}\,(z) \geq \frac{1}{2}(1+\mu)\right], \quad \text{or} \quad \underline{\sigma}[I+\hat{G}(s)^{-1}] \geq \frac{1+\mu}{2}. \tag{100}$$

By the way, here we review the result of Doyle's [*28*]:

If $\underline{\sigma}[I+\hat{G}(s)^{-1}] \geq 1_m(\omega)$, then for $\forall L(s)$ such that $\bar{\sigma}[L(s)] \leq 1_m(\omega)$ the variated system $\tilde{G}(s)=(I+L(s))G(s)$ remains stable. (101)

Equating $\hat{L}(s)=I+L(s)$, we have the following lemma.

Lemma 2.

If $\underline{\sigma}[I+\hat{G}(s)^{-1}] \geq 1_m(\omega)$, then for $\forall \hat{L}(s)$ such that $\underline{\sigma}(\hat{L}(s)) \geq 1-1_m(\omega)$ the variated system $\tilde{G}(s)=\hat{L}(s)G(s)$ remains stable. (102)

Now we apply this to the complementary condition of return difference derived lastly:

$$\underline{\sigma}[\hat{L}(s)] \geq \frac{1-\mu}{2}. \tag{103}$$

This leads to the gain margin regarding $\hat{L}(s)$ as a constant diagonal matrix:

$$\text{gain factor} \geq \frac{1-\mu}{2}. \tag{104}$$

Comparing with that of the usual L.Q. regulators, this property is improved clearly. And next, we consider the phase margin. The discussions on this are proceeded like the manners of Athans [*32*] or Kimura [*130*]. If the control input v is variated as

$$\mathscr{L}[v]=\hat{L}(s)\mathscr{L}[u], \text{ where } u \text{ is an optimal control input.} \tag{105}$$

and where the operator $\mathscr{L}(-)$ denotes the Laplace transformation, then in order that the system remains stable, it is proved to be sufficient that the following inequality is satisfied through Parseval's theorem:

$$\hat{L}(j\omega)R^{-1}+R^{-1}\hat{L}(j\omega)^* \geq (1-\mu)R^{-1}, \quad \text{or} \quad R\hat{L}(j\omega)+\hat{L}(j\omega)^*R \geq (1-\mu)R. \tag{106}$$

And the alternative forms of them can be written as

$$\tilde{L}(j\omega)+\tilde{L}(j\omega)^* \geq (1-\mu)I, \quad \text{where } \tilde{L}(s)=R^{1/2}L(s)R^{-1/2}. \tag{107}$$

This result reveals that if the operator $\tilde{L}(s)$ is

$$\tilde{L}(s)=\text{diagonal. } (\exp(\phi s)), \tag{108}$$

then the following class of phase shift is permitted:

$$\cos\phi \geq \tfrac{1}{2}(1-\mu). \tag{109}$$

Comparing with that of the usual L.Q. regulators, this phase margin is also improved clearly.

C) Sensitivity

The sensitivity which is established by Cruz [*18*] is considered and presented in the last chapter. And we show that if the ideal feedback control laws are employed, this sensitivity is reduced under the value of unity. In these "Additive Term Designs", this property is proved to be maintained. Because the modified circle condition is reformed to

$$\underline{\sigma}[I+\hat{G}(j\omega)]\geq 1 \tag{110}$$

and this implies that the sensitivity which is the inversed form of the return difference is reduced under unity with repect to the equivalent open loop system.

D) Important Results for M.C.V. or G.C.C.

So far, the general discussions on the stability margins of the systems designed by "Additive Term Designs" are shown. And the evaluations made there imply that the stability margins are improved in sufficient sense. Next we show that if the G.C.C. or M.C.V. designs are utilized, higher stability margins are realized than those in the other "Additive Term Designs". Considering the cost equation in G.C.C. or M.C.V. designs,

$$0=SA+A^TS-SBR^{-1}B^TS+Q+h(q_0,\,S) \tag{111}$$

and the upper bound feature of additive term $h(*)$,

$$h(q_0,\,S)-S\delta A(q_0)-\delta A(q_0)^TS=\Phi(>0) \tag{112}$$

we obtained the following extended type of the circle condition as previously noted:

$$\begin{aligned}&\underline{\sigma}[I+\tilde{G}(s)]\geq 1,\quad \text{where } \tilde{G}(s)=R^{1/2}\hat{G}(s)R^{-1/2},\\ &\hat{G}(s)=R^{-1}B^TS(sI-(A+q\delta A))^{-1}B,\quad |q|<q_0.\end{aligned} \tag{113}$$

Therefore, the gain margins are maintained at the same level as the usual L.Q. regulators over the specified range of parameter variations. And for the phase margins, though here we omit the detail considerations, they are also maintained like the gain margins. Thus we get the following theorem.

Theorem 6.

The systems designed by G.C.C. or M.C.V. methods have the stability margins as

gain factor $\geq\frac{1}{2}$, phase shift ≤ 60 degrees

over the specified range of parameter variations.

E) Simple Robust Realization

Lastly we refer to the simpler "Additive Term Design" methods, which are motivated by the conceps or discussions on the stability margins here. The simpler forms of the additive term $h(*)$ which may satisfy the inequality (94) are followings:

$$h=\alpha S^{m} \qquad (m=0, 1, 2, \cdots). \tag{114}$$

These types are termed "Simple Robust Realization" methods in this paper. The case $m=0$ is the analogous approach to the "Uncertainty Weighting" method. The case $m=1$ is the worst case design in which the open loop system is treated as the system with the poles shifted to the right. The case $m=2$ is a very interesting one. In this case, the μ factor in the inequality (94) is independent of the cost matrix but depends on control weighting and α factor in the equation above:

$$\mu=\alpha\underline{\sigma}(R)/\bar{\sigma}(BB^{T}). \tag{115}$$

Hence, this method is expected to improve the stability margins without complexity. The examples designed by this method are shown in numerical examples in this paper.

3–2–3. *Other Cost Properties and Their Evaluations*

In the last two sections, we looked into the improvement of cost and stability and stability margins. There we get the meaningful characteristics as to the "Additive Term designs". But the discussions are a little indirect and the intuitive interpretations are not illustrated clearly before. Hence we show in this section some characters with S_1, S_2 and S_3 surfaces introduced before and we evaluate the cost variation through the use of successive approximations and from other aspects than before we compare the M.C.V. designs with previous works. Finally through the discussions here we introduce a new design technique "Statistical Cost Expectation" (*S. E.*) method.

A) Cost Sensitivity 1—S_1, S_2, S_3 Surfaces

Now let's consider the nominal cost $S_1(q_0)$ again,

$$0=S_1(q_0)\tilde{A}_1+\tilde{A}_1^{T}S_1(q_0)+Q+K_1^{T}RK_1+S_1(q_0)\delta A(q_0)+\delta A(q_0)^{T}S_1(q_0) \tag{87}$$

and the $S_1(q)$ cost with small parameter variaton δq; $q=q_0+\delta q$,

$$\begin{aligned}0=&S_1(q_0+\delta q)\tilde{A}_1+\tilde{A}_1^{T}S_1(q_0+\delta q)+Q+K_1^{T}RK_1+S_1(q_0+\delta q)\delta A(q_0+\delta q)\\&+\delta A(q_0+\delta q)^{T}S_1(q_0+\delta q).\end{aligned} \tag{116}$$

If we define the parameter derivative $S_1'(q_0)$ of cost at q_0 as

$$S_1'(q_0)=\lim_{\delta q\to 0}\frac{1}{\delta q}(S_1(q_0+\delta q)-S_1(q_0)) \tag{117}$$

we obtain the equation that $S_1'(q_0)$, regarding $\delta A(q)$ as $q\delta A$ should satisfy the followings:

$$0=S_1'(q_0)\tilde{D}+\tilde{D}^{T}S_1'(q_0)+S_1(q_0)\delta A+\delta A^{T}S_1(q_0), \qquad \tilde{D}=A+q_0\delta A-BK. \tag{118}$$

And like the manipulations above we also have the second parameter derivative as

$$0=S_1''(q_0)\tilde{D}+\tilde{D}^T S_1''(q_0)+2(S_1'(q_0)\delta A+\delta A^T S_1'(q_0)). \tag{119}$$

Thus we prepared the local characteristics of the nominal cost surface $S_1(q)$. Well, we next attempt to compare these with those of S_2 and S_3 surfaces. If so, the intuitive features of these surfaces which have already been introduced for discussions before, will be in vision more or less clearly.

First of all we consider the S_3 surface, Before in (90) we got the result as

$$S_3(q_0)=S_1(q_0). \tag{120}$$

Here evaluating the cost derivatives defined above, we have

$$0=S_3'(q_0)\tilde{D}+\tilde{D}^T S_3'(q_0)+S_3(q_0)\delta A+\delta A^T S_3(q_0), \tag{121}$$

and

$$0=S_3''(q_0)\tilde{D}+\tilde{D}^T S_3''(q_0)+2(S_3'(q_0)\delta A+\delta A^T S_3'(q_0))+2S_3'(q_0)BR^{-1}B^T S_3'(q_0). \tag{122}$$

Comparison between these and those of S_1 shows that this S_3 surface is tangential to the S_1 nominal surface at point q_0 and that the curvature of the S_3 surface is higher than that of the S_1 nominal surface at point q_0:

$$S_3'(q_0)=S_1'(q_0), \qquad S_3''(q_0)\geq S_1''(q_0). \tag{123}$$

These characteristics are illustrated in Fig. 8. (We should note that the S_3 surfaces examined here are the surfaces which are optimized at every point with respect to these own surfaces. This is needed for eq. (120).)

And next we evaluate the cost derivative of the S_2 surface at the nominal point, for at other points the quantitative comparison like above cannot be made. Well, we get the following results through the similar procedures:

$$\left.\frac{\partial J}{\partial q}\right|_{q=q_0}=\text{tr.}\left[\left(\alpha S_0+\frac{1}{\alpha}\delta A^T S_0\delta A\right)\bar{P}\right] \quad \text{on } S_2, \quad \text{and}$$

$$\left.\frac{\partial J}{\partial q}\right|_{q=q_0}=\text{tr.}\left[(S_0\delta A+\delta A^T S_0)\bar{P}\right] \quad \text{on } S_1, \quad \text{where}$$

$$\bar{P}=\int_0^\infty \exp(Dt)P_0\exp(D^T t)dt \text{ and } S_0 \text{ is the cost at origin.} \tag{124}$$

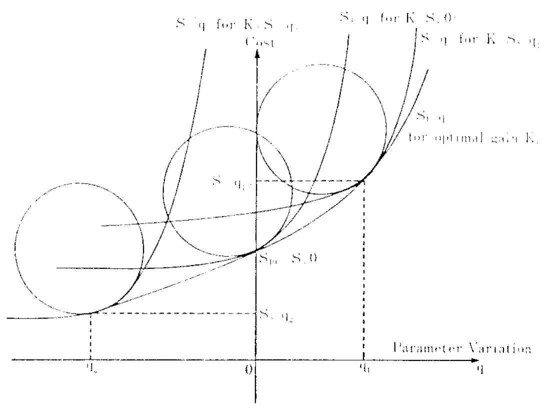


Fig. 8. Characteristics of the S_3 Surface.

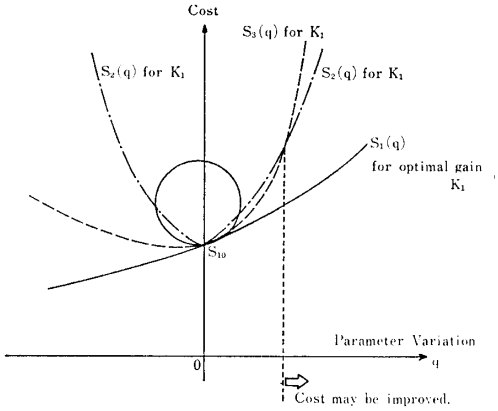


Fig. 9· Local Characteristics of S_1, S_2, S_3 Cost Surfaces.

These reveal that the cost sensitivity of the S_2 surface is higher than that of the S_1 at the nominal point. From the discussions here, we can show the microscopic structures of the S_1, S_2 and S_3 surfaces at the nominal point as illustrated in Fig. 9. Namely in the vicinity of the origin, we cannot expect that the inequality shown in eq. (85) holds, but that at the points somewhat apart from the origin such inequality may be satisfied.

B) *Cost Sensitivity 2—True Cost Surface*

The cost sensitivity with respect to the specified system variations is thus treated. But the cost sensitivity as to any elements of the system matrix is not discussed before, which is an important property expressing the unmodeled system variations. As reviewed before, Yahagi [57] devised such type of cost sensitivity. Here we derive the results by another approach as follows.

Assuming the parameter variations sufficiently small, we expand the covariance solution into the successive approximated forms as follows:

For $\dot{P}=\tilde{A}P+P\tilde{A}^T+q(\delta AP+P\delta A^T)$, $P(t_0)=P_0$, we expand P as

$$
\begin{aligned}
&P=P_0+qP_1+q^2P_2+\cdots,\\
&\dot{P}_0=\tilde{A}P_0+P_0\tilde{A}^T,\\
&\dot{P}_1=\tilde{A}P_1+P_1\tilde{A}^T+(\delta AP_0+P_0\delta A^T),\\
&\dot{P}_2=\tilde{A}P_2+P_2\tilde{A}^T+(\delta AP_1+P_1\delta A^T).
\end{aligned}
\tag{125}
$$

And we also expand the performance index (cost) J as

$$J=J_0+qJ_1+q^2J_2+\cdots. \tag{126}$$

Providing the impulsive variation of system matrix at t' such as

$$\delta A(t)=\delta A(t)\delta(t-t') \tag{127}$$

we obtain the solutions of $P_1(t)$, $P_2(t)$ and J_1, J_2 as follows:

$$P_1(t)=\begin{cases}\Phi(t,t')(\delta A(t')P_0(t')+P_0(t')\delta A(t'))\Phi^T(t,t'); & t\geq t',\\ 0; \quad t\leq t',\end{cases}$$

$$P_2(t)=\begin{cases}\Phi(t,t')(\delta A^2(t')P_0(t')+2\delta A(t')P_0(t')\delta A^T(t') \\ \qquad\qquad +P_0(t')\delta A^{T2}(t'))\Phi^T(t,t'); \quad t\geq t', \\ 0; \quad t\leq t',\end{cases}$$

$$J_1=\int_{t_0}^{t_f}\text{tr.}\,[(Q+K^TRK)P_1(s)]ds,$$

$$J_2=\int_{t_0}^{t_f}\text{tr.}\,[(Q+K^TRK)P_2(s)]ds,$$

where Φ denotes the transition matrixof A. (128)

Differentiating the cost by the system variation δA, we have

$$\frac{\partial J_1}{\partial(\delta A(t'))}=2\int_0^{t_f}\Phi^T(t,t')(Q+K^TRK)\Phi(t,t')dtP_0(t'). \tag{129}$$

And noting the following lemma

Lemma 3.

$\Phi(s,t_0)=\Phi^{*T}(t_0,s)$, where Φ^* denotes the transition matrix of $-\tilde{A}^T$, (130)

we reach to the cost sensitivity expression quite simply

$$\frac{\partial J_1}{\partial(\partial A(t'))}=2S_0(t')P_0(t'), \tag{131}$$

where $S_0(t)$ denotes the nominal adjoint cost matrix for $P_0(t)$. In time-invariant cases, a quite similar result is obtained by eq. (124). This expression implies that the cost sensitivity may increase locally if the cost is higher. But these don't deny the possibility of cost improvement, for these properties are only local ones and over the range of not so small parameter variations the higher order effects of cost variation should be considered.

At any rate, we have prepared a few kinds of cost sensitivities eqs. (118), (124), and (131). As Yahagi [57] attempted to design insensitive controllers by weighting the cost sensitivities in chapter 2 regardless of the forms of variations δA's, we also analyze the similarity of "Additive Term Designs" to the sensitivity weighting methods.

The most primitive idea is to weight the cost S_0 considering the finite feature of P_0 in eq. (131). This leads to the weighting forms as

$$\alpha S_0^m \qquad (m=0,1,2,\cdots). \tag{132}$$

As readily recognized, these relate to "Simple Robust Realization" methods in the previous section. And in view of the cost sensitivity in eq. (124), it is a natural idea to weight or evaluate the upper limit of the sensitivity at the origin. This leads to the "Additive Term Design" as

$$h\geq S_0\delta A+\delta A^TS_0. \tag{133}$$

Of cource, this is the same approach as G.C.C. and M.C.V.

C) A New Design Approach—"Statistical Cost Expectation" (S.E.) method

So far, we consider the cost sensitivities eq. (124) or (131) in this section. Finally we present the new type of design method which utilizes the statistical expectation of cost relating to the successive approximation of cost variation. For this purpose, we expand the covariance again as

$$P=P_0+qP_1+q^2P_2+\cdots. \tag{134}$$

If the parameter variation q is statistical quantity, we have the approximated expectation truncated by the second moment as

$$\tilde{P}=E[P]\cong P_0+\sigma_q^2P_2 \tag{135}$$

in which σ_q is calculated using the appropriate probability density function such as the normal distribution. Therefore, we can derive the equations that the statistical expectation $\tilde{P}$ satisfies as follows:

$$\begin{aligned}&\dot{\tilde{P}}=\tilde{A}\tilde{P}+\tilde{P}\tilde{A}^T+\sigma_q^2(\delta AP_1+P_1\delta A^T);\quad \tilde{P}(t_0)=P(t_0),\\&\dot{P}_1=\tilde{A}P_1+P_1\tilde{A}^T+\delta AP_0+P_0\delta A^T;\quad P_1(t_0)=0,\\&\dot{P}_0=\tilde{A}P_0+P_0\tilde{A}^T;\quad P_0(t_0)=P(t_0),\quad \text{where } \tilde{A}=A-BK.\end{aligned} \tag{136}$$

Namely in this case the system is augmented by three times. Correctly according to this virtual model, we can obtain the optimal control laws for this as

$$\begin{aligned}&-\dot{\tilde{S}}=\tilde{A}^T\tilde{S}+\tilde{S}\tilde{A}+Q+K^TRK;\quad \tilde{S}(t_f)=0,\\&-\dot{S}_1=\tilde{A}^TS_1+S_1\tilde{A}+\sigma_q^2(\delta A^T\tilde{S}+\tilde{S}\delta A);\quad S_1(t_f)=0,\\&-\dot{S}_0=\tilde{A}^TS_0+S_0\tilde{A}+\delta A^TS_1+S_1\delta A;\quad S_0(t_f)=0,\\&K=R^{-1}B^T(\tilde{S}+S_1P_1\tilde{P}^{-1}+S_0P_0\tilde{P}^{-1}).\end{aligned} \tag{137}$$

Although this concept expresses the cost variation very clearly and theoretically, it is not so easy to design by these procedures, for the computation is difficult and the design depends on the initial covariance very strongly. Consequently we had better use the simplified system model. For this purpose, we assume that the designed control system is stable sufficiently, that is the system has sufficiently high damping rate. So we can evaluate $P_1(t)$ and rewrite the equations as

$$\begin{aligned}&P_1(t)\cong(\delta AP_0+P_0\delta A^T)\Delta,\ \text{where } \Delta \text{ is the representative damping time constant,}\\&\dot{\tilde{P}}=\tilde{A}\tilde{P}+\tilde{P}\tilde{A}^T+\Delta\sigma_q^2(\delta A^2\tilde{P}+2\delta A\tilde{P}\delta A^T+\tilde{P}\delta A^{T2})\\&\qquad-\Delta\sigma_q^4(\delta A^2P_2+2\delta AP_2\delta A^T+P_2\delta A^{T2}).\end{aligned} \tag{139}$$

Neglecting the last term boldly, we can obtain the following approximated equation of statistical cost expectation:

$$\dot{\tilde{P}}=\tilde{A}\tilde{P}+\tilde{P}\tilde{A}^T+\Delta\sigma_q^2(\delta A^2\tilde{P}+2\delta A\tilde{P}\delta A^T+\tilde{P}\delta A^{T2});\qquad \tilde{P}(t_0)=P(t_0). \tag{140}$$

This simplified model is very crude one, but we can expect that this reflects the cost behavior qualitatively. If the virtual model eq. (140) is used, the design procedures are made as in "Additive Term Design" techniques. We show the simple example.

D) Example of S. E. Formulation

Example 2.

Consider the scalar system

$$\dot{p}=-p+qp, \quad q\sim N(0, 0.5); \quad \sigma_q^2=0.5. \tag{141}$$

The solution of this is given as

$$p(t)=p_0 \exp(-(1-q)t) \tag{142}$$

and the exact statistical expectation is

$$\tilde{p}(t)=p_0 \exp(\tfrac{1}{4}t^2-t). \tag{143}$$

Then the corresponding approximations are as follows:

$$\tilde{p}(t)=p_0(1+\tfrac{1}{2}t^2)\exp(-t) \quad \text{for eq. ((136); type I,}$$
$$\tilde{p}(t)=p_0 \exp(-\tfrac{1}{2}t) \text{ for eq. (140); type II.} \tag{144}$$

The schematic illustrations of $\tilde{P}$'s and the error ratio to the exact one are shown in Fig. 10 and Fig. 11. And the evaluations of $P_1(t)$ and $(\delta AP_0+P_0\delta A^T)$ are also

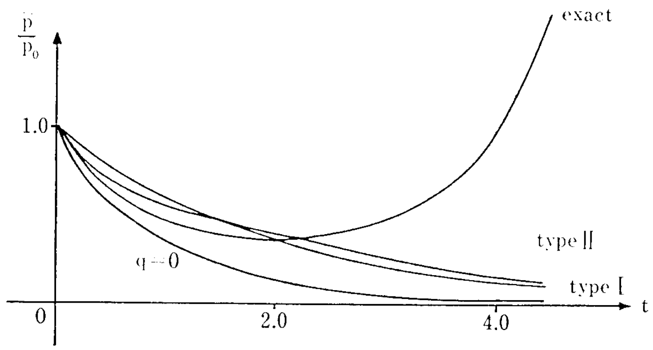


Fig. 10. Approximated Statistical Expectation.

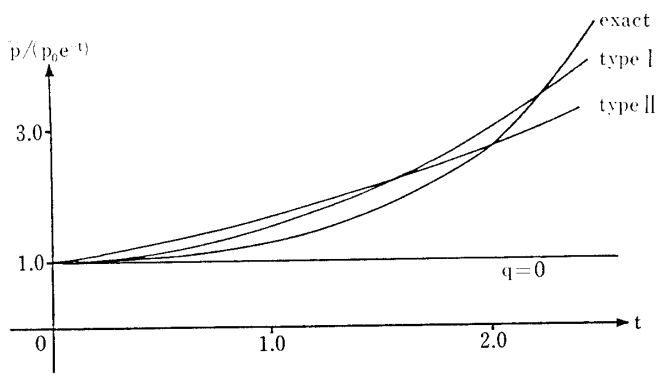


Fig. 11. Normalized Statistical Expectation.

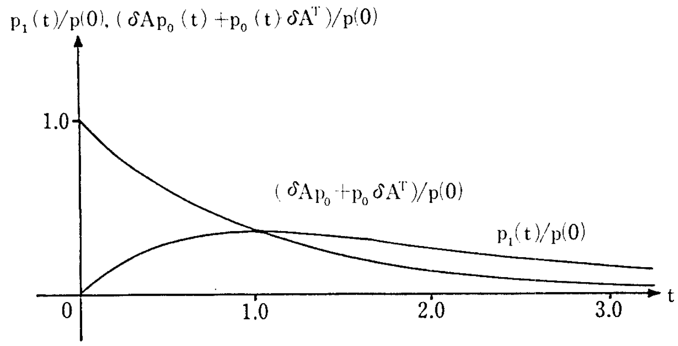


Fig. 12. Assumption in Type II Expectation.

shown in Fig. 12. These results suggest that after the transient interval, the simplified model reflects the feature of the exact solution.

Hence, we obtained the new "Additive Term Design" method as the virtual model eq. (140) with the new additive terms. The application of this is shown in numerical examples in the later chapter.

3–3. *Alternative Design Methods of "Additive Term Designs"*

We discussed so far some features of "Additive Term Designs" and through these a few of the alternative design methods of insensitive controllers are presented. In this section some robust design methods are arranged and summarized briefly, which are examined in numerical examples.

1) M.C.V. (Maximum Cost Variation) method

This is one of the "Guaranteed Cost Control" (G.C.C.) methods, both of which evaluate the upper limit of the cost variation and minimize it like the manner of mini-max designs. The difference between the M.C.V. and the G.C.C. is the uniform robust feature which is assured by the monotonous character of the additive term. The advantages of this are the absolute assurance of stability and the high level of stability margins over the specified range of parameter variations. From the practical aspects, this has the disadvantages that the calculation is a little complex and the control laws may fall into higher feedback gains which cannot be accepted.

2) U.W. (Uncertainty Weighting) method

This is the simplest one of the "Additive Term Designs" discussed before. The additive term of this is motivated by the approximation of cost variation and other points of views. It is reported in Harvey's that in spite of simpler procedures this may be a best one. Practically speaking, this has the advantage of less computational load. But there's no meaningful guarantee of cost or stability improvement.

3) S.D.N. (State Dependent Noise) method

This is derived by the random process model which contains state dependent noise in stead of the system matrix variations. The treatment made in this might be somewhat incorrrect because the variation is statistical rather than random process. But as far as such model is provided, the design is quite mathematically made. Practical applicability of this is at the same level as that of M.C.V..

4) O.D. (Off-set Design) method
This method relates to the mini-max design or the worst case design, for the point at which cost is expected to be worst and the design is made is specified a priori. The practical meaning is quite clear and the design computation is performed very easily. We can interpret that this avoids the cost deterioration by shifting the cost surface in the desired direction.

5) S.E. (Statistical cost Expectation) method
This method is formulated in the previous section, which is motivated by the approximated evaluation of the statistical cost expectation. In this establishment, it is provided that the parameter variations are sufficiently small, and some crude approximations are introduced. Practical applicability is at the same level at M.C.V. and S.D.N. methods.

6) U.W. type M.C.V. (Uncertainty Weighting type M. C.V.) method
This is one form of G.C.C, in which the monotonous feature that is required for M.C.V. is not introduced as the usual G.C.C. techniques. This is named after the form of the additive term. Of course, the advantages and the disadvantages are equivalent to the usual G.C.C.'s.

7) S.R. (Simple robust Realization) method
This is devised in the discussions on the stability margins of "Additive Term Designs", some of which are formulated in very simpler forms and can be regarded as the approximated versions of various types of "Additive Term Designs". Computational load is at the same level as M.C.V. and S.D.N. methods.

8) other methods
There are some straightforward methods to insensitive controllers. For example, to take the state weightings as larger values relative to control weightings; higher feedback gains, or to use lower feedback gains in the systems whose open loop stability is assured. A few of these are shown in numerical examples later.

Thus we obtained the robust design methods above. In the numerical examples of this paper, the qualitative evaluations and comparisons of them are considered, being applied to some aero-space systems. Next we proceed to the discussions on the applicability to more practical systems; the systems with other types of uncertainties than the system matrix variations, the discrete type systems and the dynamically compensated systems.

4. Applications to Practical Systems

In the last chapter, we established some new properties as for the systems that contain only system matrix uncertainty, assuming that they are continuous and state feedback is possible. Here we investigate the applicability of the robust designs established before particularly as for M.C.V. to more practical systems; the systems where the other types of uncertain sources exist or those that are discrete or dynamically compensated. And finally we commented the computational algorithm of robust output feedback control laws by M.C.V.

4-1. *Systems Containing Control Matrix Uncertainty (and Other Uncertainty Sources)*

So far, we considered the systems in which the system matrix uncertainties exist. But in many practical systems, there are the other types of uncertainties than those of the system matrix. The usual actuators contain the nonlinearity or time-delay more or less. In these cases, the exact values of such parameters are not known and even if those are obtained precisely, sometimes we cannot model the dynamics or characteristics due to the complexity. Hence, in many cases, we model the system as sufficiently simplified one and design it by some methods. And after the evaluation of stablity margins and the responses, we adopt the designed control laws. The other uncertainties than those of the system matrix are, in general, divided into two groups. One is the type of the control matrix uncertainties and the other is the type of the installed gain matrix uncertainties. The former is generated by nonlinearities and the latter is by installation accuracies and amplified scale errors. In this section we consider such systems and show the corresponding additive term examples of M.C.V. design methods.

Now we consider the following system:

$$\dot{x}=Ax+Bu+\delta Bu,\ u=-(K-\delta K)x. \tag{145}$$

This is also rewritten approximately as

$$\dot{x}\cong\tilde{A}x+\delta(BK)x. \tag{146}$$

Treating the ambiguous term as the system matrix uncertainty, if M.C.V. design is employed, we have the additive term $h(*)$ as

$$h=\alpha P+\frac{1}{\alpha}\delta(BK)P\delta(BK)^T, \qquad P=xx^T. \tag{147}$$

By this point, we proceeded by the same procedures as in the system matrix uncertainties. But some notes or interpretations should be made. For the systems with only gain matrix uncertainties, we don't have to devise the methods other than nominal regulators. The reason is as follows. The cost deterioration by gain matrix uncertainties is expressed as

$$\delta J=2\ \mathrm{tr.}\ (\delta K^T RKP_0)+\int_{t_0}^{t_f}\mathrm{tr.}\ [(Q+K^TRK)\Phi(t_f,\ t)$$

$$(B\delta KP_0+P_0\delta K^TB^T)\Phi^T(t_f, t)]dt. \quad (148)$$

Hence, the requirement for the cost sensitivity with repect to δK to be zero suggests

$$\frac{\partial\delta J}{\partial(\delta K)}=0, \quad \text{and} \quad K=R^{-1}B^TS_0 \quad (S_0 \text{ is the optimal cost.}) \quad (149)$$

But this result is quite trivial one. Naturally this required gain is identical to the optimal gain in the usual regulators, because the optimal cost is the stationary value with respect to gain matrix. And in the systems with only control matrix uncertainties, when the design is made by M.C.V, the resulting feedback gain is given by

$$-\dot{S}=\tilde{A}^TS+S\tilde{A}+Q+K^TRK+\alpha S+\frac{1}{\alpha}K^T\delta B^TS\delta BK,$$
$$K=\left(R+\frac{1}{\alpha}\delta B^TS\delta B\right)^{-1}B^TS. \quad (150)$$

Though the interpretation of this is not so straightforward, intuitively we recognize that the gain matrix should be lower. This fact reflects the classical approach well.

For the cases with the sensor mechanism uncertainties, we can formulate the M.C.V. design, too, through the output feedback systems as the augmented forms. And also we observe the low gain feedback control is needed in this case. (Omitting the detail discussions here.)

4–2. *Discrete Systems*

Through the discussions before, we implicitly assume the systems to be continuous. Here we show the similar results for the discrete systems.

