

Solid Propellant Burning Rate Response Functions of Higher Orders

By

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ABSTRACT: To study nonlinear behavior phenomena in a solid propellant rocket motor one should consider both nonlinear acoustic field and nonlinear interaction between acoustic field and combustion process. The former topic has been discussed in literature rather fully. The latter is not a new one but is not studied in detail. In the work presented a new concept of the higher order burning rate response function to acoustic oscillation with several modes is introduced. Analytical calculations are performed in the framework of Z-N theory. Short introduction in this theory is given. The physical-chemical basis of a very high oscillatory quality of condensed system combustion are discussed. In connection with wide use of highly metallized propellants the linear response function of such propellants is calculated. Two most important examples of the first harmonic self-interaction and the first and second modes interaction are obtained analytically. A relative role of the linear and the higher order response functions is discussed. It is shown that nonlinear effects in combustion must be included in solving solid rocket nonsteady behavior problem.

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NOMENCLATURE

c	specific heat
E	activation energy
f	temperature gradient
i	imaginary unit
k	dimensionless parameter defining the dependence of the steady-state burning rate on the initial temperature
m	mass burning rate
p	pressure
R	universal gas constant
r	dimensionless parameter defining the surface temperature dependence in the steady-state regime on the initial temperature
T	temperature
t	time
u	linear burning rate
x	space coordinate
η	dimensionless pressure
ϑ	dimensionless surface temperature
θ	dimensionless temperature
κ	thermal diffusivity
λ	dimensionless damping decrement
μ	dimensionless parameter defining the dependencies of surface temperature on pressure in the steady-state regime
ν	dimensionless parameter defining the dependence of steady-state burning rate on pressure
ξ	dimensionless space coordinate
τ	dimensionless time
φ	dimensionless temperature gradient
ω	dimensionless frequency
ϱ	density
Superscripts	
$(\bar{})$	complex conjugation
0	steady state
$*$	boundary of stability
Subscripts	
a	initial temperature
c	condensed phase
g	gas phase
i	initial condition
s	surface
φ	referred to temperature gradient
η	referred to pressure gradient
1	first harmonic
2	second harmonic

1. INTRODUCTION

In solid-propellant motors such conditions during which pressure oscillations arise are frequently realized. The problem of stability of steady-state burning in a combustion chamber was posed as far back as the early 40's. Since then it has attracted many workers in this field, and there exist several extensive reviews on this subject covering experimentally observed combustion instabilities and theoretical analyses of the problem (see, for example, Refs. 1, 2). Theoretical calculations have been performed largely with linear behavior, the main purpose being to provide results only for the initial decay or growth of small perturbations. To find the amplitude of a self-excited oscillation and to analyze the possibility of triggering an oscillation by finite size disturbances, one should go out of the framework of the linear analysis. Some recent results for nonlinear acoustics in combustion chambers are presented in Ref 3.

Most analytical nonlinear calculations performed relate the nonlinearity, as a rule, to gasdynamic behavior since a propellant is assumed to be linear, although there are some papers^{4,5)} considering nonlinear combustion modeling in numerical analyses.

It is evident, nevertheless, that combustion instability is highly dependent on propellant characteristics. To produce the nonlinear phenomena observed in experiments such as limiting amplitudes, triggering, and a mean pressure shift, the nonlinear burning rate response has to be considered. A propellant itself is a nonlinear oscillatory system with a definite natural frequency and damping decrement.

The paper discusses nonlinear oscillatory phenomena in propellant combustion which have a distinctive features as compared to similar phenomena in systems with a finite number of freedom degrees.

The lack of a consistent theory of steady-state propellant burning describing facts experimentally observed makes it difficult to develop theories for nonsteady phenomena. However, a phenomenological theory of sufficiently slow nonsteady processes can be worked out, even without a detailed pictures of steady-state processes. In many cases, we can disregard the relaxation time in the regions of chemical reactions, as well as in the regions of combustion products, compared to the time of thermal relaxation of the heated condensed layer. The quick-response regions in this approximation are considered to be quasistationary. They react quickly to changes in external conditions and a temperature gradient in the condensed substance at the interface.

Thus, to calculate a nonsteady process, it is necessary to consider rather slow variations of the temperature profile in the propellant. The theory contains the only value of time scale that determines the nonsteady processes in a propellant, i.e., thermal layer relaxation time t_c . The basic assumptions of the t_c -approximation are formulated in the second section of the paper.

From those assumptions, it follows that an instantaneous state of a quick-response region is the function of an instantaneous value of temperature gradient in the condensed phase at the interface and the instantaneous value of pressure (or some other external parameter). Therefore, we can express any quantity as functions of the instantaneous pressure and condensed-phase temperature gradient at the interface. Such functions can be obtained from the steady-state burning characteristics of a given propellant. Such an approach is known as Z-N (Zeldovich-Novozhilov) theory.⁶⁻⁹⁾ Western scientists prefer to use detailed models of burning (FM-method). Sufficiently exhaustive reviews of the papers written up within the framework of the approaches mentioned can be found in papers.¹⁰⁻¹³⁾

The third part of this paper deals with the conditions for steady-state combustion stability at constant pressure. The stability condition for a steady-state regime is obtained through linear approximation. It includes only two parameters, i.e., the derivatives of steady-state burning rate and surface temperature with respect to initial propellant temperature. The study of the perturbation asymptotic

over a long period of time makes it possible to introduce an important property of the propellant (i.e., the natural frequency of burning-rate oscillations or other time-dependent parameters).

It is shown that burning solid propellant may be considered as an oscillatory system with a very high quality. While applying oscillatory pressure to a burning solid propellant we obtain forced oscillations that can be described by a complex response function. This question is discussed in the next two sections of the paper. Section 4 gives a very well-known expression for the response function of non-metallized propellant.

Because in the ISAS' M-V program the newly developed highly aluminized BP-205J propellant is to be used, we derive in Section 5 the expression for the response function of highly metallized propellants. In spite of small value of internal radiant flux that response function differs from the usual response function very strongly.

A new concept of solid propellant burning rate response function of the higher order is introduced in Section 6. Two examples of those functions are considered in the next two sections. The first concerns nonlinear interaction of the two first harmonics of pressure and burning rate. The second is devoted to calculation of nonlinear self-interaction of the first mode.

In conclusion we underline the necessity of including nonlinear interaction between acoustic field and combustion processes in the problem of nonlinear motion in solid-propellant rocket motor.

2. Z-N THEORY

Combustion always involves a number of chemical reactions; in most cases, the burning rate depends on chemical kinetics. Therefore, practically every combustion theory essentially incorporates the kinetic characteristics of the reactions. With few exceptions, combustion kinetics are not sufficiently understood at present. Little is known, for example, about the kinetics of reactions involved in the combustion of condensed substances, hence the necessity of introducing certain reaction models into theoretical calculations that only slightly resemble the real chemical processes. It is common practice to adopt the Arrhenius dependence of the reaction rate on the temperature and the power dependence of the reaction rate on the reactant concentrations.

It is obvious that such investigations are only qualitative and hardly suitable for comparison with experimental data. For instance, the steady-state burning theory for condensed substances has been developed exclusively for the simplest types of chemical reactions. Although this type of analysis can supply a qualitative explanation of the dependence of the burning rate (say, on the pressure or the initial temperature of the propellant), it is practically impossible to compare its results with experimental results simply because it is associated with a very idealized reaction model. Real physicochemical processes are much more complicated than such theoretical ones. Moreover, it appears impossible to develop a quantitative steady-state burning theory that would hold good for a broad class of widely differing propellants.

At first glance, a nonsteady burning theory claiming quantitative agreement with experimental results should be more complicated than a steady-state theory. This is true when we deal with a theory incorporating real kinetics of chemical reactions. However, it is possible to deduce a rather good approximation, a phenomenological nonsteady theory in which the kinetics of chemical reactions and all of the complex physical processes involved in combustion would be automatically included by the introduction of data obtained from steady-state experiments.

The fundamental concept of the Z-N theory was put forward by Zeldovich in 1942. Reference 6 shows that, if we ignore the gas-phase inertia (relaxation time) compared with the thermal-layer inertia of a propellant, then the quasisteady assumption for the gas phase is applicable for processes slower than the relaxation of a gas phase. Thus, the problem can be reduced to consideration of a comparatively slow variation of the temperature profile in the propellant.

A great advantage of this theory is that it permits consideration of nonsteady burning without

involving a steady-state theory. The theory includes only the dependence of mass burning rate m on pressure p and the temperature gradient of the condensed phase at the interface f_{cs} , i.e., $m = m(f_{cs}, p)$. This dependence can be introduced into the theory through a known relationship between initial temperature T_a , pressure, and mass burning rate $m^0 = m^0(T_a, p)$ (the zero superscript corresponds to the stationary value). It should be noted that, in Ref. 6, a rather oversimplified model of burning was used. It was assumed that the interface temperature T_s was independent of the external conditions (pressure and initial temperature). When analyzing nonsteady processes, if we assume the surface temperature T_s to be constant, this would lead to a discrepancy between theoretical and experimental results; there seems to be no real system by which this condition can be fulfilled.

The author of Refs 7–9 succeeded in generalizing the preceding Zeldovich approach for more realistic situations and formulated the nonsteady burning theory with allowance for surface-temperature variations. Under unsteady operations, the surface temperature as well as the burning rate is determined by the instantaneous values of pressure and temperature gradient at the surface of the propellant side, i.e., $m(f_{cs}, p)$ and $T_s(f_{cs}, p)$. It will be shown later that these nonsteady dependencies can be deduced from steady-state relationships $m^0(T_a, p)$ and $T_s^0(T_a, p)$, obtained experimentally or theoretically from consideration of any particular combustion model.

The basic assumptions of the Z-N theory are explained below.

1) Numerous experimental data indicate that, during propellant burning, the interface remains plane (for a sufficiently large sample diameter). In the text that follows, a space coordinate system is used to move the unreacted propellant in the positive direction of the x axis with a velocity that coincides with the linear regression velocity $u(t)$. The interface surface therefore remains fixed at any combustion regime.

During combustion, chemical processes accompanied by heat transfer, reactant diffusion, and gas motion take place in the condensed phase and gas region near the interface. Accordingly, the entire space can be divided into three regions (Fig. 2.1a).

C region ($-\infty < x < x_c$): The region in which condensed-phase heating occurs. There is no chemical transformation whatsoever.

S region ($x_c < x < x_g$): Here, the condensed phase is transformed into intermediate gaseous products. The coordinate x_g corresponds to the interface.

G region ($x_g < x < \infty$): Here, as a result of gas-phase reactions, the intermediate products are transformed into the end products of burning. This transformation is accompanied by heat transfer, mass diffusion, and gas motion.

Thus, a one-dimensional problem is considered (all values depend on only one spatial variable, x). We should assume that the propellant is homogeneous and isotropic and that the boundaries between the zones are planes. These requirements are necessarily fulfilled for homogeneous compositions. For the heterogeneous propellant, such an approach is valid when the size of oxidizer and fuel particles is much less than the characteristic size of the thermal layer following the steady-state theory, i.e., the propellant thermal layer κ/u^0 , where κ is the propellant thermal diffusivity.

2) In Z-N theory, the relaxation times of S and G regions (respectively, t_s and t_g) are taken to be zero. In other words, these regions are considered to respond quickly to changing external conditions. Experimental investigations of the burning zones in propellants^{14, 15} indicate a good fulfillment of the inequalities $t_s \ll t_c$ and $t_g \ll t_c$. This result can also be obtained from simple estimates of the processes occurring in these zones. The first inequality results from the fact that the chemical transformation zone of the condensed phase is narrow. The second one is connected with a small ratio of gas and propellant densities (for details, see Ref. 12).

3) In order to derive basic relationships of nonsteady burning in the Z-N theory, it is necessary to consider the S region as infinitely thin. If it is assumed that both x_c and x_g tend to zero (Fig. 2.1b), instead of as S region, we obtain an interface S plane ($x=0$), the temperatures $T(x_c)$ and $T(x_g)$ being coincident. Henceforth, the temperature of the S plane will be called the “surface temperature,” and

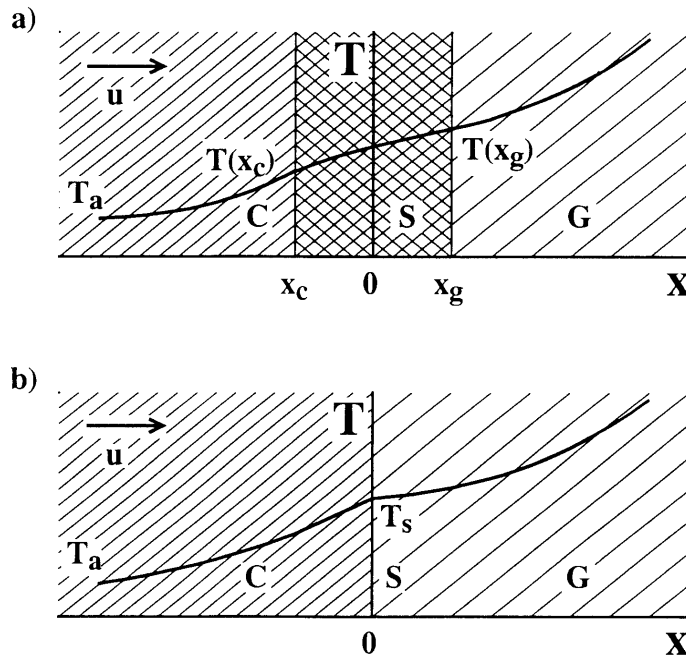


Fig. 2.1 Combustion wave structure of a homogeneous propellant.

a) C region $(-\infty < x < x_c)$, S region $(x_c < x < x_g)$, and G region $(x_g < x < \infty)$.

b) In the t_c -approximation, the S region is replaced by the S plane $(x_c = x_g = 0)$. Here $T(x_c) = T(x_g) = T_s$.

will be denoted as T_s . The approximation of an infinitely thin S region does not permit us to consider in detail physicochemical processes occurring in this region. As a result, it is necessary to provide the S plane with some definite properties. The properties of the S surface are specified when the theory is supplemented with the steady-state burning characteristics.

4) In Z-N theory, we can consider only rather slow changes of an external parameter (pressure). If the characteristic time of the changing pressure is t_p , the inequalities $t_p \gg t_s$ and $t_p \gg t_g$ should be fulfilled. In other words, S and G regions should adjust themselves without delay to a changing external parameter.

5) In order to build Z-N theory, it is necessary to know the steady-state dependencies of combustion rate and surface temperature on initial temperature and pressure $m^0(T_a, p)$ and $T_s^0(T_a, p)$. In some case, further information about the steady-state regime is also needed; for example, the dependence of the temperature of combustion products on the same parameters $T_b^0(T_a, p)$.

6) The preceding major assumptions are essential for the given approximation. In order to simplify the analysis, minor assumptions are introduced. For example, the model does not consider thermal losses, energy transfer by radiation, or the influence of external forces. Moreover, it is assumed that the density of the condensed phase, its specific heat, and the coefficient of thermal conductivity are temperature-independent.

Under steady-state conditions, the propellant burning rate, surface temperature, and any other properties depend on initial temperature and pressure. Thus, the variables T_a and p are suitable for studying steady-state burning, and we can deliberately change them to examine the dependence by systematic variation. The dependence of steady-state burning properties on these parameters, e.g., $m^0(T_a, p)$ and $T_s^0(T_a, p)$, will be considered subsequently as the steady-state laws of burning. In general, it is difficult to use the steady-state relationships directly in the theory of nonsteady burning.

In fact, the instantaneous state of S and G regions by no means depends on the temperature profile of the propellant far from these regions. At any given moment, the state of these zones can be determined only by the neighboring region of the condensed phase. Such a distortion of a temperature

profile, as plotted in Fig. 2.1a, has little effect on the burning rate at a given moment. Therefore, to consider the nonsteady processes, we should introduce, instead of T_a , some other parameter of the condensed phase that would directly affect the processes in S and G regions.

Common variables used in studying steady-state burning are pressure and initial temperature. The initial temperature in Z-N theory is by no means characteristic of the quasistationary state of a quick-response zone in the nonsteady process. From that value of temperature, we should pass to another value that directly affects the state of these zones. Such a transition was realized by Zeldovich⁶ at constant surface temperatures. That value was demonstrated to be a temperature gradient at the surface on the condensed-phase side.