First we formulate the M.C.V. designs for the discrete systems that contain not only the system matrix uncertainties but the control matrix uncertainties. Considering the variated system as

$$x_{k+1}=A_kx_k+B_ku_k+\delta A_kx_k+\delta B_ku_k \quad (151)$$

and using the state feedback control law which is assumed to be possible,

$$u_k=-K_kx_k \quad (152)$$

we have the closed loop system as

$$x_{k+1}=(A_k-B_kK_k)x_k+\delta A_kx_k-\delta B_kK_kx_k. \quad (153)$$

Introducing the covariance type matrix P_k, we get the equations of

$$P_k=x_kx_k^T, \qquad \tilde{A}_k=A_k-B_kK_k,$$
$$P_{k+1}=\tilde{A}_kP_k\tilde{A}_k^T+\delta A_kP_k\tilde{A}_k^T+\tilde{A}_kP_k\delta A_k^T+\delta A_kP_k\delta A_k^T-\delta A_kP_kK_k^T\delta B_k^T$$
$$-\delta B_kK_kP_k\delta A_k^T-\delta B_kK_kP_k\tilde{A}_k^T-\tilde{A}_kP_kK_k^T\delta B_k^T+\delta B_kK_kP_kK_k^T\delta B_k^T. \quad (154)$$

According to the M.C.V. concepts, we can observe the following inequalities hold.

$$q_1(\delta A_k P_k \tilde{A}_k^T + \tilde{A}_k P_k \delta A_k^T) \leq |q_1| \left(\alpha_1 \tilde{A}_k P_k \tilde{A}_k^T + \frac{1}{\alpha_1} \delta A_k P_k \delta A_k^T \right),$$

$$q_2(-\delta B_k K_k P_k \tilde{A}_k^T - \tilde{A}_k P_k K_k^T \delta B_k^T) \leq |q_2| \left(\alpha_2 A_k P_k A_k^T + \frac{1}{\alpha_2} \delta B_k K_k P_k K_k^T \delta B_k^T \right),$$

$$q_1 q_2(-\delta A_k P_k K_k^T \delta B_k^T - \delta B_k K_k P_k \delta A_k^T) \leq |q_1 q_2| \left(\alpha_3 \delta A_k P_k \delta A_k^T + \frac{1}{\alpha_2} \delta B_k K_k P_k K_k^T \delta B_k^T \right),$$

where $\alpha_1,\ \alpha_2,\ \alpha_3 \geq 0.$ (155)

Hence the virtual system model to be considered in M.C.V. approaches is

$$P_{k+1} = \rho_1 \tilde{A}_k P_k \tilde{A}_k^T + \rho_2 \delta A_k P_k \delta A_k^T + \rho_3 \delta B_k K_k P_k K_k^T \delta B_k^T,$$

where $\rho_1 = 1 + q_1\alpha_1 + q_2\alpha_2,\quad \rho_2 = q_1/\alpha_1 + q_3\alpha_3,\quad \rho_3 = q_2/\alpha_2 + q_3/\alpha_3.$ (156)

As the control problem, we define the performance index (cost) as

$$J = \sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k)$$

$$= \sum_{k=0}^{N-1} \mathrm{tr.}\ [(Q_k + K_k^T R_k K_k) P_k] \qquad \text{to be minimized.} \qquad (157)$$

Defining the following Hamiltonian, we obtain the optimal control laws for the virtual system (156) as follows:

$$H = \mathrm{tr.}\ [(Q_k + K_k^T R_k K_k) P_k] + \mathrm{tr.}\ [S_{k+1} P_{k+1}]; \quad \text{Hamitonian} \qquad (158)$$

$$S_k = \rho_1 \tilde{A}_k^T S_{k+1} \tilde{A}_k + Q_k + K_k^T R_k K_k + \rho_2 \delta A_k^T S_{k+1} \delta A_k + \rho_3 K_k^T \delta B_k^T S_{k+1} \delta B_k K_k; \quad S_N = 0,$$

$$K_k = (R_k + \rho_1 B_k^T S_{k+1} B_k + \rho_3 \delta B_k^T S_{k+1} \delta B_k)^{-1} \rho_1 B_k^T S_{k+1} A_k. \qquad (159)$$

Though next we would like to attempt to discuss the various properties as established in the last chapter associated with these discrete systems, unfortunately we cannot observe these so easily as in the continuous cases. In fact the stability is assured really over the specified range of parameter variations through Lyapunov's criterion. But we cannot obtain the quantitative stability margins as seen in the previous chapter, for the circle condition of the optimally controlled discrete systems is expressed as

$$(I + G(z))^* (R + B^T S B)(I + G(z)) \geq R \qquad (160)$$

as far as the author knows, and so this does not provide us the quantitative informations. And boundary surface S_3 named before, by which we can qualitatively predict the cost improvement is not represented as the simpler form. Because in discrete systems the explicit coupling between the gain and the system variation exists and so such surface which reflects the cost improvement cannot be defined clearly. Moreover, the U. W. type M.C.V. design falls into the ambiguous one in these systems due to the same reason as above, where both U.W. type and S.D.N. type additive terms exist mixedly if attempted. And the simplified "Statistical Cost Expectation" (S.E.) method in discrete systems is degenerated to the S.D.N. type design. (Details are omitted.) Consequently we cannot obtain these properties except for the stability in M.C.V. designs analytically. But we will show the numerical comparisons and evaluations among these design methods in numerical examples of this paper.

4–3. *Dynamically Compensated Systems* (*Output Feedback Systems*)

We have obtained the various types of robust controller designs in the systems where the state feedback is possible. In practical systems, there are few systems in which such control laws are accepted. Usually, only several outputs are available directly through some sensors. The modern control theory provides us so called the separation theorem and we may expect the observers or the Kalman filters to play the role of the reconstruction of states. But the total system which is constructed by these method exactly is sometimes highly complex, and in those cases we often resort to the classical simpler designs instead of the modern designs. These situations are caused by the fact that the meaningful or rational designs for the dynamical compensators are not provided uniquely; there exist the full or reduced order or function observers and the Kalman filters and the appropriate compensators which accomplish the desired pole locations. Hence, the designers have to adopt trial-and-error approaches between the gains and compensators. Here we confine ourselves to the discussions on the optimal gains in robust sense assuming that the appropriate type of compensator is installed a priori. (Of course actually these procedures should be repeated many times consulting the compensator designs.)

Now we consider the system containing the system and control matrix uncertainties as

$$\dot{x}=Ax+Bu+\delta Ax+\delta Bu, \qquad y=Cx. \tag{161}$$

And we assume that the following compensators are used a priori:

$$\dot{z}=Fz+Gy. \tag{162}$$

Here the gains to be determined are K_1 and K_2 as

$$u=K_1y+K_2z. \tag{163}$$

These systems are manipulated and arranged into the following augmented systems:

$$\begin{pmatrix}\dot{x}\\ \dot{z}\end{pmatrix}=\begin{pmatrix}A & 0\\ GC & F\end{pmatrix}\begin{pmatrix}x\\ z\end{pmatrix}+\begin{pmatrix}B\\ 0\end{pmatrix}u+\begin{pmatrix}\delta A & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\ z\end{pmatrix}+\begin{pmatrix}\delta B\\ 0\end{pmatrix}u,$$
$$\begin{pmatrix}y\\ z\end{pmatrix}=\begin{pmatrix}C & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}x\\ z\end{pmatrix}, \qquad u=(K_1, K_2)\begin{pmatrix}y\\ z\end{pmatrix}. \tag{164}$$

Thus the dynamically compensated system like above is reduced to the output feedback system. Hereafter we redefine the system as eq. (161) and use the same nomenclature as those in (161). Applying the M.C.V. concept to this, we have the virtual system model to be considered as follows:

$$P = xx^T, \qquad P(t_0) = P_0,$$
$$\dot{P} = \tilde{A}P + P\tilde{A}^T + q_1\left(\alpha_1 P + \frac{1}{\alpha_1}\delta AP\delta A^T\right) + q_2\left(\alpha_2 P + \frac{1}{\alpha_2}\delta BKCPC^TK^T\delta B^T\right). \quad (165)$$

If we settle the usual perfomance index (cost) as

$$J = \int_{t_0}^{t_f} (x^TQx + u^TRu)dt \quad \text{to be minimized,} \quad (166)$$

we have the optimal feedback laws in M.C.V. sense for this virtual system as

$$-\dot{S} = \tilde{A}^TS + S\tilde{A} + Q + C^TK^TRKC + \rho_1 S + \rho_2\delta A^TS\delta A + \rho_3 C^TK^T\delta B^TS\delta BKC; \quad S(t_f) = 0,$$
$$\dot{P} = \tilde{A}P + P\tilde{A}^T + \rho_1 P + \rho_2\delta AP\delta A^T + \rho_3\delta BKCPC^TK^T\delta B^T; \quad P(t_0) = P_0,$$
$$K = -(R + \rho_3\delta B^TS\delta B)^{-1}B^TSPC^T(CPC^T)^{-1},$$

with cost $J_{\min} = \text{tr.}\,[S(t_0)P(t_0)]$, where $\rho_1 = q_1\alpha_1 + q_2\alpha_2$, $\rho_2 = q_1/\alpha_1$, $\rho_3 = q_2/\alpha_2$. (167)

But the derived equation above is so to say the two point boundary value problem of matrix type, which cannot be solved at any rate except for the time-invariant infinite time problems. In this case the problem is equivalent to the following by McLane [*61*] and Levine [*88*], through the reformations of

$$\hat{H} = \text{tr.}\,[(Q + C^TK^TRKC)\hat{P}] + \text{tr.}\,[\hat{S}\dot{\hat{P}}]; \quad \text{modified Hamiltonian}$$
$$\hat{P} = \int_{t_0}^{t} P(s)ds,$$
$$\dot{\hat{P}} = \tilde{A}\hat{P} + \hat{P}\tilde{A}^T + \rho_1 P + \rho_2\delta A\hat{P}\delta A^T + \rho_3\delta BKC\hat{P}C^TK^T\delta B^T + P_0; \quad \hat{P}(t_0) = 0,$$
$$J = \text{tr.}\,[(Q + C^TK^TRKC)\hat{P}] \quad \text{to be minimized.} \quad (168)$$

Therefore we can obtain the following results evaluating the stationary case,

$$0 = \tilde{A}^T\hat{S} + \hat{S}\tilde{A} + Q + C^TK^TRKC + \rho_1\hat{S} + \rho_2\delta A^T\hat{S}\delta A + \rho_3 C^TK^T\delta B^T\hat{S}\delta BKC,$$
$$0 = \hat{P}\tilde{A}^T + \tilde{A}\hat{P} + \rho_1\hat{P} + \rho_2\delta A\hat{P}\delta A^T + \rho_3\delta BKC\hat{P}C^TK^T\delta B^T + P_0,$$
$$K = -(R + \rho_3\delta B^T\hat{S}\delta B)^{-1}B^T\hat{S}\hat{P}C^T(C\hat{P}C^T)^{-1}. \quad (169)$$

And in discrete systems, we can also derive the analogous equations in time-invariant infinite time problems. As to the system with system and control matrix uncertainties

$$x_{k+1} = Ax_k + Bu_k + \delta Ax_k + \delta Bu_k, \quad u_k = -Ky_k = -KCx_k,$$
$$P_{k+1} = \rho_1\tilde{A}P_k\tilde{A}^T + \rho_2\delta AP_k\delta A^T + \rho_3\delta BKCP_kC^TK^T\delta B^T, \quad P_k = x_kx_k^T,$$
$$J = \sum_{k=0}^{\infty} \text{tr.}\,[(Q + C^TK^TRKC)P_k] \quad \text{to be minimized,} \quad (170)$$

we have the necessary conditions for optimal control in M.C.V. sense as follows:

$$\hat{S} = \rho_1\tilde{A}^T\hat{S}\tilde{A} + Q + C^TK^TRKC + \rho_2\delta A^T\hat{S}\delta A + \rho_3 C^TK^T\delta B^T\hat{S}\delta BKC,$$
$$\hat{P} = \rho_1\tilde{A}\hat{P}\tilde{A}^T + \rho_2\delta A\hat{P}\delta A^T + \rho_3\delta BKC\hat{P}C^TK^T\delta B^T + P_0,$$
$$K = (R + \rho_1 B^T\hat{S}B + \rho_3\delta B^T\hat{S}\delta B)^{-1}\rho_1 B^T\hat{S}\delta A\hat{P}C^T(C\hat{P}C^T)^{-1}. \quad (171)$$

Thus we have preparations for the practical applications of M.C.V. to the output feedback or dynamically compensated systems. In the next section, we show the calculation algorithms concerning the dynamically compensated systems, which will be utilized for numerical examples and other practical systems.

4–4. *Computation Algorithm of Robust Output Feedback Systems with Additive Terms*

Here we show the computation algorithms of M.C.V. type robust output feedback systems with additive terms, which are derived in the last section. Firstly we present the overall algorithm and next prove some characteristics which assure the validity of them. And finally the algorithms for discrete systems are commented.

Theorem 7.

[Algorithm—1]

1) Find K_1 that guarantees the existence of S_1, P_1.

2) $$0=(A+BK_iC)^TS_i+S_i(A+BK_iC)+Q+C^TK_i^TRK_iC$$
$$+\rho_1S_i+\rho_2\delta A^TS_i\delta A+\rho_3C^TK_i^T\delta B^TS_i\delta BK_iC,$$
$$0=(A+BK_iC)P_i+P_i(A+BK_iC)^T+\rho_1P_i$$
$$+\rho_2\delta AP_i\delta A^T+\rho_3\delta BK_iCP_iC^TK_i^T\delta B^T+P_0,$$
$$K_{i+1}=-(R+\rho_3\delta B^TS_i\delta B)^{-1}B^TS_iP_iC^T(CP_iC^T)^{-1}. \tag{172}$$

Though the detail proof is given later, we can show that this algorithm is only a Newton's method. The relation that the gain matrix K should satisfy is

$$0=(R+\rho_3\delta B^TS\delta B)K(CPC^T)+B^TSPC^T. \tag{173}$$

When the approximated solutions S_i, P_i are found at i-th iteration, the residual error E_i is written as

$$E_i=(R+\rho_3\delta B^TS_i\delta B)K_i(CP_iC^T)+B^TS_iP_iC^T. \tag{174}$$

If $(i+1)$-th approximated solution is searched for as the form of

$$K_{i+1}=K_i+\Delta K_i, \tag{175}$$

ΔK_i should satisfy the following relation:

$$(R+\rho_3\delta B^TS_i\delta B)\Delta K_i(CP_iC^T)+E_i=0. \tag{176}$$

Therefore K_{i+1} is calculated by

$$K_{i+1}=-(R+\rho_3\delta B^TS_i\delta B)^{-1}B^TS_iP_iC^T(CP_iC^T)^{-1}. \tag{177}$$

But this equation is identical to the eq. (172) in the algorithm.

In order to prove the theorem we show some lemmas. First of all, if the algorithm (172) is employed we can observe that the cost is decreasing monotonously along the iterations. For this purpose we define a certain matrix function f as

$$f=(A+BK_iC)^TS_{i-1}+S_{i-1}(A+BK_iC)+Q+C^TK_i^TRK_iC$$
$$+\rho_1S_{i-1}+\rho_2\delta A^TS_{i-1}\delta A+\rho_3C^TK_i^T\delta B^TS_{i-1}\delta BK_iC. \tag{178}$$

Applying the algorithm to this, finally we have

$$(CP_{i-1})f(CP_{i-1})^T = -(CP_{i-1}C^T)(K_i - K_{i-1})^T(R + \rho_3 \delta B^T S_{i-1} \delta B)$$
$$(K_i - K_{i-1})(CP_{i-1}C^T). \tag{179}$$

Providing that $CP_{i-1}C^T$ is nonsingular, which is required for this algorithm, we obtain

$$f < 0. \tag{180}$$

Next subtracting the eq. (172) in the algorithm from the eq. (178) which defines f, we reach to the following equations:

$$(A + BK_iC)^T(S_{i-1} - S_i) + (S_{i-1} - S_i)(A + BK_iC) + \rho_1(S_{i-1} - S_i)$$
$$+ \rho_2 \delta A^T(S_{i-1} - S_i)\delta A + \rho_3 C^T K_i^T \delta B^T (S_{i-1} - S_i)\delta BK_iC = f < 0. \tag{181}$$

By the way, from the definition of f, equating $\delta S = r\ S_{i-1}$ ($r > 0$),

$$(A + BK_iC)^T \delta S + \delta S(A + BK_iC) + \rho_1 \delta S + \rho_2 \delta A^T \delta S \delta A + \rho_3 C^T K_i^T \delta B^T \delta S \delta BK_iC$$
$$= rf - r(Q + C^T K_i^T RK_iC) < 0. \tag{182}$$

Hence observing that there may exist r (>0) such that

$$r(-f + Q + C^T K_i^T RK_iC) \leq -f \tag{183}$$

we have, comparing eq. (181) with eq. (182),

$$S_i \leq (1-r)S_{i-1}. \tag{184}$$

(Obviously r is lower than unity, because if $r > 1$, then the inequality

$$-f \leq -\frac{r}{r-1}(Q + C^T K_i^T RK_iC) < 0 \tag{185}$$

must be satisfied. But this is contradiction.) And denoting the optimal cost matrix as S_0, we can show

$$(S_i - S_0) \leq (1-r)(S_{i-1} - S_0). \tag{186}$$

Thus, by the eqs. (184) and (186) we easily find out that the cost is monotonously decreasing along the iterations if this algorithm is employed.

These discussions guarantee the validity of the algorithm because of the lower boundedness of the cost. The readers may ask how the covariance matrix P_i behaves, though which does not lead to cost directly. Unfortunately, the behavior of the covariances is not so clear as that of the cost along the iterations. But as for the special case in which the open loop system is stable, we can observe the covariance behavior a little. Considering the system in which the open loop system is stable, we can begin with $K_1 = 0$, and

$$J_1 = \text{tr.}\,[S_1 P(t_0)] = \text{tr.}\,[QP_1],$$
$$J_2 = \text{tr.}\,[S_2 P(t_0)] = \text{tr.}\,[(Q + C^T K_2^T R K_2 C) P_2],$$
$$\vdots$$
$$J_i = \text{tr.}\,[S_i P(t_0)] = \text{tr.}\,[(Q + C^T K_i^T R K_i C) P_i]. \tag{187}$$

We can observe

$$S_i \leq S_1, \quad \text{or} \tag{188}$$
$$\text{tr.}\,[Q(P_1 - P_i)] \geq 0 \qquad (i = 2, 3, \cdots). \tag{189}$$

In case Q is a unit or scalar weighting matrix,

$$\text{tr.}\,[P_i] \leq \text{tr.}\,[P_1] \qquad (i = 2, 3, \cdots). \tag{190}$$

Hence, the following lemmas are proved.

Lemma 4.

By this algorithm, cost matrix S_i are monotonously decreasing. And so the algorithm converges to the true one, if the solution exists.

Lemma 5.

By this algorithm, if the open loop is stable, with $K_1 = 0$, then the relation of eq. (189) holds.

In order to obtain the control laws by this algorithm, we must solve the two Lyapunov type equations in the M.C.V. design as in eq. (172),

$$\begin{aligned} 0 &= (A + BK_iC)^T S_i + S_i(A + BK_iC) + \rho_1 S_i + \rho_2 \delta A^T S_i \delta A + \rho_3 C^T K_i^T \delta B^T S_i \delta B K_i C \\ &\quad + Q + C^T K_i^T R K_i C, \\ 0 &= (A + BK_iC) P_i + P_i (A + BK_iC)^T + \rho_1 P_i + \rho_2 \delta A P_i \delta A^T + \rho_3 \delta B K_i C P_i C^T K_i^T \delta B^T + P_0. \end{aligned} \tag{191}$$

Next we discuss the solving technique of these. Considering the transposed forms of them, the problem is settled to that of solving the following equation:

$$0 = \tilde{A}^T S + S\tilde{A} + \delta A^T S \delta A + \delta B^T S \delta B + S + M \qquad (M > 0). \tag{192}$$

For this problem, we present the following algorithm.

Theorem 8.

[Algorithm—2]

1) Setting S_0 as 0,
2) $M_j = M + \delta A^T S_{j-1} \delta A + \delta B^T S_{j-1} \delta B + S_{j-1}$,
$$0 = \tilde{A}^T S_j + S_j \tilde{A} + M_j \qquad (j = 1, 2, \cdots). \tag{193}$$

Now we consider the characteristic of this algorithm. As easily recognized, we have the following successive equations.

$$\tilde{A}^T(S_i - S_{i-1}) + (S_i - S_{i-1})\tilde{A}$$
$$= -[\delta A^T(S_{i-1} - S_{i-2})\delta A + \delta B^T(S_{i-1} - S_{i-2})\delta B + (S_{i-1} - S_{i-2})] < 0. \quad (194)$$

And observing that $S_1 > S_0$ and that the cost matrix to be solved is finite, we conclude the theorem above.

In discrete systems, the effective algorithms of solving the control laws as to the M.C.V. type robust output feedback systems are derived as in continuous systems. The discussions are made by the analogous manners, but the details are omitted here. The resulted algorithms for M.C.V. type robust controller designs are as follows.

Theorem 9.

[Algorithm—3]

1) Find K_1 that guarantees the existence of S_1, P_1 (>0),

2) $S_i = \rho_1(A - BK_iC)^T S_i(A - BK_iC) + Q + C^T K_i^T R K_i C + \rho_2 \delta A^T S_i \delta A + \rho_3 C^T K_i^T \delta B^T S_i \delta B K_i C,$

$P_i = \rho_1(A - BK_iC)P_i(A - BK_iC)^T + \rho_2 \delta A P_i \delta A^T + \rho_3 \delta B K_i C P_i C^T K_i^T \delta B^T + P_0,$

$K_{i+1} = (R + \rho_1 B^T S_i B + \rho_3 \delta B^T S_i \delta B)^{-1} B^T S_i A P_i C^T (C P_i C^T)^{-1}.$ (195)

Theorem 10,

[Algorithm—4]

To solve the $S = \rho_1 \tilde{A}^T S \tilde{A} + \delta A^T S \delta A + \delta B^T S \delta B + M$, $M > 0$

1) Setting S_0 as 0,

2) $M_j = M + \delta A^T S_{j-1} \delta A + \delta B^T S_{j-1} \delta B,$

$S_j = \rho_1(A - BK_iC) S_j (A - BK_iC) + M_j$ $(j = 1, 2, 3, \cdots).$ (196)

5. Numerical Examples

In this chapter we show three numerical examples, to which the robust controller design methods discussed so far are applied. The systems to be considered are not only the continuous ideal system but also the discrete output feedback system and the nonlinear estimation problem. Firstly we show the longitudinal autopilot system and next the radar tracking system of a re-entry vehicle and finally the attitude control system of a flexible booster.

5-1. *Longitudinal Autopilot*

In aircraft control systems, there are some ambiguous parameters in their dynamics. One type of these is the completely unknown uncertainty before the real identifications in flight, and the other is the uncertainty associated with the circumstances under which the system is driven. The former is the dynamic damping derivative at the initial design phase and the latter is the cruising velocity or the dynamic pressure or the static margin shift caused by payload unbalance. Here we restrict our attention on the system with two types of uncertainties; the dynamic damping derivative and the static margin uncertainties.

The longitudinal dynamics of the aircraft can be expressed as

$$\dot{u}=\frac{qs_0}{mU_0}(C_{xu}u+C_{x\alpha}\alpha+C_w\cos\theta_0\theta).$$

$$-C_{zu}u+\left(\frac{mU_0}{qS_0}-\frac{c}{2U_0}C_{z\dot{\alpha}}\right)\dot{\alpha}-C_{z\alpha}\alpha-\left(\frac{mU_0}{qS_0}+\frac{c}{2U_0}C_{zq}\right)\dot{\theta}-C_w\sin\theta_0\theta=C_{z\delta e}\delta_e,$$

$$\frac{I_y}{qS_0c}\ddot{\theta}-\frac{c}{2U_0}C_{mq}\dot{\theta}-\frac{c}{2U_0}C_{m\dot{\alpha}}\dot{\alpha}-C_{m\alpha}\alpha=C_{m\delta e}\delta_e. \tag{197}$$

Here we adopt some assumptions as follows:

1) short period approximation,

2) $\left|\frac{qS_0c}{2mU_0^2}C_{z\dot{\alpha}}\right|,\ \left|\frac{qS_0c}{2mU_0^2}C_{zq}\right|,\ |\theta_0|\ll 1,$

3) $\left|\frac{qS_0c}{2mU_0^2}\frac{C_{m\dot{\alpha}}C_{z\alpha}}{C_{m\alpha}}\right|,\ \left|\frac{qS_0c}{2mU_0^2}\frac{C_{m\dot{\alpha}}C_{z\delta e}}{C_{m\delta e}}\right|\ll 1.$ (198)

Using these and regarding the elevator servo mechanism as the first order system, we have the following system:

$$\dot{\alpha}=pC_{z\alpha}\alpha+\dot{\theta}+pC_{z\delta e}\delta_e,$$

$$\ddot{\theta}=p'C_{m\alpha}\alpha+p'q\dot{\theta}+p'C_{m\delta e}\delta_e,$$

$$\dot{\delta}_e=-\frac{1}{\tau_0}\delta_e+\frac{1}{\tau_0}u \qquad (u;\ \text{command input}),$$

$$\text{where } p=\frac{qS_0}{mU_0},\quad p'=\frac{qS_0c}{I_y},\quad q=\frac{c}{2U_0}(C_{mq}+C_{m\dot{\alpha}}). \tag{199}$$

Moreover nondimensionalizing these by the nominal short period of the system,

$$t=\frac{1}{\omega_0}\tau,\quad \dot{\theta}=\omega_0\cdot\omega,\quad \text{where } \omega_0^2=-C_{m\alpha}\frac{qS_0c}{I_y}, \tag{200}$$

the following equations are derived.

$$\frac{d}{d\tau}\begin{bmatrix}\theta\\ \omega\\ \alpha\\ \delta_e\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & \phi\tilde{c} & \mu C_{m\alpha} & \mu C_{m\delta e}\\ 0 & 1 & \phi C_{z\alpha} & \phi C_{z\delta e}\\ 0 & 0 & 0 & -\lambda\end{bmatrix}\begin{bmatrix}\theta\\ \omega\\ \alpha\\ \delta_e\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 0\\ \lambda\end{bmatrix}u,$$

$$\text{where } \phi=p/\omega_0,\ \mu=p'/\omega_0^2,\ \lambda=1/\omega_0\tau_0,\ \tilde{c}=\frac{mc^2}{2I_y}(C_{mq}+C_{m\dot{\alpha}}). \tag{201}$$

The system matrix uncertainties are expressed as follows respectively:

$$\delta A_1=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & \delta(\mu C_{m\alpha}) & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\quad \delta A_2=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & \delta(\phi\tilde{c}) & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}. \tag{202}$$

Where δA_1 and δA_2 denote the static margin and damping derivative variations respectively. Here as the control problem we consider the level flight autopilot system. The corresponding performance index (cost) is expressed as

$$J=\int_0^\infty \left[\frac{1}{J_1}\delta e^2+\frac{1}{J_2}(\theta-\alpha)^2\right]dt \text{ to be minimized.} \tag{201}$$

Referring to the Blakelock's [*132*], we use the following nominal parameters:

$$\begin{aligned}&p=0.07257\ (1/\mathrm{sec}),\ p'=1.94553\ (1/\mathrm{sec}^2),\\&C_{m\alpha}=-3.714,\ C_{z\alpha}=-4.46,\ C_{m\delta e}=-0.710,\ C_{z\delta e}=-0.246,\\&(C_{mq}+C_{m\dot\alpha})=-14.67,\ \omega_0=1.0974\ (1/\mathrm{sec}),\ \tau_0=0.5\ (1/\mathrm{sec}).\end{aligned} \tag{204-a}$$

And we specify the range of parameter variations concerning their nominal values as

$$\delta(\mu C_{m\alpha})=\pm 300\%\ \text{full scale},\ \delta(\phi\tilde{c})=\pm 83\%\ \text{full scale}. \tag{204-b}$$

Under the parameters above, we show some numerical results. In this problem the parameters q used in M.C.V. and U.W. type M.C.V. are normalized to unity; i.e. the largest design range is 1. And the design parameters in the other methods have little meaning and should be considered as the scale of the additive terms. Firstly the evaluations for only δA_1 variations are shown in Fig. 13—Fig. 26, both the projection and the contour lines of the cost srrfaces are displayed in order; Nominal ones in Fig. 13, 14 and the results concerning various factors of M.C.V. design techniques in Fig. 15, 16, and those of S.R. method in Fig. 17–22, and those of intuitive high and low feedback approaches are displayed in Fig. 23–26. We can easily recognize that some of those are valid for the systems to be robust. But the quantitative comparison of these "Additive Term Design" techniques is not clear from these figures, so we arrange the results in Table 1—Table 16 and Fig. 27 involving the results that are made by the other methods than those illustrated in

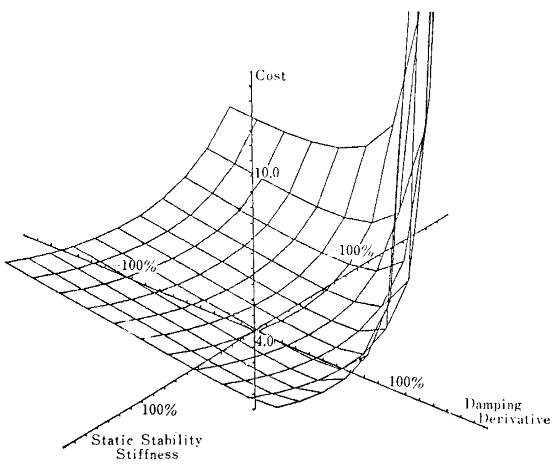


Fig. 13. Projection of the Nominal Cost Surface—Autopilot.

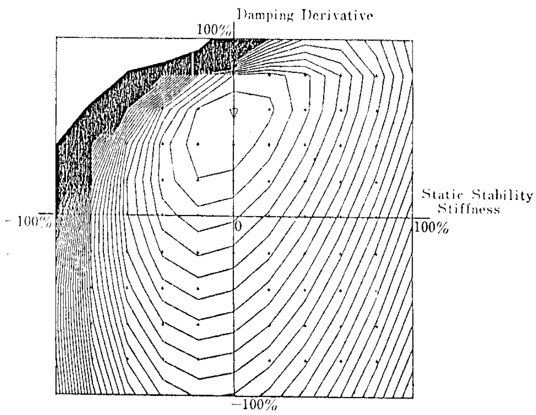
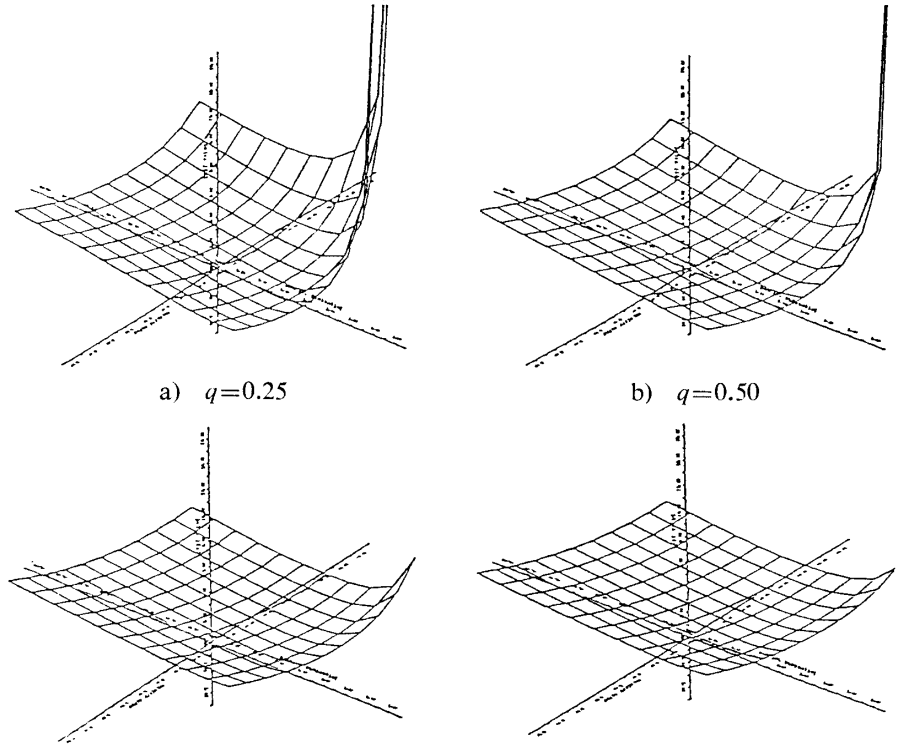


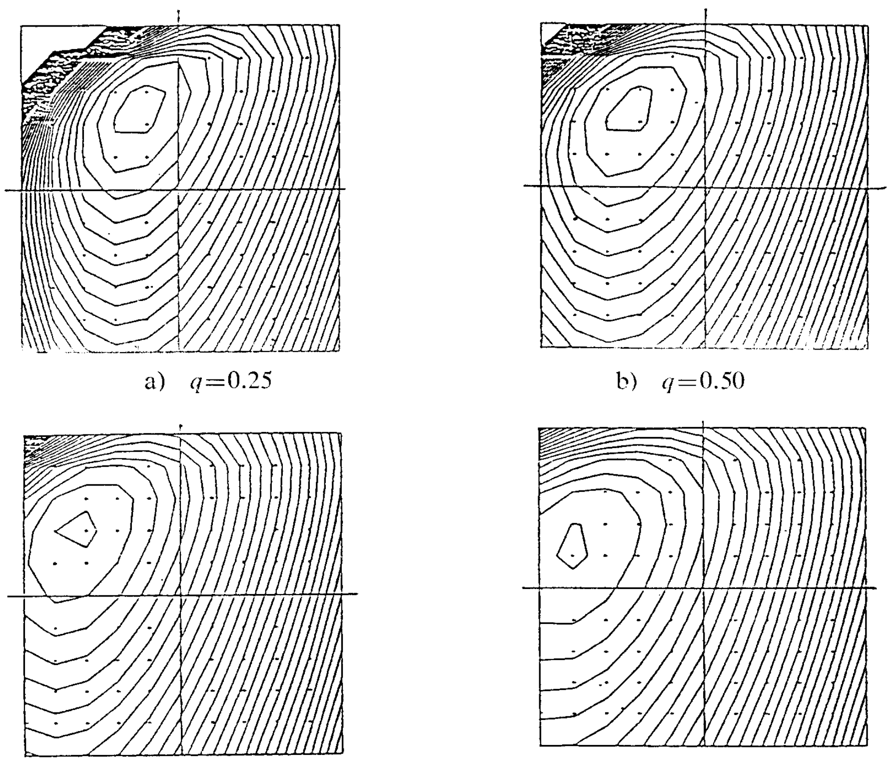
Fig. 14. Contour Lines of the Nominal Cost Surface—Autopilot.



a) $q=0.25$ b) $q=0.50$

c) $q=0.75$ d) $q=1.00$

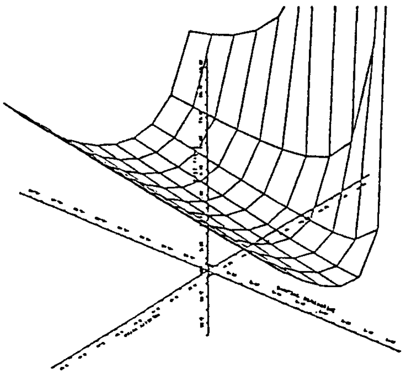
Fig. 15. Projection of Robustly Designed Cost Surface—Autopilot (M.C.V. $\alpha=0.5$, δA_1 only).