Surface temperature being variable, the nontrivial moment of this procedure is that this value belongs to the inertial condensed phase on the one hand and to the region considered to be without inertia on the other hand. However it has been found⁷) that not only the burning rate but also the surface temperature are determined by instant values of the pressure and temperature gradient at the surface on the condensed-phase side. Thus, a new concept of non-steady burning laws is introduced. Those connect the burning rate and surface temperature from one side and the pressure and temperature gradient f at the interface from the other side.

These nonsteady laws of burning $u(f, p)$ and $T_s(f, p)$ can be obtained from the steady-state combustion laws $u^0(T_a, p)$ and $T_s^0(T_a, p)$ by eliminating the initial temperature with the aid of the steady state relation

$$f^0 = u^0(T_s^0 - T_a)/\kappa \quad (2.1)$$

which follows from the Michelson temperature distribution in the steady-state regime

$$T^0(x) = T_a + (T_s^0 - T_a)\exp(u^0 x/\kappa) \quad (2.2)$$

The proof of this statement has been done in Ref. 13.

Two aspects must be considered in the theory of nonsteady burning. The first is associated with determining the burning rate with given external conditions (pressure or tangential flow velocity). Let us call this an "internal" problem of the nonsteady burning theory. Solution of problems in nonsteady burning with given external conditions opens a way to study combustion with a variable burning rate in combustion chambers. In investigating the second class of problems, the pressure-time relationship should be replaced by equations relating the pressure and temperature in the combustion chamber to the nonsteady burning rate and the temperature of the gases formed. As a result, we must find both the burning rate and pressure, besides the temperature inside the combustion chamber. This problem (let us call it "external") can be solved only if the internal problem has been investigated. Solution of the internal problem, which is basic for all kinds of applied problems, should be considered the main problem in the nonsteady combustion theory. Therefore, we will restrict the following to consideration of the internal problem only.

The problem of finding the nonsteady burning rate in Z-N theory is reduced to accounting for the thermal inertia of the propellant by solving the heat-conduction equation:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - u \frac{\partial T}{\partial x} \quad (2.3)$$

with boundary conditions:

$$x \rightarrow -\infty, T = T_a; \quad x = 0, T = T_s \quad (2.4)$$

at the given initial state and pressure in time:

$$t=0, T(x, 0)=T_i(x); \quad p=p(t). \quad (2.5)$$

The laws of nonsteady combustion are known, i.e., the relationships between the mass burning rate, surface temperature, temperature gradient, and pressure:

$$m=m(f, p), \quad T_s=T_s(f, p) \quad (2.6)$$

where

$$f=\left(\frac{\partial T}{\partial x}\right)_{x=0}. \quad (2.7)$$

Dimensionless variables are used in successive developments. In any problem, it is possible to determine a basic steady-state regime. Let u^0 be the linear rate of the steady-state combustion at pressure p^0 , and introduce a dimensionless space coordinate, time, pressure, and burning rate with the help of the following definitions:

$$\xi=u^0 x/\kappa, \quad \tau=(u^0)^2 t/\kappa, \quad \eta=p/p^0, \quad v=u/u^0. \quad (2.8)$$

The dimensionless space coordinate and time are expressed in terms of characteristic length and characteristic time of the propellant.

The temperature in the condensed phase, the gradient, and the temperature at the surface can be conveniently expressed in the form of

$$\theta=T/\Delta, \quad \theta_s=T_s/\Delta, \quad \varphi=f/f^0, \quad \Delta=T_s^0-T_a. \quad (2.9)$$

The internal problem, in terms of these variables, is formulated in the following manner. Find the burning rate $v(\tau)$ from the heat-conduction equation, which takes into account the thermal inertia of the propellant:

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \xi^2} - v \frac{\partial \theta}{\partial \xi} \quad (2.10)$$

with initial and boundary conditions

$$\theta(\xi, 0)=\theta_i(\xi), \quad \theta(-\infty, \tau)=\theta_a, \quad \theta(0, \tau)=\theta_s \quad (2.11)$$

where $\theta_a=T_a/(T_s^0-T_a)$.

The following relationships are also given:

$$v=v(\varphi, \eta), \quad \theta_s=\theta_s(\varphi, \eta) \quad (2.12)$$

where

$$\varphi=\left(\frac{\partial \theta}{\partial \xi}\right)_{\xi=0}$$

and the pressure dependence on time $\eta(\tau)$.

At the steady-state regime, for $\eta=1$, we have

$$v=1, \quad \varphi=1, \quad \theta=\theta_a+e^\xi, \quad \vartheta \equiv \theta_s^0=1+\theta_a. \quad (2.13)$$

3. BURNING SOLID PROPELLANT AS AN OSCILLATORY SYSTEM

A heat-conduction equation (2.10), with boundary conditions (2.11), at constant pressure leads to the steady-state regime (2.13). Such a solution has to be valid for any external steady-state conditions, for the given values of pressure and initial temperature. However, not every steady-state regime can be realized in practice. The solution should not only satisfy the equation and the boundary conditions but should also be stable relative to small random perturbations. The stability of the regime means that, once small perturbations appear, they should attenuate in time. Conversely, the regime is unstable if the perturbations increase in time. Thus, the problem of the combustion stability is connected to investigation of small perturbations in time. Of course, this problem can be solved only in terms of the theory of nonsteady phenomena.

Because of the mathematical difficulties of studying the nonlinear equations, stability is often investigated in the case of small-amplitude perturbations. This makes it possible to linearize the initial problem and to obtain an analytical expression for the stability boundary. Note that linear approximation is also effective in solving other interesting and important problems of the nonsteady burning theory (e.g., combustion at oscillatory varying pressure that will be considered in the next section).

When considering problems of the nonsteady combustion theory, in the linear approximation one needs only the four dimensionless parameters

$$\begin{aligned} k &= \Delta \left(\frac{\partial \ln u^0}{\partial T_a} \right)_p, & r &= \left(\frac{\partial T_s^0}{\partial T_a} \right)_p \\ \nu &= \left(\frac{\partial \ln u^0}{\partial \ln p} \right)_{T_a}, & \mu &= \frac{1}{\Delta} \left(\frac{\partial T_s^0}{\partial \ln p} \right)_{T_a}. \end{aligned} \quad (3.1)$$

These derivatives fully characterize the behavior of propellant at small deviations from the steady-state regime.

The stability of a steady-state regime in the linear approximation is usually investigated by superposing small perturbations to the steady-state solution and subsequently analyzing their behavior in time. The right way to investigate stability of a system is to see how an initial disturbance depends on time. If at $t \rightarrow \infty$ it disappears, the regime is stable and vice versa. This may lead even in a linear approximation to very complex expressions and it was applied in a few articles (for example, see Refs. 16, 17). A much simpler way is to presume the asymptotic behavior of a disturbance by a special form $\exp(\Omega\tau)$, where Ω is the dimensionless complex frequency. If these perturbations increase in time for, at least, one value Ω ($\text{Re}\Omega > 0$), the combustion process would be unstable. The stability boundary is determined by the condition $\text{Re}\Omega = 0$.

The stability conditions for the steady-state regime of propellant combustion at constant pressure are formulated in the following way:⁷⁾

$$\begin{aligned} &\text{at } k < 1, \text{ the regime is always stable} \\ &\text{at } k > 1, \text{ the regime is stable only if} \\ &\quad r > \frac{(k-1)^2}{k+1} \end{aligned} \quad (3.2)$$

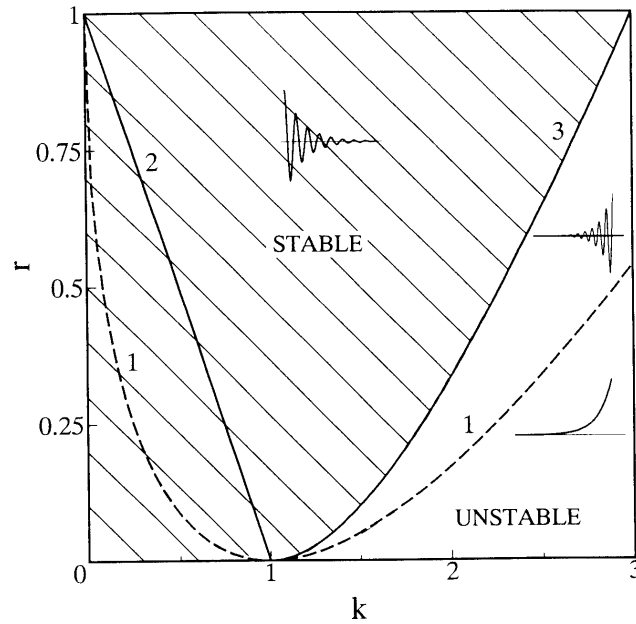


Fig. 3.1 Boundary of steady-state combustion stability at constant pressure.
1) $r = (k^{1/2} - 1)^2$, 2) $r = 1 - k$, 3) $r = (k - 1)^2 / (k + 1)$.

and are illustrated in Fig. 3.1, where three curves are shown. The first curve ($r = (\sqrt{k} - 1)^2$) corresponds to the bounding curve whether the disturbance varies oscillatory or monotonously. The second one ($r = 1 - k$) corresponds to the boundary of possibility of considering the perturbations of the present special form. The third one ($r = (k - 1)^2 / (k + 1)$) is the stability boundary, but the inequality $r > 1 - k$ means that we can consider only the right branch of the function $r = (k - 1)^2 / (k + 1)$ as the stability boundary. Below the line $r = 1 - k$ we are not able to consider the perturbations with $\exp(\Omega\tau)$ dependence on time.

From the characteristic equations for the frequency, we obtain a value of frequency at the stability boundary. Denote $\omega = \text{Im}\Omega$; then,

$$\omega^* = (k + 1)k^{1/2} / (k - 1)^2. \quad (3.3)$$

The real part of Ω is zero. Outside the stability boundary, the frequency has both imaginary and real parts $\Omega = -\lambda_n \pm i\omega_n$, where

$$\lambda_n = [r(k + 1) - (k - 1)^2] / 2r^2, \quad \omega_n = (k/r^2 - \lambda_n^2)^{1/2}. \quad (3.4)$$

By analogy with the oscillations of a system with one degree of freedom, we call λ_n a damping decrement. In the region of stability, $\lambda_n > 0$; i.e., the oscillations are damped, and conversely, $\lambda_n < 0$ in the region of instability. The value ω_n is called a natural frequency of oscillation.

In the unstable region, $\lambda_n < 0$ and, with decreasing r apart from the stability boundary, the imaginary part of frequency decreases. At $r = (k^{1/2} - 1)^2$, curve 1 in Fig. 3.1, it becomes zero. If r is less than this value, the perturbations increase exponentially in time without oscillations. The asymptotic change in time in different parts of plane (k, r) is shown in Fig. 3.1.

A very important question is in what region of the (k, r) plane real systems are. From experimental data given in Refs. 14, 15, 18, and 19, one can find

$$k \approx 1.3 - 2.0 \quad r \approx 0.15 - 0.3.$$

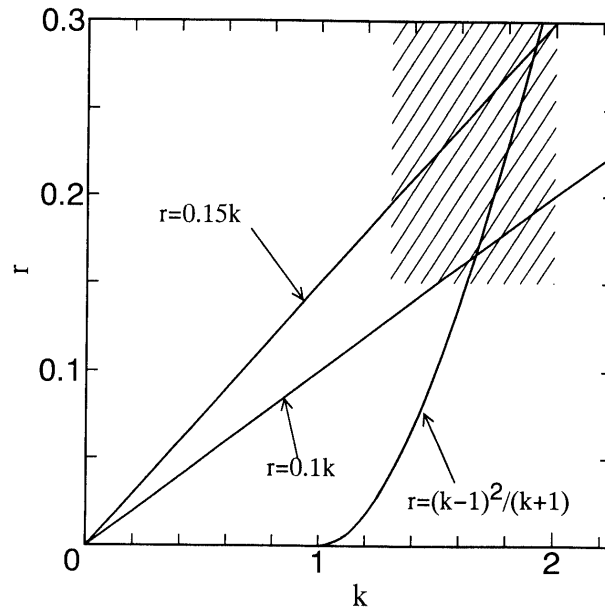


Fig. 3.2 The parameters k and r of real systems lie in the hatched region.

From the other side the burning rate u usually depends on the surface temperature very strongly

$$u \sim \exp(-E/RT_s).$$

From this relation we find

$$r = \frac{R(T_s^0)^2}{E\Delta} k.$$

For combustion processes, the value of RT_s^0/E usually is very small, $RT_s^0/E \ll 1$. In Fig. 3.2 region 1 represents the place where any real system is. We see that this region is close to the stability boundary. In this region

$$\lambda_n \ll \omega_n$$

and the propellant may be considered as an oscillatory system with a very high quality.

4. BURNING RATE RESPONSE FUNCTION TO OSCILLATORY PRESSURE

In the framework of the Z-N approach, the solid-propellant burning rate response function has been found in Ref. 20. Before that several investigations (see, for example, Refs. 16, 21, 22) were performed in the framework of F-M method. Those studies were reviewed in Ref. 23.

If the pressure changes in harmonic way

$$p = p^0 + p_1 \cos \omega \tau$$

the burning rate in linear approximation will be

$$u = u^0 + u_1 \cos(\omega \tau + \psi).$$

Using the complex amplitude method we have

$$\eta = 1 + \eta_1 e_1 + \bar{\eta}_1 \bar{e}_1, \quad v = 1 + v_1 e_1 + \bar{v}_1 \bar{e}_1 \quad (4.1)$$

where

$$\begin{aligned} \eta_1 &= \frac{p_1}{2p^0} \\ v_1 &= \frac{u_1}{2u^0} e^{i\psi} \\ e_1 &= e^{i\omega\tau} \end{aligned} \quad (4.2)$$

and $\bar{\eta}_1$, \bar{v}_1 , and \bar{e}_1 are the complex conjugations of η_1 , v_1 , and e_1 .

From the solution of the heat-conduction equation (2.10) we have two relations between the amplitudes of ϑ_1 , v_1 , and φ_1 .

$$\vartheta_1 = C_1 + \frac{iv_1}{\omega}, \quad \varphi_1 = C_1 z_1 + \frac{iv_1}{\omega}. \quad (4.3)$$

The additional two relations follow from the nonsteady burning laws

$$\begin{aligned} v_1 &= \frac{k}{D} \varphi_1 + \frac{\delta - \nu}{D} \eta_1, & \vartheta_1 &= \frac{r}{D} \varphi_1 - \frac{\delta + \mu}{D} \eta_1 \\ D &= k + r - 1, & \delta &= \nu r - \mu k. \end{aligned} \quad (4.4)$$

The set of Eqs. (4.3) – (4.4) contains four unknown values: v_1 , ϑ_1 , φ_1 , and C_1 , and four equations. So we can find any of these four unknown values. The most interesting is of course the burning rate amplitude²⁰⁾

$$v_1 = \frac{\nu + (\nu r - \mu k)(z_1 - 1)}{1 + r(z_1 - 1) - k(z_1 - 1)/z_1} \eta_1.$$

At this point the definition of the burning rate response function to oscillatory pressure is introduced

$$U = \frac{v_1}{\eta_1}. \quad (4.6)$$

Therefore, in the framework of Z–N linear theory we have

$$U = \frac{\nu + (\nu r - \mu k)(z_1 - 1)}{1 + r(z_1 - 1) - k(z_1 - 1)/z_1}. \quad (4.7)$$

Fig. 4.1 illustrates the real and imaginary parts of this function. The parameters k , r , ν , and μ are taken as in Ref. 3.