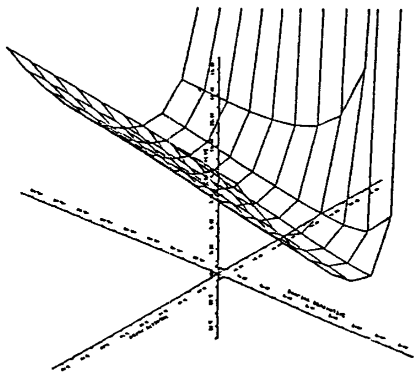
a) $q=0.25$ b) $q=0.50$

c) $q=0.75$ d) $q=1.00$

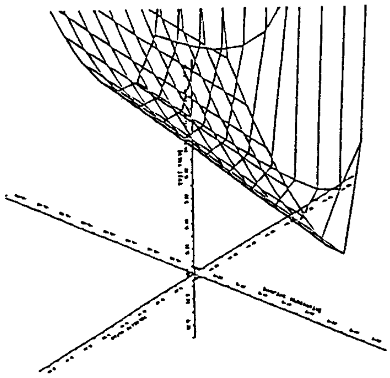
Fig. 16. Contour Lines of Robustly Designed Cost Surface—Autopilot (M.C.V. $\alpha=0.5$, δA_1 only).



a) $\alpha=0.1$

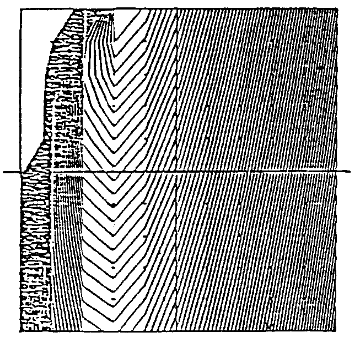


b) $\alpha=0.2$

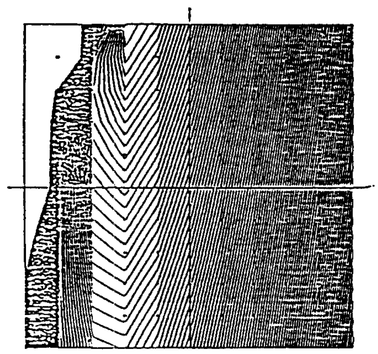


c) $\alpha=1.0$

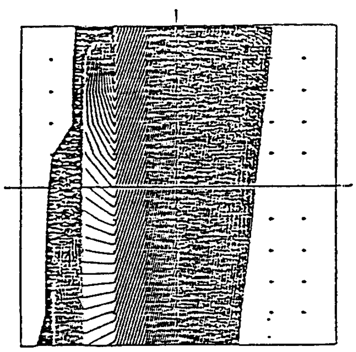
Fig. 17. Projection of Robustly Designed Cost Surface—Autopilot (S.R. $m=0$).



a) $\alpha=0.1$

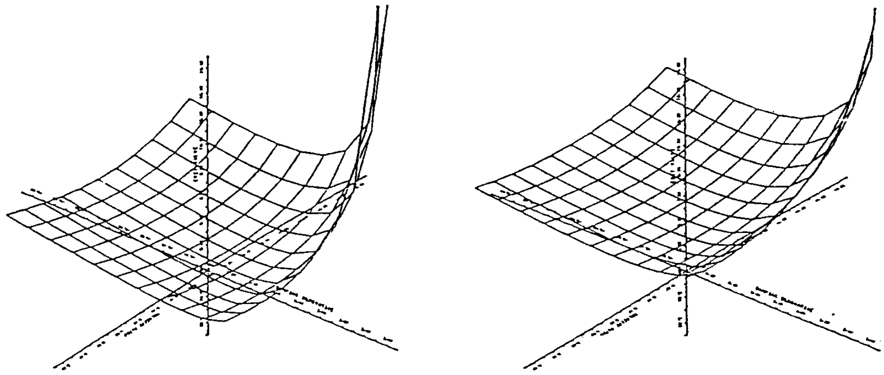


b) $\alpha=0.2$



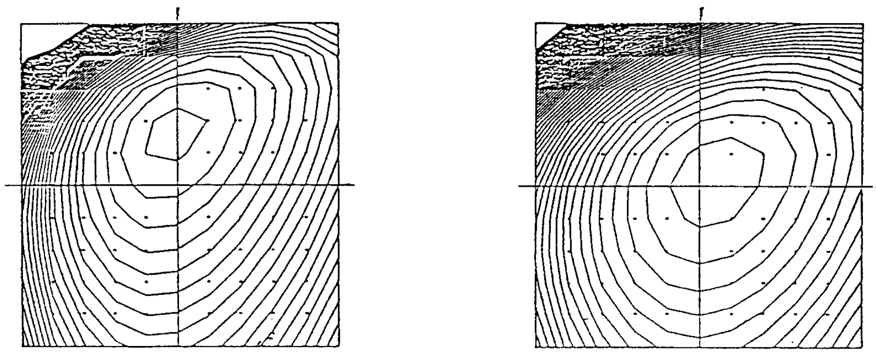
c) $\alpha=1.0$

Fig. 18. Contour Lines of Robustly Designed Cost Surface—Autopilot (S.R. $m=0$).



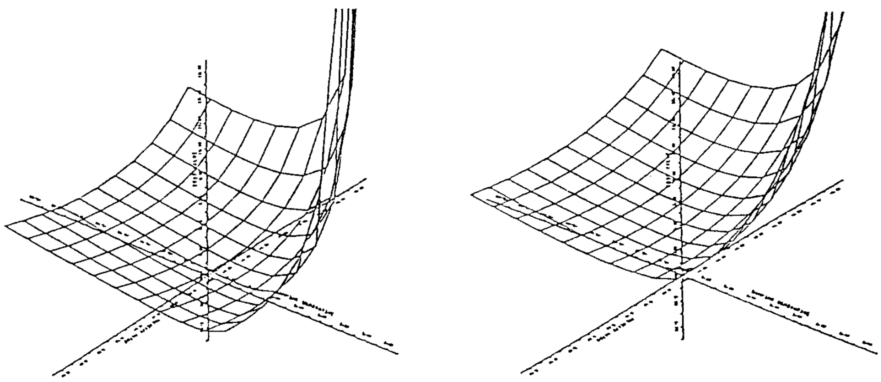
a) $\alpha=1.0$ b) $\alpha=2.0$

Fig. 19. Projection of Robustly Designed Cost Surface—Autopilot (S.R. $m=1$).



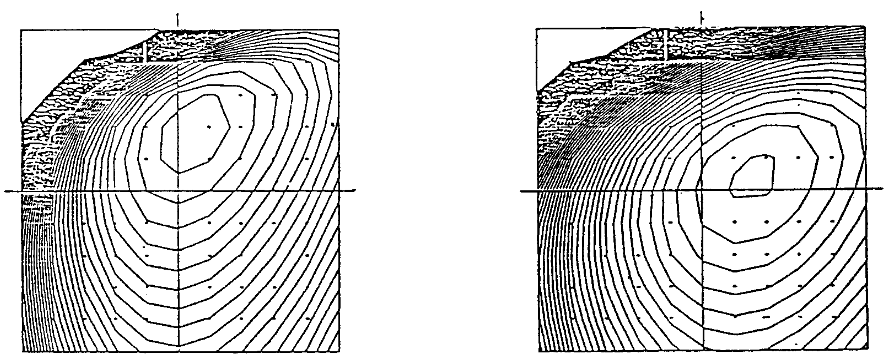
a) $\alpha=1.0$ b) $\alpha=2.0$

Fig. 20. Contour Lines of Robustly Designed Cost Surface—Autopilot (S.R. $m=1$).



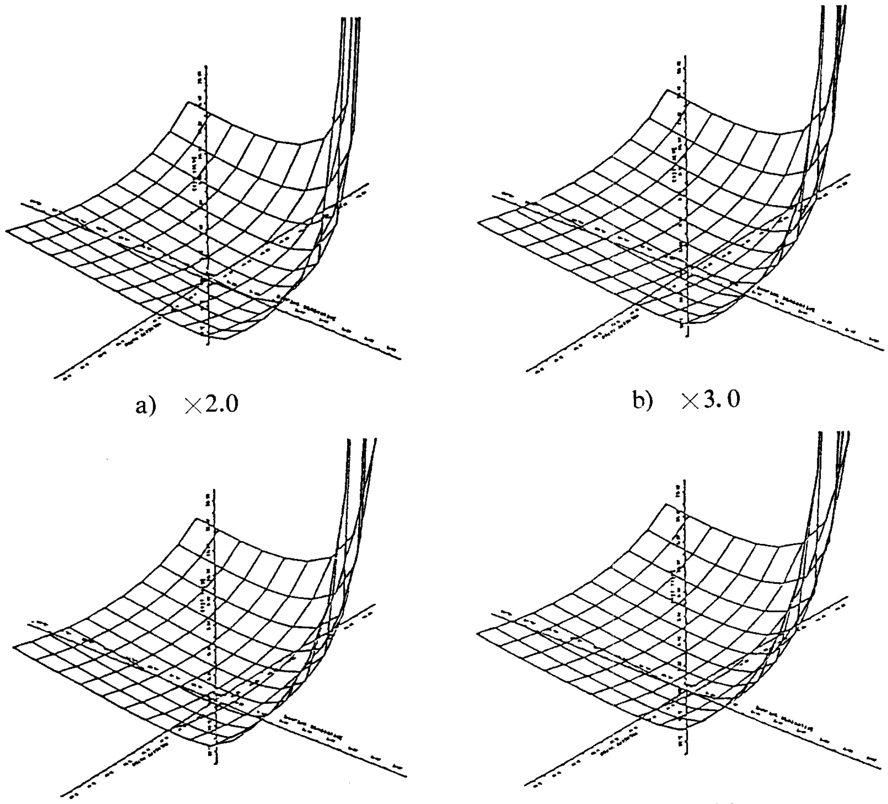
a) $\alpha=0.1$ b) $\alpha=0.2$

Fig. 21. Projection of Robustly Designed Cost Surface—Autopilot (S.R. $m=2$).



a) $\alpha=0.1$ $\alpha=0.2$

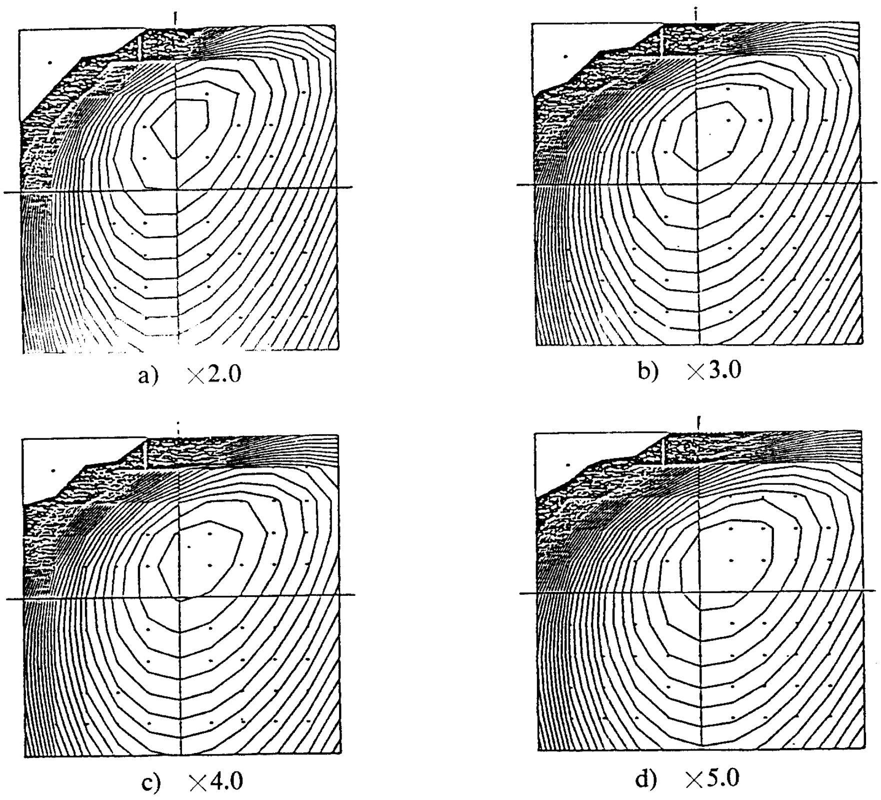
Fig. 22. Contour Lines of Robustly Designed Cost Surface—Autopilot (S. R. $m=2$).



a) ×2.0 b) ×3.0

c) ×4.0 d) ×5.0

Fig. 23. Projection of Robustly Designed Cost Surface—Autopilot (High Gain Feedback).



a) ×2.0 b) ×3.0

c) ×4.0 d) ×5.0

Fig. 24. Contour Lines of Robustly Designed Cost Surface—Autopilot (High Gain Feedback).

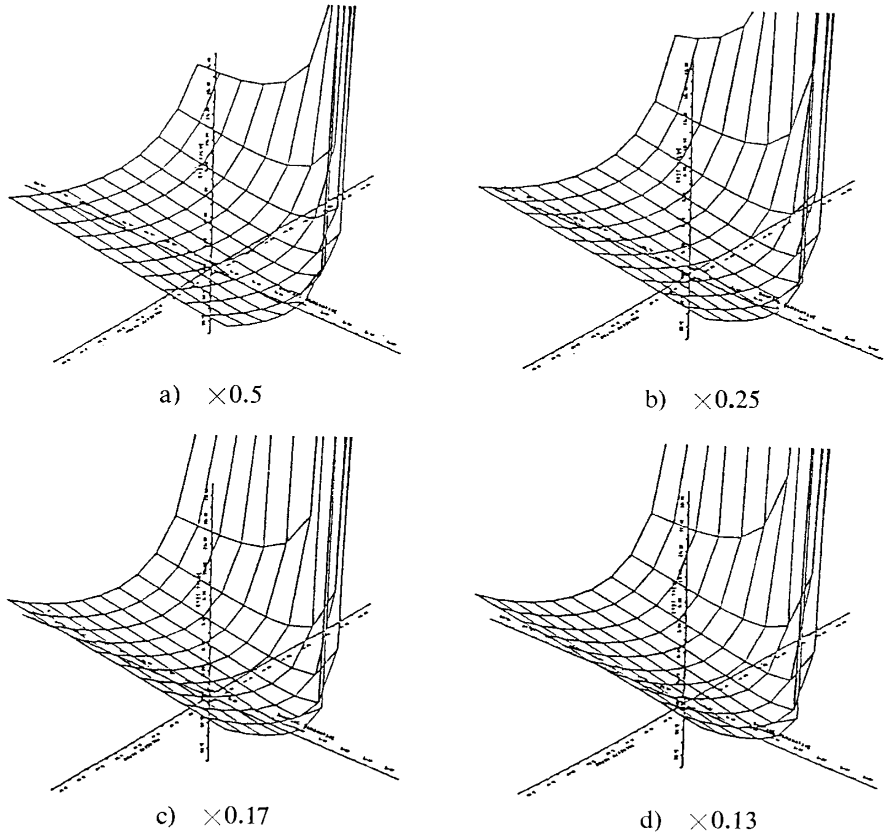


Fig. 25. Projection of Robustly Designed Cost Surface—Autopilot (Low Gain Feedback).

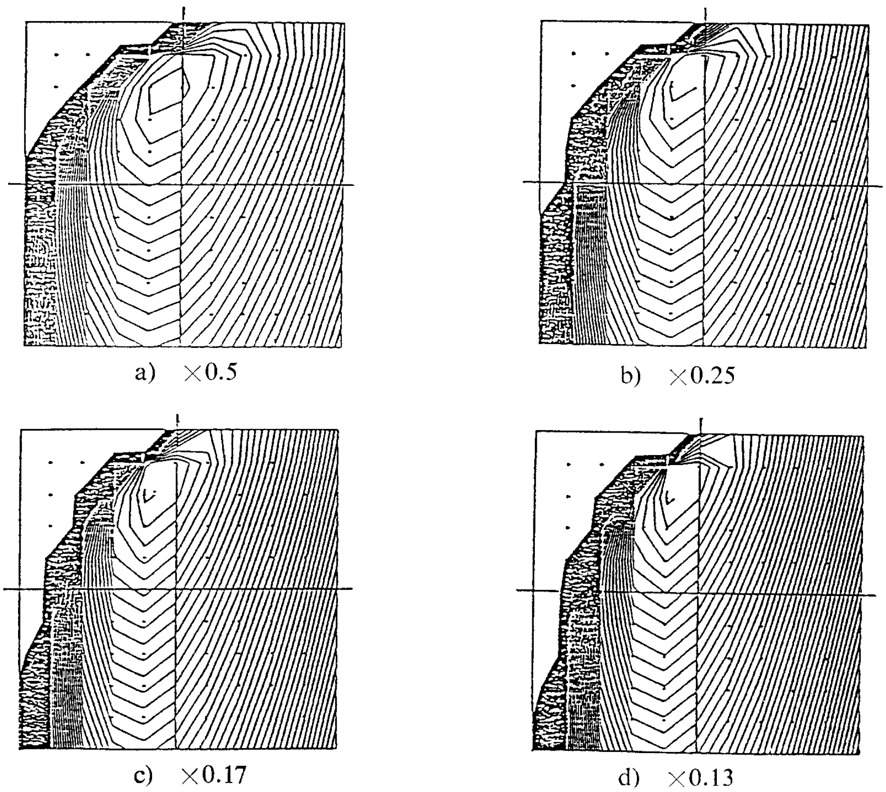


Fig. 26. Contour Lines of Robustly Designed Cost Surface—Autopilot (Low Gain Feedback).

Table 1. Robust Design—Autopilot (M.C.V. $\alpha=0.1$, δA_1 only)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$q=0.0$ (nominal)	4.975	∞ (8.483)	∞ (3.508)	66 deg
$q=0.25$	5.996 (9.468)	9.032 (8.448)	3.036 (2.452)	68 deg
$q=0.50$	8.077 (17.408)	11.823 (11.317)	3.746 (3.240)	66 deg

Table 2. Robust Design—Autopilot (M.C.V. $\alpha=0.25$, δA_1 only)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$q=0.0$ (nominal)	4.975	∞ (8.483)	∞ (3.508)	66 deg
$q=0.25$	5.156 (6.783)	∞ (6.972)	∞ (1.816)	69 deg
$q=0.50$	5.592 (9.859)	8.300 (7.704)	2.708 (2.112)	68 deg
$q=0.75$	6.088 (15.168)	8.888 (8.347)	2.800 (2.259)	67 deg
$q=1.00$	6.471 (24.692)	9.196 (8.705)	2.725 (2.234)	66 deg

Table 3. Robust Design—Autopilot (M.C.V. $\alpha=0.5$, δA_1 only)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$q=0.0$ (nominal)	4.975	∞ (8.483)	∞ (3.508)	66 deg
$q=0.25$	5.023 (6.265)	∞ (6.580)	∞ (1.557)	69 deg
$q=0.50$	5.154 (8.500)	20.783 (6.808)	15.624 (1.654)	69 deg
$q=0.75$	5.343 (12.658)	7.597 (6.998)	2.254 (1.655)	69 deg
$q=1.00$	5.614 (21.030)	7.698 (7.152)	2.084 (1.538)	68 deg

Table 4. Robust Design—Autopilot (M.C.V. α=0.75, δA_1 only)

Case	Cost	max. Cost	Δ Cost	Stability Margin
q=0.0 (nominal)	0.495	∞ (8.483)	∞ (3.508)	66 deg
q=0.25	5.006 (6.283)	∞ (6.451)	∞ (1.445)	68 deg
q=0.50	5.105 (8.622)	∞ (6.556)	∞ (1.451)	69 deg
q=0.75	5.314 (13.304)	10.850 (6.688)	5.536 (1.374)	69 deg
q=1.00	5.814 (23.818)	9.330 (7.000)	3.516 (1.186)	69 deg

Table 5. Robust Design—Autopilot (M.C.V. α=1.00, δA_1 only)

Case	Cost	max. Csot	Δ Cost	Stability Margin
q=0.0 (nominal)	4.975	∞ (8.483)	∞ (3.508)	66 deg
q=0.25	5.009 (6.432)	∞ (6.508)	∞ (1.499)	68 deg
q=0.50	5.136 (9.133)	∞ (6.454)	∞ (1.318)	69 deg
q=0.75	5.470 (14.970)	13.390 (6.650)	7.920 (1.180)	70 deg
q=1.00	6.447 (29.612)	12.218 (7.364)	5.771 (0.917)	69 deg

Table 6. Robust Design—Autopilot (U.W. δA_1 only)

Case	Cost	max. Cost	Δ Cost	Stability Margin
q=0.0 (nominal)	4.975	∞ (8.483)	∞ (3.508)	66 deg
q=5.0	4.984 (5.233)	∞ (7.224)	∞ (2.240)	67 deg
q=10.0	5.077 (5.912)	∞ (6.820)	∞ (1.743)	69 deg
q=15.0	5.302 (6.857)	18.522 (7.315)	13.220 (2.013)	69 deg
q=20.0	5.642 (7.960)	8.525 (7.914)	2.883 (2.272)	69 deg

Table 7. Robust Design—Autopilot (S.D.N. δA_1 only)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$q=0.0$ (nominal)	4.975	∞ (8.483)	∞ (3.508)	66 deg
$q=0.5$	5.025 (5.604)	∞ (6.666)	∞ (1.062)	68 deg
$q=1.0$	5.166 (6.330)	∞ (7.036)	∞ (1.870)	69 deg
$q=1.5$	5.387 (7.151)	10.225 (7.473)	4.838 (2.086)	69 deg
$q=2.0$	5.677 (8.064)	8.580 (7.971)	2.903 (2.294)	69 deg

Table 8. Robust Design—Autopilot (Offset Design)

Case	Cost	max. Cost	Δ Cost	Stability Margin
0% (nominal)	4.975	∞ (8.483)	∞ (3.508)	66 deg
20%	5.013 (4.887)	∞ (6.616)	∞ (1.603)	67 deg
40%	5.117 (4.818)	∞ (6.911)	∞ (1.794)	67 deg
60%	5.277 (4.765)	∞ (7.248)	∞ (1.971)	66 deg
80%	5.488 (4.727)	12.422 (7.625)	6.934 (2.137)	66 deg
100%	5.742 (4.702)	8.633 (8.039)	2.891 (2.297)	65 ged

Table 9. Robust Design—Autopilot (S.E. δA_1 only)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$q=0.0$ (nominal)	4.975	∞ (8.483)	∞ (3.508)	66 deg
$q=0.1$	5.100 (6.028)	∞ (6.800)	∞ (1.780)	69 deg
$q=0.2$	5.440 (7.327)	8.472 (7.567)	3.032 (2.127)	69 deg
$q=0.3$	5.953 (8.856)	9.001 (8.410)	3.048 (2.457)	68 deg
$q=0.4$	6.603 (10.591)	9.944 (9.383)	3.341 (2.780)	67 deg

Table 10. Robust Design—Autopilot (U.W.M.C.V. $\alpha=\sqrt{2}/2$, δA_1 only)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$q=0.0$ (nominal)	4.975	∞ (8.483)	∞ (3.508)	66 deg
$q=0.014$ ($\sqrt{2}/100$)	5.188 (7.055)	166.738 (7.031)	161.550 (1.843)	69 deg
$q=0.028$ ($\sqrt{2}/50$)	5.334 (9.520)	8.090 (7.193)	2.756 (1.859)	68 deg
$q=0.042$ ($3\sqrt{2}/100$)	5.355 (14.000)	8.440 (6.906)	3.085 (1.551)	67 deg

Table 11. Robust Design—Autopilot (U.W.M.C.V. $\alpha=\sqrt{2}$, δA_1 only)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$q=0.0$ (nominal)	4.975	∞ (8.483)	∞ (3.508)	66 deg
$q=0.014$ ($\sqrt{2}/100$)	5.033 (6.430)	∞ (6.603)	∞ (1.570)	68 deg
$q=0.028$ ($\sqrt{2}/50$)	5.103 (8.607)	∞ (6.555)	∞ (1.452)	67 deg
$q=0.042$ ($3\sqrt{2}/100$)	5.530 (13.552)	86.315 (6.536)	80.785 (1.006)	66 deg

Table 12. Robust Design—Autopilot (Simple Realization m=0)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$q=0.0$ (nominal)	4.975	∞ (8.483)	∞ (3.508)	66 deg
$q=0.1$	6.641 (8.907)	∞ (11.525)	∞ (4.884)	95 deg
$q=0.2$	7.528 (10.801)	∞ (26.267)	∞ (18.739)	95 deg

Table 13. Robust Design—Autopilot (Simple Realization m=1)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$q=0.0$ (nominal)	4.975	∞ (8.483)	∞ (3.508)	66 deg
$q=1.0$	5.459 (9.943)	∞ (6.980)	∞ (1.521)	71 deg
$q=2.0$	7.342 (20.910)	20.536 (8.740)	13.194 (1.398)	73 deg

Table 14. Robust Design—Autopilot (Simple Realization, m=2)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$q=0.0$ (nominal)	4.975	∞ (8.483)	∞ (3.508)	66 deg
$q=0.1$	5.248 (6.987)	∞ (8.277)	∞ (3.029)	66 deg
$q=0.2$	7.465 (13.236)	∞ (10.150)	∞ (2.685)	71 deg

Table 15. Robust Design—Autopilot (High Gain Feedback)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$Q\times1.0$ (nominal)	4.975	∞ (8.483)	∞ (3.508)	66 deg
$Q\times2.0$	5.102 (9.199)	∞ (7.718)	∞ (2.616)	65 deg
$Q\times3.0$	5.329 (13.213)	∞ (7.752)	∞ (2.423)	65 deg
$Q\times4.0$	5.584 (17.100)	∞ (7.966)	∞ (2.382)	64 deg
$Q\times5.0$	5.851 (20.899)	∞ (8.245)	∞ (2.394)	64 deg

Table 16. Robust Design—Autopilot (Low Gain Feedback)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$R\times1.0$	4.975	∞ (8.483)	∞ (3.508)	66 deg
$R\times2.0$	5.070 (5.415)	∞ (11.076)	∞ (6.006)	68 deg
$R\times4.0$	5.318 (5.937)	∞ (20.729)	∞ (15.411)	70 deg
$R\times6.0$	5.520 (6.291)	∞ (54.793)	∞ (49.273)	71 deg
$R\times8.0$	5.687 (6.569)	∞ (∞)	∞ (∞)	72 deg

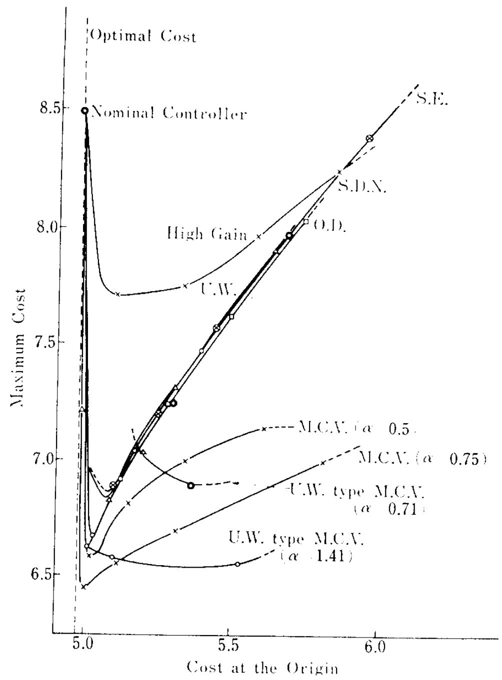


Fig. 27. Robust Controller Designs—Autopilot (δA_1)

Fig. 13–26. (In these Tables, the values in parentheses in Cost column denote the design costs and those in max. Cost and Δ Cost column indicate the values evaluated only in the direction of δA_1, moreover if not commented the gain margins are infinite.) Especially the Fig. 27 represents the characteristics and differences between the design methods. Naturally the method which reduces not only the cost at origin but the highest cost over the specified range of parameter variations is more preferable. From this point of view, we can conclude that the M.C.V. and U.W. type M.C.V. methods are superior to any other design method considered here, and that the methods except for them have as same effectiveness as one another. The S.R. methods and high or low gain feedback techniques have less validity for insensitivity. And S.E. design established in this paper has as same effectiveness to it as U.W. and O.D, S.D.N. designs. The best result of the M.C.V. designs has higher cost than the optimally designed cost by only several percents, but the cost surface is improved very drastically. Here we should note that the stability margins are improved not so considerably as in the cost surface from Tables 1–16. This fact can be interpreted that the stability margins at the origin do not reflect the overall improvement well. The robust feature of the stability margins in M.C.V. designs in chapter 3; the stability margins are maintained at the same level as in the usual L.Q. regulators over the specified range of parameters, is observed in Table 17 together with the other effective designs. It is clear that the M.C.V. design actually guarantees the sufficient stability margins over the specified region as considered in chapter 3. This property can be expected to play the role of robustness for the unmodeled uncertainties other than δA_1. In fact, Fig. 28 demonstrates that the M.C.V. and U.W. type M.C.V. methods that are based on only

Table 17. Robust Design—Autopilot (Stability Margins)

	nominal	MCV $\alpha=0.5$ $q=1.0$	UW ($q=20.0$)	offset (100%)
−100%	29 deg (−5.6 dB)	68 deg (−19.9 dB)	67 deg (−16.0 dB)	62 deg (−16.0 dB)
−80%	42 deg (−11 dB)	68 deg (−26.1 dB)	67 deg (−22.6 dB)	63 deg (−23.4 dB)
−60%	52 deg (−36 dB)	68 deg (−55.0 dB)	68 deg (−52.0 dB)	63 deg (−52.0 dB)
−40%	58 deg	68 deg	68 deg	64 deg
−20%	63 deg	68 deg	68 deg	65 deg
0%	66 deg	68 deg	69 deg	65 deg
20%	69 deg	68 deg	69 deg	66 deg
40%	71 deg	68 deg	69 deg	66 deg
60%	73 deg	68 deg	69 deg	66 deg
80%	74 deg	68 deg	70 deg	67 deg
100%	76 deg	68 deg	79 deg	67 deg
Cost	4.975	5.614	5.642	5.742

δA_1 variation have higher robustness than other methods in the case that not only δA_1 variation but δA_2 variation exist. Fig. 29 illustrates the S_1, S_2 and S_3 surfaces discussed in chapter 3 for this case. From the considerations made there, we can expect qualitatively that over the left part of about −30% line the cost may be improved. This perspective is true roughly for the cost is really improved over the left part of about −60% line. The two S_2 surfaces corresponding to the optimal and the robust gain are also shown in this figure. We should note that though the $S_2(q, K_2)$ surface is not below the $S_1(q, K_1)$ surface, which is sufficient for cost improvement as discussed in chapter 3, the true cost surface is actually improved. This shows that the design range need not be the maximum boundary value of parameters.

Next, were show the results when the two uncertainties δA_1 and δA_2 are considered in insensitive controller designs. They are illustrated in Fig. 30—Fig. 35, in which the projection and the contour lines of the cost are also shown in order concerning various types of designs. In Fig. 30, 31 the results by M.C.V. and in Fig. 32, 33 the improved region in detail, and in Fig. 34, 35 those of U.W. type M.C.V. are illustrated. Particularly in Fig. 32, 33 the qualitative discussions in chapter 3 are shown, where one point dashed lines correspond to the region of eq. (72) and two point dashed lines denote that of eq. (85) respectively. As evaluated before, the quantitative arrangements are displayed in Table 18—Table 27, and in Fig. 36,

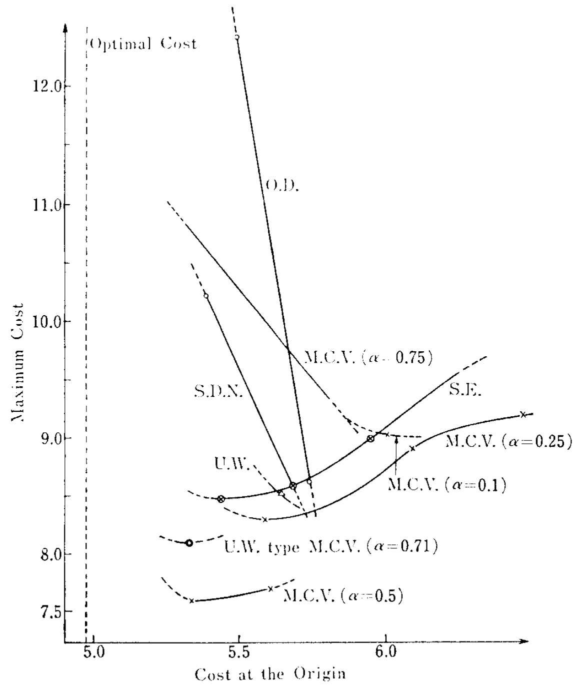


Fig. 28. Robust Controller Designs—Autopilot (δA_1, δA_2)

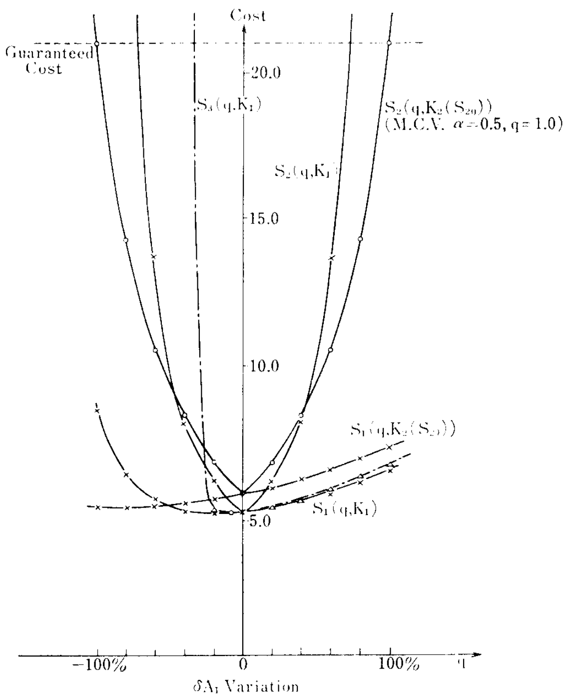
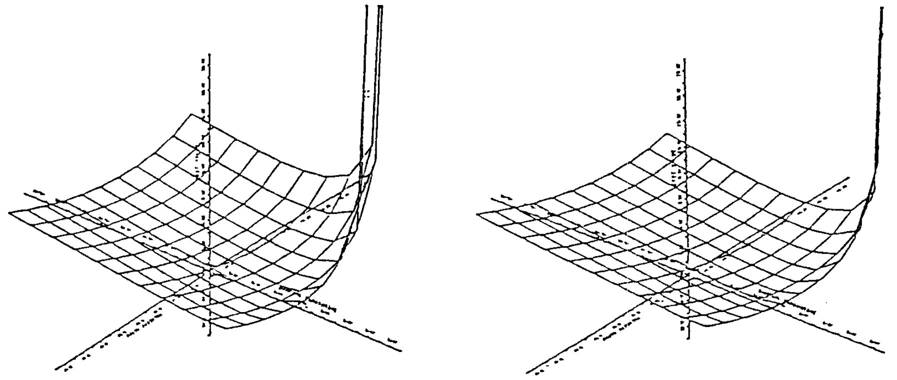
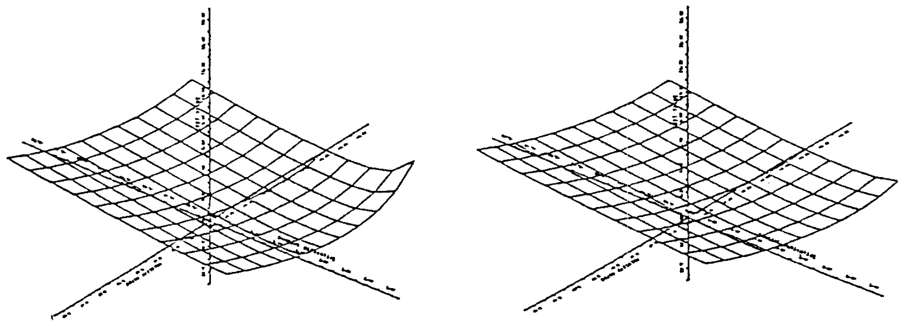


Fig. 29. S_1, S_2, S_3 Cost Surfaces—Autopilot.

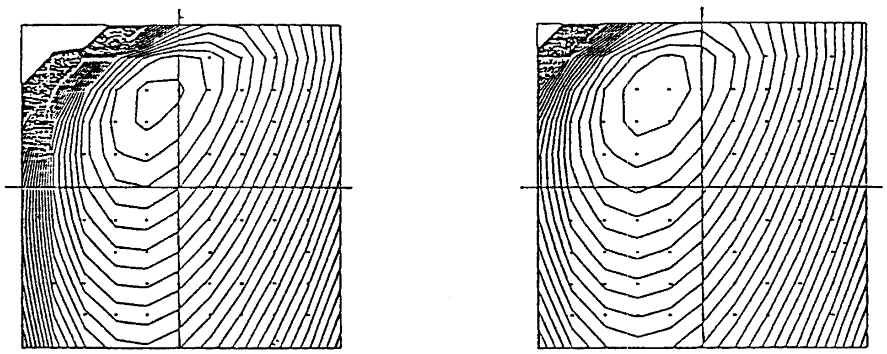


a) $q=0.1$ b) $q=0.2$

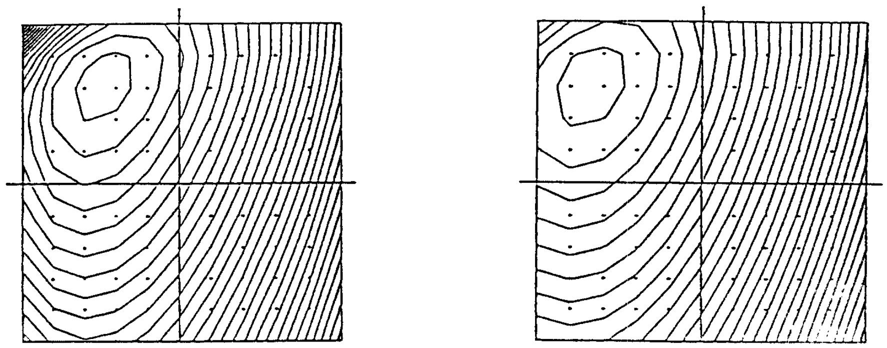


c) $q=0.3$ d) $q=0.4$

Fig. 30. Projection of Robustly Designed Cost Surface—Autopilot (M.C.V. $\alpha=0.75$).

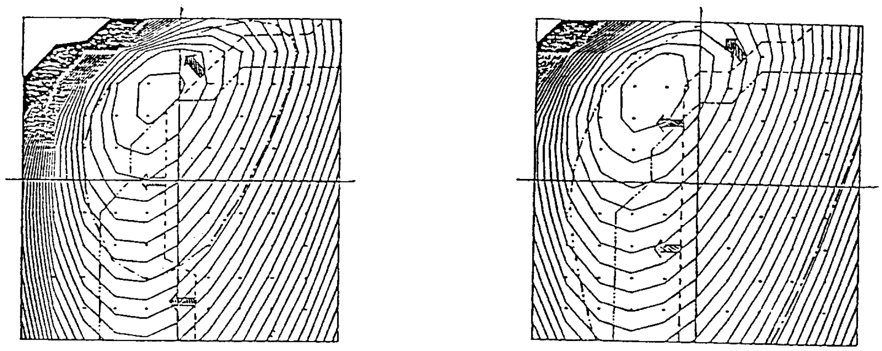


a) $q=0.1$ b) $q=0.2$



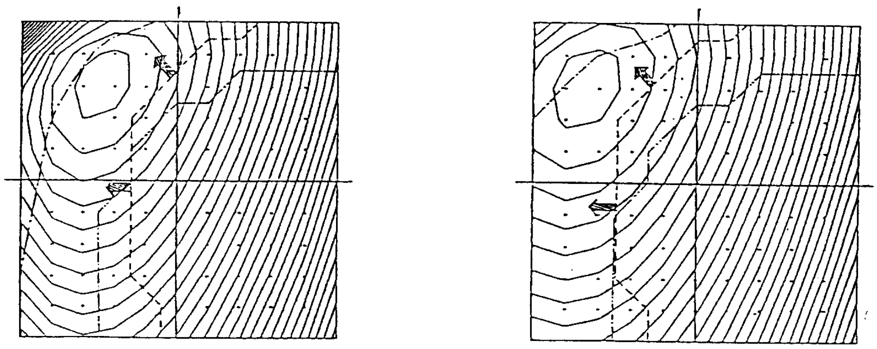
c) $q=0.3$ d) $q=0.4$

Fig. 31. Contour Lines of Robustly Designed Cost Surface—Autopilot (M.C.V. $\alpha=0.75$).



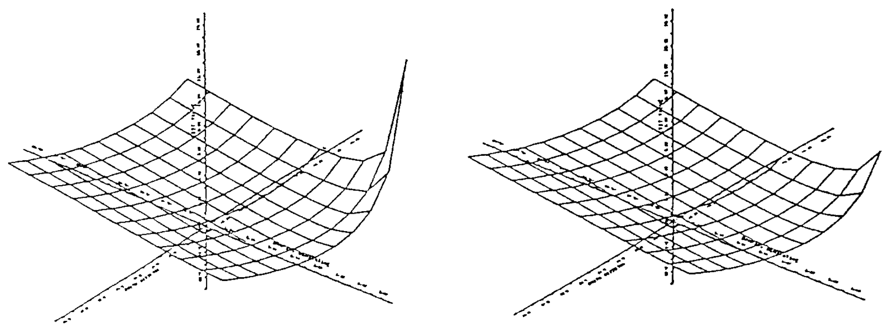
a) $q=0.1$ b) $q=0.2$

Fig. 32. Cost Improved Region—Autopilot (M.C.V. $\alpha=0.75$).

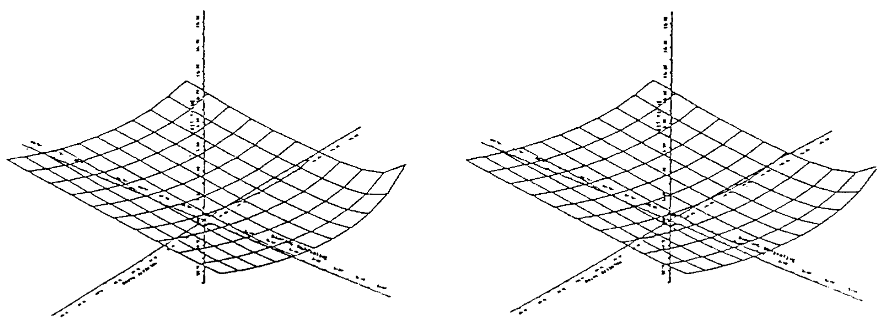


a) $q=0.3$ b) $q=0.4$

Fig. 33. Cost Improved Region—Autopilot (M.C.V. $\alpha=0.75$).

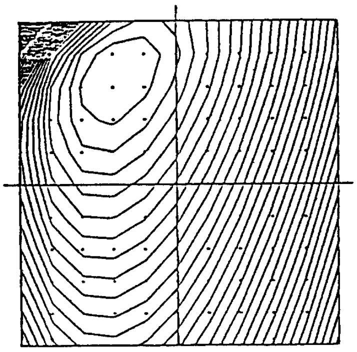


a) $q=0.01$ b) $q=0.015$

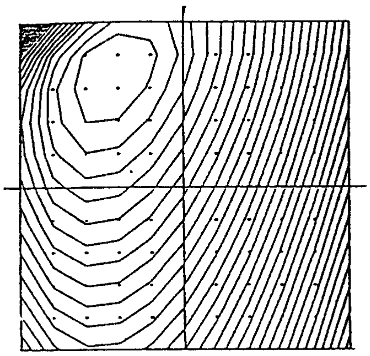


c) $q=0.020$ d) $q=0.025$

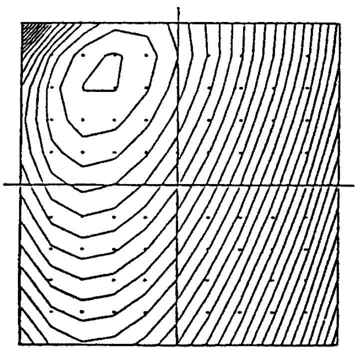
Fig. 34. Projection of Rebustly Designed Cost Surface—Autopilot (U.W type M.C.V. $\alpha=1.0$).



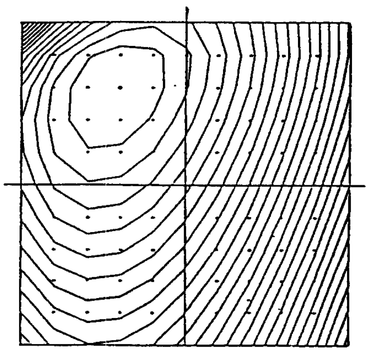
a) $q=0.01$



b) $q=0.015$



c) $q=0.020$



d) $q=0.025$

Fig. 35. Contour Lines of Robustly Designed Cost Surface—(U.W. type M.C.V. $\alpha=1.0$).

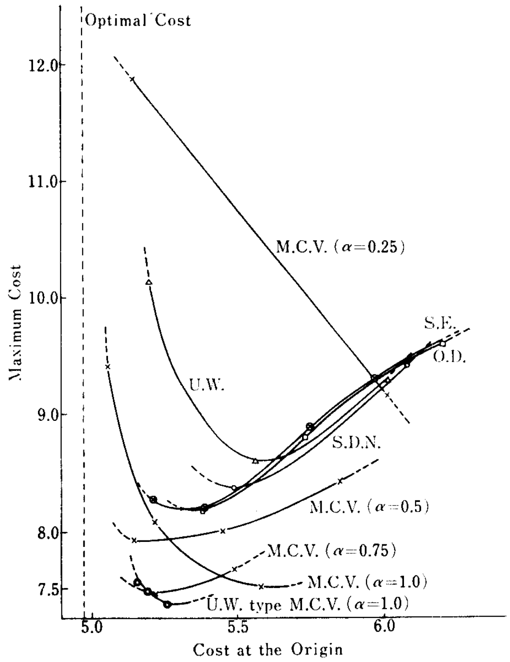


Fig. 36. Robust Controller Designs—Autopilot.

Table 18. Robust Design—Autopilot (M.C.V. $\alpha=0.1$)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$q=0.0$ (nominal)	4.975	∞	∞	66 deg
$q=0.1$	7.003 (11.600)	10.827	3.824	74 deg

Table 19. Robust Design—Autopilot (M.C.V. $\alpha=0.25$)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$q=0.0$ (nominal)	4.975	∞	∞	66 deg
$q=0.1$	5.153 (6.381)	11.881	6.728	76 deg
$q=0.2$	6.011 (10.375)	9.147	3.136	75 deg
$q=0.3$	7.603 (22.427)	11.344	3.741	72 deg

Table 20. Robust Design—Autopilot (M.C.V. $\alpha=0.5$)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$q=0.0$ (nominal)	4.975	∞	∞	66 deg
$q=0.1$	5.010 (5.868)	∞	∞	72 deg
$q=0.2$	5.153 (7.545)	7.918	2.765	75 deg
$q=0.3$	5.449 (11.155)	8.004	2.555	75 deg
$q=0.4$	5.854 (20.038)	8.416	2.562	73 deg

Table 21. Robust Design—Autopilot (M.C.V. α=0.75)

Case	Cost	max. Cost	Δ Cost	Stability Margin
q=0.0 (nominal)	4.975	∞	∞	66 deg
q=0.1	4.993 (5.883)	∞	∞	71 deg
q=0.2	5.063 (7.421)	23.271	18.208	74 deg
q=0.3	5.216 (10.432)	7.448	2.232	74 deg
q=0.4	5.494 (17.349)	7.658	2.164	73 deg

Table 22. Robust Design—Autopilot (M.C.V. α=1.0)

Case	Cost	max. Cost	Δ Cost	Stability Margin
q=0.0 (nominal)	4.975	∞	∞	66 deg
q=0.1	4.994 (5.999)	∞	∞	70 deg
q=0.2	5.063 (7.691)	9.398	4.335	73 deg
q=0.3	5.221 (10.963)	8.068	2.847	73 deg
q=0.4	5.578 (18.594)	7.506	1.928	73 deg

Table 23. Robust Design—Autopilot (U.W.)

Case	Cost	max. Cost	Δ Cost	Stability Margin
q=0.0 (nominal)	4.975	∞	∞	66 deg
q=5.0	5.003 (5.342)	∞	∞	72 deg
q=10.0	5.201 (6.207)	10.120	4.919	76 deg
q=15.0	5.561 (7.297)	8.509	2.948	76 deg
q=20.0	6.013 (8.501)	9.283	3.276	76 deg

Table 24. Robust Design—Autopilot (S.D.N.)

Case	Cost	max. Cost	Δ Cost	Stability Margin
q=0.0 (nominal)	4.975	∞	∞	66 deg
q=0.2	5.012 (5.403)	∞	∞	72 deg
q=0.4	5.158 (6.057)	15.606	10.448	76 deg
q=0.6	5.485 (7.083)	8.372	2.887	77 deg
q=0.8	6.084 (8.686)	9.402	3.318	76 deg

Table 25. Robust Design—Autopilot (O.D.)

Case	Cost	max. Cost	Δ Cost	Stability Margin
0% (nominal)	4.975	∞	∞	66 deg
20%	5.017 (4.749)	∞	∞	72 deg
40%	5.149 (4.571)	20.042	14.893	75 deg
60%	5.384 (4.439)	8.180	2.796	76 deg
80%	5.733 (4.352)	8.802	3.069	76 deg
100%	6.240 (4.306)	9.579	3.375	75 deg

Table 26. Robust Design—Autopilot (S.E.)

Case	Cost	max. Cost	Δ Cost	Stability Margin
q=0.0 (nominal)	4.975	∞	∞	66 deg
q=0.02	5.015 (5.231)	∞	∞	76 deg
q=0.04	5.214 (5.854)	8.275	3.061	79 deg
q=0.06	5.749 (7.188)	8.878	3.129	78 deg
q=0.08	6.805 (9.652)	10.690	3.885	75 deg

Table 27. Robust Design—Autopilot (U.W.M.C.V. $\alpha=1.0$)

Case	Cost	max. Cost	Δ Cost	Stability Margin
$q=0.0$ (nominal)	4.975	∞	∞	66 deg
$q=0.01$	5.115 (6.819)	11.395	6.280	75 deg
$q=0.015$	5.161 (7.973)	7.553	2.392	75 deg
$q=0.020$	5.188 (9.550)	7.467	2.279	74 deg
$q=0.025$	5.262 (12.023)	7.341	2.079	73 deg

Table 28. Robust Design—Autopilot (Stability Margins)

	nominal	MCV $\alpha=.75$ $q=.4$	UW ($q=13.0$)	offset (70%)
Cost	4.975	5.494	5.403	5.544
$0.4\delta A_1+0.4\delta A_2$	82 deg	77 deg	83 deg	82 deg
$0.4\delta A_1-0.4\delta A_2$	55 deg	68 deg	68 deg	68 deg
$-0.4\delta A_1+0.4\delta A_2$	70 deg	77 deg	83 deg	82 deg
$-0.4\delta A_1-0.4\delta A_2$	40 deg	69 deg	69 deg	69 deg

involving more results than Fig. 30–35. The superiority of the M.C.V. and U.W. type M.C.V. designs is also recognized in Fig. 36, while the other methods except for them construct the almost same robust systems as one another. The robust property of stability margins is also depicted in Table 28, which also supports the discussions in chapter 3. The numerical examples of S_1, S_2 and S_3 surfaces are shown in Fig. 37, which demonstrates the qualitative features of them considered in chapter 3. In this figure, the mathematical discussions in that chapter are illustrated particularly for gain K_3, where the S_3 surface suggests the cost improvement over the left part of -40% line, the true surface is really improved over the left part of -80% line. At the points where the $S_2(q, K_3)$ is below the $S_1(q, K_1)$ surface, the true surface is really improved, which is treated as sufficient condition in chapter 3. But it is also found out that the required gain need not be so "large" one as K_3, because even the "smaller" gain K_2 actually improves the true cost surface drastically, though the mathematical discussions as for $S_2(q, K_2)$ and $S_1(q, K_1)$ are not applied. Here the readers should note that the S_2 surfaces for K_2, K_3 are much higher than that in the case with only δA_1 variation in Fig. 29, because the double additive terms in M.C.V. or U.W. type M.C.V. make the design cost higher. Therefore we dare not have M.C.V. designs and U.W. type M.C.V. designs with correct

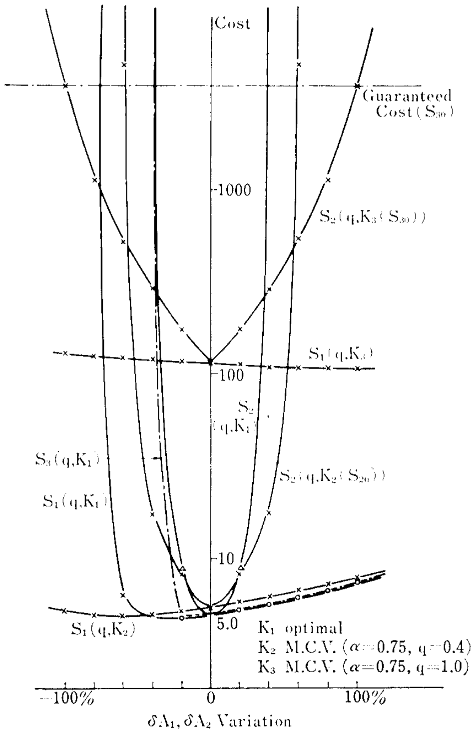


Fig. 37. S_1, S_2, S_3, Cost Surfaces—Autopilot.

Table 29. Statistical Cost Exspectation—Autopilot (Quasi Normal Distribution)

Method	Ceas	Cost	Method	Case	Cost
MCV (α=0.75)	q=0.2	5.131	SDN	q=0.4	5.223
	q=0.3	5.269		q=0.6	5.539
	q=0.4	5.539		q=0.8	6.133
UWMCV (α=1.0)	q=0.01	5.178	OD	40%	5.214
	q=0.015	5.218		60%	5.439
	q=0.02	5.242		80%	5.783
	q=0.025	5.316		100%	6.621
UW	q=10.0	5.263	SE	q=0.03	5.163
	q=15.0	5.613		q=0.04	5.283
	q=20.0	6.061		q=0.06	5.809
				q=0.08	6.863

design range for parameters and treat this as somewhat confined and appropriate narrower one. Next we attempt to compare the statistical cost expectations between some robust design methods. Table 29 shows these based on the quasi normal distribution which regards the specified parameter boundaries as 3σ levels. And Table 30 shows these based on the uniform distribution; i.e. the simple expectation.

Table 30. Statistical Cost Expectation—Autopilot (Uniform Distribution)

Method	Case	Cost	Method	Case	Cost
MCV (α=0.75)	q=0.2	5.783	SDN	q=0.4	5.767
	q=0.3	5.607		q=0.6	5.858
	q=0.4	5.795		q=0.8	6.396
UWMCV (α=1.0)	q=0.01	5.661	OD	40%	5.806
	q=0.015	5.595		60%	5.782
	q=0.02	5.583		80%	6.063
	q=0.025	5.654		100%	6.503
UW	q=10.0	5.724	SE	q=0.03	6.529
	q=15.0	5.919		q=0.04	5.766
	q=20.0	6.327		q=0.06	6.152
				q=0.08	7.172

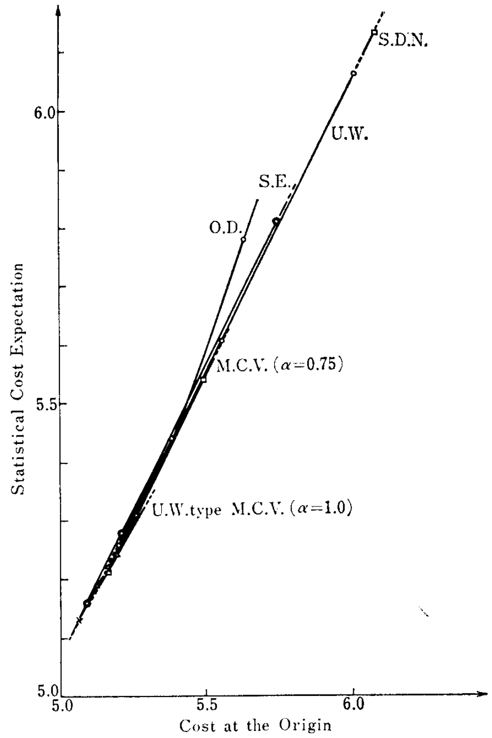


Fig. 38. Statistical Cost Expectation—Autopilot (Normal Distribution).

The schematic illustrations cosresponding to these are made in Fig. 38 and Fig. 39 respectively. These results indicate that against our anticipation if the system is only stable over the region then the cost expectation depends almost only on the cost at the origin regardless of the design methods. But only in Fig. 39, the differences between the methods are observed clearly, which suggest that the M.C.V.

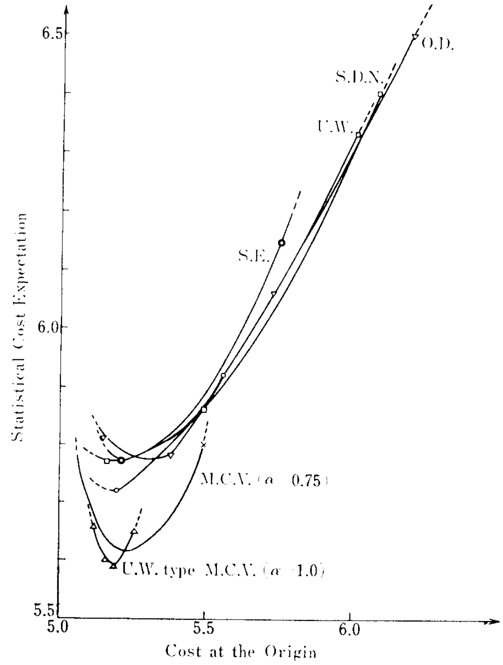


Fig. 39. Statistical Cost Expectation—Autopilot (Uniform Distribution).

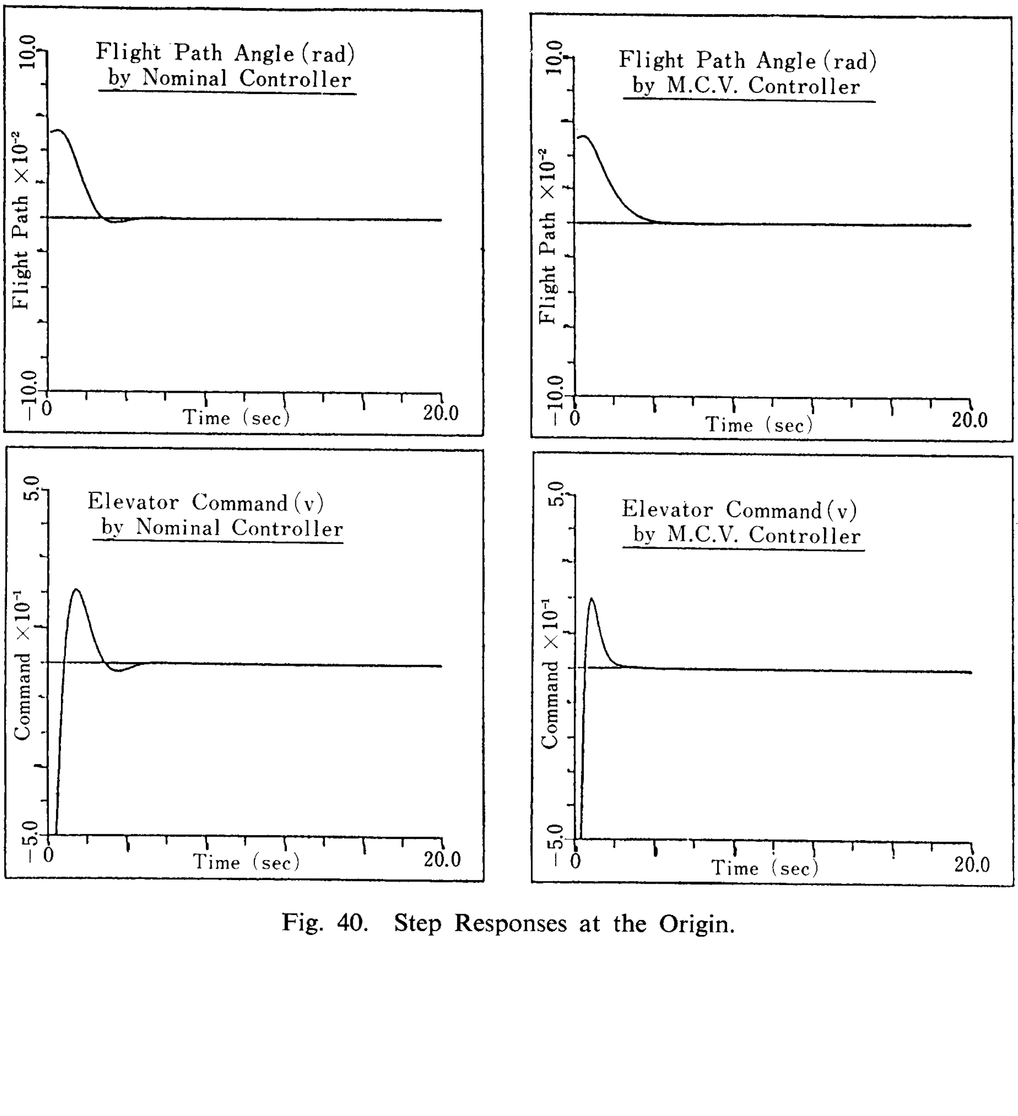


Fig. 40. Step Responses at the Origin.

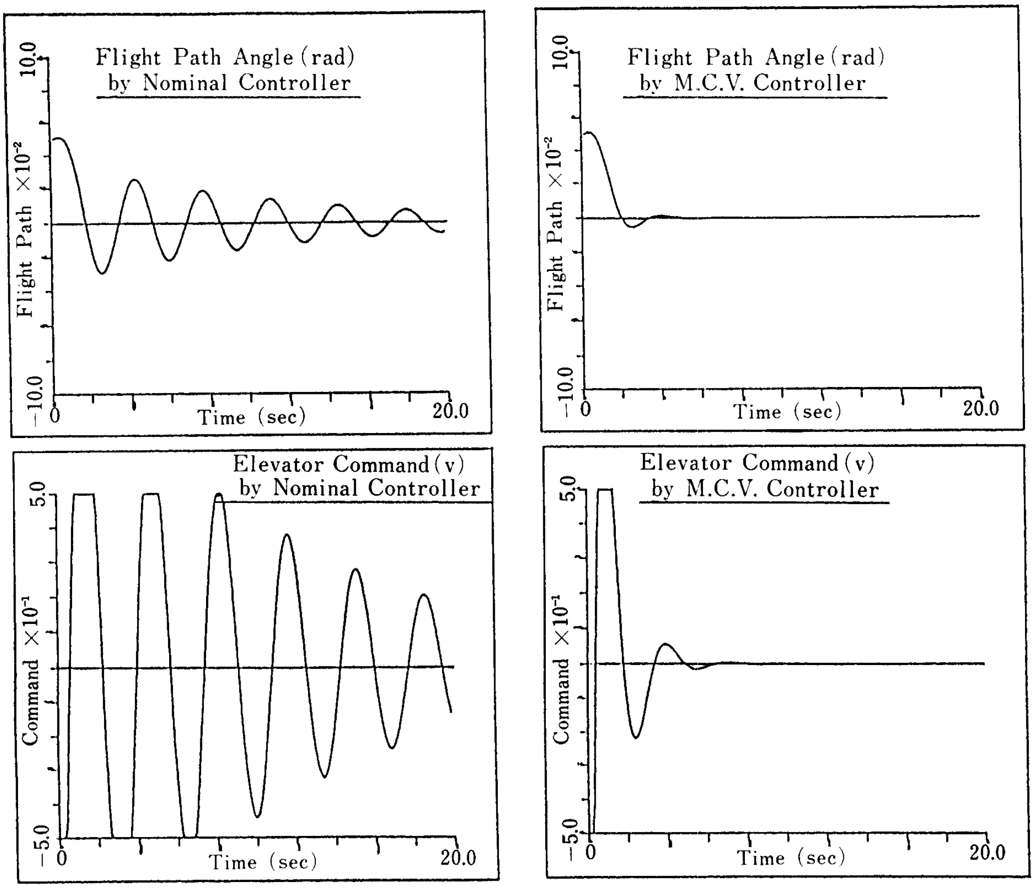


Fig. 41. Step Responses at 70% Variated Point.

and U.W. type M.C.V. designs are superior to any other design, which is also revealed in Fig. 36. Finally we compare the step responses between the nominal design and the robust M.C.V. design at the point where the nominal surface produces considerably high cost. These are summarized in Fig. 40, 41. While the response is quite bad in the nominal system, much better response is obtained by the robust M.C.V. system.

5–2. *Nonlinear Filtering*

Now we apply the robust design techniques to the estimation problem. The problem is the radar tracking one with unknown dynamical parameters in the model. Here we implicitly assume that the systems to be considered are extended Kalman filters, because of nonlinearity. We consider the two-dimensional dynamics of the re-entry vehicle as in Fig .42. The equations of motion are expressed as

$$\dot{x}=V_x,\ \dot{y}=V_y,$$

$$\dot{V}_x=-\frac{D}{m}\cos\gamma-\frac{L}{m}\sin\gamma,$$

$$\dot{V}_y=\frac{L}{m}\cos\gamma-\frac{D}{m}\sin\gamma-g,$$

$$D=\frac{1}{2}\rho V^2C_DS,\ L=\frac{1}{2}\rho V^2C_LS,\ \cos\gamma=V_x/V,\ \sin\gamma=V_y/V. \tag{205}$$

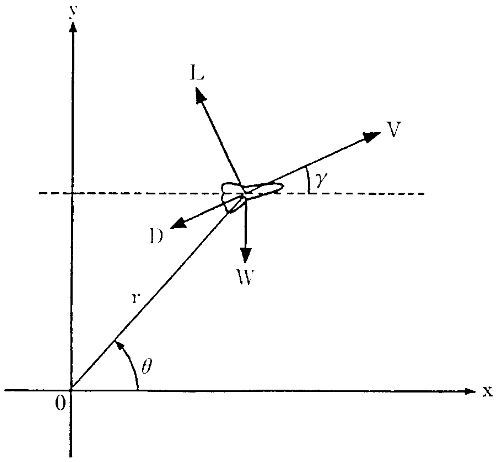


Fig. 42. Tracking of a Vehicle.