One can see that the real part of U may reach values much greater than its steady-state value $ReU(0) = \nu$. It is a consequence of the high quality of our oscillatory system. The maximum of ReU is at a frequency close to the natural frequency $\omega_n \sim \sqrt{k}/r$. In this frequency region the denominator of Eq. (4.7) is much less than unity. So any small perturbation of the system can strongly influence

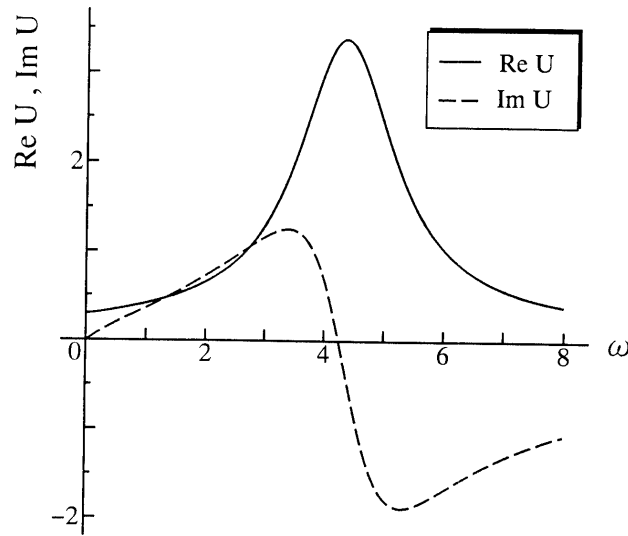


Fig. 4.1 Solid propellant burning rate response function.
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\nu r - \mu k = 0$.

the response function.

As an example of such a change we consider in the next section the burning rate response function of highly metallized propellants.

5. RESPONSE FUNCTION OF HIGHLY METALLIZED SOLID PROPELLANTS

Metallized solid propellants are widely used in rocket motors^{24,25)} because of their high density and high heat of reaction. A peculiar feature of metallized propellant is the intensive internal radiant flux. The densities of radiant fluxes directed toward the combustion surface are given in Ref. 14. Though the values of radiant flux are small in comparison to the conductive heat flux, it may be seen that its presence significantly changes the burning rate response function.

The aim of this section is to obtain the burning rate response function of highly metallized solid propellant. Note that there are a large number of papers (see, for example, Refs. 11, 26–29) devoted to solid propellant combustion under external radiation flux.

To obtain the response function by use of Z-N theory, we should consider the heat conduction equation in the condensed phase with the radiant flux source as well as the thermal inertia,

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - u \frac{\partial T}{\partial x} + \frac{I(t)}{\rho_c C_c L} e^{x/L} \quad (5.1)$$

where T is the temperature, u the burning rate, κ the thermal diffusivity, ρ_c the mass density, C_c the specific heat, $I(t)$ the internal radiant flux that may change in time at oscillatory pressure, and L the absorption length of the condensed phase. The space coordinate system is taken so that it moves the unreacted propellant in the positive direction of the x axis with a velocity that coincides with the linear regression velocity, that is, $u(t)$. The interface surface therefore remains fixed at any combustion regime.

Boundary conditions are

$$x \rightarrow -\infty, T = T_a; \quad x = 0, T = T_s \quad (5.2)$$

where T_a and T_s are the initial and the surface temperatures, respectively, at the given initial state and the pressure

$$t=0, T(x, 0)=T_i(x); \quad p=p(t). \quad (5.3)$$

The laws of nonsteady combustion are known, i.e., the relationships between the mass burning rate, surface temperature, temperature gradient, and pressure:

$$m=m(f, p), \quad T_s=T_s(f, p) \quad (5.4)$$

where

$$f=\left(\frac{\partial T}{\partial x}\right)_{x=0}.$$

We assume the law of nonsteady internal radiant flux is known in the form of $I=I(f, p)$.

Dimensionless variables are used in successive developments. In any problem, it is possible to determine a basic steady-state regime. Let u^0 be the linear rate of the steady-state combustion at pressure p^0 , and introduce a dimensionless space coordinate, time, pressure, and burning rate with the help of the following definitions:

$$\xi=u^0x/\kappa, \quad \tau=(u^0)^2t/\kappa, \quad \eta=p/p^0, \quad v=u/u^0. \quad (5.5)$$

The dimensionless space coordinate and time are expressed in terms of characteristic length and characteristic time of the propellant.

The temperature in the condensed phase, the gradient, and the temperature at the surface can be conveniently expressed in the form of

$$\theta=T/\Delta, \quad \theta_s=T_s/\Delta, \quad \varphi=f/f^0, \quad \Delta=T_s^0-T_a. \quad (5.6)$$

The internal radiant flux and the absorption length are expressed in the following form,

$$S(t)=I(t)/(\rho_c C_c u^0 \Delta), \quad l=Lu^0/\kappa. \quad (5.7)$$

Using the above nondimensionalization, Eq. (5.1) is written in the following non-dimensional form,

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \xi^2} - v \frac{\partial \theta}{\partial \xi} + \frac{S(t)}{l} e^{\xi/l} \quad (5.8)$$

$$\xi = -\infty, \theta = \theta_a; \quad \xi = 0, \theta = \vartheta$$

where

$$\theta_a=T_a/\Delta, \quad \vartheta=\theta_s^0=1+\theta_a.$$

The following relationships are also given:

$$v=v(\varphi, \eta), \quad \theta_s=\theta_s(\varphi, \eta), \quad S=S(\varphi, \eta) \quad (5.9)$$

where

$$\varphi = \left(\frac{\partial \theta}{\partial \xi} \right)_{\xi=0}$$

and the pressure dependence on time $\eta(\tau)$.

The steady-state solution of Eq. (5.8) has the form

$$\theta(\xi) = \theta_a + (1-a)e^\xi + ae^{\xi/l}, \quad a = S^0 l / (l-1) \quad (5.10)$$

where S^0 is the nondimensional radiant heat flux at the steady-state combustion. Note that the temperature gradient at the surface decreases in comparison with the non-radiant case.

$$f^0 = u^0 \Delta / \kappa - I^0 / (\rho_c C_c \kappa), \quad \varphi^0 = 1 - S^0. \quad (5.11)$$

Using the method of complex amplitude, let us represent any value in the form $X = X^0 + X_1 e^{i\omega\tau}$, where $|X^0| \gg |X_1 e^{i\omega\tau}|$ for all τ . That is,

$$\begin{aligned} \theta &= \theta^0 + \theta_1 e^{i\omega\tau} \\ v &= v^0 + v_1 e^{i\omega\tau} \\ S &= S^0 + S_1 e^{i\omega\tau}. \end{aligned} \quad (5.12)$$

Substituting these into Eq. (5.8), we have the following linear equation.

$$\theta_1'' - \theta_1' - i\omega\theta_1 = v_1(1-a)e^\xi + \frac{1}{l}(v_1 a - S_1)e^{\xi/l}. \quad (5.13)$$

with its solution

$$\theta_1 = C_1 e^{z_1 \xi} + \frac{iv_1(1-a)}{\omega} e^\xi + \frac{(v_1 a - S_1)l}{(1-lz_1)[1+l(z_1-1)]} e^{\xi/l}. \quad (5.14)$$

A root of the characteristic equation z_1 is connected with the frequency by the following correlations:

$$2z_1 = 1 + (1 + 4i\omega)^{1/2}, \quad i\omega = z_1(z_1 - 1). \quad (5.15)$$

We should neglect the second solution of the homogeneous equation, which corresponds to another value of the root, because it increases infinity at $\xi \rightarrow -\infty$.

The first order corrections to the steady-state condition for the surface temperature and temperature gradient from Eq. (5.14) are

$$\begin{aligned} \vartheta_1 &= \theta_1|_{\xi=0} = C_1 + \frac{iv_1(1-a)}{\omega} + \frac{(v_1 a - S_1)l}{(1-lz_1)[1+l(z_1-1)]} \\ \varphi_1 &= \left(\frac{\partial \theta_1}{\partial \xi} \right)_{\xi=0} = C_1 z_1 + \frac{iv_1(1-a)}{\omega} + \frac{(v_1 a - S_1)}{(1-lz_1)[1+l(z_1-1)]}. \end{aligned} \quad (5.16)$$

With no radiation ($a=0$, $S_1=0$) Eqs. (5.16) coincide with those for the non-radiant case.

$$\begin{aligned}\vartheta_1^{(NR)} &= C_1 + \frac{iv_1^{(NR)}}{\omega} \\ \varphi_1^{(NR)} &= C_1 z_1 + \frac{iv_1^{(NR)}}{\omega}.\end{aligned}\quad (5.17)$$

Now we must obtain three additional relations for v_1 , ϑ_1 , S_1 , φ_1 , and η_1 from the nonsteady combustion laws $v(\varphi, \eta)$, $\vartheta(\varphi, \eta)$, and $S(\varphi, \eta)$. Using the chain rule, the perturbation terms may be written as

$$\begin{aligned}v_1 &= v_\varphi \varphi_1 + v_\eta \eta_1 \\ \vartheta_1 &= \vartheta_\varphi \varphi_1 + \vartheta_\eta \eta_1 \\ S_1 &= S_\varphi \varphi_1 + S_\eta \eta_1.\end{aligned}\quad (5.18)$$

The derivatives v_φ , v_η , ϑ_φ , ϑ_η , S_φ , and S_η can be represented in the following form (see Appendix A).

$$\begin{aligned}v_\varphi &= \frac{k}{D - sS^0}, & v_\eta &= \frac{\delta - \nu + S^0(\sigma k - s\nu)}{D - sS^0}, & D &= k + r - 1 \\ \vartheta_\varphi &= \frac{r}{D - sS^0}, & \vartheta_\eta &= \frac{-\delta - \mu + S^0(\sigma r - s\mu)}{D - sS^0}, & \delta &= \nu r - \mu k \\ S_\varphi &= \frac{sS^0}{D - sS^0}, & S_\eta &= \frac{S^0[\sigma D - s(\nu + \mu)]}{D - sS^0}\end{aligned}\quad (5.19)$$

where

$$\begin{aligned}k &= \Delta \left(\frac{\partial \ln u^0}{\partial T_a} \right)_p, & r &= \left(\frac{\partial T_s^0}{\partial T_a} \right)_p, & s &= \Delta \left(\frac{\partial \ln I^0}{\partial T_a} \right)_p \\ \nu &= \left(\frac{\partial \ln u^0}{\partial \ln p} \right)_{T_a}, & \mu &= \frac{1}{\Delta} \left(\frac{\partial T_s^0}{\partial \ln p} \right)_{T_a}, & \sigma &= \left(\frac{\partial \ln I^0}{\partial \ln p} \right)_{T_a}.\end{aligned}$$

Again with no radiation ($S^0=0$, $\sigma=0$, $s=0$), equations (5.18, 5.19) coincides with those for the non-radiant case.

The set of equations (5.16, 5.18) contains five unknown values; v_1 , ϑ_1 , φ_1 , C_1 , and S_1 , and five equations. So we can find any of these five unknown values. The most interesting is, of course, the burning rate amplitude

$$v_1 = \frac{\nu + \delta(z_1 - 1) - S^0(\sigma k - s\nu)F(l, \omega)}{1 + r(z_1 - 1) - k(z_1 - 1)/z_1 + S^0F(l, \omega)(s - k/z_1)} \eta_1 \quad (5.20)$$

where

$$F(l, \omega) = l(z_1 - 1)/[1 + l(z_1 - 1)].$$

At this point the definition of the burning rate response function to oscillatory pressure with radiant flux is introduced

$$U_r = \frac{v_1}{\eta_1}. \quad (5.21)$$

Therefore, in the framework of Z-N linear theory, we have

$$U_r = \frac{\nu + \delta(z_1 - 1) - S^0(\sigma k - s\nu)F(l, \omega)}{1 + r(z_1 - 1) - k(z_1 - 1)/z_1 + S^0F(l, \omega)(s - k/z_1)}. \quad (5.22)$$

In the case of combustion of propellants with low transparency ($l \rightarrow 0$, $F \rightarrow 0$), the response function is the same as without radiation (Eq. (4.7)).

For a highly transparent propellants ($l \rightarrow \infty$, $F \rightarrow 1$) we have

$$U_r = \frac{\nu + \delta(Z_1 - 1) - S^0(\sigma k - s\nu)}{1 + r(z_1 - 1) - k(z_1 - 1)/z_1 + S^0(s - k/z_1)}. \quad (5.23)$$

The experimental data give the maximum value of S^0 at about ten percent.¹⁴⁾ In regard to values σ and s , there are no experimental results for those. We may only suppose that they are too small to affect response function. They are connected with changing of internal heat flux when the pressure and initial temperature vary. Because the internal heat flux is linked to burning particle radiation, it is too much to expect that p and T_a can greatly change the particle temperature. For this reason in our calculation we put $\sigma=0$, $s=0$.

In Figs. 5.1 and 5.2 the dependencies of the real and imaginary parts of the burning rate response function on frequency are illustrated for various S^0 at $l \rightarrow \infty$. The parameters k , r , ν , and μ are taken as in Ref. 3. One can see that consideration of the internal radiant heat flux gives only a slight shift of the resonant frequency but greatly changes the response function. For the set of the parameters considered, the peak value becomes more than twice as large as that of the non-radiant case. In Figs. 5.3 and 5.4, ReU_r and ImU_r are illustrated for various l at $S^0=0.1$. The dependencies of ReU_r on the absorption length l at several frequencies are depicted in Fig. 5.5, where we can see two types of transition from the left half to the right. One is monotone decrease which occurs at higher frequencies, and the other is non-monotone increase which occurs at lower frequencies. Near the resonant frequency ($\omega=4$) overshoot is observed at around $l=1$. Figure 5.6 illustrates the contours of constant $\log_{10} ReU_r$ in $\omega-l$ plane. The resonant frequency slightly decreases as l increases. These figures show that the major change of the response function occurs in the region between $l=10^{-1}$ and 10.

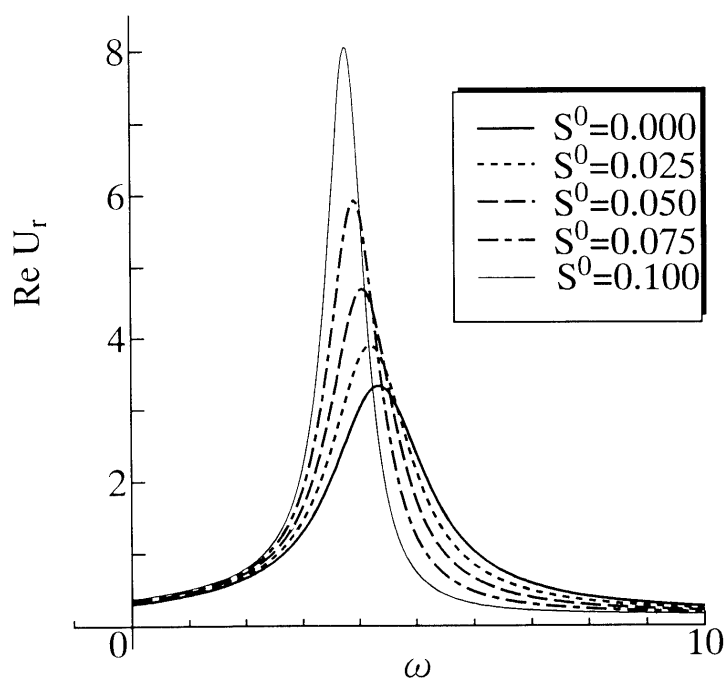


Fig. 5.1 Real part of burning-rate response function with radiant heat flux.
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $l \rightarrow \infty$.

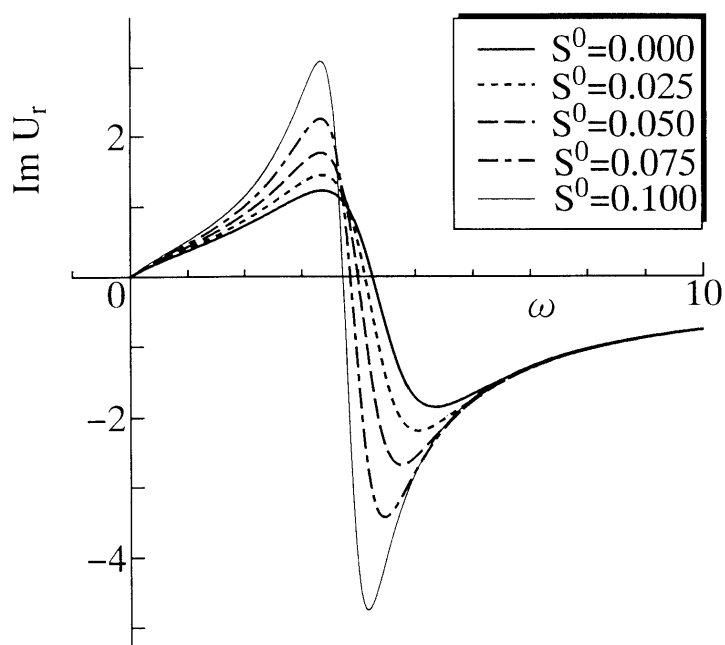


Fig. 5.2 Imaginary part of burning-rate response function with radiant heat flux.
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $l \rightarrow \infty$.

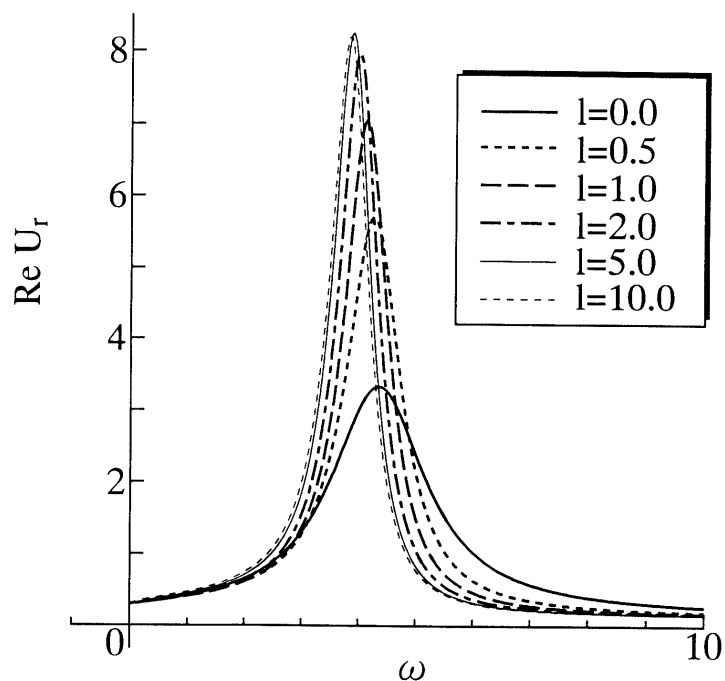


Fig. 5.3 Real part of burning-rate response function with radiant heat flux.
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $S^0=0.10$.

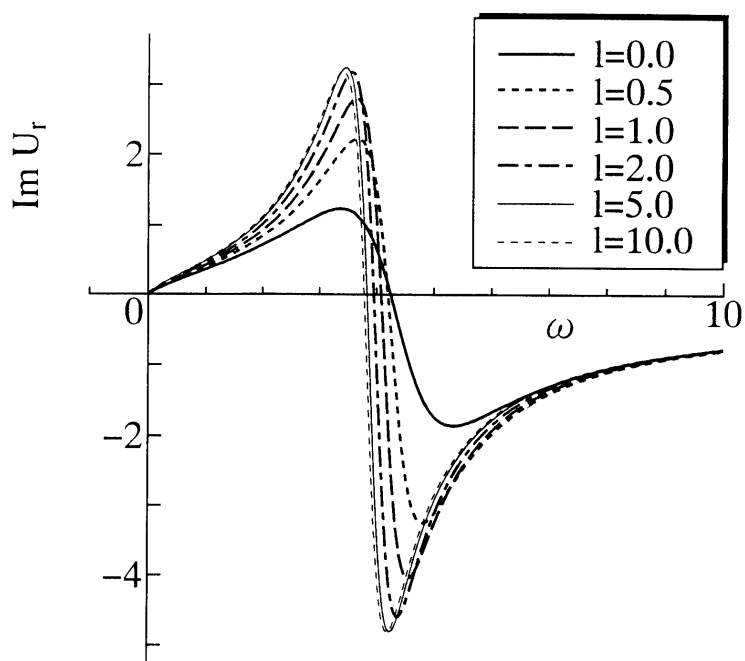


Fig. 5.4 Imaginary part of burning-rate response function with radiant heat flux.
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $S^0=0.10$.

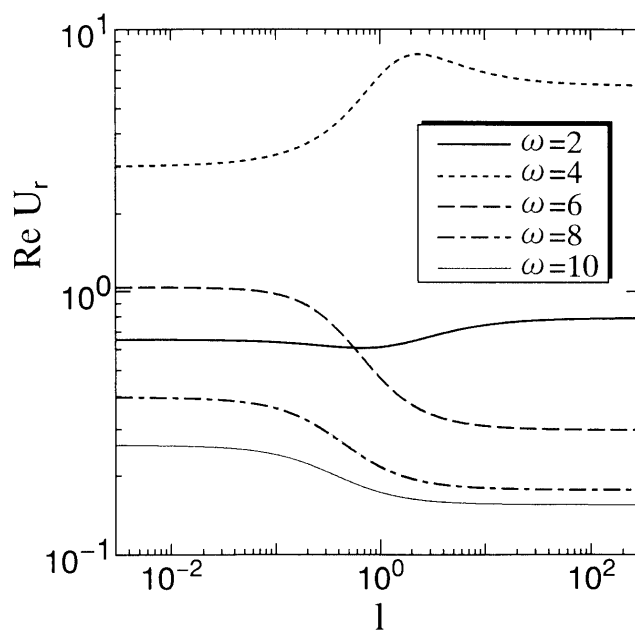


Fig. 5.5 Dependencies of $\text{Re } U_r$ on l at several frequencies.

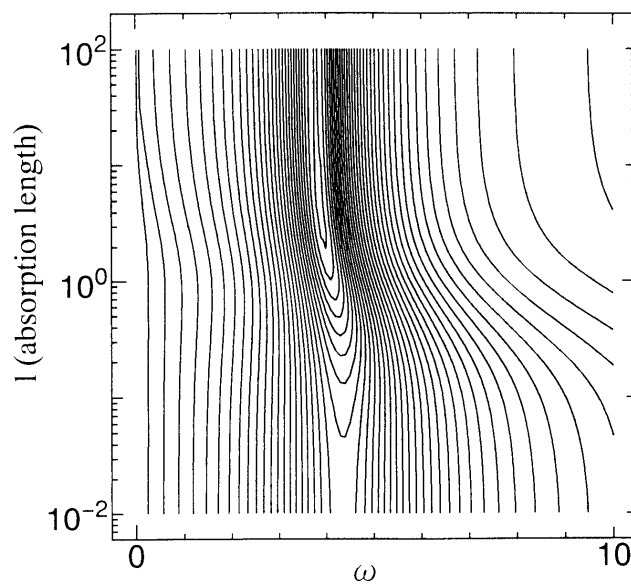


Fig. 5.6 Contours of constant $\log_{10} \text{Re } U_r$, $\Delta \log_{10} \text{Re } U_r = 0.05$.

6. RESPONSE FUNCTION OF THE HIGHER ORDERS

Suppose that the acoustic field contains several (or all) harmonics so the pressure can be represented as

$$p = p^0 + p_1 \cos \omega_1 \tau + p_2 \cos(\omega_2 \tau + \psi_2) + p_3 \cos(\omega_3 \tau + \psi_3) + \dots \quad (6.1)$$

where $\omega_k = k\omega_1$.

The burning rate in this case should be written as

$$u = u^0 + u_1 \cos(\omega_1 \tau + \psi_{u1}) + u_2 \cos(\omega_2 \tau + \psi_{u2}) + u_3 \cos(\omega_3 \tau + \psi_{u3}) + \dots \quad (6.2)$$

We introduce here the definition of response function of the higher orders. Nonlinearity of combustion process can give the oscillation of the burning rate with frequency, for example, ω_k by interaction of two (or several) other modes. The simplest examples are:

1) self-interaction of the first harmonic may give the second harmonic, because the product $\cos \omega_1 \tau \times \cos \omega_1 \tau$ contains the oscillation with $2\omega_1$ frequency,

2) interaction of the first and second modes give the second order correction to the linear response function for the first or third modes because the product $\cos \omega_1 \tau \times \cos 2\omega_1 \tau$ contains the oscillations with ω_1 and $3\omega_1$ frequencies.

Such nonlinear interaction may produce a large effect if one of interacting harmonics are close to the natural frequency ω_n .

Fig. 6.1 illustrates the two examples above pointed. In the case of Fig. 6.1a the first mode is near the natural frequency so it may produce a large effect to ω_2 response function. In the case of Fig. 6.1b the second mode is close to ω_n , so its interaction with the first mode can give not a small correction to the ω_1 linear response function.

Let us introduce nondimensional complex amplitude of the pressure and the burning rate

$$\eta_k = \frac{p_k}{2p^0} e^{i\psi_k}, \quad v_k = \frac{u_k}{2u^0} e^{i\psi_{u_k}}. \quad (6.3)$$

The Eqs. (6.1)–(6.2) have the following forms

$$\begin{aligned} \eta &= 1 + \sum_{k=1}^{\infty} \eta_k e_k + c.c. \\ v &= 1 + \sum_{k=1}^{\infty} v_k e_k + c.c. \end{aligned} \quad (6.4)$$

where $e_k = \exp(i\omega_k \tau)$ and *c.c.* denotes the complex conjugation.

To distinguish the linear and nonlinear response function we shall supply the former to one subscript and the latter to several subscripts.

So the linear response function for the first mode is

$$U_1 = v_1 / \eta_1, \quad U_1 \equiv U(\omega_1)$$

where $U(\omega_1)$ is given by Eq. (4.7). The linear response function of the second mode is

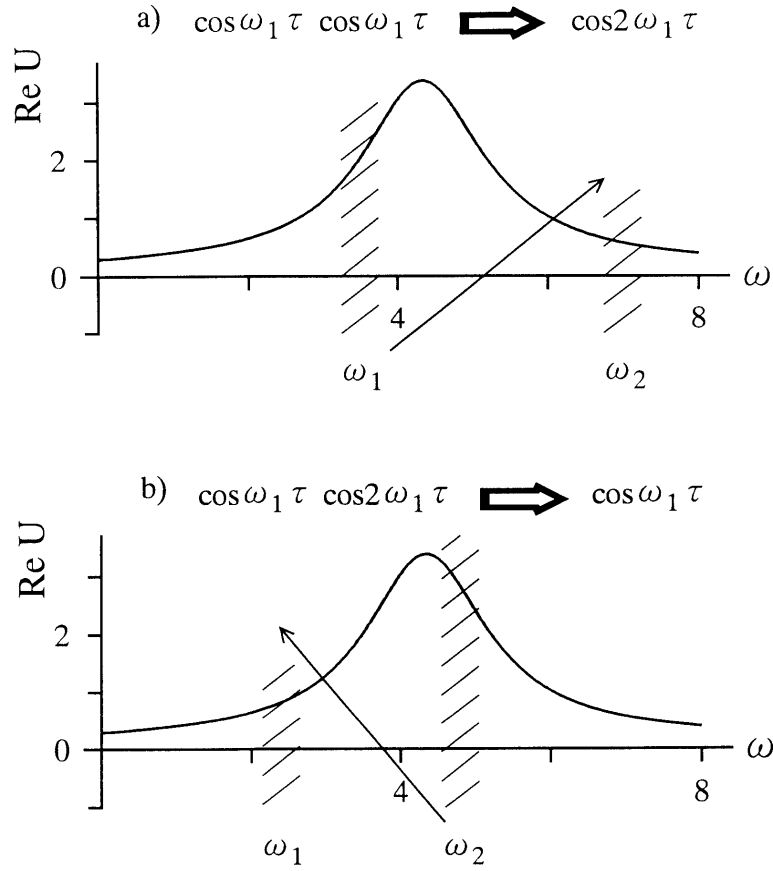


Fig. 6.1 Second order interactions of the pressure harmonics.

a) First harmonic self-interaction. The first harmonic is close to the natural frequency.

b) Interaction of the first and second harmonics. The second harmonic is close to the natural frequency.

$\nu=0.3$, $k=1.82$, $r=0.303$, $\nu r - \mu k = 0$.

$$U_2 = v_2 / \eta_2$$

so that $U_2 = U(2\omega_1)$ and

$$U_2 = \frac{\nu + \delta(z_2 - 1)}{1 + r(z_2 - 1) - k(z_2 - 1)/z_2} \quad (6.5)$$

where $z_2 = z_1(2\omega)$ or

$$z_2 = \frac{1}{2} [1 + \sqrt{1 + 8i\omega}]. \quad (6.6)$$

In common case the linear response function for k-th mode is

$$U_k = v_k / \eta_k, \quad U_k = U(k\omega)$$

or

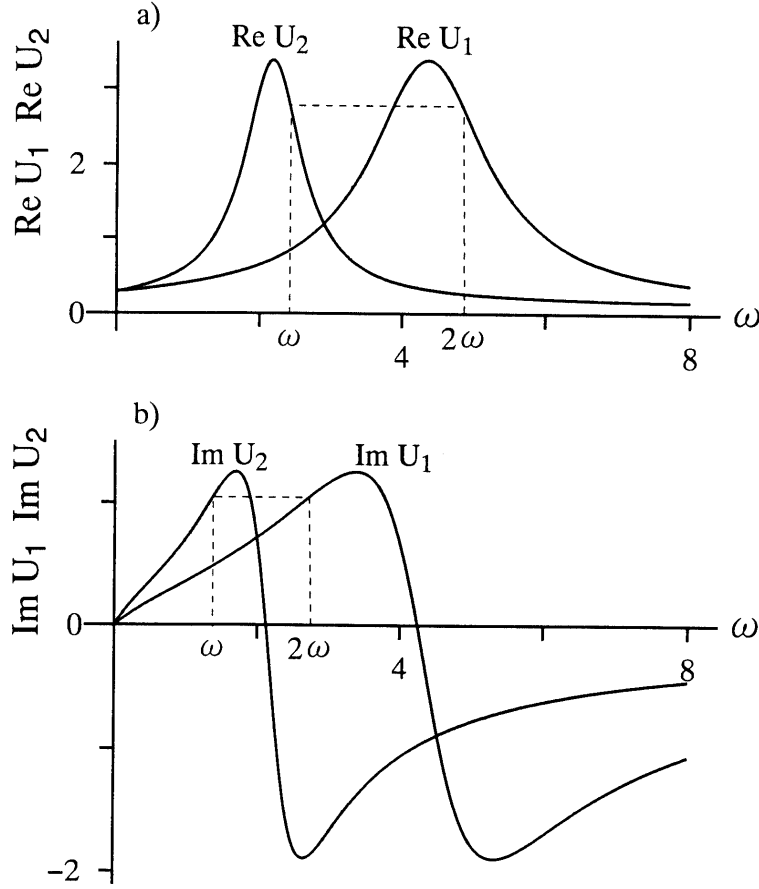


Fig. 6.2 Response functions $U_1(\omega)$ and $U_2(\omega)$.
a) real parts, b) imaginary parts. $\nu=0.3$, $k=1.82$, $r=0.303$, $\nu r - \mu k = 0$.

$$U_k = \frac{\nu + \delta(z_k - 1)}{1 + r(z_k - 1) - k(z_k - 1)/z_k} \quad (6.7)$$

where $z_k = z_1(k\omega)$.

Fig. 6.2 illustrates the linear response function of the first and second modes. To go from $U_1(\omega)$ to $U_2(\omega)$ we should only remember $U_2(\omega) = U_1(2\omega)$.

Now we pass to nonlinear interaction of different modes. Note that some nonlinear effects of solid propellant combustion near the steady-state stability boundary were considered before in Refs. 12, 30, 31.

Let us consider the second order nonlinear correction to the first harmonic response function. The oscillation of the frequency of ω_1 may be obtained by interaction of two neighboring harmonics ω_k and ω_{k+1} , because the product of $\cos \omega_k \tau$ and $\cos \omega_{k+1} \tau$ has two components $\cos(\omega_{k+1} - \omega_k) \tau$ and $\cos(\omega_{k+1} + \omega_k) \tau$. In our complex notations it means

$$\bar{\eta}_k e^{-i\omega_k \tau} \eta_{k+1} e^{i\omega_{k+1} \tau} = \bar{\eta}_k \eta_{k+1} e^{i(\omega_{k+1} - \omega_k) \tau} = \bar{\eta}_k \eta_{k+1} e^{i\omega_1 \tau}.$$

So the first harmonic of the burning rate will have the following form

$$v_1 = U_1 \eta_1 + U_{-1,2} \bar{\eta}_1 \eta_2 + U_{-2,3} \bar{\eta}_2 \eta_3 + \dots$$

or

$$v_1 = U_1 \eta_1 + \sum_{k=1}^{\infty} U_{-k, k+1} \bar{\eta}_k \eta_{k+1}. \quad (6.8)$$

We may say that the second order interaction to give the nonlinear addition to burning rate

$$(v_1)_{-k, k+1} = U_{-k, k+1} \bar{\eta}_k \eta_{k+1}$$

due to the interaction of k and $k+1$ modes is proportional to the product of $\bar{\eta}_k \eta_{k+1}$. Thus, the proportionality coefficient

$$U_{-k, k+1} = \frac{(v_1)_{-k, k+1}}{\bar{\eta}_k \eta_{k+1}}$$

as the response function of the second order. Note that the sum of its subscripts is equal to the number of the harmonics in question.