Rewriting these, we have

$$\dot{V}_x = -\alpha V_x V - \beta V_y V,$$
$$\dot{V}_y = \beta V_x V - \alpha V_y V - g,$$
$$\alpha = \frac{\rho S}{2m} C_D, \quad \beta = \frac{\rho S}{2m} C_L \tag{206-a}$$

with two measurements range and elevation angle:

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \cos^{-1}[x/\sqrt{x^2+y^2}]. \tag{206-b}$$

Here we assume the gravity is constant and the earth is flat and the vehicle does not have any thrusters. More solidly speaking, this is modeled as a re-entry vehicle under the hypersonic flight condition, with the following parameters:

$m = 50$ (ton), $S = 200$ (m^2), C_L (true) $= 0.40$, C_D (true) $= 0.15$,
$\sigma_R = 5$ (m), $\sigma_\theta = 0.005$ (deg), measurement interval $= 1.0$ (sec),
$V_{x0} = 3.0$ (km/sec), $V_{y0} = -0.9$ (km/sec), $X_0 = -200$ (km), $Y_0 = 60$ (km),
$\hat{V}_{x0} = 3.3$ (km/sec), $\hat{V}_{y0} = 0.0$ (km/sec), $\hat{X}_0 = -220$ (km), $\hat{Y}_0 = 66$ (km).
(206-c)

where σ_R and σ_θ denote the standard deviation of range and elevation angle measurement errors respectively and ($\hat{}$) indicates the estimated value.

In application of the M.C.V. and other "Additive Term Design" techniques to the filtering problem, we dare to assume that the modification is confined to the covariance propagation. Because these design techniques are derived in control problems providing that the system structure is not changed and so the design procedure is reduced to the selection or calculation of gains (in this case Kalman gain). For example, from the standpoint of the statistical expectation, we should modify not only the covariance propagation equations but the state estimate propagation equation in Kalman filtering. But from the practical view points such systems may produce worse estimates between the measurements or updates and so they cannot be adopted. Hence we have the evaluations confined to the systems with modified covariance propagation equations.

In Table 31 and Fig. 43, 44, 45, 46, we show the estimation results (error covariances and actual errors) of the nominal Kalman filter in various cases where the parameters are exactly modeled or mismodeled to the true ones. From these it is easily observed that the theoretical nominal Kalman filter is highly sensitive to the system parameters and may diverge. In practical applications, the other estimation techniques than Kalman filters such as least square methods or some methods are usually used in order to keep the Kalman gain high; the methods of resetting the covariance matrix or introducing the virtual system noise. Or if the system is permitted to be complex, the real time parameter identification technique is used to adjust the installed model to the true dynamics. The readers should note that this approach is similar to the "Mismatch Estimation" concept. The results of this taking the virtual noise covariances as much larger values are shown also in Table 31 and Fig. 47, 48. These reveal that if the system structure is permitted to be higher order one or complex one, in this exmaple from 4-th order to 6-th order, it is possible for us to improve the estimates as demonstrated. But our approach here is quite different one from this. We seek for the methods that may not change

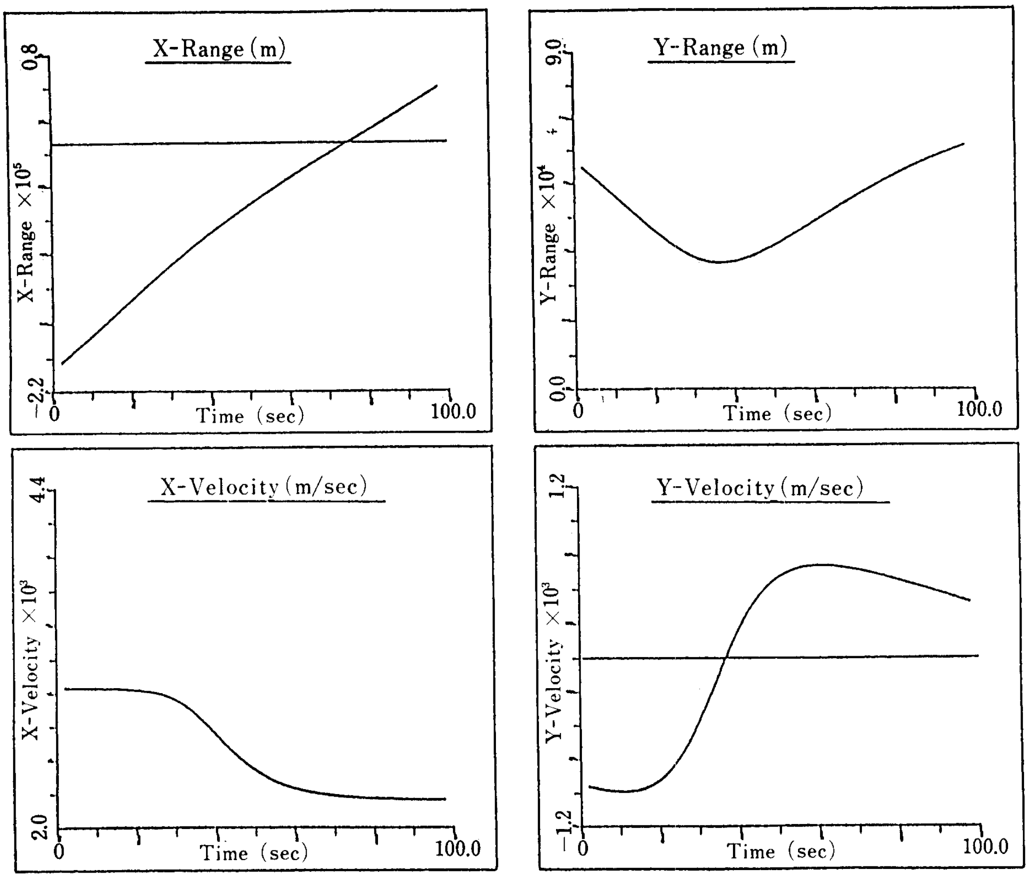


Fig. 43. Nominal Tralectory Data I.

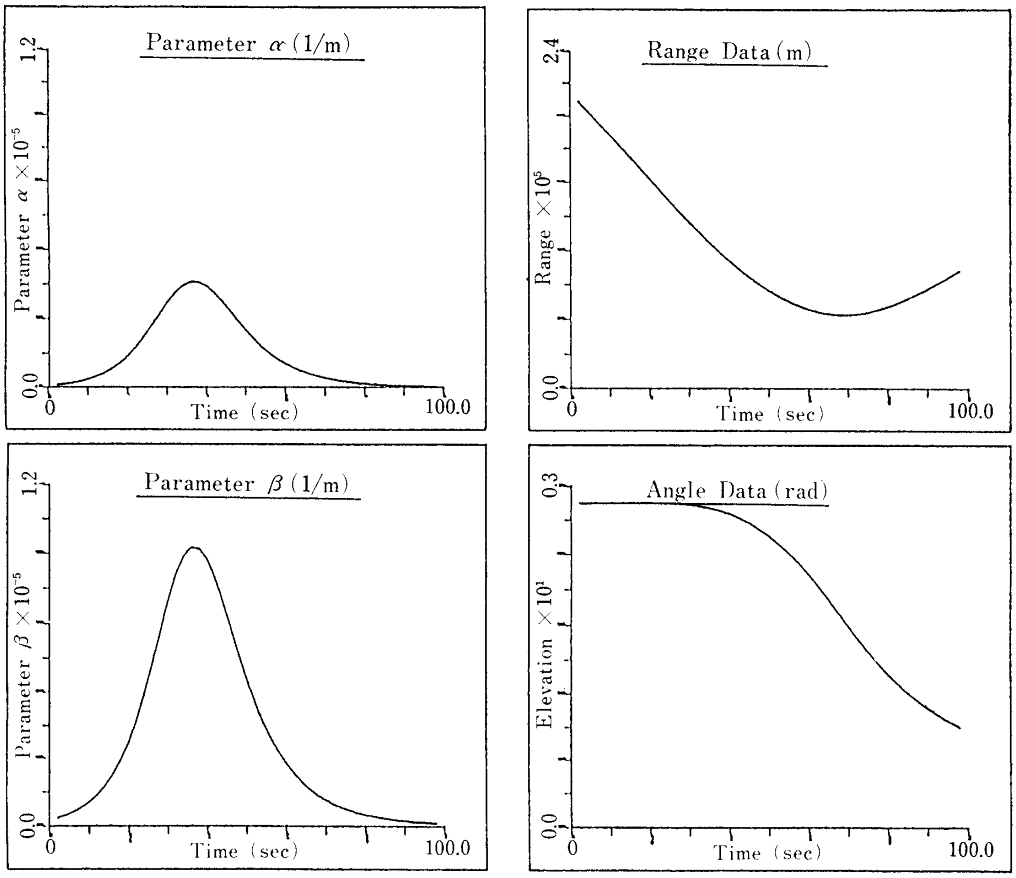


Fig. 44. Nominal Trajectory Data II.

Table 31. Estimation by the Models with Various Parameters

	T=40 (sec)				T=100 (sec)			
Case	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)
×0.8	337.1	68.6	556.3	122.8	1569.0	57.1	872.2	32.6
×0.9	164.2	33.8	274.3	57.9	750.6	27.9	409.1	15.3
×1.0 (true)	2.9	0.4	6.7	0.1	5.5	0.1	0.7	0.02
×1.1	164.9	33.1	247.1	52.4	700.4	25.7	371.6	13.4
×1.2	334.7	66.2	481.1	101.0	1353.1	49.7	713.2	25.3
Real Time Identification	1.7	0.001	14.7	0.2	7.2	13.5	4.1	7.9

the system structure as in the discussion on control problems before. Essentially our robust designs are equivalent to the methods of keeping the Kalman gain high. But while the practical or intuitive approaches are formulated without any theoretical foundations, ours are justified by the discussions in chapter 3. Fig. 49—Fig. 58 illustrate the results (covariances and errors) by some "Additive Term Design"

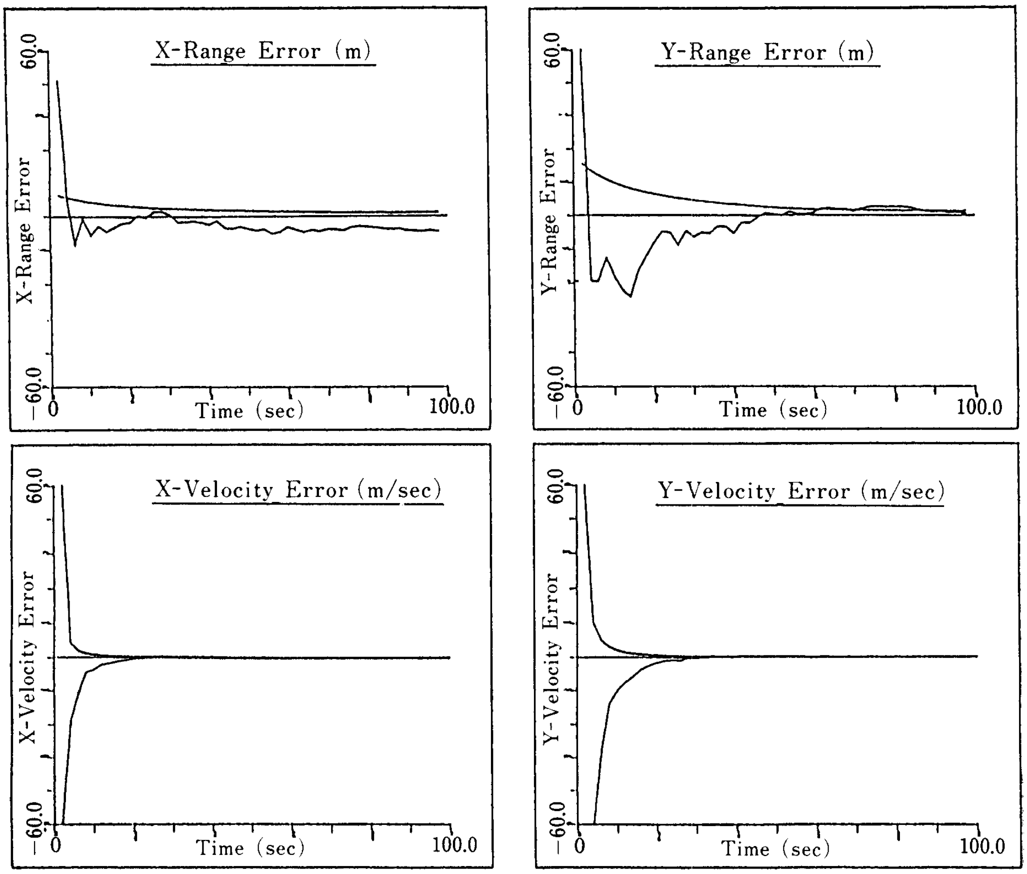


Fig. 45. Estimation by Exact Model (±60 m, m/sec scale).

Table 32. Estimation by the Robust Filter (M.C.V. $\alpha=1.0$)

	$T=40$ (sec)				$T=100$ (sec)			
Case	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)
$q=0.1$	82.1	48.0	94.3	76.1	7.6	1.6	2.1	1.8
$q=0.2$	19.9	28.0	25.0	45.9	1.9	0.4	0.4	0.4
$q=0.3$	3.0	15.8	22.3	36.1	2.6	0.04	4.4	1.5
$q=0.4$	1.2	12.2	4.7	21.6	12.0	2.4	3.4	1.7
$q=0.5$	2.3	12.7	1.3	19.7	2.6	2.5	9.4	0.9

techniques. In Fig. 49–53 M.C.V. designs are shown and in Fig. 54–57 U.W. type M.C.V. designs are demonstrated, and the attempt to introduce the apparent system noises is made in Fig. 58. Here the true boundary values of variations or the largest design range in M.C.V. or U.W. type M.C.V. are 0.2, because these results are evaluated in the 80% mismodeled filter. In the other cases than M.C.V. or U.W. type M.C.V, this design range has little meaning and should be taken as the scale of the additive term. It is obvious that considerable improvement of

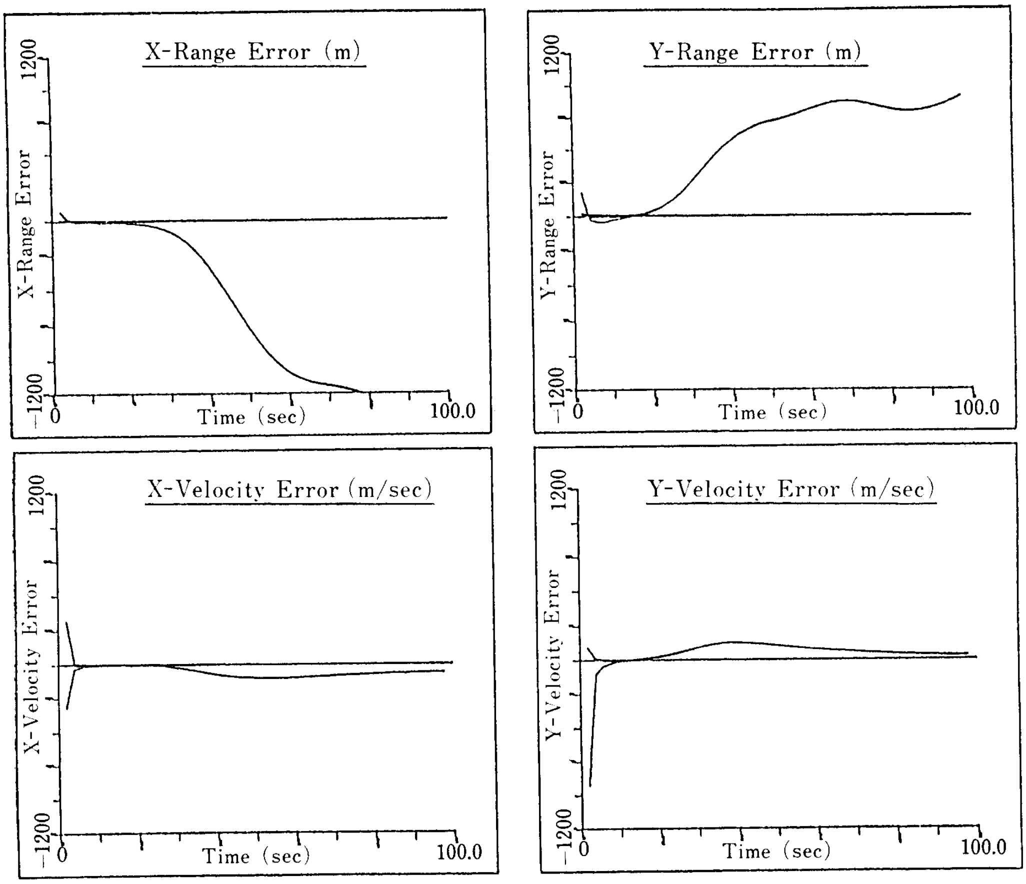


Fig. 46. Estimation by the model with 80% parameters. (±1200 m, m/sec scale).

Table 33. Estimation by the Robust Filter (M.C.V. $q=0.3$)

	$T=40$ (sec)				$T=100$ (sec)			
Case	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)
$\alpha=0.5$	39.4	35.8	36.8	56.3	3.0	0.2	2.9	1.5
$\alpha=1.0$	3.0	15.8	22.3	36.1	2.6	0.04	4.4	1.5
$\alpha=2.0$	0.5	9.2	2.8	12.5	4.5	3.3	0.2	0.3
$\alpha=3.0$	2.4	5.6	1.2	11.1	5.5	3.9	9.9	6.2
$\alpha=4.0$	3.1	3.5	2.8	2.6	3.5	0.9	3.1	2.4

estimates is made by appropriate selection of these techniques. The quantitative evaluations of actual estimation errors are rearranged and summarized in Table 32—Table 37, and the schematic comparison is made as histograms in Fig. 59—Fig. 62 for various types of robust estimation techniques. These show that any "Additive Term Design" method is very effective to the estiamtion improvement more or less. Particularly, the M.C.V. or U.W. type M.C.V. methods are assured to produce highly improved estimates. (Here the deterioration of estimate at time 40 (sec) is

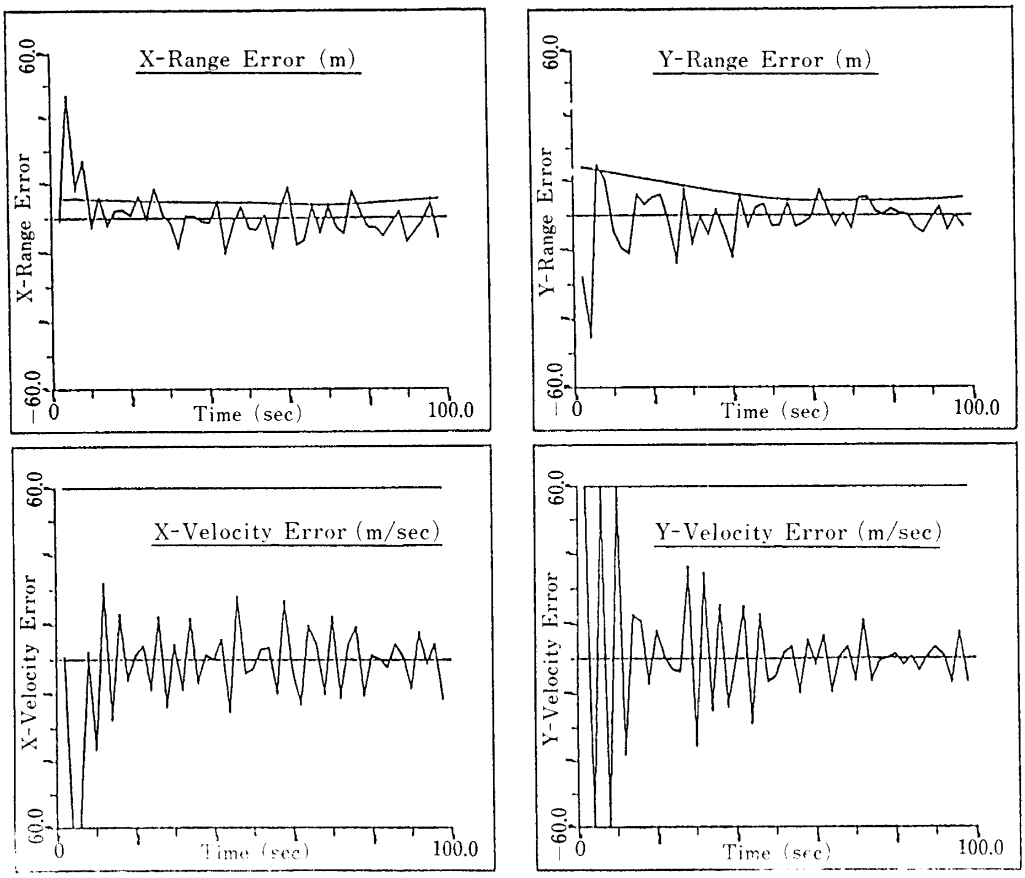
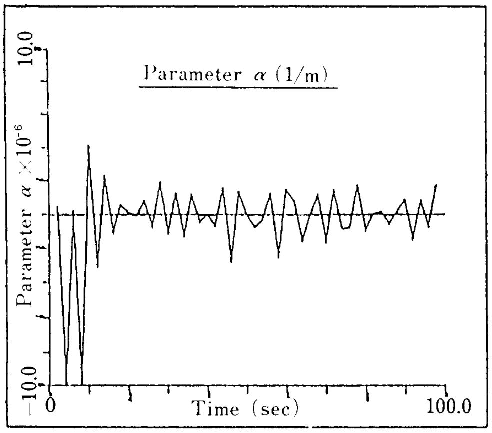


Fig. 47. Estimation by Real Time Parameter Identification (±60 m, m/sec scale).

Table 34. Estimation by the Robust Filter (U.W.)

	$T=40$ (sec)				$T=100$ (sec)			
Case	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)
$q^2=10^6$	1.6	10.9	507.5	116.3	72.4	5.2	150.7	10.3
$q^2=10^7$	20.9	5.7	221.2	64.5	27.7	2.8	71.3	6.5
$q^2=10^8$	6.7	0.6	59.4	37.0	6.6	1.3	28.8	3.6
$q^2=10^6$	2.8	0.4	14.6	15.9	3.7	9.4	6.9	1.6

caused by the fact that the difference between the installed model and the true dynamics is large at that time. And the fact that the design ranges in M.C.V. and U.W. type M.C.V. are treated over 0.2 in some cases is due to the nonlinear features under which the inequality eq. (31) cannot be assured easily here.) From the results obtained here, we observe that while the error covariances that reflect the magnitudes of Kalman gains settle to stationary values regardless of the actual errors by apparent system noise concept as in Fig. 58, the covariances in U.W. type M.C.V. reflect



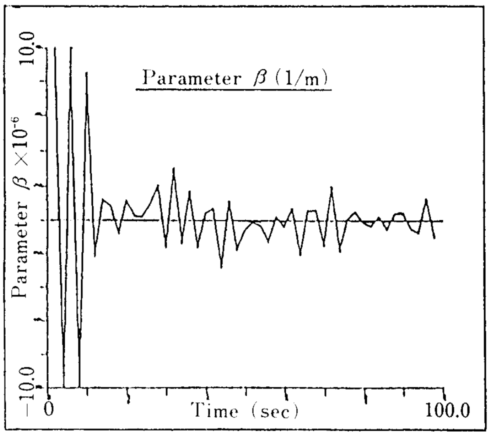


Fig. 48. Estimation by Real Time Parameter Identification.

Table 35. Estimation by the Robust Filter (S.D.N.)

	T=40 (sec)				T=100 (sec)			
Case	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)
$q^2=10^4$	328.0	67.6	562.7	121.8	1350.6	49.5	828.2	31.8
$q^2=10^5$	302.3	63.6	553.5	122.6	691.0	27.0	708.3	30.0
$q^2=10^6$	147.0	40.1	554.3	125.0	213.7	11.0	433.4	21.6
$q^2=10^7$	37.1	5.3	250.0	55.0	75.8	5.7	118.3	9.7
$q^2=10^8$	1.2	20.4	1.8	199.8	7.3	1.5	11.2	2.4
$q^2=5\times10^7$	7.3	5.7	13.7	7.6	24.6	2.9	43.8	5.2

the actual errors well as in Fig. 54–57 and so that mathematical basis can be justified qualitatively for this U.W. type M.C.V. It should be noted that the results by M.C.V. also indicate these properties more or less. But unfortunately, we cannot obtain more informations concerning the difference between these methods as in the previous numerical example. Because while the high feedback gains indicate

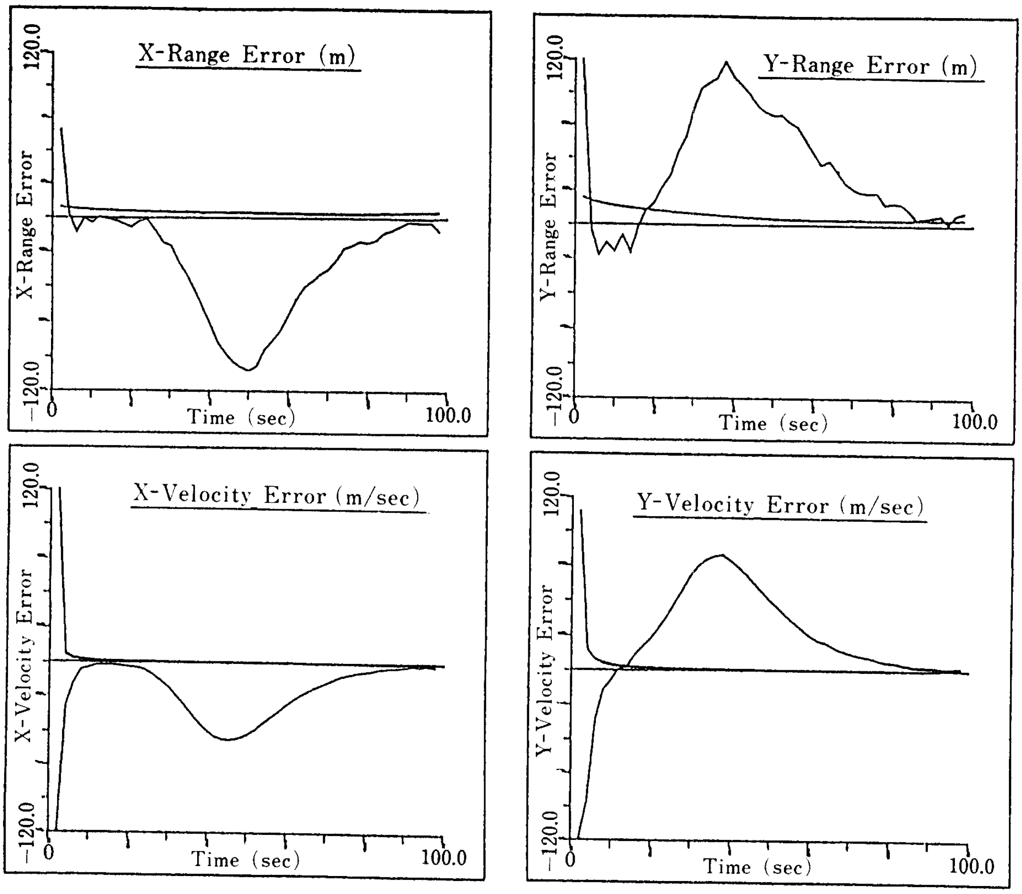


Fig. 49. Estimation by M.C.V. (q=0.1) (±120 m, m/sec scale).

Table 36. Estimation by the Robust Filter (S.E.)

	T=40 (sec)				T=100 (sce)			
Case	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)
$q^2=10^5$	267.5	58.9	557.7	122.2	493.3	20.4	630.5	28.1
$q^2=10^6$	60.5	23.7	464.2	111.4	148.7	9.2	237.7	14.9
$q^2=10^7$	2.9	1.4	16.6	11.3	41.9	4.0	72.5	7.0
$q^2=2\times10^7$	1.7	4.7	2.0	42.4	21.8	2.7	31.3	4.2

the much effort of actuators and these are evaluated as the penalty of cost in control problems, in estimation problems high Kalman gains are almost independent of the resulted estimation error and can be realized without difficulty actually. Hence we cannot discuss further as to this comparison. Finally the results of the cases where the modeled parameters are different from the cases before are presented in Table 38. The M.C.V. methods are fairly effective also to these cases with highly mismatched models.

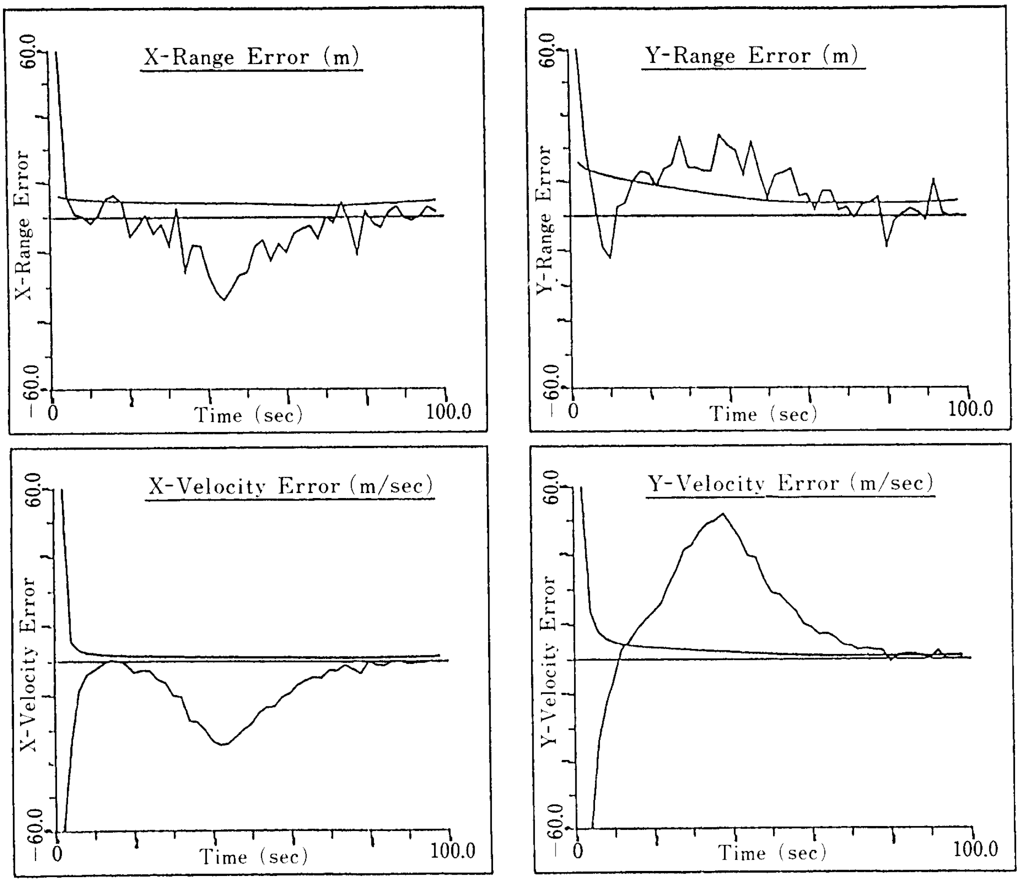


Fig. 50. Estimation by M.C.V. (q=0.2) (±60 m, m/sec scale).