Another example. The second harmonic of the burning rate may be obtained by self-interaction of the first harmonic or harmonics with frequencies ω_k and ω_{k+2} because

$$\eta_1 e^{i\omega_1 \tau} \eta_1 e^{i\omega_1 \tau} = \eta_1^2 e^{i\omega_2 \tau}$$

and

$$\bar{\eta}_k e^{-i\omega_k \tau} \eta_{k+2} e^{i\omega_{k+2} \tau} = \bar{\eta}_k \eta_{k+2} e^{i\omega_2 \tau}.$$

The burning rate complex amplitude of the second mode, therefore, has the following form

$$v_2 = U_2 \eta_2 + U_{1,1} \eta_1^2 + \sum_{k=1}^{\infty} U_{-k, k+2} \bar{\eta}_k \eta_{k+2}. \quad (6.9)$$

The functions $U_{1,1}$ and $U_{-k, k+2}$ are the second order response functions for the second harmonic.

It is easy to understand that we can consider the interaction of higher than the second order. For example the self-interaction of the first mode in the third order gives the correction to the response function of the first harmonic

$$v_1 = U_1 \eta_1 + U_{-1,1,1} \eta_1 |\eta_1|^2 \quad (6.10)$$

because

$$\eta_1 e^{i\omega_1 \tau} \bar{\eta}_1 e^{-i\omega_1 \tau} \eta_1 e^{i\omega_1 \tau} = \eta_1 |\eta_1|^2 e^{i\omega_1 \tau}.$$

The value

$$U_{-1,1,1} = \frac{(v_1)_{-1,1,1}}{\eta_1 |\eta_1|^2} \quad (6.11)$$

can be named as the third order solid propellant burning rate response function for the first mode.

In the two following sections we show how to calculate the response function of the second order. As examples we take the most important cases of $U_{-1,2}$ and $U_{1,1}$ functions.

7. THE SECOND ORDER INTERACTION OF THE FIRST AND SECOND HARMONICS

As it has been shown in the previous section, the interaction of the first two harmonics with frequencies ω_1 and $2\omega_1$ may give the oscillations with frequency ω_1 . In this section the second order response function

$$U_{-1,2}(\omega) = \frac{(v_1)_{-1,2}}{\bar{\eta}_1 \eta_2} \quad (7.1)$$

will be obtained.

The pressure $p(\omega)$ and burning rate $u(\omega)$

$$\begin{aligned} p &= p^0 + p_1 \cos \omega \tau + p_2 \cos(2\omega \tau + \psi_2) \\ u &= u^0 + u_1 \cos(\omega \tau + \psi_{u1}) + u_2 \cos(2\omega \tau + \psi_{u2}) \end{aligned}$$

are represented in the form

$$\begin{aligned} \eta &= 1 + \eta_1 e_1 + \bar{\eta}_1 \bar{e}_1 + \eta_2 e_2 + \bar{\eta}_2 \bar{e}_2 \\ v &= 1 + v_1 e_1 + \bar{v}_1 \bar{e}_1 + v_2 e_2 + \bar{v}_2 \bar{e}_2 \end{aligned} \quad (7.2)$$

where

$$\begin{aligned} \eta_1 &= \frac{p_1}{2p^0}, & \eta_2 &= \frac{p_2}{2p^0} e^{i\psi_2}, & v_1 &= \frac{u_1}{2u^0} e^{i\psi_{u1}}, & v_2 &= \frac{u_2}{2u^0} e^{i\psi_{u2}} \\ e_1 &= e^{i\omega\tau}, & e_2 &= e^{2i\omega\tau} \end{aligned}$$

and the $\bar{\eta}_n$, \bar{v}_n , and \bar{e}_n are the complex conjugations of η_n , v_n , and e_n .

To solve the heat-conduction equation in the condensed phase

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \xi^2} - v \frac{\partial \theta}{\partial \xi} \quad (7.3)$$

the space-time temperature distribution should also be represented as a set of different modes

$$\theta(\xi, \tau) = \theta^0(\xi) + \theta_1(\xi) e_1 + \bar{\theta}_1(\xi) \bar{e}_1 + \theta_2(\xi) e_2 + \bar{\theta}_2(\xi) \bar{e}_2 \quad (7.4)$$

where

$$\theta^0(\xi) = \theta_a + e^\xi$$

is the steady-state solution.

The derivatives $\partial \theta / \partial \tau$ and $\partial^2 \theta / \partial \xi^2$ are

$$\begin{aligned} \frac{\partial \theta}{\partial \tau} &= i\omega \theta_1 e_1 + 2i\omega \theta_2 e_2 + c.c. \\ \frac{\partial^2 \theta}{\partial \xi^2} &= \theta^{0''} + \theta_1'' e_1 + \theta_2'' e_2 + c.c. \end{aligned}$$

where c.c. are corresponding complex conjugate values.

The second term in the right-hand side of Eq. (7.3) is nonlinear. In the product

$$-v \frac{\partial \theta}{\partial \xi} = -(1 + v_1 e_1 + v_2 e_2 + c.c.) (\theta^{0'} + \theta_1' e_1 + \theta_2' e_2 + c.c.)$$

we should keep only the terms that oscillate with frequency of ω . After multiplying two series in the previous expression and using the expression for $\partial \theta / \partial \tau$ and $\partial^2 \theta / \partial \xi^2$ we have the following equation for the first mode

$$\theta_1'' - \theta_1' - i\omega \theta_1 = v_1 \theta^{0'} + \bar{v}_1 \theta_2' + v_2 \bar{\theta}_1' \quad (7.5)$$

which contains the nonlinear terms $\bar{v}_1 \theta_2' + v_2 \bar{\theta}_1'$.

In the linear approximation there are two equations for the first and second harmonics

$$\theta_1'' - \theta_1' - i\omega \theta_1 = v_1 e^\xi \quad (7.6)$$

$$\theta_2'' - \theta_2' - 2i\omega \theta_2 = v_2 e^\xi. \quad (7.7)$$

The solutions of Eqs. (7.6) and (7.7) are

$$\theta_1 = C_1 e^{z_1 \xi} + \frac{i}{\omega} v_1 e^\xi \quad (7.8)$$

$$\theta_2 = C_2 e^{z_2 \xi} + \frac{i}{2\omega} v_2 e^\xi \quad (7.9)$$

with

$$z_1 = \frac{1}{2} [1 + \sqrt{1 + 4i\omega}], \quad z_2 = \frac{1}{2} [1 + \sqrt{1 + 8i\omega}]. \quad (7.10)$$

Using Eqs. (7.8) and (7.9) we can write Eq. (7.5) in the form:

$$\theta_1'' - \theta_1' - i\omega \theta_1 = \left(v_1 - \frac{i}{2\omega} \bar{v}_1 v_2 \right) e^\xi + \bar{v}_1 C_2 z_2 e^{z_2 \xi} + \bar{C}_1 v_2 \bar{z}_1 e^{\bar{z}_1 \xi}. \quad (7.11)$$

It is easy to solve this equation to obtain

$$\theta_1 = D_1 e^{z_1 \xi} + \frac{i}{\omega} \left(v_1 - \frac{i}{2\omega} \bar{v}_1 v_2 \right) e^\xi - \frac{i}{\omega} \bar{v}_1 C_2 z_2 e^{z_2 \xi} + \frac{i}{2\omega} \bar{C}_1 v_2 \bar{z}_1 e^{\bar{z}_1 \xi}. \quad (7.12)$$

From this expression we can find the surface temperature ϑ_1 and gradient of the temperature at the interface φ_1 corrections to the steady-state conditions $\vartheta^0 = \theta_a$ and $\varphi = 1$:

$$\vartheta_1 = D_1 + \frac{iv_1}{\omega} + a, \quad a = \frac{1}{2\omega^2} \bar{v}_1 v_2 - \frac{i}{\omega} \bar{v}_1 C_2 z_2 + \frac{i}{2\omega} \bar{C}_1 v_2 \bar{z}_1 \quad (7.13)$$

$$\varphi_1 = D_1 z_1 + \frac{iv_1}{\omega} + b, \quad b = \frac{1}{2\omega^2} \bar{v}_1 v_2 - \frac{i}{\omega} \bar{v}_1 C_2 z_2^2 + \frac{i}{2\omega} \bar{C}_1 v_2 \bar{z}_1^2. \quad (7.14)$$

The additional two relationships for v_1 , ϑ_1 , and φ_1 are known from the nonsteady burning laws $v(\varphi, \eta)$ and $\vartheta(\varphi, \eta)$ written to the second order perturbations.

From Appendix B (See Eqs. (B.3)) we have

$$\begin{aligned} v_1 &= v_\varphi \varphi_1 + v_\eta \eta_1 + N_{-1,2} \\ N_{-1,2} &= v_{\varphi\varphi} \bar{\varphi}_1 \varphi_2 + v_{\varphi\eta} (\bar{\varphi}_1 \eta_2 + \bar{\eta}_1 \varphi_2) + v_{\eta\eta} \bar{\eta}_1 \eta_2 \end{aligned} \quad (7.15)$$

$$\begin{aligned} \vartheta_1 &= \vartheta_\varphi \varphi_1 + \vartheta_\eta \eta_1 + M_{-1,2} \\ M_{-1,2} &= \vartheta_{\varphi\varphi} \bar{\varphi}_1 \varphi_2 + \vartheta_{\varphi\eta} (\bar{\varphi}_1 \eta_2 + \bar{\eta}_1 \varphi_2) + \vartheta_{\eta\eta} \bar{\eta}_1 \eta_2. \end{aligned} \quad (7.16)$$

From Eqs. (7.13)–(7.16) one can obtain the burning rate amplitude v_1 as a function of frequency and, therefore, the response function $U_{-1,2}(\omega)$.

Substitution of Eq. (7.13) into Eq. (7.14) gives

$$\varphi_1 = \left(\vartheta_1 - \frac{iv_1}{\omega} - a \right) z_1 + \frac{iv_1}{\omega} + b.$$

Take ϑ_1 from Eq. (7.16) and put it in this expression. That gives

$$\varphi_1(1 - \vartheta_\varphi z_1) = \vartheta_\eta z_1 \eta_1 + \frac{v_1}{z_1} + z_1(M_{-1,2} - a) + b.$$

Multiplying Eq. (7.15) by $(1 - \vartheta_\varphi z_1)$ and using the previous relationship we have

$$\begin{aligned} v_1(1 - \vartheta_\varphi z_1 - \frac{v_\varphi}{z_1}) &= (v_\varphi \vartheta_\eta z_1 + v_\eta - \vartheta_\varphi v_\eta z_1) \eta_1 + \\ &+ v_\varphi [z_1(M_{-1,2} - a) + b] + (1 - \vartheta_\varphi z_1) N_{-1,2} \end{aligned} \quad (7.17)$$

For any propellant model v_φ , v_η , ϑ_φ , and ϑ_η may be expressed through parameters k , μ , ν , and r given by equations

$$\begin{aligned} v_\varphi &= \frac{k}{D}, & \vartheta_\varphi &= \frac{r}{D} \\ v_\eta &= \frac{\delta - \nu}{D}, & \vartheta_\eta &= -\frac{\delta + \mu}{D} \end{aligned}$$

where $D = k + r - 1$.

Now one can write Eq. (7.17) in the form:

$$v_1 = \frac{\nu + \delta(z_1 - 1)}{1 + r(z_1 - 1) - k(z_1 - 1)/z_1} \eta_1 + \frac{g_{-1,2}}{1 + r(z_1 - 1) - k(z_1 - 1)/z_1} \bar{\eta}_1 \eta_2. \quad (7.18)$$

The first item of Eq. (7.18) represents the well-known linear response of the burning rate to oscillatory pressure and the second term gives us the second order response function

$$U_{-1,2} = \frac{g_{-1,2}}{1 + r(z_1 - 1) - k(z_1 - 1)/z_1} \quad (7.19)$$

where $g_{-1,2}$ has the following expression

$$\begin{aligned} g_{-1,2} &= g_h + g_l \\ g_h &= k(z_1 a - b) / \bar{\eta}_1 \eta_2 \\ g_l &= D[z_1(\vartheta_\varphi N_{-1,2} - v_\varphi M_{-1,2}) - N_{-1,2}] / \bar{\eta}_1 \eta_2. \end{aligned} \quad (7.20)$$

Here g_h is connected with nonlinearity of the heat-conduction equation, and g_l with nonlinearity of the nonsteady burning laws.

To calculate g_h we write using Eqs. (7.13) – (7.14)

$$z_1 a - b = \frac{1}{2\omega^2} (z_1 - 1) \bar{v}_1 v_2 - \frac{i}{\omega} \bar{v}_1 C_2 z_2 (z_1 - z_2) + \frac{i}{2\omega} \bar{C}_1 v_2 \bar{z}_1 (z_1 - \bar{z}_1).$$

Equations (7.8) and (7.9) give

$$\bar{C}_1 \bar{z}_1 = \bar{\varphi}_1 + \frac{i}{\omega} \bar{v}_1, \quad C_2 z_2 = \varphi_2 - \frac{i}{2\omega} v_2.$$

The corrections φ_1 and φ_2 to the steady-state temperature gradient can be expressed in terms of the corrections of the pressure and velocity (see App. B)

$$\bar{\varphi}_1 = \frac{1}{v_\varphi} (\bar{v}_1 - v_\eta \bar{\eta}_1), \quad \varphi_2 = \frac{1}{v_\varphi} (v_2 - v_\eta \eta_2).$$

Thus

$$\begin{aligned} z_1 a - b &= \frac{1}{2\omega^2} (z_1 - 1) \bar{v}_1 v_2 + \frac{i}{2\omega} \frac{v_\eta}{v_\varphi} [2 \bar{v}_1 \eta_2 (z_1 - z_2) - v_2 \bar{\eta}_1 (z_1 - \bar{z}_1)] + \\ &+ \frac{i}{2\omega} \bar{v}_1 v_2 \left\{ \frac{1}{v_\varphi} [-2(z_1 - z_2) + (z_1 - \bar{z}_1)] + \frac{i}{\omega} [(z_1 - z_2) + (z_1 - \bar{z}_1)] \right\}. \end{aligned} \quad (7.21)$$

Denote

$$v_1 = U(\omega), \quad U_2 = U(2\omega)$$

so that

$$\bar{v}_1 = \bar{U}_1 \bar{\eta}_1, \quad v_2 = U_2 \eta_2$$

and Eq. (7.21) may be written as

$$\begin{aligned} \frac{2\omega(z_1 a - b)}{i \bar{\eta}_1 \eta_2} &= \bar{U}_1 U_2 \left\{ \frac{1}{v_\varphi} [(z_2 - z_1) + (z_2 - \bar{z}_1)] + \frac{i}{\omega} (z_1 - \bar{z}_1 - z_2 + 1) \right\} + \\ &+ \frac{v_\eta}{v_\varphi} [2 \bar{U}_1 (z_1 - z_2) - U_2 (z_1 - \bar{z}_1)]. \end{aligned}$$

As

$$z_1(z_1 - 1) = i\omega, \quad \bar{z}_1(\bar{z}_1 - 1) = -i\omega, \quad z_2(z_2 - 1) = 2i\omega$$

one can represent g_h term in the following form

$$g_h = \frac{i}{2\omega} \left\{ \bar{U}_1 U_2 \left[(z_2 - z_1) \left(D - \frac{k}{z_1 z_2} \right) + (z_2 - \bar{z}_1) \left(D - \frac{k}{\bar{z}_1 z_2} \right) \right] + (\nu - \delta) [2(z_2 - z_1) \bar{U}_1 + (z_1 - \bar{z}_1) U_2] \right\}. \quad (7.22)$$

Now we pass to calculation of the g_l terms. From App. B we can obtain using the linear relationship

$$\frac{\varphi_1}{\eta_1} = \frac{v_1 - v_\eta \eta_1}{v_\eta \eta_1}, \quad \frac{\varphi_2}{\eta_2} = \frac{v_2 - v_\eta \eta_2}{v_\eta \eta_2}$$

or

$$\frac{\varphi_1}{\eta_1} = \frac{1}{v_\varphi} (U_1 - v_\eta), \quad \frac{\varphi_2}{\eta_2} = \frac{1}{v_\varphi} (U_2 - v_\eta).$$

We obtain the next expressions for $N_{-1,2}$ and $M_{-1,2}$

$$\begin{aligned} \frac{N_{-1,2}}{\bar{\eta}_1 \eta_2} &= \frac{v_{\varphi\varphi}}{v_\varphi^2} [\bar{U}_1 U_2 - v_\eta (\bar{U}_1 + U_2) + v_\eta^2] + \frac{v_{\varphi\eta}}{v_\varphi} (\bar{U}_1 + U_2 - 2v_\eta) + v_{\eta\eta} \\ \frac{M_{-1,2}}{\bar{\eta}_1 \eta_2} &= \frac{\vartheta_{\varphi\varphi}}{v_\varphi^2} [\bar{U}_1 U_2 - v_\eta (\bar{U}_1 + U_2) + v_\eta^2] + \frac{\vartheta_{\varphi\eta}}{v_\varphi} (\bar{U}_1 + U_2 - 2v_\eta) + \vartheta_{\eta\eta}. \end{aligned} \quad (7.23)$$

The complex $(\vartheta_{\varphi} N_{-1,2} - v_\varphi M_{-1,2}) / \bar{\eta}_1 \eta_2$ that exists in the expression for g_l may be represented using Eq. (7.23) as

$$\begin{aligned} \frac{\vartheta_{\varphi} N_{-1,2} - v_\varphi M_{-1,2}}{\bar{\eta}_1 \eta_2} &= \frac{L_{\varphi\varphi}}{v_\varphi^2} \bar{U}_1 U_2 + \left(\frac{L_{\varphi\eta}}{v_\varphi} - \frac{v_\eta}{v_\varphi^2} L_{\varphi\varphi} \right) (\bar{U}_1 + U_2) + \\ &+ \frac{v_\eta^2}{v_\varphi^2} L_{\varphi\varphi} - 2 \frac{v_\eta}{v_\varphi} L_{\varphi\eta} + L_{\eta\eta} \end{aligned} \quad (7.24)$$

where

$$\begin{aligned} L_{\varphi\varphi} &= \vartheta_{\varphi} v_{\varphi\varphi} - v_\varphi \vartheta_{\varphi\varphi} \\ L_{\varphi\eta} &= \vartheta_{\varphi} v_{\varphi\eta} - v_\varphi \vartheta_{\varphi\eta} \\ L_{\eta\eta} &= \vartheta_{\varphi} v_{\eta\eta} - v_\varphi \vartheta_{\eta\eta}. \end{aligned}$$

As an example we choose the simplest model of a propellant whose steady-state laws are

$$u^0 = A p^n e^{\beta T_a}, \quad u^0 = B \exp(-E/RT_s^0). \quad (7.25)$$

For this model App. C gives the next expressions

$$L_{\varphi\varphi} = \frac{k^2 r}{D^3} (1 - \epsilon), \quad L_{\varphi\eta} = \frac{k r v}{D^3} (\epsilon - 1), \quad L_{\eta\eta} = \frac{r v^2}{D^3} (1 - \epsilon) \quad (7.26)$$

where

$$\epsilon = \frac{2RT_s^0}{E}.$$

Thus we can obtain

$$(\vartheta_\varphi N_{-1,2} - v_\varphi M_{-1,2}) / \bar{\eta}_1 \eta_2 = \frac{r}{D} (1 - \epsilon) \bar{U}_1 U_2 \quad (7.27)$$

and

$$\frac{N_{-1,2}}{\bar{\eta}_1 \eta_2} = \frac{1}{D} [1 - r(1 + \epsilon)] \bar{U}_1 U_2 - \frac{v}{D} (\bar{U}_1 + U_2). \quad (7.28)$$

Combining Eqs. (7.27) and (7.28) to obtain g_l (See Eq. (7.20)) we have the final expression for g_l

$$g_l = \bar{U}_1 U_2 [z_1 r(1 - \epsilon) - 1 + r(1 + \epsilon)] - v(\bar{U}_1 + U_2) \quad (7.29)$$

and from Eqs. (7.22) and (7.29) the final expression for $g_{-1,2} = g_h + g_l$

$$\begin{aligned} g_{-1,2} = & \bar{U}_1 U_2 \left\{ \frac{i}{2\omega} \left[(z_2 - z_1) \left(D - \frac{k}{z_1 z_2} \right) + (z_2 - \bar{z}_1) \left(D - \frac{k}{\bar{z}_1 z_2} \right) \right] + \right. \\ & \left. + z_1 r(1 - \epsilon) - 1 + r(1 + \epsilon) \right\} + \\ & + v \{ \bar{U}_1 [2(z_2 - z_1) - 1] + U_2 (z_1 - \bar{z}_1 - 1) \}. \end{aligned} \quad (7.30)$$

Now we can calculate the response function $U_{-1,2}$

$$U_{-1,2} = \frac{g_{-1,2}}{1 + r(z_1 - 1) - \frac{k(z_1 - 1)}{z_1}} \quad (7.31)$$

with $g_{-1,2}$ given by Eq. (7.30).

Figures 7.1 and 7.2 are the real and imaginary parts of the burning rate response function $U_{-1,2}$, which describes the second order interaction of the first and second harmonics. As it was noted in Section 6 we can see a great change of response function close to $\omega = \omega_n/2$.

The burning rate up to the second order may be written as

$$v_1 = U_1 \eta_1 + U_{-1,2} \bar{\eta}_1 \eta_2$$

and the response function that includes both a linear term and nonlinear one is

$$V_1 = U_1 + U_{-1,2} \frac{\bar{\eta}_1 \eta_2}{\eta_1}. \quad (7.32)$$

To calculate V_1 we notice that

$$\frac{\bar{\eta}_1 \eta_2}{\eta_1} = \frac{p_2}{2p^0} e^{i\psi_2}.$$

To show the influence of the nonlinear effect Figs. 7.3 and 7.4 are pictured for the real and imaginary part of the response function V_1 . The necessary parameters are taken from Ref. 3 and given in the figure captions. A new aspect of the problem should be noticed. The response function V_1 depends on the phase shift ψ_2 between the first and the second harmonics. Figures 7.5 and 7.6 illustrate that.

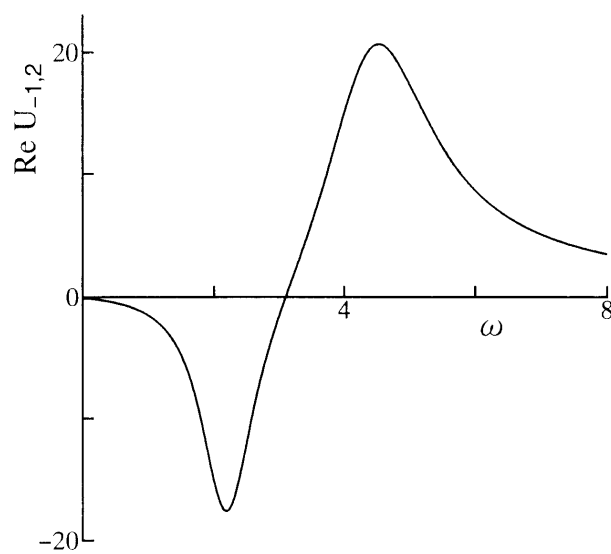


Fig. 7.1 Real part of second order burning rate response function.
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $\epsilon=0.2$.

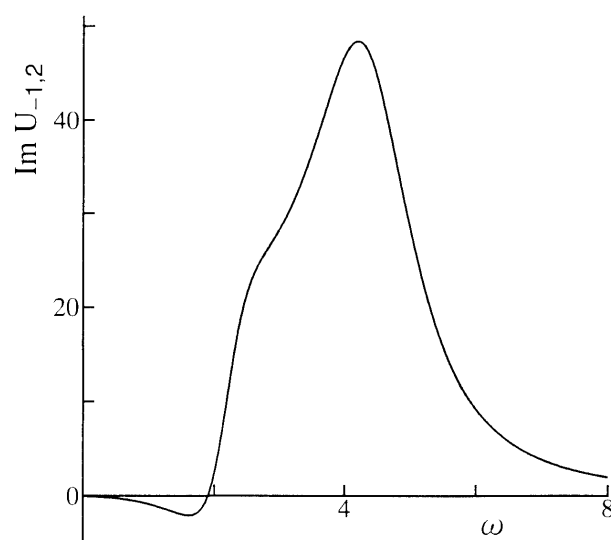


Fig. 7.2 Imaginary part of second order burning rate response function.
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $\epsilon=0.2$.

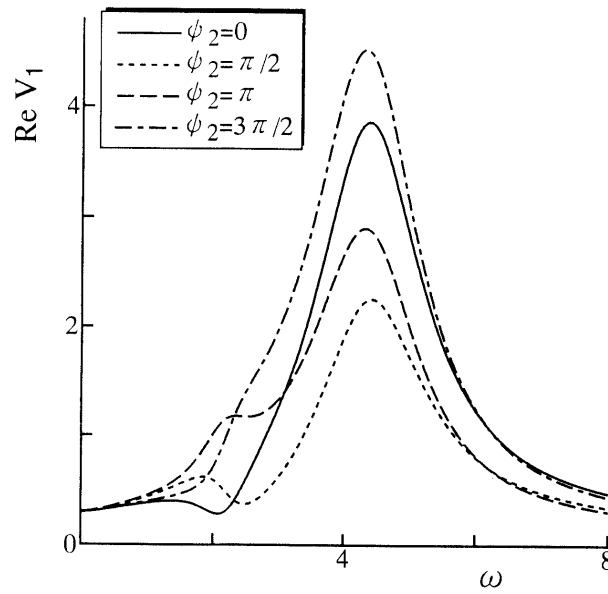


Fig. 7.3 Real part of burning rate response function V_1 .
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $p_2/p^0=0.048$, $\epsilon=0.2$.

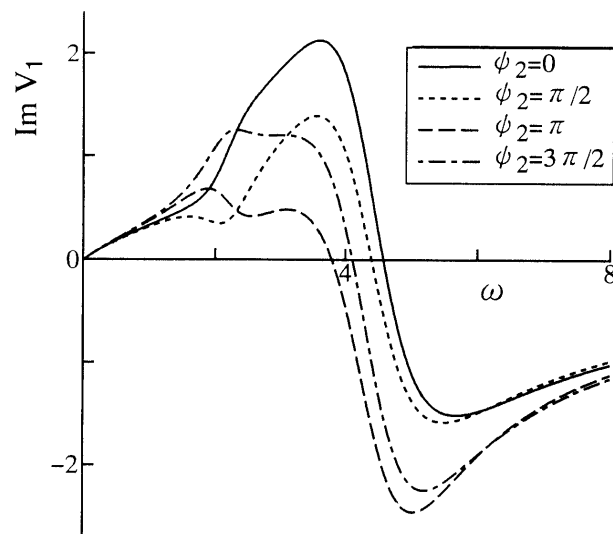


Fig. 7.4 Imaginary part of burning rate response function V_1 .
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $p_2/p^0=0.048$, $\epsilon=0.2$.

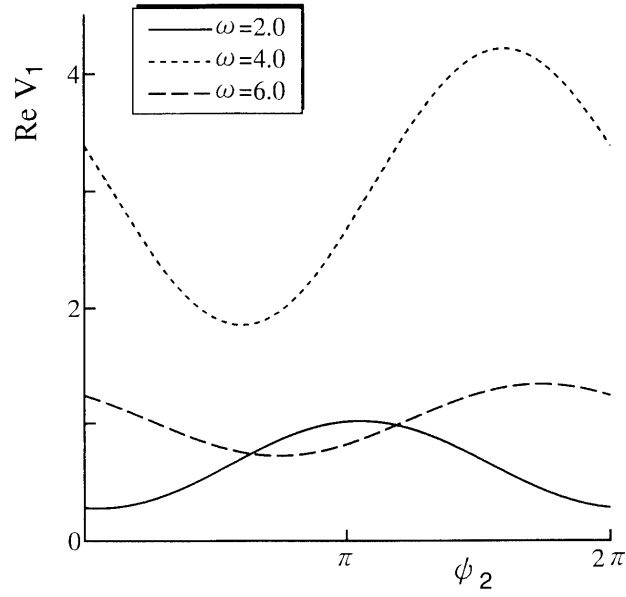


Fig. 7.5 Real part of V_1 as a function of phase shift ψ_2 .
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $p_2/p^0=0.048$, $\epsilon=0.2$.

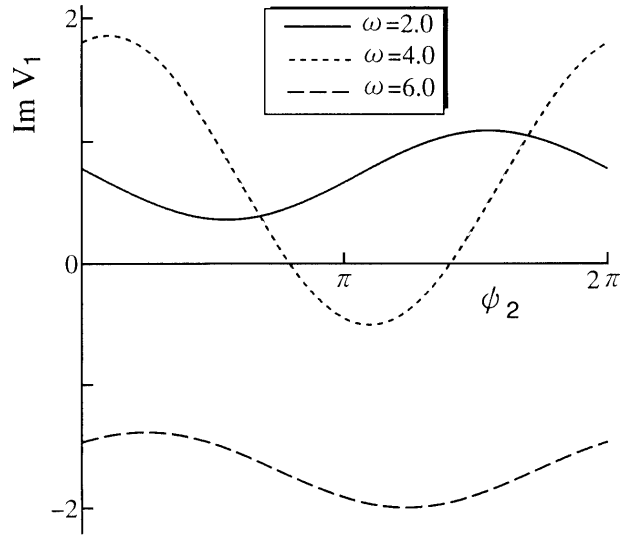


Fig. 7.6 Imaginary part of V_1 as a function of phase shift ψ_2 .
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $p_2/p^0=0.048$, $\epsilon=0.2$.

8. THE SECOND ORDER SELF-INTERACTION OF THE FIRST HARMONIC

The second order interaction of the first harmonic with the frequency of ω may give the oscillation of 2ω frequency. In this section the second order solid propellant burning rate response function

$$U_{1,1} = \frac{(v_2)_{1,1}}{\eta_1^2} \quad (8.1)$$

will be obtained.

The pressure $p(\omega)$ and burning rate $u(\omega)$

$$\begin{aligned} p &= p^0 + p_1 \cos \omega \tau + p_2 \cos(2\omega \tau + \psi_2) \\ u &= u^0 + u_1 \cos(\omega \tau + \psi_{u1}) + u_2 \cos(2\omega \tau + \psi_{u2}) \end{aligned}$$

are represented in the form

$$\begin{aligned} \eta &= 1 + \eta_1 e_1 + \bar{\eta}_1 \bar{e}_1 + \eta_2 e_2 + \bar{\eta}_2 \bar{e}_2 \\ v &= 1 + v_1 e_1 + \bar{v}_1 \bar{e}_1 + v_2 e_2 + \bar{v}_2 \bar{e}_2 \end{aligned} \quad (8.2)$$

where

$$\begin{aligned} \eta_1 &= \frac{p_1}{2p^0}, & \eta_2 &= \frac{p_2}{2p^0} e^{i\psi_2} \\ v_1 &= \frac{u_1}{2u^0} e^{i\psi_{u1}}, & v_2 &= \frac{u_2}{2u^0} e^{i\psi_{u2}} \\ e_1 &= e^{i\omega\tau}, & e_2 &= e^{2i\omega\tau}. \end{aligned}$$

and the $\bar{\eta}_n$, \bar{v}_n and \bar{e}_n are the complex conjugations of η_n , v_n , and e_n .