Table 37. Estimation by the Robust Filter (U.W.M.C.V. $\alpha=10^{-10}$)

	T=40 (sec)				T=100 (sec)			
Case	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)
q=0.01	4.9	2.8	37.8	31.6	13.8	1.6	24.8	3.4
q=0.1	9.7	0.6	10.9	11.3	5.3	1.2	11.0	2.1
q=1.0	4.6	6.8	2.3	4.6	1.1	0.3	4.8	1.0
q=5.0	2.8	1.1	16.2	14.5	1.6	0.3	3.8	0.7

Table 38. Estimation by the Robust Filter (to other Models)

	T=40 (sec)				T=100 (sec)			
Case	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)	x (m)	$\dot{x}$ (m/s)	y (m)	$\dot{y}$ (m/s)
×1.2	7.3	15.0	5.7	11.2	16.1	7.0	5.2	1.7
×0.5	3.1	20.5	14.1	49.3	6.2	2.1	3.0	2.6
×0.5 (q=0.5)	6.6	20.1	6.9	3.2	1.3	1.4	0.3	0.2

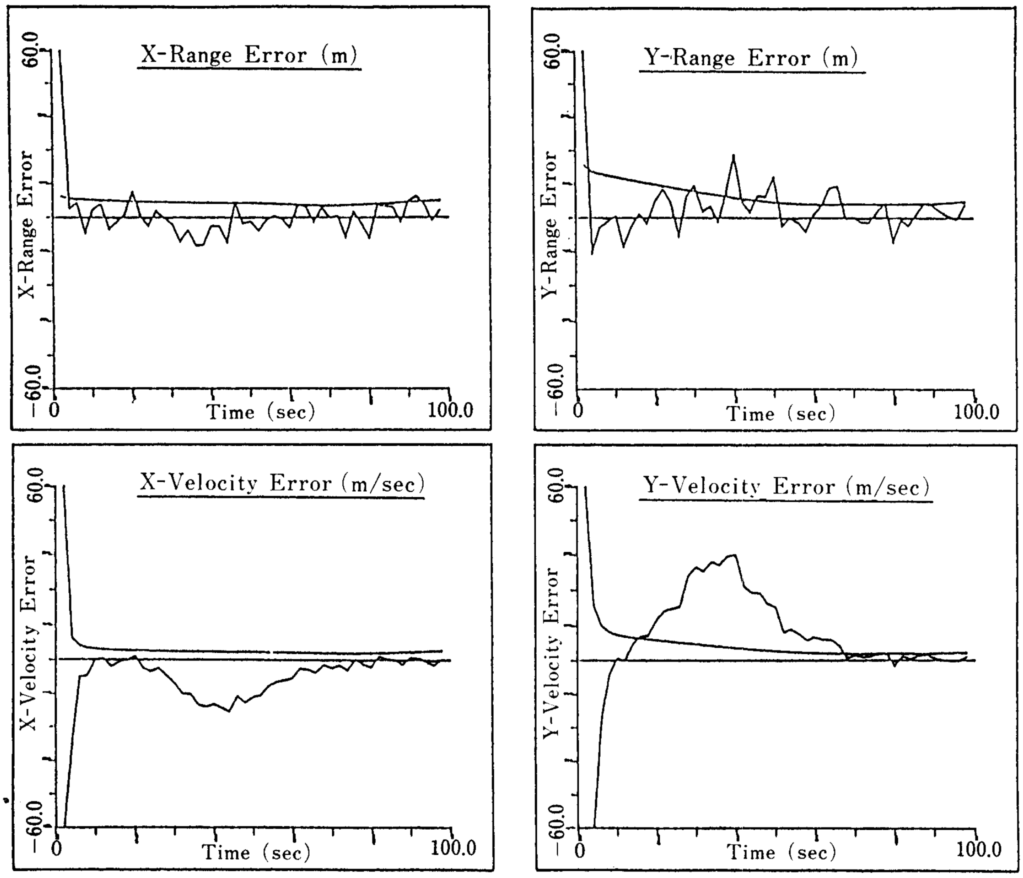

X-Range Error (m)
X-Range Error
Y-Range Error (m)
Y-Range Error
X-Velocity Error (m/sec)
X-Velocity Error
Y-Velocity Error (m/sec)
Y-Velocity Error
60.0
−60.0
0
Time (sec)
100.0

Fig. 51.

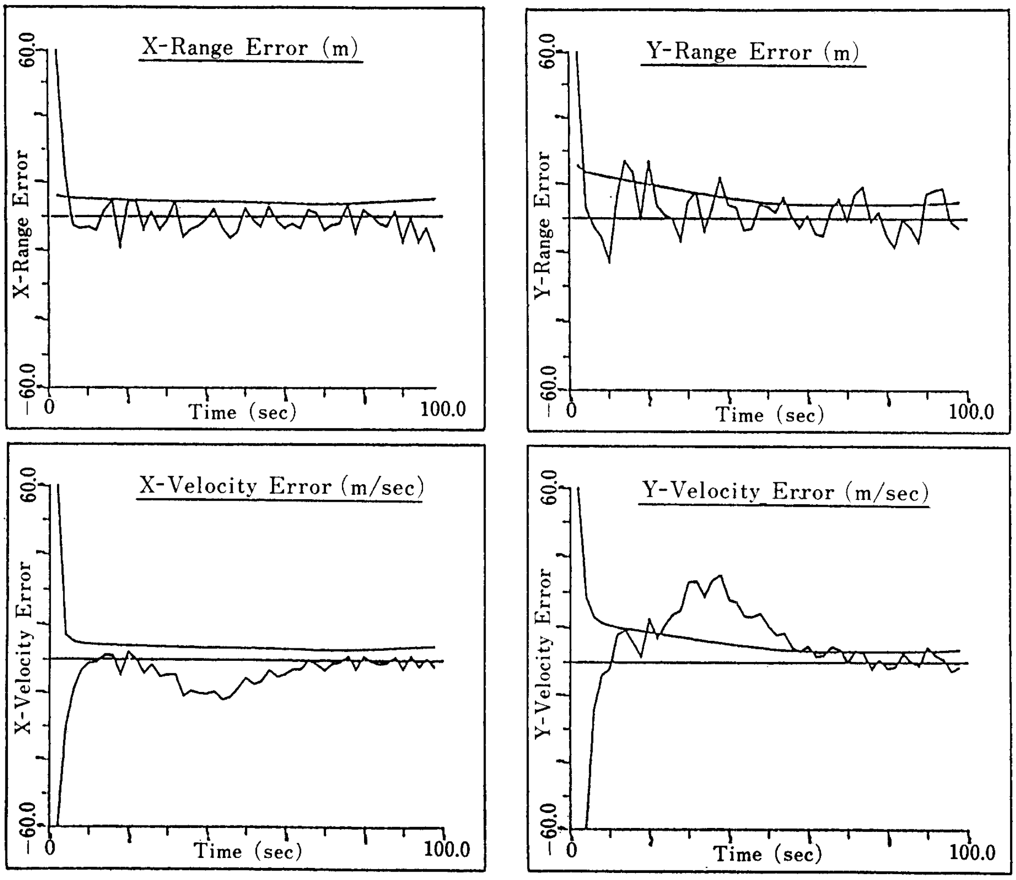

X-Range Error (m)
X-Range Error
Y-Range Error (m)
Y-Range Error
X-Velocity Error (m/sec)
X-Velocity Error
Y-Velocity Error (m/sec)
Y-Velocity Error
60.0
−60.0
0
Time (sec)
100.0

Fig. 52.

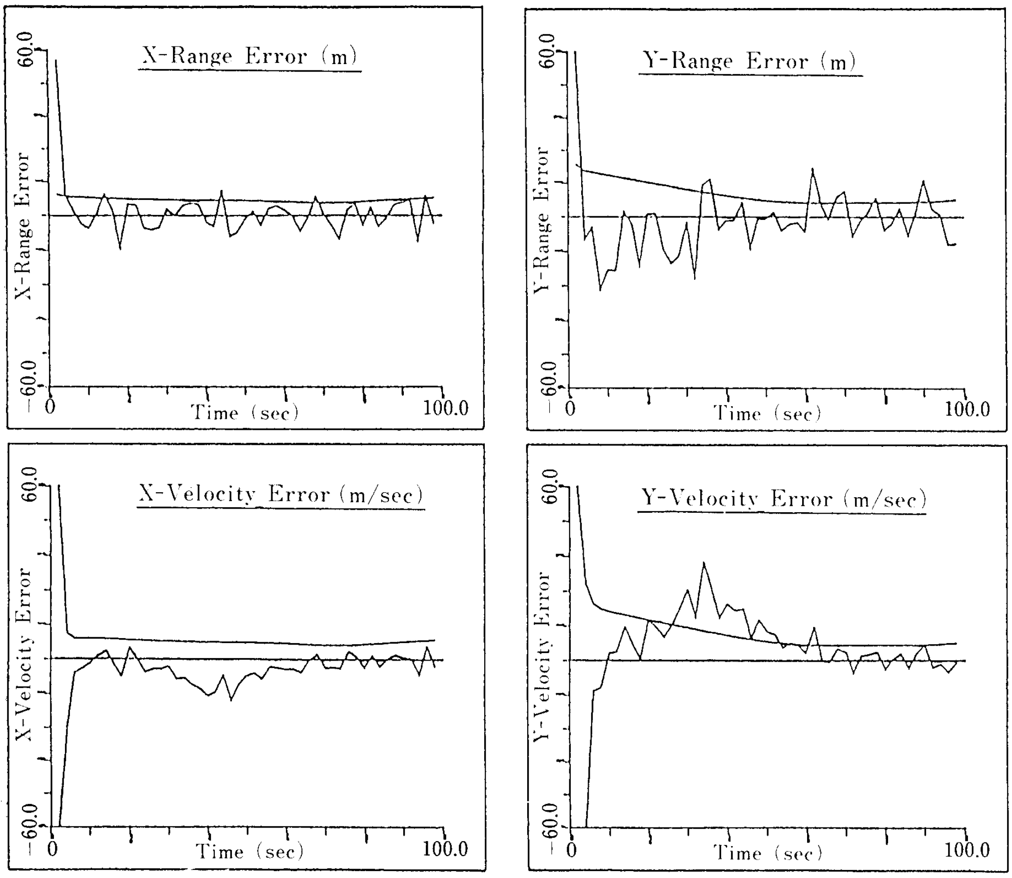

X-Range Error (m)
X-Range Error
60.0
−60.0
0
Time (sec)
100.0
Y-Range Error (m)
Y-Range Error
60.0
−60.0
0
Time (sec)
100.0
X-Velocity Error (m/sec)
X-Velocity Error
60.0
−60.0
0
Time (sec)
100.0
Y-Velocity Error (m/sec)
Y-Velocity Error
60.0
−60.0
0
Time (sec)
100.0

Fig. 53.

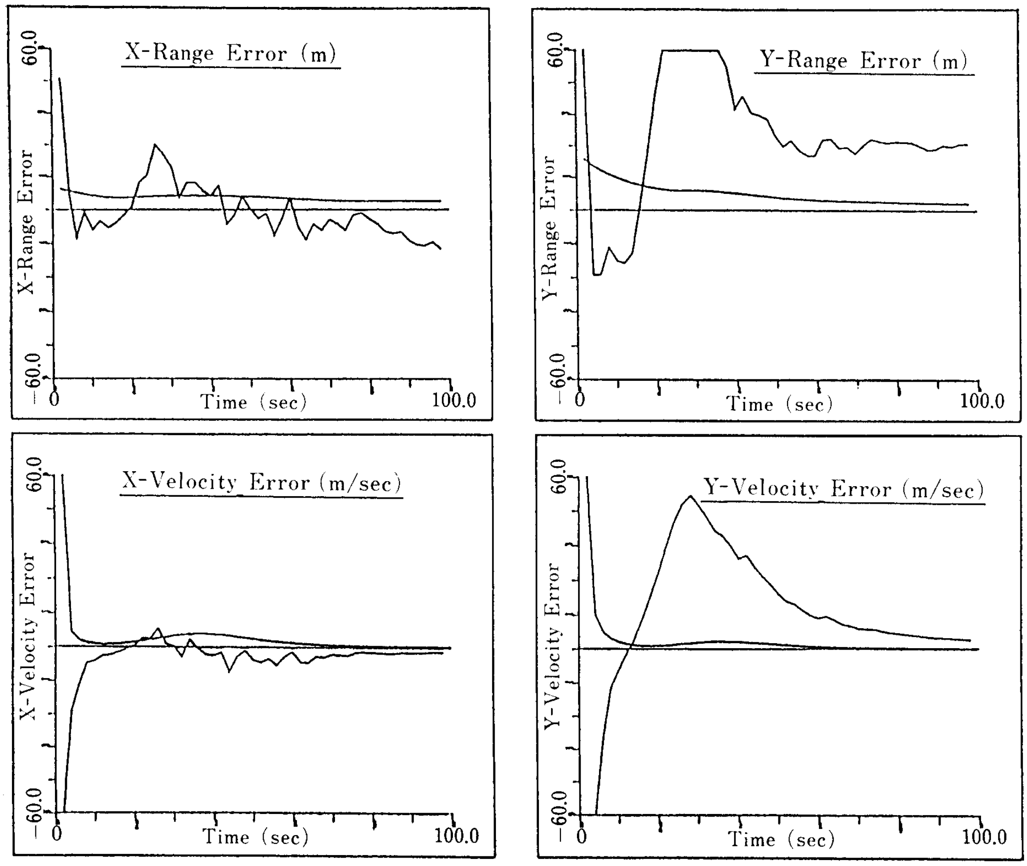

X-Range Error (m)
X-Range Error
60.0
−60.0
0
Time (sec)
100.0
Y-Range Error (m)
Y-Range Error
60.0
−60.0
0
Time (sec)
100.0
X-Velocity Error (m/sec)
X-Velocity Error
60.0
−60.0
0
Time (sec)
100.0
Y-Velocity Error (m/sec)
Y-Velocity Error
60.0
−60.0
0
Time (sec)
100.0

Fig. 54.

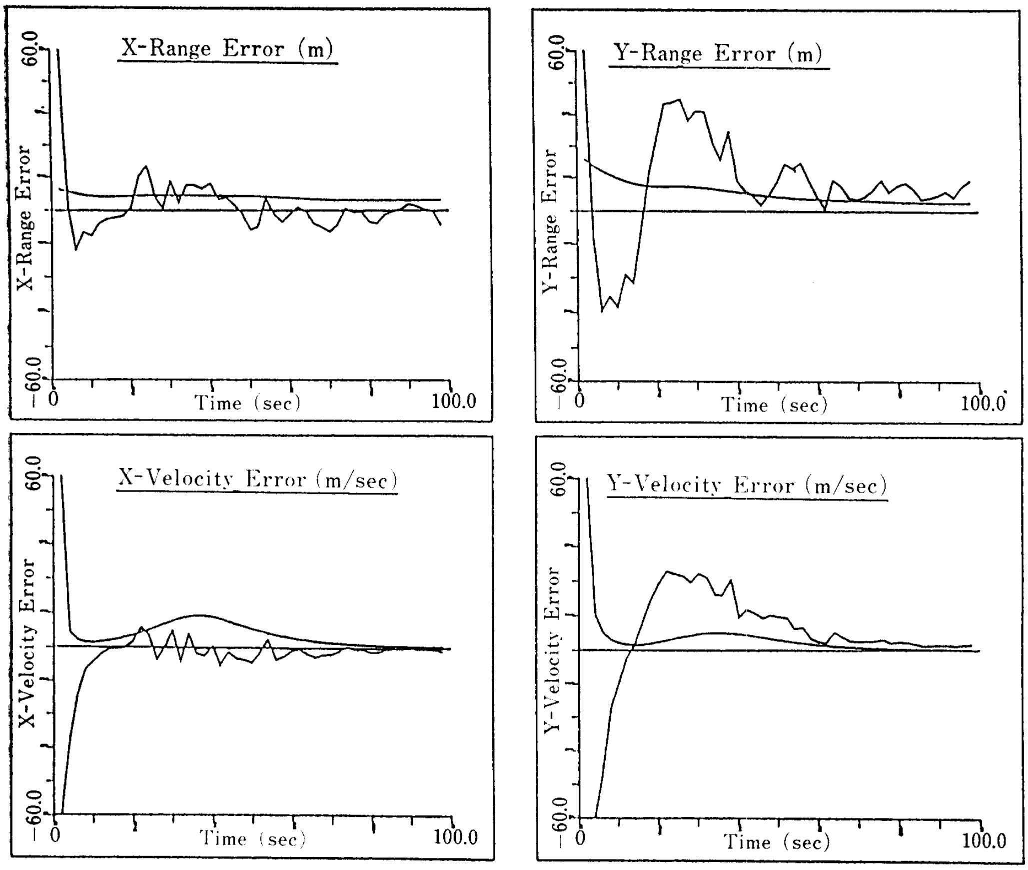

X-Range Error (m)
X-Range Error
60.0
−60.0
0
Time (sec)
100.0
Y-Range Error (m)
Y-Range Error
60.0
−60.0
0
Time (sec)
100.0
X-Velocity Error (m/sec)
X-Velocity Error
60.0
−60.0
0
Time (sec)
100.0
Y-Velocity Error (m/sec)
Y-Velocity Error
60.0
−60.0
0
Time (sec)
100.0

Fig. 55.

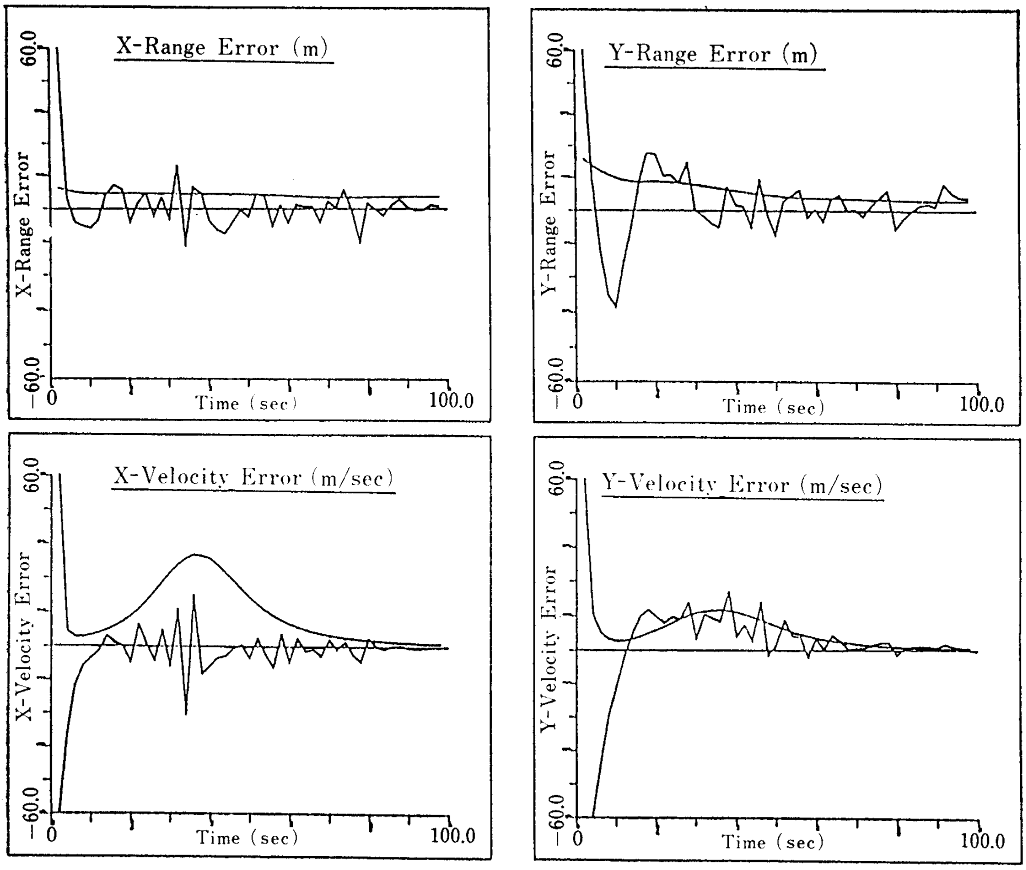

X-Range Error (m)
X-Range Error
60.0
−60.0
0
Time (sec)
100.0
Y-Range Error (m)
Y-Range Error
60.0
−60.0
0
Time (sec)
100.0
X-Velocity Error (m/sec)
X-Velocity Error
60.0
−60.0
0
Time (sec)
100.0
Y-Velocity Error (m/sec)
Y-Velocity Error
60.0
−60.0
0
Time (sec)
100.0

Fig. 56.

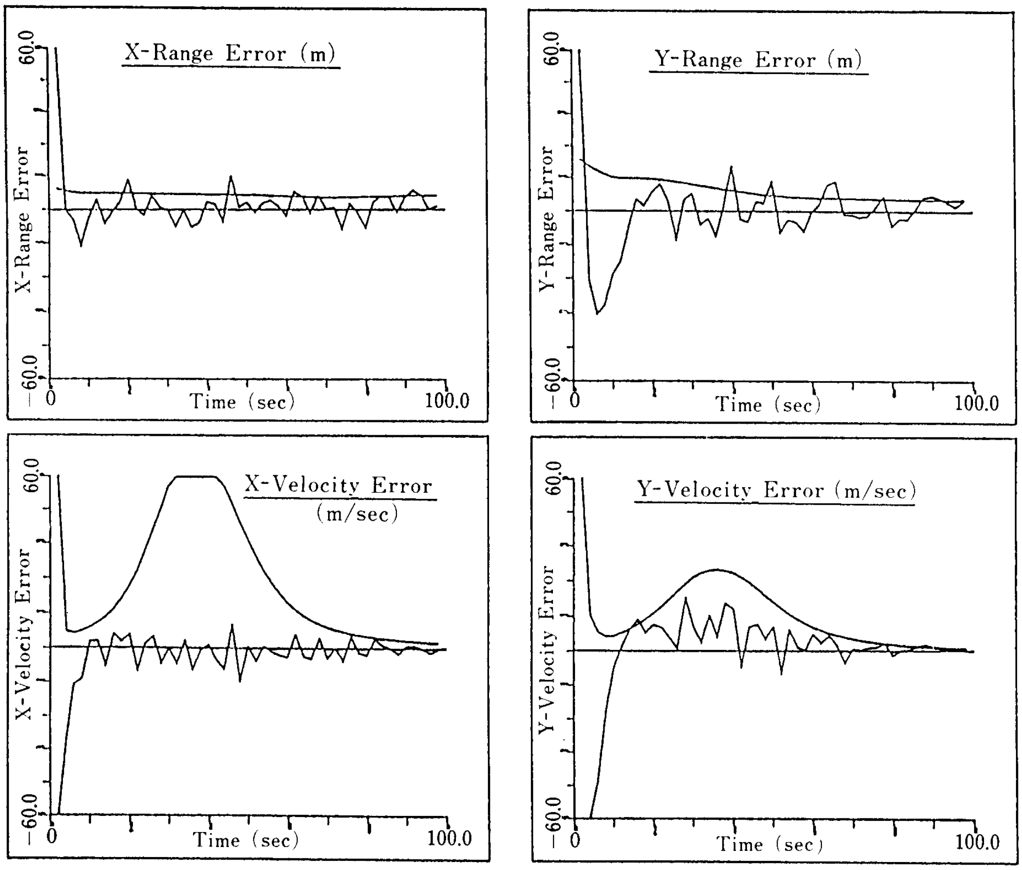

X-Range Error (m)
X-Range Error
60.0
−60.0
0
Time (sec)
100.0
Y-Range Error (m)
Y-Range Error
60.0
−60.0
0
Time (sec)
100.0
X-Velocity Error (m/sec)
X-Velocity Error
60.0
−60.0
0
Time (sec)
100.0
Y-Velocity Error (m/sec)
Y-Velocity Error
60.0
−60.0
0
Time (sec)
100.0

Fig. 57

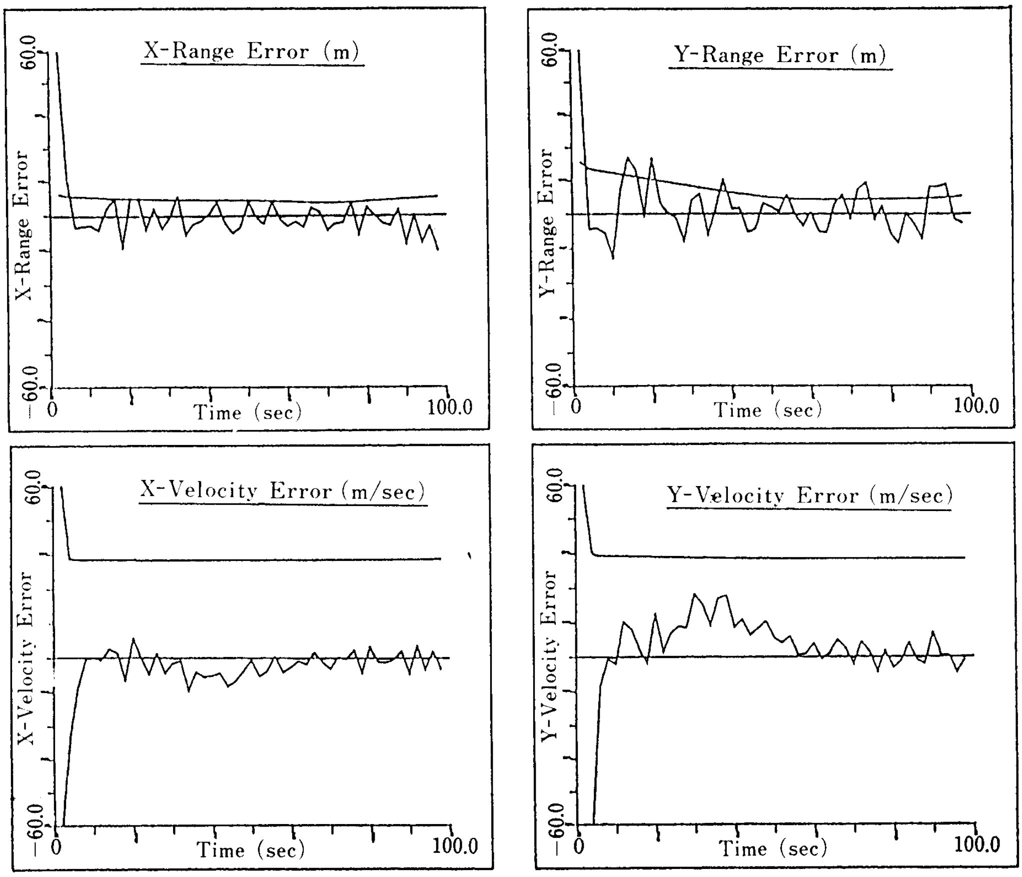

X-Range Error (m)
X-Range Error
60.0
−60.0
0
Time (sec)
100.0
Y-Range Error (m)
Y-Range Error
60.0
−60.0
0
Time (sec)
100.0
X-Velocity Error (m/sec)
X-Velocity Error
60.0
−60.0
0
Time (sec)
100.0
Y-Velocity Error (m/sec)
Y-Velocity Error
60.0
−60.0
0
Time (sec)
100.0

Fig. 8.

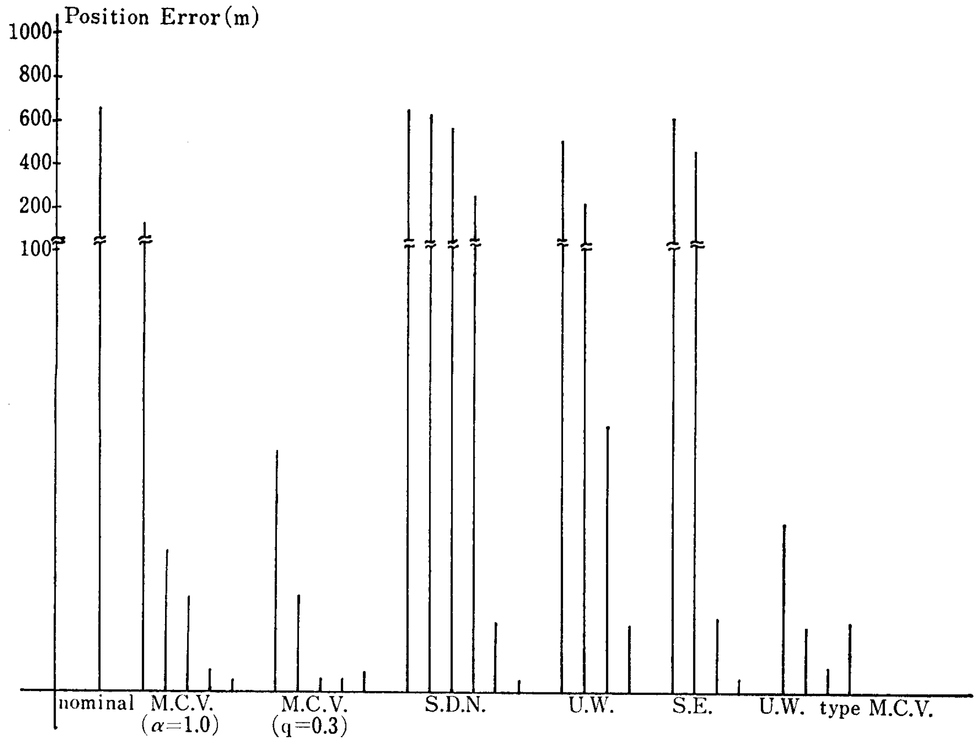


Fig. 59.

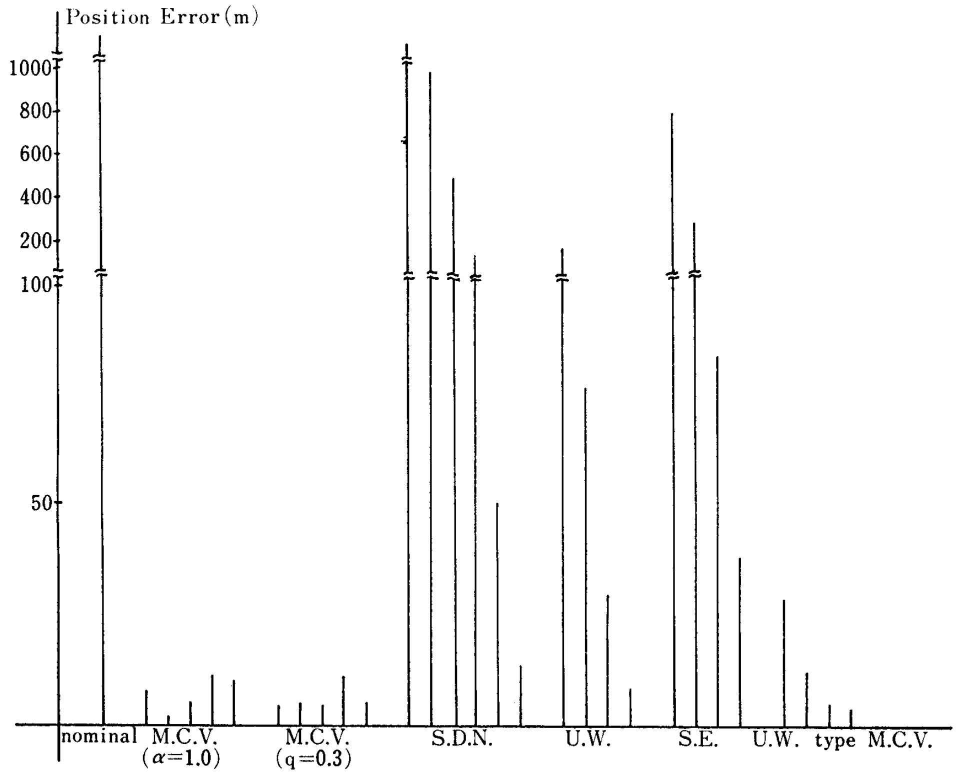


Fig. 60.

Fig. 51. Estimation by M.C.V. (q=0.3) (±60 m, m/sec scale).

Fig. 52. Estimation by M.C.V. (q=0.4) (±60 m, m/sec scale).

Fig. 53. Estimation by M.C.V. (q=0.5) (±60 m, m/sec scale).

Fig. 54. Estimation by U.W. type M.C.V. (q=0.01) (±60 m, m/sec scale).

Fig. 55. Estimation by U.W. type M.C.V. (q=0.1) (±60 m, m/sec scale).

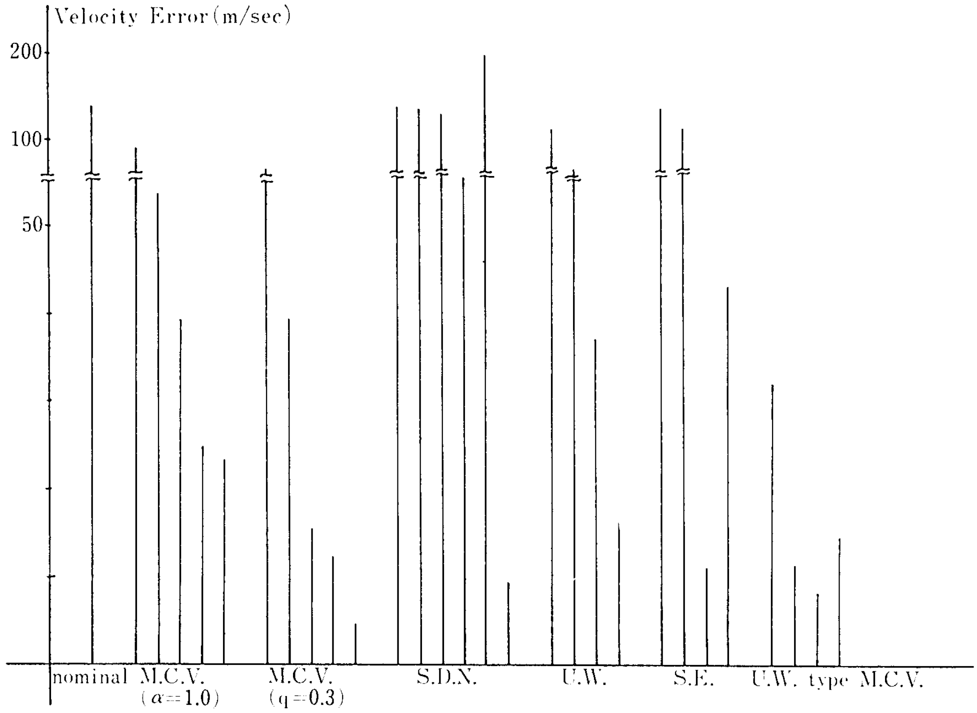


Fig. 61.