To solve the heat conduction equation for the condensed phase

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \xi^2} - v \frac{\partial \theta}{\partial \xi}, \quad \theta(\xi, \tau)|_{\xi \rightarrow -\infty} = \theta_a, \quad \theta(0, \tau) = \theta_s(\tau) \quad (8.3)$$

the space-time temperature distribution should be also represented as a set of two modes

$$\theta(\xi, \tau) = \theta^0(\xi) + \theta_1(\xi) e_1 + \bar{\theta}_1(\xi) \bar{e}_1 + \theta_2(\xi) e_2 + \bar{\theta}_2(\xi) \bar{e}_2 \quad (8.4)$$

where

$$\theta^0(\xi) = \theta_a + e^\xi$$

is the steady-state solution.

The derivatives $\partial \theta / \partial \tau$ and $\partial^2 \theta / \partial \xi^2$ are

$$\begin{aligned} \frac{\partial \theta}{\partial \tau} &= i\omega \theta_1 e_1 + 2i\omega \theta_2 e_2 + c.c. \\ \frac{\partial^2 \theta}{\partial \xi^2} &= \theta^{0''} + \theta_1'' e_1 + \theta_2'' e_2 + c.c. \end{aligned}$$

where *c.c.* are corresponding complex conjugate values.

The second term in the right-hand side of Eq. (8.3) is nonlinear. In the product

$$-v \frac{\partial \theta}{\partial \xi} = -(1 + v_1 e_1 + v_2 e_2 + c.c.) (\theta^{0'} + \theta_1' e_1 + \theta_2' e_2 + c.c.)$$

we should keep only the terms that oscillate with frequency of 2ω . After multiplying the two series of the previous expression and using the expression for $\partial \theta / \partial \tau$ and $\partial^2 \theta / \partial \xi^2$ we have the following equation for the second mode

$$\theta_2'' - \theta_2' - 2i\omega \theta_2 = v_2 \theta^{0'} + v_1 \theta_1' \quad (8.5)$$

which contains the nonlinear term $v_1 \theta_1'$.

In the linear approximation there are two equations for the first and second harmonics

$$\theta_1'' - \theta_1' - i\omega \theta_1 = v_1 e^\xi \quad (8.6)$$

$$\theta_2'' - \theta_2' - 2i\omega \theta_2 = v_2 e^\xi. \quad (8.7)$$

The solutions of Eqs. (8.6) and (8.7) are

$$\theta_1 = C_1 e^{z_1 \xi} + \frac{i}{\omega} v_1 e^\xi \quad (8.8)$$

$$\theta_2 = C_2 e^{z_2 \xi} + \frac{i}{2\omega} v_2 e^\xi \quad (8.9)$$

with

$$z_1 = \frac{1}{2} [1 + \sqrt{1 + 4i\omega}], \quad z_2 = \frac{1}{2} [1 + \sqrt{1 + 8i\omega}]. \quad (8.10)$$

Using Eq. (8.8) we can write Eq. (8.5) in the form

$$\theta_2'' - \theta_2' - 2i\omega \theta_2 = \left(v_2 + \frac{i}{\omega} v_1^2 \right) e^\xi + C_1 v_1 z_1 e^{z_1 \xi}. \quad (8.11)$$

It is easy to solve this equation to obtain

$$\theta_2 = D_2 e^{z_2 \xi} + \frac{i}{2\omega} \left(v_2 + \frac{i}{\omega} v_1^2 \right) e^\xi + \frac{i}{\omega} C_1 v_1 z_1 e^{z_1 \xi}. \quad (8.12)$$

From this expression we can find the surface temperature ϑ_2 and gradient of the temperature at the interface φ_2 corrections to the steady-state conditions $\vartheta^0 = \theta_a$ and $\varphi = 1$

$$\vartheta_2 = D_2 + \frac{i}{2\omega} v_2 + a, \quad a = -\frac{v_1^2}{2\omega^2} + \frac{i}{\omega} C_1 v_1 z_1 \quad (8.13)$$

$$\varphi_2 = D_2 z_2 + \frac{i}{2\omega} v_2 + b, \quad b = -\frac{v_1^2}{2\omega^2} + \frac{i}{\omega} C_1 v_1 z_1^2. \quad (8.14)$$

The additional two relationships for v_2 , ϑ_2 , and φ_2 are known from the nonsteady burning laws $v(\varphi, \eta)$ and $\vartheta(\varphi, \eta)$ written to the second order perturbations.

From Appendix B (see Eqs. (B.4)) we have

$$v_2 = v_\varphi \varphi_2 + v_\eta \eta_2 + N_{1,1}, \quad N_{1,1} = \frac{1}{2} v_\varphi \varphi_1^2 + v_{\varphi\eta} \varphi_1 \eta_1 + \frac{1}{2} v_{\eta\eta} \eta_1^2 \quad (8.15)$$

$$\vartheta_2 = \vartheta_\varphi \varphi_2 + \vartheta_\eta \eta_2 + M_{1,1}, \quad M_{1,1} = \frac{1}{2} \vartheta_\varphi \varphi_1^2 + \vartheta_{\varphi\eta} \varphi_1 \eta_1 + \frac{1}{2} \vartheta_{\eta\eta} \eta_1^2. \quad (8.16)$$

Equations (8.13 – 16) permit to obtain the burning rate amplitude v_2 as a function of frequency and, therefore, the second order response function $U_{1,1}$.

Substitution of Eq. (8.13) into Eq. (8.14) gives

$$\varphi_2 = \left(\vartheta_2 - \frac{i}{2\omega} - a \right) z_2 + \frac{i}{2\omega} v_2 + b.$$

Take ϑ_2 from Eq. (8.16) and put it in this expression that gives

$$\varphi_2(1 - \vartheta_\varphi z_2) = \vartheta_\eta \eta_2 z_2 + \frac{v_2}{z_2} + z_2(M_{1,1} - a) + b.$$

Multiplying Eq. (8.15) by $(1 - \vartheta_\varphi z_2)$ and using the previous relationship we have

$$v_2 \left(1 - \vartheta_\varphi z_2 - \frac{v_\varphi}{z_2} \right) = (v_\varphi \vartheta_\eta z_2 + v_\eta - \vartheta_\varphi v_\eta z_2) \eta_2 + v_\varphi [z_2(M_{1,1} - a) + b] + (1 - \vartheta_\varphi z_2) N_{1,1}. \quad (8.17)$$

For any propellant model v_φ , v_η , ϑ_φ , and ϑ_η may be expressed through parameters k , μ , ν , and r given by equations:

$$v_\varphi = \frac{k}{D}, \quad \vartheta_\varphi = \frac{r}{D}$$

$$v_\eta = \frac{\delta - \nu}{D}, \quad \vartheta_\eta = -\frac{\delta + \mu}{D}$$

where $D = k + r - 1$.

Now one can write Eq. (8.17) in the form

$$v_2 = \frac{\nu + \delta(z_2 - 1)}{1 + r(z_2 - 1) - k(z_2 - 1)/z_2} \eta_2 + \frac{g_{1,1}}{1 + r(z_2 - 1) - k \frac{(z_2 - 1)}{z_2}} \eta_1^2 \quad (8.18)$$

The first term of Eq. (8.18) represents the well-known linear response of the burning rate to oscillatory pressure and the second term gives us the second order response function

$$U_{1,1} = \frac{g_{1,1}}{1 + r(z_2 - 1) - k(z_2 - 1)/z_2} \quad (8.19)$$

where $g_{1,1}$ has the following expression

$$\begin{aligned} g_{1,1} &= g_h + g_l \\ g_h &= k(z_2 a - b)/\eta_1^2, \quad g_l = D[z_2(\vartheta_\varphi N_{1,1} - v_\varphi M_{1,1}) - N_{1,1}]/\eta_1^2 \end{aligned} \quad (8.20)$$

Here g_h is connected with nonlinearity of the heat-conduction equation, and g_l with nonlinearity of the nonsteady burning laws.

To calculate g_h we write using Eqs. (8.13 – 14)

$$z_2 a - b = -\frac{v_1^2}{2\omega^2} (z_2 - 1) + \frac{i}{\omega} C_1 v_1 z_1 (z_2 - z_1).$$

Equation (8.8) gives

$$C_1 z_1 = \varphi_1 - \frac{i}{\omega} v_1.$$

The correction φ_1 to the steady-state temperature gradient can be expressed in terms of the corrections of the pressure and velocity (see App. B)

$$\varphi_1 = \frac{1}{v_\varphi} (v_1 - v_\eta \eta_1).$$

Thus

$$z_2 a - b = -\frac{v_1^2}{2\omega^2} (z_2 - 1) + \frac{i}{\omega} v_1 (z_2 - z_1) \left[\frac{1}{v_\varphi} (v_1 - v_\eta \eta_1) - \frac{i}{\omega} v_1 \right]. \quad (8.21)$$

Because $v_1 = U_1 \eta_1$ we can rewrite Eq. (8.21) in the following form

$$\omega(z_2 a - b)/i\eta_1^2 = U_1^2 \left[-\frac{z_2 - 1}{2i\omega} + (z_2 - z_1) \left(\frac{1}{v_\varphi} - \frac{i}{\omega} \right) \right] - \frac{v_\eta}{v_\varphi} (z_2 - z_1) U_1.$$

As

$$z_1(z_1 - 1) = i\omega, \quad z_2(z_2 - 1) = 2i\omega$$

one can represent g_h term in the form

$$g_h = \frac{i}{\omega} (z_2 - z_1) \left[\left(D - \frac{k}{z_1 z_2} \right) U_1^2 + (\nu - \delta) U_1 \right]. \quad (8.22)$$

The same procedure as in previous section gives

$$g_l = \frac{D}{2v_\varphi^2} \{ z_1 [L_{\varphi\varphi} U_1^2 + 2(v_\varphi L_{\varphi\eta} - v_\eta L_{\varphi\varphi}) U_1 + (v_\eta^2 L_{\varphi\varphi} - 2v_\varphi v_\eta L_{\varphi\eta} + v_\varphi^2 L_{\eta\eta})] - [v_{\varphi\varphi} U_1^2 + 2(v_\varphi v_{\varphi\eta} - v_\eta v_{\varphi\varphi}) U_1 + (v_\eta^2 v_{\varphi\varphi} - 2v_\varphi v_\eta v_{\varphi\eta} + v_\varphi^2 v_{\eta\eta})] \}. \quad (8.23)$$

For the propellant model introduced in Section 7 we have

$$g_l = \frac{1}{2} [z_1 r(1 - \epsilon) - 1 + r(\epsilon + 1)] U_1^2 + \nu U_1 \quad (8.24)$$

and for $g_{1,1} = g_h + g_l$ the following expression

$$g_{1,1} = \left\{ \frac{2i}{\omega} (z_2 - z_1) \left(D - \frac{k}{z_1 z_2} \right) + r[z_1 + 1 - \epsilon(z_1 - 1)] - 1 \right\} \frac{U_1^2}{2} + \nu \left[1 + \frac{i}{\omega} (z_2 - z_1) \right] U_1. \quad (8.25)$$

Now we can calculate the response function $U_{1,1}$

$$U_{1,1} = \frac{g_{1,1}}{1 + r(z_2 - 1) - k(z_2 - 1)/z_2} \quad (8.26)$$

with $g_{1,1}$ given by Eq. (8.25).

The real and imaginary part of the burning response function $U_{1,1}$, which describes the second order self-interaction of the first harmonic, are pictured in Fig. 8.1 and 8.2.

The burning rate up to the second order may be written as

$$v_2 = U_2 \eta_2 + U_{1,1} \eta_1^2$$

and the response function has the form

$$V_2 = U_2 + U_{1,1} \frac{\eta_1^2}{\eta_2}. \quad (8.27)$$

It is easy to show that

$$\frac{\eta_1^2}{\eta_2} = \frac{p_1^2}{2p_0^0 p_2} e^{-i\psi_2}.$$

Figures 8.3 and 8.4 are pictured for the real and imaginary parts of V_2 . This response function also depends on the phase shift ψ_2 . Figures 8.5 and 8.6 illustrate this dependency.

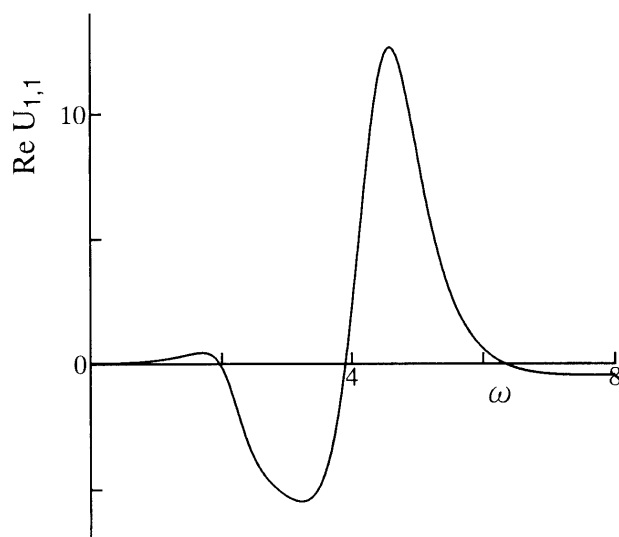


Fig. 8.1 Real part of second order burning rate response function.
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $\epsilon=0.2$.

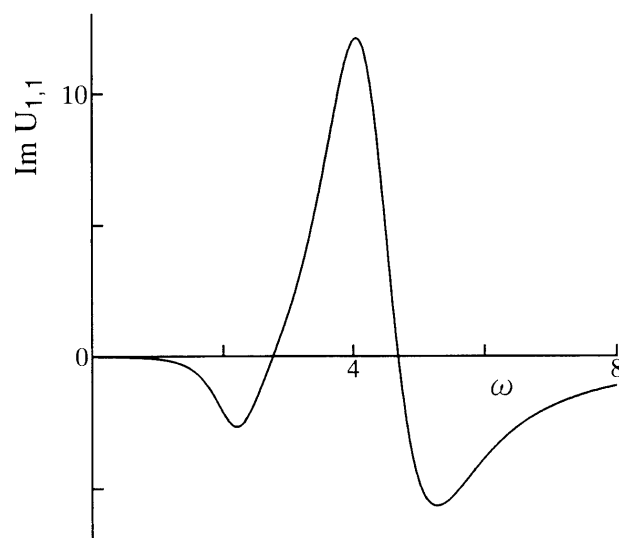


Fig. 8.2 Imaginary part of second order burning rate response function.
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $\epsilon=0.2$.

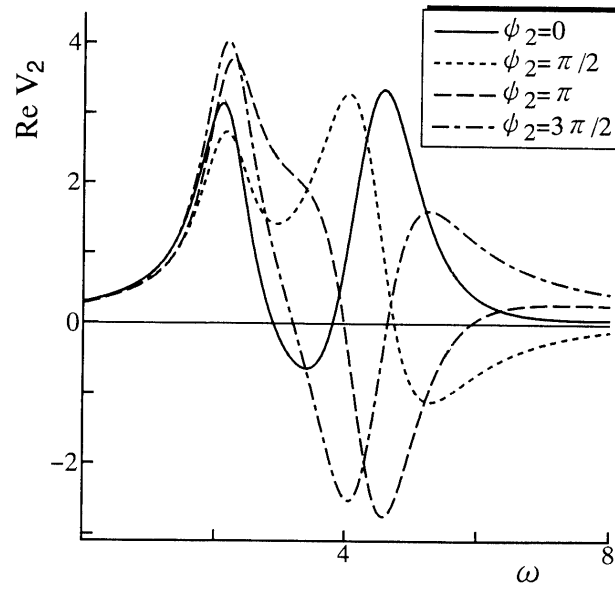


Fig. 8.3 Real part of burning rate response function V_2 .
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $p_1^2/2p^0p_2=0.24$, $\epsilon=0.2$.

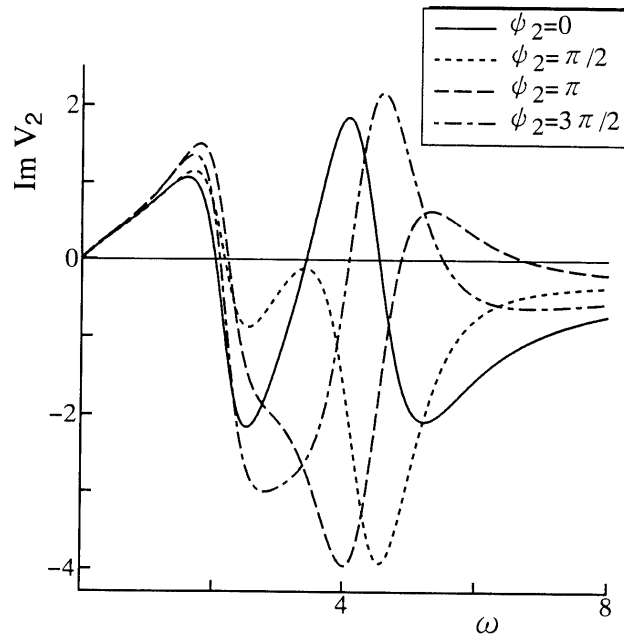


Fig. 8.4 Imaginary part of burning rate response function V_2 .
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $p_1^2/2p^0p_2=0.24$, $\epsilon=0.2$.