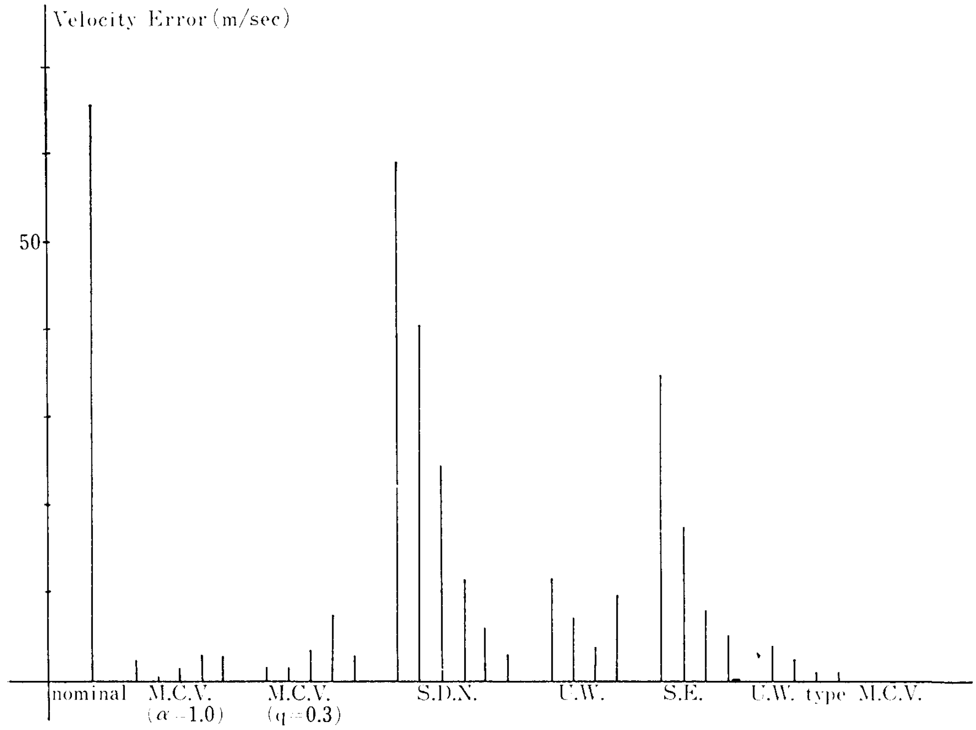


Fig. 62.

Fig. 56. Estimation by U.W. type M.C.V. ($q=1.0$) (±60 m, m/sec scale).
Fig. 57. Estimation by U.W. type M.C.V. ($q=5.0$) (±60 m, m/sec scale).
Fig. 58. Estimation by Apparent System Noise (±60 m, m/sec scale).
Fig. 59. Estimation by Robust Filters—Position Error at 40 (sec)
Fig. 60. Estimation by Robust Filters—Position Error at 100 (sec).
Fig. 61. Estimation by Robust Filters—Velocity Error at 40 (sec).
Fig. 62. Estimation by Robust Filters—Velocity Error at 100 (sec).

5-3. *The Attitude Control System of a Large Flexible Booster*

In this section we consider the applicability of the "Additive Term Designs" to the robust attitude control system of a large flexible booster. The system is highly complex and a large scale system and so we can insist that this is a typical example of practical application. The block diagram of this system is illustrated in Fig. 63. The body dynamics are derived by the structural transfer matrix method considering the aeroelastic interactions. (The detailed discussions as to this subject are omitted here.) The equations of motion of the body are written as follows:

$$\begin{aligned} &\dot{\gamma}+a\gamma=b\dot{\theta}+a\theta+\sum_j (c_j\dot{\xi}_j+d_j\xi_j)+k_\gamma T_c, \\ &\ddot{\theta}+e\dot{\theta}+f\theta=f\gamma+\sum_j (g_j\dot{\xi}_j+h_j\xi_j)+k_\theta T_c, \\ &\ddot{\xi}_i+2\zeta_i\omega_i\dot{\xi}_i+\omega_i^2\xi_i=p_i(\theta-\gamma)+q_i\dot{\theta}+\sum_j (r_{ij}\dot{\xi}_j+s_{ij}\xi_j)+k_{\xi i}T_c. \end{aligned} \tag{207}$$

In this example, bending vibrations are truncated into 5-th degrees, and so the body dynamics are modeled as 13-th order one. The actuator is a Thrust Vector Control device (T.V.C.) which is modeled as the second order system. The rate gyro is installed at the foot of the booster body which is an analogue type device and modeled as the fourth order system. The rate integration gyro is located at the second stage of the booster which is a discrete type device and modeled as third order system. Thus the overall system is modeled as 22-nd order one. The informations obtained directly are outputs of these sensors; two rate informations containing the bending vibrations and the measured attitude. The on-board controller is constructed in the digital computer which is of course a discrete system with the computation time-delay. The structure of this controller is assumed to be as follows:

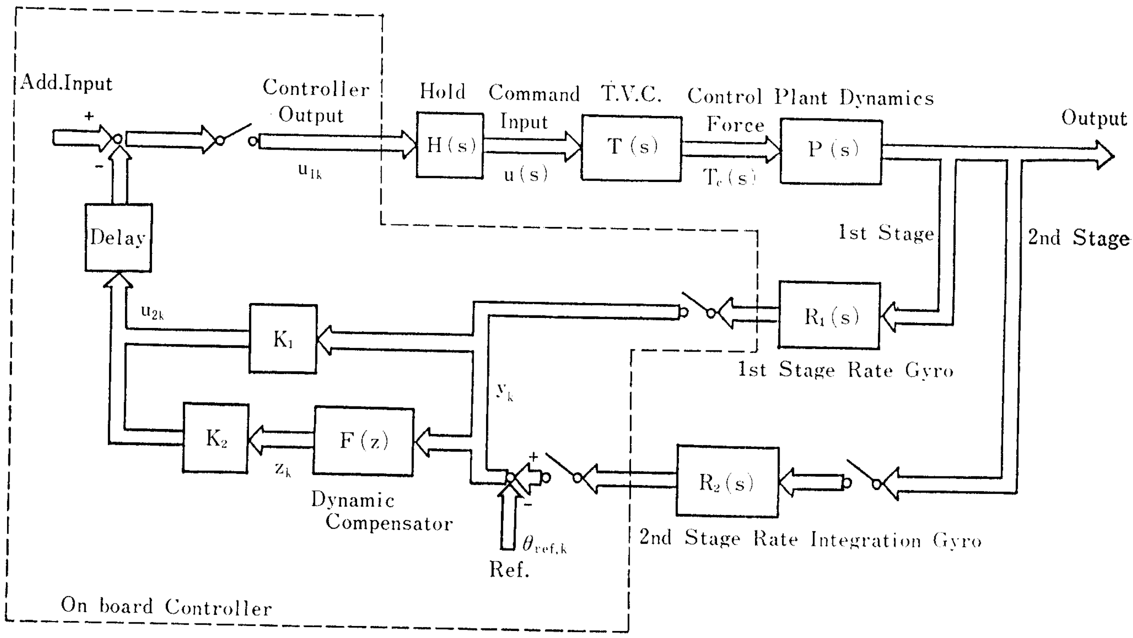


Fig. 63. Block Diagram of the Attitude Control System of a Flexible Booster.

$$\begin{aligned} &z_{k+1}=Fz_k+Gy_k && \text{(compensator),}\\ &u_{2k}=K_1y_k+K_2z_k, &&\\ &u_{1k+1}=u_{2k} && \text{(delay),}\\ &y_k^T=(\omega_{1k},\ \omega_{2k},\ \theta_k) && \text{(direct outputs).} \end{aligned} \tag{208}$$

The overall system is the order of $(23+p)$-th system considering the delay mechanism, where p denotes the order of the controller.

There are many uncertainties in this system as in many large scale systems. They are first of all structural uncertainties of the bending mode shapes or frequencies and the aerodynamic coefficients of each part of the booster which affect the system through the aeroelastic interactions, and second the characteristic frequency of the T.V.C. device which depends on gas consumption before and others. In this consideration, we confine the uncertainties to the structural one and that of T.V.C. device for simplicity. Namely the structural uncertainty considered here is that of bending mode shapes at the second stage gyro where the slopes of higher mode shapes are changing considerably. Here we assume this ambiguity is equivalent to the uncertainty of the second stage gyro location, which is empirically proved to be possible.

As well known, there's no unique and rational technique to construct that type of conrollers (208), which are much smaller than the open loop full system. For this purpose we adopt the model reduction technique and the Kalman filter type design approach. There have been reported many considerations as to the model reduction techniques. Aoki [*112*, *113*] devised the aggregation technique and Davison [*114*, *115*] and Rao [*116*] considered the method based on the representative pole locations, which are utilizing so to speak the singular perturbation technique. Here we used the similar method to Devison and Rao's, which is an intuitive and simple truncation technique as follows:

The system:

$\dot{x}=Ax+Bu,\ y=Cx$

can be transformed into

$\dot{\hat{x}}=\hat{A}\hat{x}+\hat{B}u,\ y=\hat{C}\hat{x},$

where

$\hat{A}=$block diagonal $\begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$ and some real diagonal elements,

$\lambda_{i,i+1}(A)=a_i\pm b_i j$; complex conjugate pair, eigen values of A then

the smaller system is reduced after truncation. (209)

Using this we prepared the simplified structure of the controller like the manners taken in usual discrete type Kalman filters as follows: (z_k in eq. (208) is exchanged with $\hat{x}_k$ here)

$$\begin{aligned} &\bar{x}_{k+1}=\tilde{F}\bar{x}_k+Bu_k+\tilde{G}y; && \text{propagation (as compensation),}\\ &\hat{x}_k=H\bar{x}_k+Ky_k; && \text{update (used for feedback).} \end{aligned} \tag{201}$$

Here we omit the covariance propagation equations, regarding the covariance as constant and assumed to be identical to the initial one which is evaluated qualitatively. Thus the matrices in the controller are modeled as

$$K=APA^TC^T\left(C\left(\frac{1}{sn^2}P+APA^T\right)C^T\right)^{-1};\ \text{Kalman gain},$$

$$\tilde{F}=A(I-KC),\quad \tilde{G}=AK,\quad H=(I-KC),$$

where sn denotes the adjustable parameter and P is a virtual covariance matrix fixed a priori, and A and C are the system and measurement matrices of the reduced smaller model. (211)

Through these discussions, we have preparations for the control problem. The objectives of this control system are to control the attitude to the reference one and to suppress the first bending vibration which is required to reduce the structual load during the transient state. Hence the performance index (cost) to be minimized is expressed as (concerning the states)

$$J=\sum_{k=0}^{\infty}[q_\theta\theta_k^2+q_\omega\omega_k^2+q_\xi\xi_{1,k}^2+q_{\dot\xi}\dot\xi_{1,k}^2+q_u\cdot u_{2k}^2]. \tag{212}$$

The weighting parameters are determined by the expected cost ratios of each state as

$$q_\theta=(1-r_\theta^2)/\theta_0^2,\quad r_\theta=\exp(-\Delta\tau/\tau_\theta)\ \text{etc},$$

where θ_0 is an assumed initial attitude error and $\Delta\tau$ is a sample interval, τ_θ is a desired damping time-constant. (213)

The readers should note that some design parameters are left undetermined in order to accopmlish the actual pole locations or other properties and to obtain the nominal system after the trial-and-error synthesis. Particularly the initial covariance matrix which is required for both the optimal output feedback design as weighting factors and the controller structure design, is calculated based on the evaluations of characteristic frequencies and the interactions, except for a few parameters to be adjusted.

Based on these preparations, we present the numerical evaluations in Table 39—

Table 39. Design Parameter(θ_0)—Attitude Control System of a Booster

Case	θ-pole	ξ_1-pole	θ-gain	Cost	Gain Margin	Phase Margin
$\theta_0=0.025$	−2.685 ±4.238 j	−2.533 ±25.157 j	118.1	6.932×10^2	1.2 dB	44 deg −3 deg
$\theta_0=0.050$	−2.721 ±3.738 j	−2.690 ±24.819 j	107.9	7.078×10^2	1.9 dB	45 deg −5 deg
$\theta_0=0.075$	−2.671 ±3.119 j	−2.883 ±24.368 j	93.9	7.266×10^2	2.7 dB	46 deg −8 deg
$\theta_0=0.100$	−2.460 ±2.571 j	−3.088 ±23.940 j	82.5	7.495×10^2	3.4 dB	47 deg −8 deg

Table 45 in order to select the nominal system without considerations of uncertainties. In these some factors are adjusted; initial attitude error (Table 39), admissible initial T.V.C. command level (Table 40), mixing ratio of the first bending state (Table 41), virtual factor in the controller (Table 42) and the order and type of the controller (Table 43, 44, 45). As this system is reduced to an output feedback one, the resulted system is not so satifactory one naturally. But here we select the case with 5-th degrees controller as nominal one and proceed to the robust design.

Table 40. Design Parameter (TVC_0)—Attitude Control System of a Booster

Case	θ-pole	ξ_1-pole	θ-gain	Cost	Gain Margin	Phase Margin
$TVC_0=1.0$	−1.305 ±1.400 j	−1.968 ±20.470 j	39.2	9.632×10^2	5.8 dB	53 deg −21 deg
$TVC_0=3.0$	−2.460 ±2.571 j	−3.088 ±23.940 j	82.5	7.495×10^2	3.4 dB	47 deg −8 deg
$TVC_0=5.0$	−2.726 ±3.469 j	−2.780 ±24.619 j	100.8	7.142×10^2	1.9 dB	45 deg −6 deg

Table 41. Design Parameter (K_1; Mixing Ratio of 1st Bending Mode)—Attitude Control System of a Booster

Case	θ-pole	ξ_1-pole	θ-gain	Cost	Gain Margin	Phase Margin
$K_1=0.0$	−0.910 ±1.013 j	−0.344 ±20.240 j	23.5	1.832×10^1	9.6 dB	57 deg
$K_1=0.5$	−2.070 ±2.095 j	−3.408 ±23.220 j	68.6	3.988×10^2	4.8 dB	48 deg −5 deg
$K_1=1.0$	−2.460 ±2.571 j	−3.088 ±23.940 j	82.5	7.495×10^2	3.4 dB	47 deg −8 deg
$K_1=2.0$	−2.710 ±3.246 j	−2.866 ±24.453 j	96.0	1.440×10^3	3.0 dB	45 deg −7 deg

Table 42. Design Parameter (SNR; Compensator Parameter)—Attitude Control System of a Booster

Case	θ-pole	ξ_1-pole	θ-gain	Cost	Gain Margin	Phase Margin
SNR=0.1	−1.694 ±3.064 j	−1.152 ±20.047 j	157.3	8.842×10^2	0.3 dB	46 deg −2 deg
SNR=1.0	−2.460 ±2.571 j	−3.088 ±23.940 j	82.5	7.495×10^2	3.4 dB	47 deg −8 deg
SNR=10.0	−1.775 ±2.317 j	−2.960 ±22.552 j	77.3	7.657×10^2	3.3 dB	46 deg −8 deg
SNR=100.0	−1.769 ±2.311 j	−2.945 ±22.531 j	77.2	7.664×10^2	3.3 dB	46 deg −8 deg

Table 43. Feedback Types—Attitude Control System of a Booster

Case	θ-pole	ξ_1-pole	θ-gain	Cost	Gain Margin	Phase Margin
State Feedback	−2.657 ±1.641 j	−6.418 ±20.433 j	20.6	4.293×10^2	7.1 dB	40 deg
Output Feedback	−0.439 ±1.398 j	−0.659 ±20.681 j	30.0	1.099×10^3	0.7 dB	35 deg −16 deg

Table 44. 2-input Kalman—Attitude Control System of a Booster

Case	θ-pole	ξ_1-pole	θ-gain	Cost	Gain Margin	Phase Margin
3rd	−1.625 ±2.701 j	−1.009 ±21.259 j	143.5	9.05×10^2	0.9 dB	46 deg −5 deg
5th	−2.460 ±2.571 j	−3.088 ±23.940 j	82.5	7.495×10^2	3.4 dB	47 deg −8 deg
7th	−2.080 ±2.178 j	−4.445 ±24.768 j	76.0	7.304×10^2	5.7 dB	44 deg −13 deg
9th	−1.949 ±2.057 j	−4.579 ±26.042 j	80.4	7.131×10^2	7.0 dB	47 deg

Table 45. 3-input Kalman—Attitude Control System of a Booster

Case	θ-pole	ξ_1-pole	θ-pole	Cost	Gain Margin	Phase Margin
3rd	−0.785 ±2.208 j	−0.667 ±21.321 j	53.5	9.455×10^2	1.0 dB	33 deg −4 deg
5th	−0.845 ±1.888 j	−0.716 ±21.071 j	34.9	9.457×10^2	1.8 dB	41 deg −5 deg
7th	−0.921 ±1.837 j	−0.819 ±21.131 j	48.6	9.272×10^2	0.8 dB	44 deg −4 deg
9th	−1.233 ±2.355 j	−1.289 ±21.762 j	95.7	8.300×10^2	3.6 dB	42deg −12dəg

The readers should note that this nominal system has the following properties:

$$\text{cost}=7.495\times10^2,\ \text{cost variation (at }60\%)=\infty,$$
$$\text{cost variation (at }100\%)=\infty,\ \text{gain margin }3.4\text{ dB},$$
$$\text{phase margin}=47\text{ deg or }-8\text{ deg.} \tag{214}$$

To speak solidly, we have specified the range of two unceıtainties as

$$\omega_s;\ (55.35\text{—}135.35)\ (1/\text{sec}),\quad \omega_{s0}=95.35\ (1/\text{sec}),$$
$$\Delta r_{G2};\ (-70.0\text{—}+70.0)\ (\text{cm}),\quad \Delta r_{G20}=0.0\ (\text{cm}). \tag{215}$$

Where ω_s denotes the characteristic frequency of T.V.C. device and Δr_{G2} indicates the location uncertainty of the second stage gyro, which reflects the ambiguities of higher mode shapes mentioned before. Here these uncertainties are linearized approximatedly and used for robust controller designs. In this case the largest design range in M.C.V. is adjusted as 3.1. In the other cases than M.C.V. this range has little meaning and should be considered as the scale of the additive term. In Fig. 64—Fig. 71, we show the numerical evaluations made here; projections and the contour lines of the cost surfaces. In Fig. 64, 65, the resulted cost surfaces of the nominal system are shown, and in Fig. 66, 67 those of the M.C.V. design, in Fig. 68, 69 those of the U.W. design, in Fig. 70, 71 those of the S.D.N. design are illustrated. From these, we can see that some "Additive Term Design" methods improve the cost surfaces drastically. And the quantitative evaluations are rearranged and shown in Table 46—Table 51 involving the other results than those demonstrated in Fig. 64—71. And schematic comparisons of these are summarized in Fig. 72. As noted in chapter 4, in these discrete systems, the U.W. type M.C.V. and S.E. methods are degenerated to M.C.V. or S.D.N. methods respectively. In view of Fig. 72, we easily find out that the M.C.V. method is slightly inferior to the other methods such as O.D, U.W. and S.D.N.'s against our anticipation. This is caused by the fact that while the uncertainty models in this system are approximated ones which is constructed and modeled equivalently through the corresponding continuous models, the evaluations presented here are based on the exact parameter variations as for (215). It is a very cynical nature that though we want to design insensitive controllers, the M.C.V. designs are sensitive to the ambiguities of the uncerainty modeling. And the fact that the design range in M.C.V. cannot be taken here arbitrarily for parameter α due to the difficulty of first step in eq. (195). If this is circumvented, the better results are produced than those here. But at any rate it may be insisted that we can obtain the insensitive systems through the "Additive Term Design" techniques involving M.C.V. thus. The representative properties of these systems are as follows:

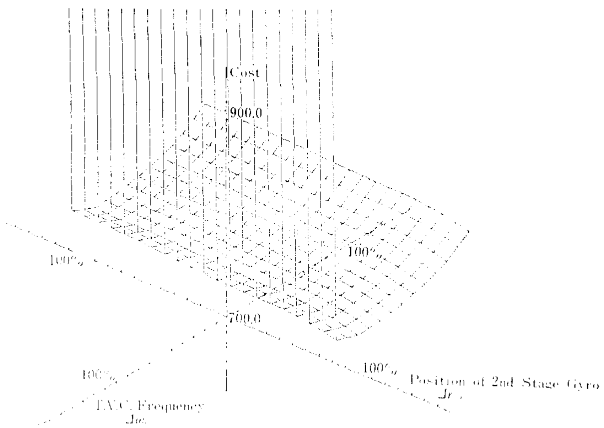


Fig. 64. Projection of the Nominal Cost Surface—Attitude Control System of a Booster

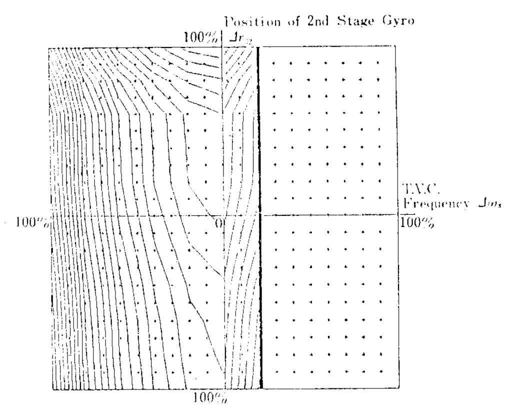


Fig. 65. Contour Lines of the NominalCost Surface—Attitude Control System of a Booster.

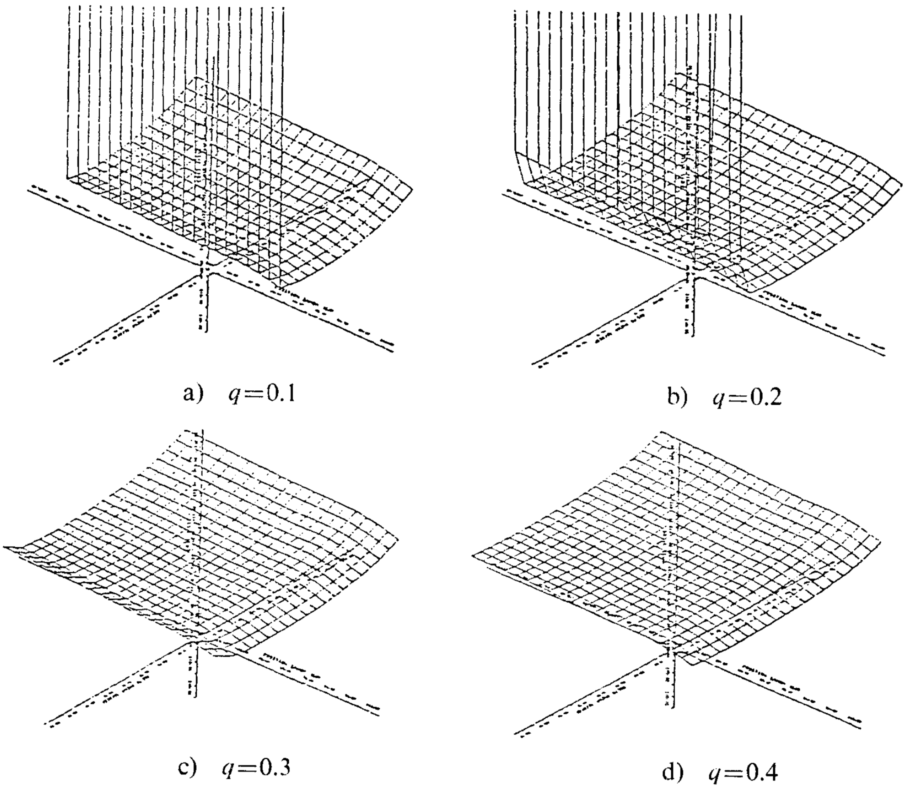


Fig. 66. Projection of Robustly Designed Cost Surface—Attitude Control System of a Booster (M.C.V. $\alpha=400^{-1}$).

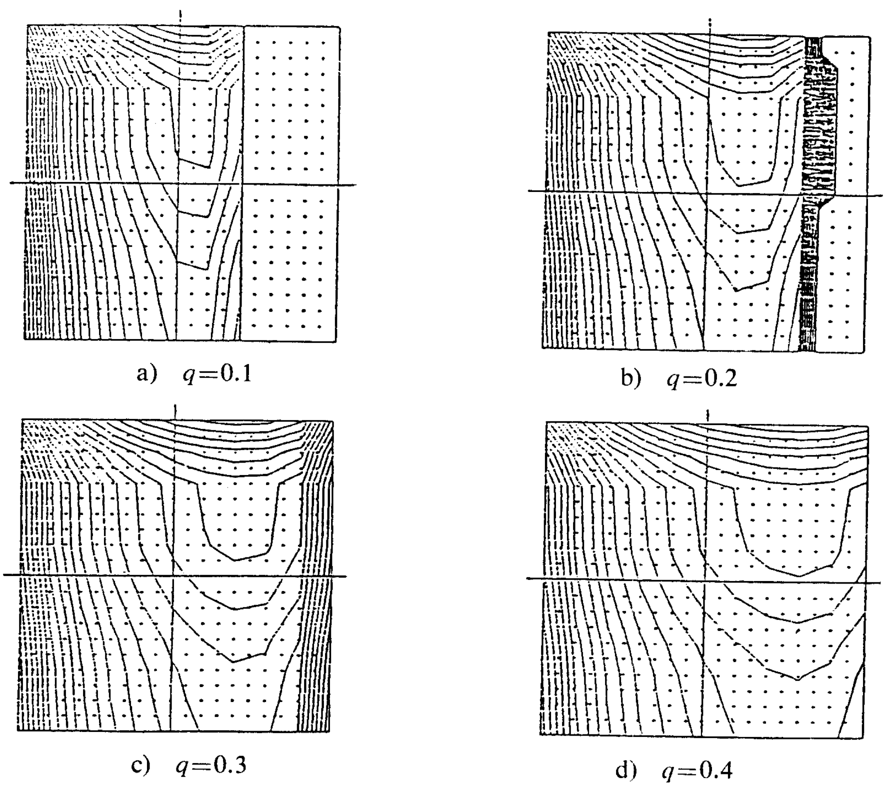
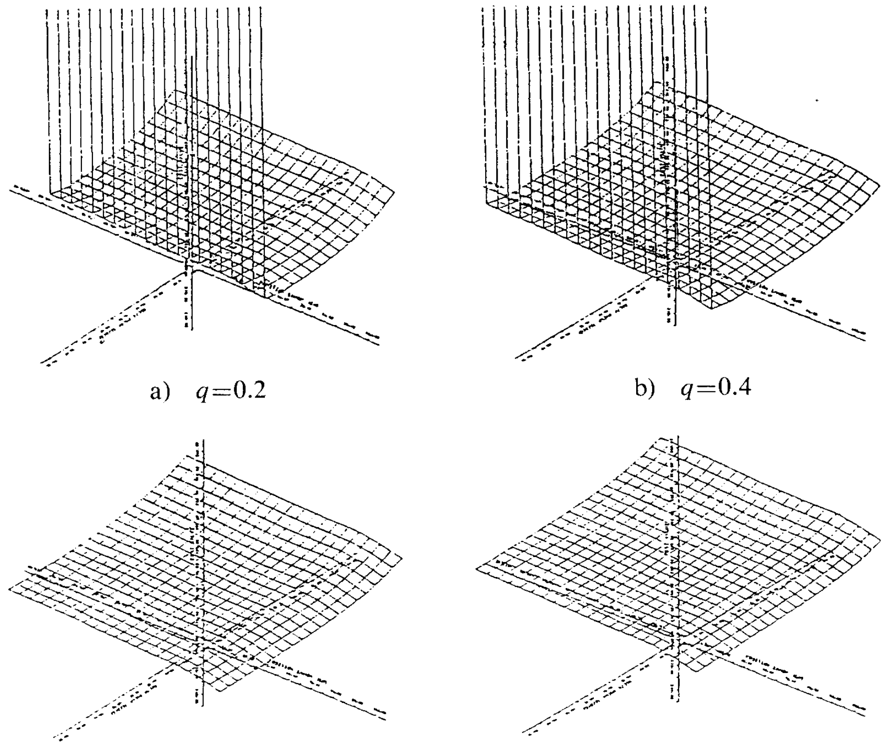


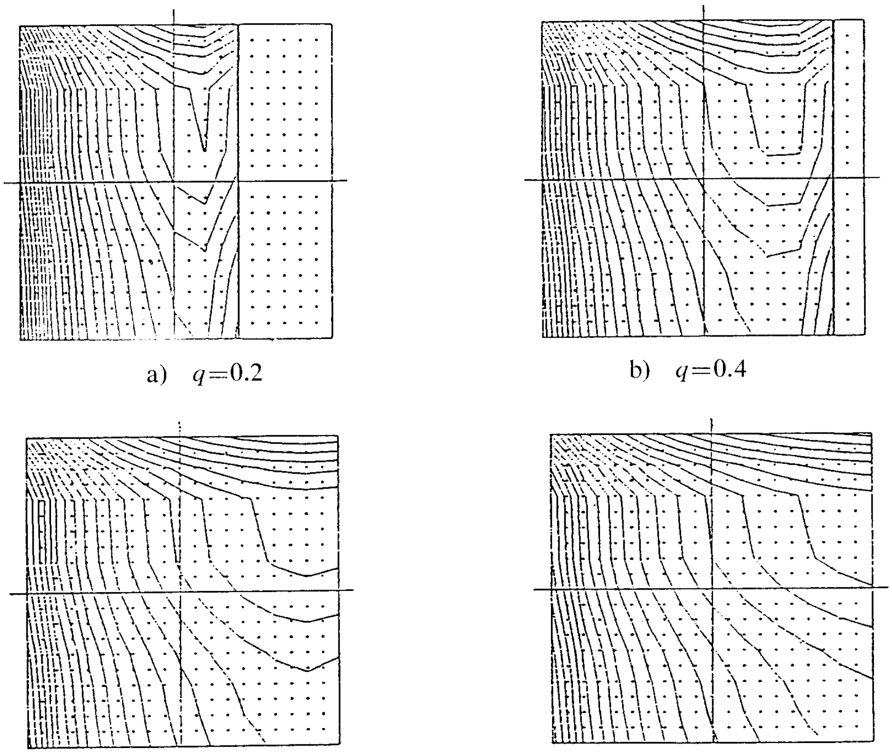
Fig. 67. Contour Lines of Robustly Designed Cost Surface—Attitude Control System of a Booster (M.C.V $\alpha=400^{-1}$).



a) $q=0.2$ b) $q=0.4$

c) $q=0.6$ d) 0.8

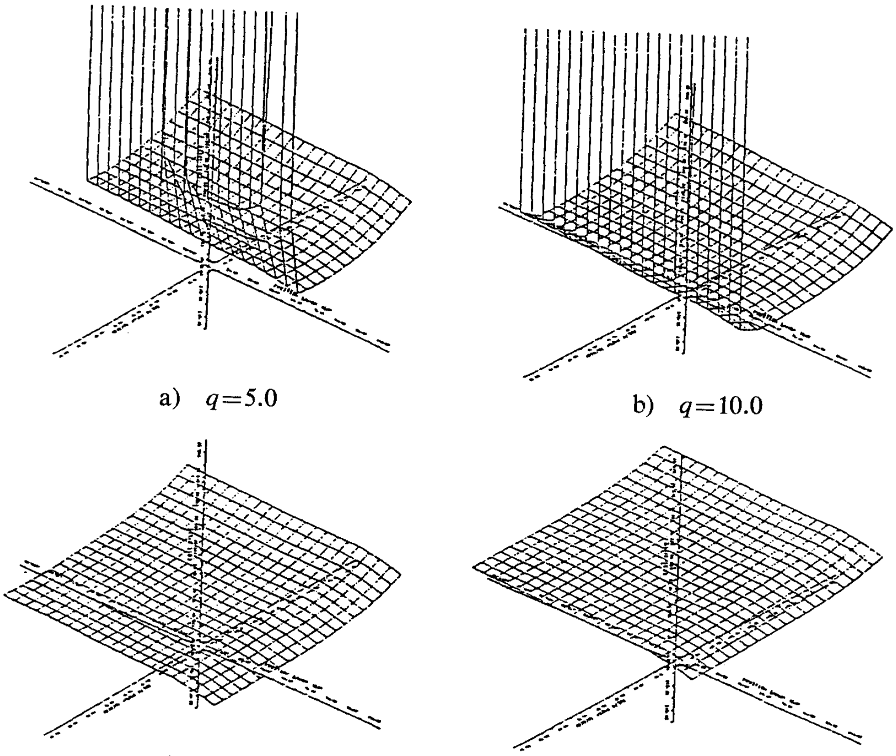
Fig. 68. Projection of Robustly Designed Cost Surface—Attitude Control System of a Booster (U.W.).



a) $q=0.2$ b) $q=0.4$

c) $q=0.6$ d) $q=0.8$

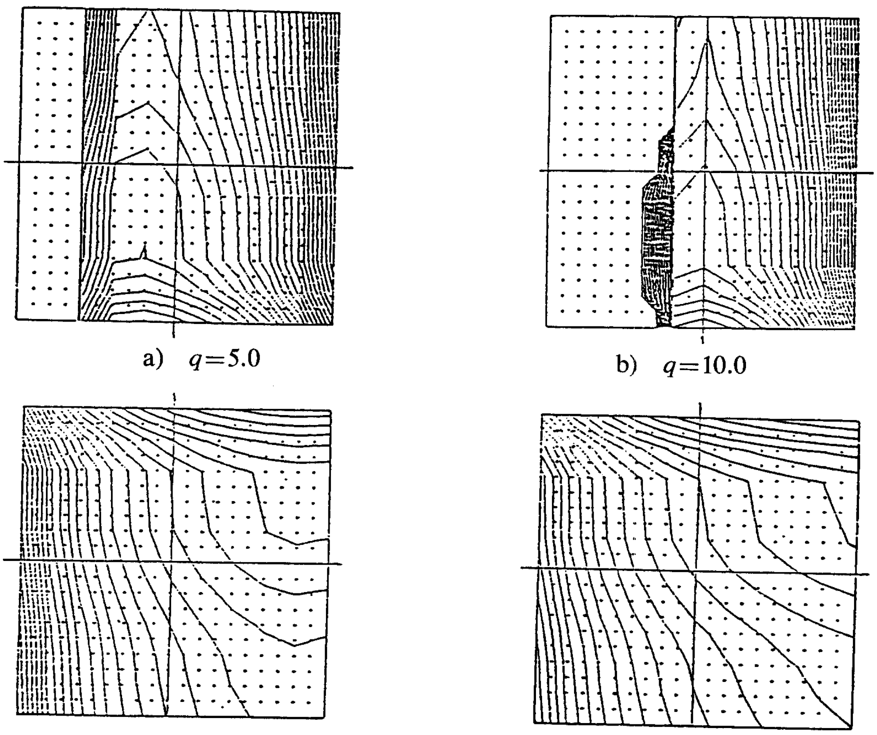
Fig. 69. Contour Lines of Robustly Designed Cost Surface—Attitude Control System of a Booster (U.W.).



a) $q=5.0$ b) $q=10.0$

c) $q=15.0$ d) $q=20.0$

Fig. 70. Projection of Robustly Designed Cost Surface—Attitude Control System of a Booster (S.D.N.).



a) $q=5.0$ b) $q=10.0$

c) $q=15.0$ d) $q=20.0$

Fig. 71. Contour Lines of Robustly Designed Cost Surface—Attitude Control System of a Booster (S.D.N.).

Table 46-a. Robust Design (M.C.V. $\alpha=200^{-1}$)—Attitude Control System of a Booster

Case	θ-pole	ξ_1-pole	θ-gain	Cost	Δ Cost (60%)	Δ Cost (100%)
$q=0.0$	-2.460 ± 2.571 j	-3.088 ± 23.490 j	82.5	7.495×10^2	∞	∞
$q=0.1$	-2.341 ± 1.889 j	-3.191 ± 23.866 j	30.5	7.614×10^2 (8.245×10^2)	∞	∞
$q=0.2$	-2.363 ± 1.597 j	-3.280 ± 23.828 j	19.0	7.712×10^2 (9.005×10^2)	24.38	∞
$q=0.3$	-2.385 ± 1.339 j	-3.373 ± 23.771 j	13.8	7.810×10^2 (9.909×10^2)	15.25	∞

Table 46-b. Classical Stability Margins—Attitude Control System of a Booster

	$q=0.0$	$q=0.1$	$q=0.2$	$q=0.3$
Gain Margin	3.4 dB	4.2 dB	5.1 dB	6.3 dB
Phase Margin	47 deg −8 deg	50 deg −12 deg	52 deg −16 deg	53 deg −18 deg

Table 47-a. Robust Design (M.C.V. $\alpha=300^{-1}$)—Attitude Control System of a Booster

Case	θ-pole	ξ_1-pole	θ-gain	Cost	Δ Cost (60%)	Δ Cost (100%)
$q=0.0$	-2.460 ± 2.571 j	-3.088 ± 23.940 j	82.5	7.495×10^2	∞	∞
$q=0.1$	-2.335 ± 1.745 j	-3.251 ± 23.858 j	24.5	7.651×10^2 (8.177×10^2)	∞	∞
$q=0.2$	-2.316 ± 1.354 j	-3.382 ± 23.756 j	15.0	7.769×10^2 (8.768×10^2)	14.95	∞
$q=0.3$	-2.293 ± 1.036 j	-3.522 ± 23.627 j	10.5	7.885×10^2 (9.406×10^2)	14.39	∞
$q=0.4$	-2.253 ± 0.747 j	-3.684 ± 23.483 j	7.8	8.006×10^2 (10.126×10^2)	13.72	33.44

Table 47-b. Classical Stability Margins—Attitude Control System of a Booster

	$q=0.0$	$q=0.1$	$q=0.2$	$q=0.3$	$q=0.4$
Gain Margin	3.4 dB	4.3 dB	5.4 dB	6.5 dB	7.2 dB
Phase Margin	47 deg −8 deg	51 deg −15 deg	53 deg −17 deg	55 deg −21 deg	56 deg −24 deg

Table 48-a. Robust Design (M.C.V. $\alpha=400^{-1}$)—Attitude Control System of a Booster

Case	θ-pole	ξ_1-pole	θ-gain	Cost	Δ Cost (60%)	Δ Cost (100%)
$q=0.0$	-2.460 ± 2.571 j	-3.008 ± 23.940 j	82.5	7.495×10^2	∞	∞
$q=0.1$	-2.315 ± 1.603 j	-3.300 ± 23.825 j	20.1	7.688×10^2 (8.183×10^2)	∞	∞
$q=0.2$	-2.270 ± 1.159 j	-3.475 ± 23.662 j	12.0	7.835×10^2 $(8.736\times10_z)$	14.55	∞
$q=0.3$	-2.202 ± 0.801 j	-3.670 ± 23.462 j	8.3	7.986×10^2 (9.314×10^2)	13.79	33.59
$q=0.4$	-2.107 ± 0.473 j	-3.923 ± 23.232 j	5.9	8.147×10^2 (9.951×10^2)	12.89	30.65

Table 48-b. Classical Stability Margins—Attitude Control System of a Booster

	$q=0.0$	$q=0.1$	$q=0.2$	$q=0.3$	$q=0.4$
Gain Margin	3.4 dB	4.4 dB	6.0 dB	7.3 dB	8.2 dB
Phase Margin	47 deg -8 deg	52 deg -14 deg	54 deg -18 deg	56 deg -24 deg	58 deg -27 deg

Table 49-a. Robust Design (U.W.)—Attitude Control System of a Booster

Case	θ-pole	ξ_1-pole	θ-gain	Cost	Δ Cost (60%)	Δ Cost (100%)
$q=0.0$	-2.460 ± 2.571 j	-3.088 ± 23.940 j	82.5	7.495×10^2	∞	8
$q=0.2$	-2.157 ± 2.438 j	-3.302 ± 23.416 j	77.3	7.519×10^2 (7.670×10^2)	∞	∞
$q=0.4$	-1.739 ± 2.264 j	-3.877 ± 22.303 j	68.6	7.635×10^2 (8.015×10^2)	15.1	∞
$q=0.6$	-1.463 ± 2.148 j	-5.114 ± 21.624 j	61.7	7.808×10^2 (8.402×10^2)	13.9	32.2
$q=0.8$	-1.272 ± 2.035 j	-5.868 ± 21.495 j	55.6	8.011×10^2 (8.793×10^2)	13.4	29.9
$q=1.0$	-1.136 ± 1.926 j	-6.354 ± 21.403 j	50.1	8.225×10^2 (9.179×10^2)	12.5	28.4

Table 49-b. Classical Stability Margins—Attitude Control System of a Booster

	$q=0.0$	$q=0.2$	$q=0.4$	$q=0.6$	$q=0.8$	$q=1.0$
Gain Margin	3.4 dB	4.7 dB	6.8 dB	8.4 dB	10.1 dB	11.2 dB
Phase Margin	47 deg -8 deg	46 deg -14 deg	45 deg -22 deg	44 deg -28 deg	43 deg -35 deg	43 deg -40 deg

Table 50-a. Robust Design (S.D.N.)—Attitude Control System of a Booster

Case	θ-pole	ξ_1-pole	θ-gain	Cost	Δ Cost (60%)	Δ Cost (100%)
$q=0.0$	−2.460 ±2.571 j	−3.088 ±23.940 j	82.5	7.495×10^2	∞	∞
$q=5.0$	−2.385 ±2.557 j	−3.134 ±23.839 j	80.6	7.497×10^2 (7.530×10^2)	∞	∞
$q=10.0$	−2.136 ±2.441 j	−3.305 ±23.488 j	75.1	7.527×10^2 (7.656×10^2)	18.39	∞
$q=15.0$	−1.688 ±2.202 j	−3.804 ±22.553 j	62.1	7.672×10^2 (7.984×10^2)	14.83	35.31
$q=20.0$	−1.117 ±1.956 j	−1.757 ±20.262 j	46.6	8.177×10^2 (8.847×10^2)	12.07	26.58

Table 50-b. Classical Stability Margins—Attitude Control System of a Booster

	$q=0.0$	$q=5.0$	$q=10.0$	$q=15.0$	$q=20.0$
Gain Margin	3.4 dB	3.6 dB	5.4 dB	8.9 dB	13.6 dB
Phase Margin	47 deg −8 deg	47 deg −10 deg	46 deg −17 deg	45 deg −29 deg	42 deg −48 deg

Table 51-a. Robust Design (O.D. Quasi Offset Design)—Attitude Control System of a Booster

Case	θ-pole	ξ_1-pole	θ-gain	Cost	Δ Cost (60%)	Δ Cost (100%)
0%	−2.460 ±2.571 j	−3.088 ±23.940 j	82.5	7.495×10^2	∞	∞
25%	−2.447 ±2.630 j	−2.917 ±23.879 j	84.0	7.511×10^2 (7.499×10^2)	72.13	∞
50%	−2.354 ±2.561 j	−2.821 ±23.755 j	82.8	7.538×10^2 (7.506×10^2)	21.09	∞
75%	−2.299 ±2.539 j	−2.716 ±23.672 j	82.9	7.569×10^2 (7.514×10^2)	22.60	59.89
100%	−2.237 ±2.516 j	−2.641 ±23.587 j	82.5	7.595×10^2 (7.522×10^2)	23.68	62.85

Table 51-b. Classical Stability Margins—Attitude Control System of a Booster

	0%	25%	50%	75%	100%
Gain Margin	3.4 dB	4.9 dB	6.5 dB	7.9 dB	8.9 dB
Phase Margin	47 deg −8 deg	47 deg −15 deg	47 deg −21 deg	47 deg −25 deg	47 deg −29 deg

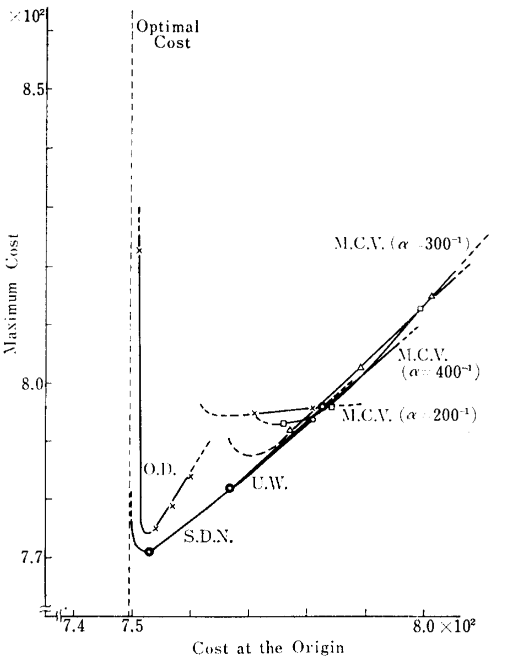


Fig. 72. Robust Controller Designs—Attitude Control System of a Booster (at 60% level).

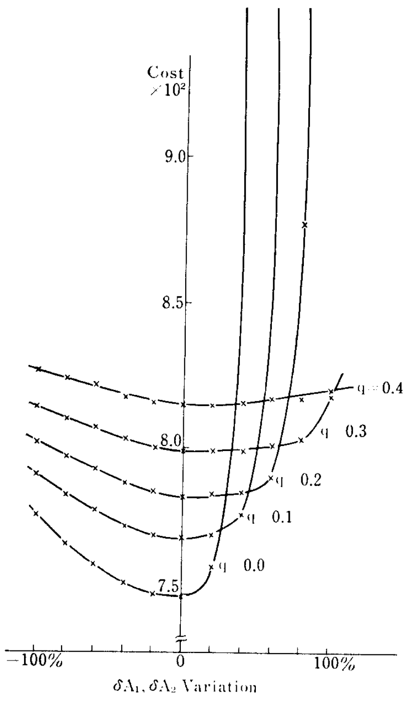


Fig. 73. Cost Surfaces in M.C.V. ($\alpha=400^{-1}$) Design

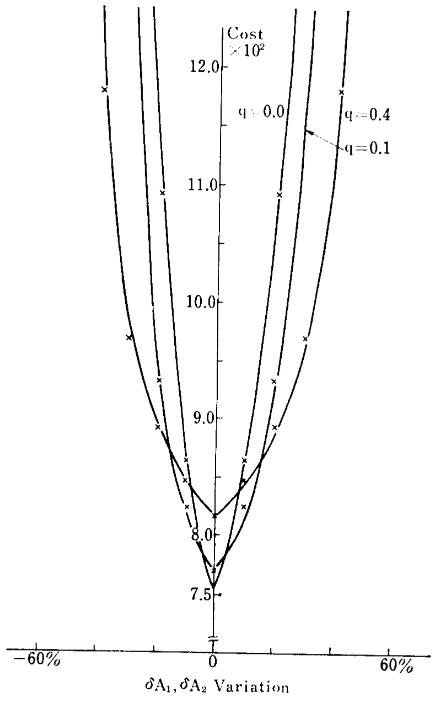


Fig. 74. S_2 Surfaces in M.C.V. ($\alpha=400^{-1}$) Design

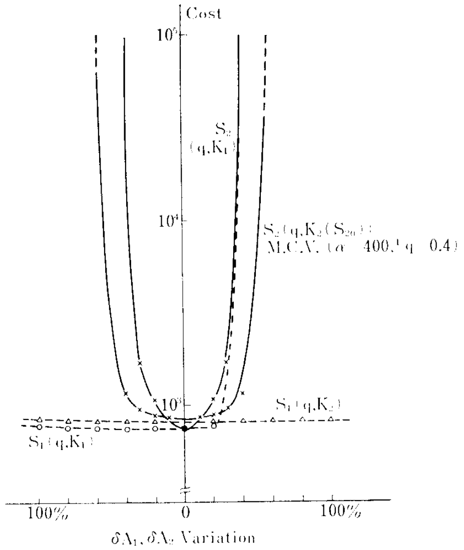


Fig. 75. S_1, S_2, Surfaces—Attitude Control System of a Booster.

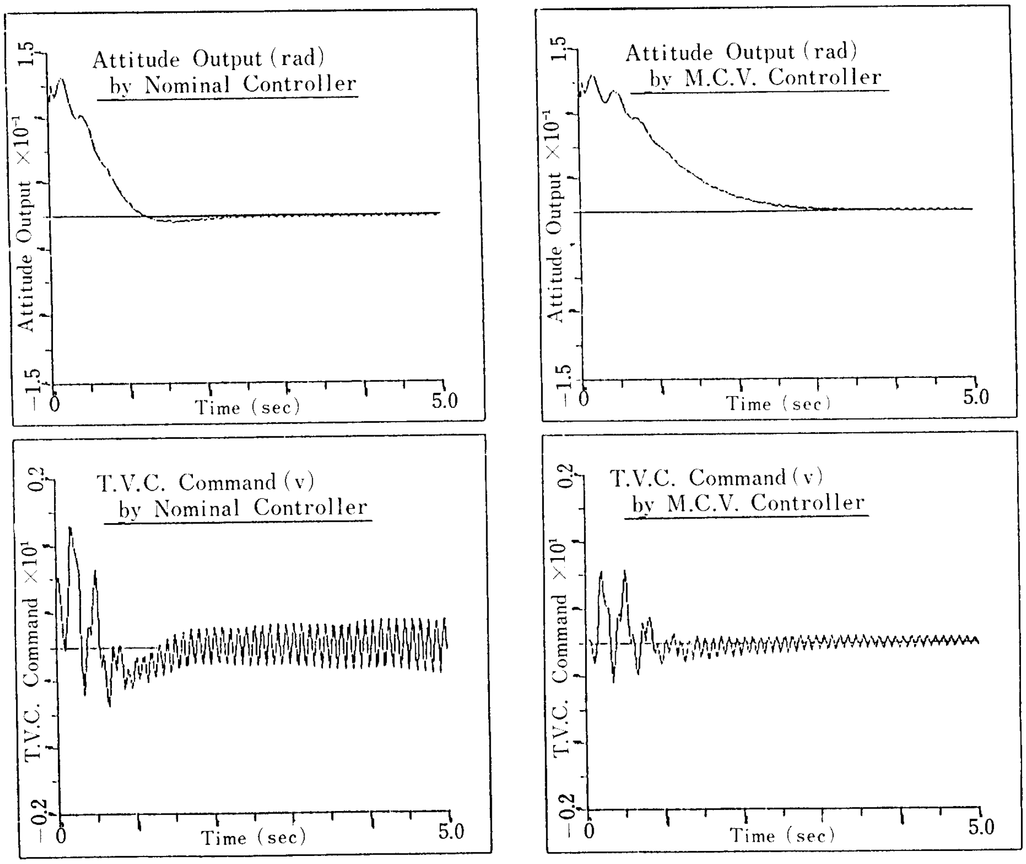


Fig. 76. Step Responses at the Origin.

$$\text{cost} = 8.147 \times 10^2, \text{ cost variation (at } 60\%) = 12.9,$$
$$\text{cost variation (at } 100\%) = 30.7,$$
$$\text{gain margin} = 8.2 \text{ dB, phase margin} = 58 \text{ deg or } -27 \text{ deg.} \tag{216}$$

From this we observe that cost and stability margins are fairly improved compared with (214).

We show the improvement behavior of the cost surfaces by successive utilization of M.C.V. in Fig. 73, 74 schematically. And the S_1, S_2 surfaces for these are presented in Fig. 75. These demonstarate the validity of M.C.V. methods also in this system in spite of ambiguous modeling of uncertainties. Because over the region where the $S_2(q, K_2)$ surface is below the $S_2(q, K_1)$ the true cost surface is really improved.

Next we present the statistical cost expectation as to these resulted systems in Table 52—Table 54, where as commented in the example of the autopilot, these expectations depend almost on the cost at the origin and meaningful differences are not clarified. Finally we demonstarate the step responses of these systems in Fig. 76 and Fig. 77. We should note that in the nominal system the instability occurs at the relatively higher frequency, which is essentially out of our concern and in fact the output to be controlled is not almost influenced. In our insensitive designs,

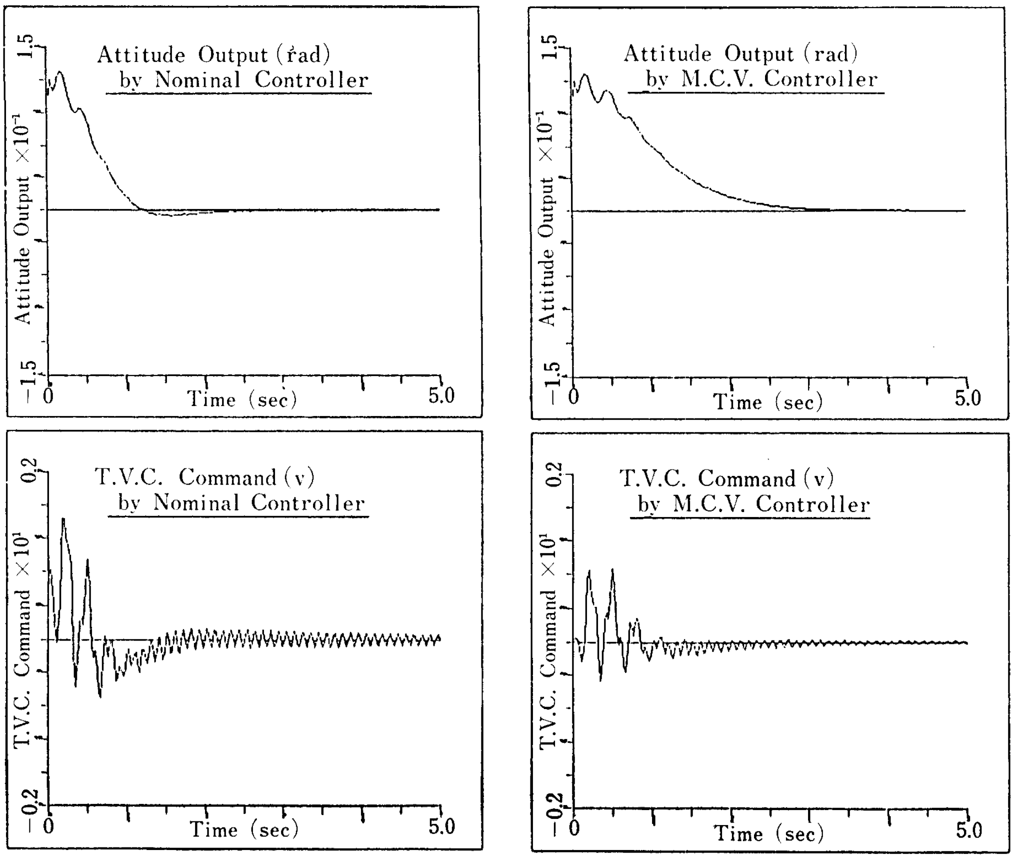


Fig. 77. Step Responses at 60% Variated Point

Table 52. Statistical Cost Expectation (60% level, Quasi Normal Distribution [1σ])—Attitude Control System of a Booster

Method	Case	Cost × 10^2	Method	Case	Cost × 10^2
MCV ($\alpha=400^{-1}$)	$q=0.2$	7.846	UW	$q=0.4$	7.644
	$q=0.3$	7.995		$q=0.6$	7.815
	$q=0.4$	8.155		$q=0.8$	8.018
SDN	$q=10.0$	7.542		$q=1.0$	8.232
	$q=15.0$	7.680	OD	25%	7.535
	$q=20.0$	8.183		50%	7.551
				75%	7.581
				100%	7.607

Table 53. Statistical Cost Expectation (60% level, Quasi Normal Distribution [2σ])—Attitude Control System of a Booster

Method	Case	Cost × 10^2	Method	Case	Cost × 10^2
MCV ($\alpha=400^{-1}$)	$q=0.2$	7.840	UW	$q=0.4$	7.640
	$q=0.3$	7.990		$q=0.6$	7.811
	$q=0.4$	8.151		$q=0.8$	8.015
SDN	$q=10.0$	7.534		$q=1.0$	8.228
	$q=15.0$	7.676	OD	25%	7.520
	$q=20.0$	8.180		50%	7.544
				75%	7.574
				100%	7.601

Table 54. Statistical Cost Expectation (60% level, Uniform Distribution)—Attitude Control System of a Booster

Method	Case	Cost	Method	Case	Cost
MCV ($\alpha=400^{-1}$)	$q=0.2$	7.868	UW	$q=0.4$	7.659
	$q=0.3$	8.009		$q=0.6$	7.827
	$q=0.4$	8.167		$q=0.8$	8.029
SDN	$q=10.0$	7.577		$q=1.0$	8.242
	$q=15.0$	7.693	OD	25%	7.614
	$q=20.0$	8.193		50%	7.573
				75%	7.601
				100%	7.627

though this critical state is not considered explicitly and not weighted, the influence of this state on the response is only a little. (Of course, the weighting of this state will improve the responses. But here we dare to avoid this approach for comparison, because such a state is not reguired to be damped well but desired not to grow up, and from the truncation problem of higher modes we had better suppress the loop gain over the range of higher frequencies.)

6. Concluding Remarks

In this paper, we show first in chapter 3 that the "Additive Term Design" methods can improve the cost surfaces and stability and stability margins and the sensitivity with respect to the equivalent open loop system. In the discussions on cost and stability improvement, we showed these mechanisms in detail and established the sufficient condition of cost imrpovement, for which the useful properties are introduced via some new cost surfaces. Particularly, the M.C.V. design techniques which introduce the monotonous feature into the G.C.C. method established by Peng, are revealed to guarantee almost the same amount of stability margins over the specified range of parameter variations as in the usual L.Q. regulators. And through the discussions there we devised a few of new "Additive Term Design" techniques such as the "Simple Robust Realization" and the "Statistical Cost Expectation" methods. While the main investigations are confined to the continuous systems where the state feedback control is possible, we consider the applicability of such techniques to the more practical systems; the systems that contain the control matrix uncertainty and the discrete systems and the dynamically compensated systems. And the computational algorithms which are also utilized for the other types of systems, are established for the general form of the M.C.V. type robust output feedback systems that are continuous or discrete. In numerical examples, we have the satisfactory results, though the designed systems have as simple structures as the nominal system, which do not require the additional internal states as in the adaptive systems. These illustrations are shown in the longitudinal autopilot system and the estimation problem of the radar tracking system and the attitude control problem of a large flexible booster. The first example; the autopilot system demonstrates the superiority of the M.C.V. method to the other "Additive Term Design" techniques, for the additional cost at the origin is only about several percents of the nominal cost in order to accomplish the satisfactory improvement of the cost surface and stability margins. And the last example; the attitude control system of a booster which is a very practical one and is a discrete type system containing the output feedback, reveals that the M.C.V. method is slightly inferior to the U.W. or O.D. or S.D.N. methods, for the ambiguous modeling of the uncertainty matrices but that these methods really improve the cost surfaces and the classical stability margins. From these numerical examples we can surely recognize that without making the system compelx it is possible to construct the robust systems by these "Additive Term Design" techniques particularly by the M.C.V. design with a slight amount of the additional cost at the origin.

There are many problems to be studied in future as to these designs; In the methods investigated in this paper, we must adjust not only the weighting factors in the performance index but the other factors for the additive terms. These are essentially arbitrary ones and we have no rational procedures to determine the forms of them. And we should note that these methods are not always accepted, for usually there are no possibility that the stability is assured over infinite range of parameter variations and these may require the much higher feedback gains that cannot be realized for the actual devices. In these cases, we must consult with the other quite new design techniques.

ACKNOWLEDGEMENT

The author would like to acknowledge the helpful guidance of prof. R. Akiba. And he wishes to express his gratitude to Dr. H. Matsuo for his sincere advices.

APPENDIX 1. The Equations of Attitude Motion of a Flexible Booster

Employing modal analyses and quantitizing the integration, we have the following ordinary differential equations of attitude motion as

$$
\begin{aligned}
&MV\dot{\gamma}=qS\sum_i \alpha_i C_{N\alpha i}+T_t(\theta-\gamma+\sum_j Y'_{jt}\xi_j)+T_c-Mg\cos\gamma,\\
&I\ddot{\theta}=qS\sum_i \alpha_i C_{N\alpha i}x_i+T_t(x_t\sum_j Y'_{jt}\xi_j-\sum_j Y_{jt}\xi_j)+T_c x_t,\\
&M(\ddot{\xi}_k+2\zeta_k\omega_k\dot{\xi}_k+\omega_k^2\xi_k)=qS\sum_i C_{N\alpha i}\alpha_i Y_{ki}+T_t Y_{kt}\sum_j Y'_{jt}\xi_j+T_c Y_{kt},\\
&\alpha_i=\theta-\gamma+\sum_j Y'_{jt}\xi_j-\frac{1}{V}(x_i\dot{\theta}+\sum_j Y_{ji}\dot{\xi}_j+V_w),\\
&\int\rho(x)Y_j(x)Y_k(x)dx=M\delta_{jk}.
\end{aligned}
\tag{A-1}
$$

And introducing some definitions, we have the following equations of motion in chapter 5:

$$
\begin{aligned}
&c_1=qS\sum_i C_{N\alpha i},\quad c_{2j}=qS\sum_i C_{N\alpha i}Y'_{ji},\quad c_3=qS\sum_i C_{N\alpha i}x_i,\\
&c_{4j}=qS\sum_i C_{N\alpha i}Y'_{ji},\quad c_{5j}=qS\sum_i C_{N\alpha i}Y_{ji}x_i,\quad c_5=qS\sum_i C_{N\alpha i}x_i^2,\\
&c_{7j}=qS\sum_i C_{N\alpha i}Y'_{ji}x_i,\quad c_{8jk}=qS\sum_i C_{N\alpha i}Y_{ji}Y'_{ki},\\
&c_{9jk}=qS\sum_i C_{N\alpha i}Y_{ji}Y_{ki},
\end{aligned}
\tag{A-2}
$$

$$
\begin{aligned}
&a=(c_1+T_t)/(MV),\quad b=-c_3/(MV^2),\quad \tilde{c}_j=-c_{4j}/(MV^2),\\
&d_j=(c_{2j}+T_tY'_{jt})/(MV),\quad k_\gamma=1/(MV),\quad e=c_6/(IV),\quad f=-c_3/I,\\
&g_j=-c_{5j}/(IV),\quad h_j=(c_{7j}+T_t(x_tY'_{jt}--Y_{jt}))/I,\quad k_\theta=x_t/I,\\
&p_k=c_{4k}/M,\quad q_k=-c_{5k}/(MV),\quad r_{kj}=-c_{9kj}/(MV)\\
&s_{kj}=(c_{8kj}+T_tY_{kt}Y'_{jt})/M,\quad k_{\xi k}=Y_{kt}/M,
\end{aligned}
\tag{A-3}
$$

$$\dot{\gamma}+a\gamma=b\dot{\theta}+a\theta+\sum_j(\tilde{c}_j\dot{\xi}_j+d_j\xi_j)+k_\gamma T_c,$$
$$\ddot{\theta}+e\dot{\theta}+f\theta=f\gamma+\sum_j(g_j\dot{\xi}_j+h_j\xi_j)+k_\theta T_c,$$
$$\ddot{\xi}_k+2\zeta_k\omega_k\dot{\xi}_k+\omega_k^2\xi_k=p_k(\theta-\gamma)+q_k\dot{\theta}+\sum_j(r_{kj}\dot{\xi}_j+s_{kj}\xi_j)+k_{\xi k}T_c. \quad (A\text{–}4)$$

APPENDIX 2. The Model Simplifying Technique

Here, we present the model simplifying technique which is used for the construction of the smaller compensator in chapter 5. The basic idea of this is the truncation principle which neglects the rapidly damped states after the decoupling of the states. We consider the following system:

$$\dot{\hat{x}}=\hat{A}\hat{x}+\hat{B}u, \quad y=\hat{C}\hat{x}.$$

If the eigen values of A-matrix are arranged in order of slowly damped states as

$$\dot{x}=Ax+Bu, \quad y=Cx, \quad (A\text{–}5)$$
$$\sigma(A)=(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n), \quad \text{where } \lambda_1, \lambda_2, \cdots, \lambda_i; \quad \text{real},$$
$$\lambda_j, \lambda_{j+1} \ (j=i+1, i+3, \cdots); \quad \text{complex conjugates } (a_j \pm b_j\sqrt{-1}). \quad (A\text{–}6)$$

where some real eigen values are assumed to be not rapidly damped. Thus these are rewritten and we denote the corresponding eigen vectors as

$$e_1, e_2, e_3, \cdots \text{ for } \lambda_1, \lambda_2, \lambda_3, \cdots \quad \text{respectively.} \quad (A\text{–}7)$$

then the following transformations are derived:

$$\tilde{Y}=[e_1, e_2, e, \cdots, e_n]^{-1},$$
$$Z=\left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & \\ 0 & -1 & 0 & 0 & \\ 0 & 0 & -1 & 0 & 0 \\ 0 & & \ddots & & \\ 0 & & & & \\ 0 & & & -0 & \\ \hline & & 0 & & \text{block diagonal} \begin{pmatrix} -j & j \\ 1 & 1 \end{pmatrix} \end{array}\right],$$
$$Y=Z\tilde{Y}, \quad j=\sqrt{-1},$$
$$\dot{\hat{x}}=\hat{A}\hat{x}\hat{B}u, \quad y=\hat{C}\hat{x},$$
$$\hat{x}=Yx, \qquad \hat{B}=YB, \quad \hat{C}=CY^{-1},$$
$$\hat{A}=YAY^{-1}=\text{block diagonal} \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}. \quad (A\text{–}8)$$

Therefore we have the approximated smaller systems through neglecting higher order systems:

$$\dot{\tilde{x}}=\tilde{A}\tilde{x}+\tilde{B}u, \quad y=\tilde{C}\tilde{x}. \quad (A\text{–}9)$$

APPENDIX 3. Nomenclature

a; certain coefficients
b; certain coefficients
c; closed loop suffix or chord length
d; dimensions of some systems
e; error vector or elevator deflection angle
f; some vectors or matrix functions
g; gravity constant or a certain coefficient
h; additive terms
i; indices
j; imaginary unit
k; indices in discrete systems
l; some functions
m; mass
n; nominal suffix or dimension
o; open loop suffix
p; parameter vectors
q; parameter vectors or their norms or dynamic pressure
r; some factors
s; Laplace variables or some state vectors
t; time
u; control input vectors or velocity variation
v; variated input vectors
x; state vectors
y; output vectors
z; some state vectors

A; system matrices
B; control matrices
C; observation matrices
D; some matrices
E; expectation operator
F; compensator dynamics or matrices
G; certain systems or matrices or loop transfer functions
H; Hamiltonian
I; unit matrix or moment of inertia
J; performance indices (costs)
K; gain matrices or certain systems
L; certain systems
M; some positive matrices or mass
N; normal distribution
P; covariance matrices or plant dynamics or some matrices
Q; state weighting matrices

R; control weighting matrices
S; cost matrices or sensitivity matrices or area
T; some transformation matrices or transpose operator
U; cruising velocity
V; some positive matrices or Lyapunov function
X; some systems
Y; Laplace transformations of outputs
Z; some positive matrices

α; some adjustable parameters or angle of attack
β; some adjustable parameters
γ; flight path angle or some matrices
δ; variation
ζ; damping constant
η; some noises
θ; attitude angle
λ; eigen values or random noises
μ; some factors
ξ; exogeneous states or bending states or noises
ρ; air density
δ; spectrum radius or standard deviations
τ; representative time
ϕ; phase angle
ω; frequency
Λ; diagonal matrices

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