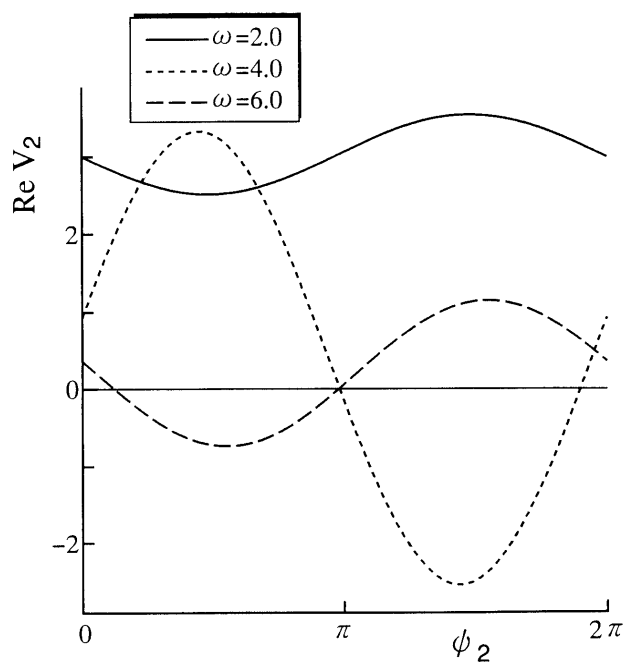


Fig. 8.5 Real part of V_2 as a function of phase shift ψ_2 .
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $p_1^2/2p^0p_2=0.24$, $\epsilon=0.2$.

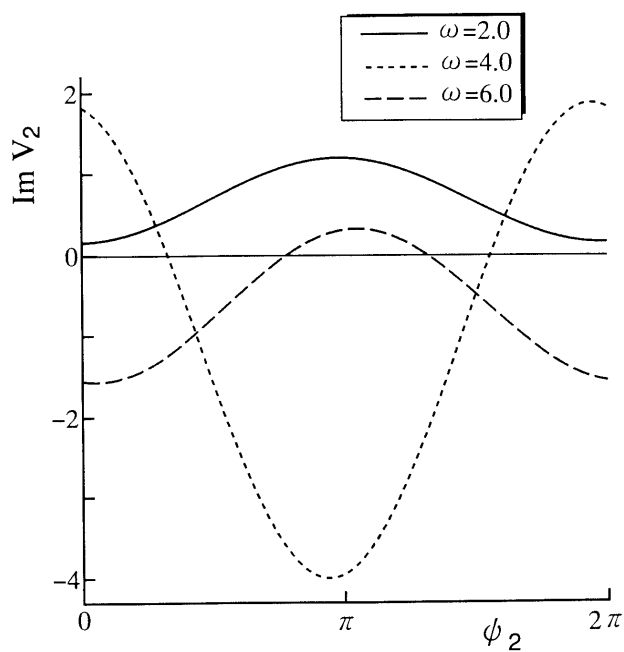


Fig. 8.6 Imaginary part of V_2 as a function of phase shift ψ_2 .
 $\nu=0.3$, $k=1.82$, $r=0.303$, $\delta=0$, $p_1^2/2p^0p_2=0.24$, $\epsilon=0.2$.

9. CONCLUSION

Research on the phenomena called combustion instabilities in a solid propellant rocket motor has been developed since 40's. At present there exist both analytical and numerical approaches to predict stability of solid motors and describe nonlinear nonsteady motions if steady-state combustion regime is unstable. Some computing programs, especially in the USA, pretend to be quantitative. The aim of this paper is to show that there are some aspects that should be investigated in this problem.

The primary, and the most complicated, cause of combustion instability is the propellant burning response to the pressure oscillations. It is enough for prediction of the stability condition to describe the burning response in the linear approximation. One should remember, however, that a burning propellant is an oscillator with a very high quality. So, any small perturbation in its parameters can greatly change the burning response to oscillatory pressure. As an example, we investigated in Section 5 the influence of internal radiant flux on the oscillatory behavior of a solid propellant. To describe a nonsteady gas motion and propellant combustion while the steady-state regime is unstable the nonlinear approach must be considered. As a rule, in the programs above mentioned only nonlinear acoustics is considered. In Sections 7 and 8 we show that the nonlinear effects in propellant combustion are of great importance. They cannot be neglected in the problem of nonsteady motion description. Only the simplest examples of nonlinear interactions between combustion and pressure oscillations are considered. They show, however, that there are a large number of problems to be solved in this field.

APPENDIX A

The relationships between derivatives with respect to the initial temperature from the one side and with respect to the temperature gradient are derived in this Appendix.

The steady-state relation Eq. (5.11)

$$f^0 = \frac{u^0}{\kappa} (T_s^0 - T_a) - \frac{I^0}{\rho_c C_c \kappa} \quad (\text{A.1})$$

should be used.

From Eq. (A.1) we have

$$\left(\frac{\partial f^0}{\partial T_a} \right)_p = \frac{(T_s^0 - T_a)}{\kappa} \left(\frac{\partial u^0}{\partial T_a} \right)_p + \frac{u^0}{\kappa} \left(\left(\frac{\partial T_s^0}{\partial T_a} \right)_p - 1 \right) - \frac{1}{\rho_c C_c \kappa} \left(\frac{\partial I^0}{\partial T_a} \right)_p$$

and introducing

$$s = \Delta \left(\frac{\partial \ln I^0}{\partial T_a} \right)_p, \quad \sigma = \left(\frac{\partial \ln I^0}{\partial \ln p} \right)_{T_a} \quad (\text{A.2})$$

we obtain

$$\left(\frac{\partial f^0}{\partial T_a} \right)_p = \frac{u^0}{\kappa} (D - sS^0), \quad D = k + r - 1. \quad (\text{A.3})$$

The derivatives of f^0 with respect to the pressure is

$$\left(\frac{\partial f^0}{\partial \ln p}\right)_{T_a} = \frac{u^0}{\kappa} \Delta(\nu + \mu - \sigma S^0). \quad (\text{A.4})$$

Because

$$\left(\frac{\partial u^0}{\partial f^0}\right)_p = \frac{(\partial u^0 / \partial T_a)_p}{(\partial f^0 / \partial T_a)_p}$$

from Eq. (A.3) we have

$$\left(\frac{\partial u^0}{\partial f^0}\right)_p = \frac{\kappa}{\Delta} \frac{k}{D - sS^0}. \quad (\text{A.5})$$

For

$$\left(\frac{\partial T_s^0}{\partial f^0}\right)_p = \frac{(\partial T_s^0 / \partial T_a)_p}{(\partial f^0 / \partial T_a)_p}$$

we obtain

$$\left(\frac{\partial T_s^0}{\partial f^0}\right)_p = \frac{\kappa}{u^0} \frac{r}{D - sS^0}. \quad (\text{A.6})$$

Similarly, because

$$\left(\frac{\partial I^0}{\partial f^0}\right)_p = \frac{(\partial I^0 / \partial T_a)_p}{(\partial f^0 / \partial T_a)_p}$$

we obtain

$$\left(\frac{\partial I^0}{\partial f^0}\right)_p = \frac{\kappa s S^0 \varrho_c C_c}{D - sS^0}. \quad (\text{A.7})$$

To calculate the derivatives with respect to the pressure at constant temperature gradient, it is useful to apply Jacobian-method. We write the set of equations

$$\begin{aligned} \left(\frac{\partial u^0}{\partial \ln p}\right)_{f^0} &= \frac{\partial(u^0, f^0)}{\partial(\ln p, f^0)} = \frac{\partial(u^0, f^0) / \partial(\ln p, T_a)}{\partial(\ln p, f^0) / \partial(\ln p, T_a)} = \\ &= \frac{(\partial u^0 / \partial \ln p)_{T_a} (\partial f^0 / \partial T_a)_p - (\partial u^0 / \partial T_a)_p (\partial f^0 / \partial \ln p)_{T_a}}{(\partial f^0 / \partial T_a)_p}. \end{aligned}$$

Using Eqs. (A.3, A.4) we have

$$\left(\frac{\partial u^0}{\partial \ln p}\right)_{f^0} = \frac{u^0 [\delta - \nu + S^0 (\sigma k - s\nu)]}{D - sS^0}. \quad (\text{A.8})$$

The same procedure gives

$$\left(\frac{\partial T_s^0}{\partial \ln p} \right)_{f^0} = -\Delta \frac{[\mu + \delta - S^0(\sigma r - S\mu)]}{D - S S^0} \quad (\text{A.9})$$

and for the radiation intensity

$$\left(\frac{\partial I^0}{\partial \ln p} \right)_{f^0} = S^0 \varrho_c C_c u^0 \Delta \frac{\sigma D - s(\nu + \mu)}{D - S S^0}. \quad (\text{A.10})$$

From Eqs. (A.5 – 10) in nondimensional form we have the relationships Eq. (5.19) used in Section 5.

APPENDIX B

The relations between complex amplitude of burning rate, temperature gradient, and pressure are obtained to second order terms in this appendix.

Expanding the nonsteady combustion laws $v(\varphi, \eta)$ and $\vartheta(\varphi, \eta)$ in a Taylor series up to second order terms we have

$$\begin{aligned} v_e &= v_\varphi \varphi_e + v_\eta \eta_e + \frac{1}{2} v_{\varphi\varphi} \varphi_e^2 + v_{\varphi\eta} \varphi_e \eta_e + \frac{1}{2} v_{\eta\eta} \eta_e^2 \\ \vartheta_e &= \vartheta_\varphi \varphi_e + \vartheta_\eta \eta_e + \frac{1}{2} \vartheta_{\varphi\varphi} \varphi_e^2 + \vartheta_{\varphi\eta} \varphi_e \eta_e + \frac{1}{2} \vartheta_{\eta\eta} \eta_e^2 \end{aligned} \quad (\text{B.1})$$

where subscript e denotes extra-terms to the steady-state conditions:

$$\begin{aligned} \eta &= 1 + \eta_e, & \varphi &= 1 + \varphi_e \\ v &= 1 + v_e, & \vartheta &= \vartheta^0 + \vartheta_e. \end{aligned}$$

Subscripts φ and η relate to derivatives with respect to temperature gradient and pressure for any function Y

$$\begin{aligned} \left(\frac{\partial Y}{\partial \varphi} \right)_\eta &= Y_\varphi, & \left(\frac{\partial Y}{\partial \eta} \right)_\varphi &= Y_\eta \\ \left(\frac{\partial^2 Y}{\partial \varphi^2} \right)_\eta &= Y_{\varphi\varphi}, & \frac{\partial^2 Y}{\partial \varphi \partial \eta} &= Y_{\varphi\eta}, & \left(\frac{\partial^2 Y}{\partial \eta^2} \right)_\varphi &= Y_{\eta\eta}. \end{aligned}$$

In Section 7 the interaction between first and second modes is investigated. In that case

$$\begin{aligned} \eta_e &= \eta_1 e_1 + \bar{\eta}_1 \bar{e}_1 + \eta_2 e_2 + \bar{\eta}_2 \bar{e}_2 \\ \varphi_e &= \varphi_1 e_1 + \bar{\varphi}_1 \bar{e}_1 + \varphi_2 e_2 + \bar{\varphi}_2 \bar{e}_2. \end{aligned} \quad (\text{B.2})$$

It is very easy to obtain for the first mode oscillation

$$\begin{aligned} \varphi_e^2 &\Rightarrow 2 \bar{\varphi}_1 \varphi_2 e_1, & \eta_e^2 &\Rightarrow 2 \bar{\eta}_1 \eta_2 e_1 \\ \varphi_e \eta_e &\Rightarrow (\bar{\varphi}_1 \eta_2 + \bar{\eta}_1 \varphi_2) e_1. \end{aligned}$$

Thus Eqs. (B.1) can be written as

$$\begin{aligned} v_1 &= v_\varphi \varphi_1 + v_\eta \eta_1 + N_{-1,2} \\ \vartheta_1 &= \vartheta_\varphi \varphi_1 + \vartheta_\eta \eta_1 + M_{-1,2} \end{aligned}$$

where

$$\begin{aligned} N_{-1,2} &= v_{\varphi\varphi} \bar{\varphi}_1 \varphi_2 + v_{\varphi\eta} (\bar{\varphi}_1 \eta_2 + \bar{\eta}_1 \varphi_2) + v_{\eta\eta} \bar{\eta}_1 \eta_2 \\ M_{-1,2} &= \vartheta_{\varphi\varphi} \bar{\varphi}_1 \varphi_2 + \vartheta_{\varphi\eta} (\bar{\varphi}_1 \eta_2 + \bar{\eta}_1 \varphi_2) + \vartheta_{\eta\eta} \bar{\eta}_1 \eta_2 \end{aligned} \quad (\text{B.3})$$

are nonlinear corrections that were used in calculation of the burning rate response function $U_{-1,2}$. In Section 8 the self-interaction of the first mode is investigated. In this case

$$\begin{aligned} \eta_e &= \eta_1 e_1 + \bar{\eta}_1 \bar{e}_1 \\ \varphi_e &= \varphi_1 e_1 + \bar{\varphi}_1 \bar{e}_1 \end{aligned}$$

and

$$\varphi_e^2 \Rightarrow \varphi_1^2 e_2, \quad \eta_e^2 \Rightarrow \eta_1^2 e_2, \quad \varphi_e \eta_e \Rightarrow \varphi_1 \eta_1 e_2.$$

Therefore, Eqs. (B.1) may be written in the form

$$\begin{aligned} v_2 &= v_\varphi \varphi_2 + v_\eta \eta_2 + N_{1,1} \\ \vartheta_2 &= \vartheta_\varphi \varphi_2 + \vartheta_\eta \eta_2 + M_{1,1} \end{aligned}$$

where

$$\begin{aligned} N_{1,1} &= \frac{1}{2} v_{\varphi\varphi} \varphi_1^2 + v_{\varphi\eta} \varphi_1 \eta_1 + \frac{1}{2} v_{\eta\eta} \eta_1^2 \\ M_{1,1} &= \frac{1}{2} \vartheta_{\varphi\varphi} \varphi_1^2 + \vartheta_{\varphi\eta} \varphi_1 \eta_1 + \frac{1}{2} \vartheta_{\eta\eta} \eta_1^2 \end{aligned} \quad (\text{B.4})$$

are nonlinear corrections that were used in calculation of the burning rate response function $U_{1,1}$.

APPENDIX C

Let a model of a propellant is defined by the next steady-state laws

$$u = A p^n e^{\beta T_a}, \quad u = B e^{-E/RT_s^0} \quad (\text{C.1})$$

where A , B , n , β , and E are constant values, and R is the universal gas constant.

The first derivatives Eqs. (3.1) with respect to pressure and initial temperature are easily calculated:

$$\begin{aligned} k &= \beta \Delta, & r &= \frac{\beta (T_s^0)^2 R}{E} \\ \nu &= n, & \mu &= \nu \frac{R (T_s^0)^2}{E \Delta}. \end{aligned} \quad (\text{C.2})$$

Note that for this model

$$\nu r - \mu k = 0.$$

In this appendix we obtain the second order derivatives of the burning rate and surface temperature with respect to pressure and surface gradient.

Using steady-state relationship

$$f^0 = \frac{u^0}{\kappa} (T_s^0 - T_a)$$

we obtain the nonsteady laws of combustion

$$u = Ap^n \exp \beta \left(T_s - \frac{\kappa f}{u} \right), \quad u = B \exp(-E/RT_s) \quad (\text{C.3})$$

Denoting $l = \ln p$ we have

$$\begin{aligned} \ln u &= \ln A + nl + \beta (T_s - \kappa f/u) \\ \ln u &= \ln B + \frac{E}{R} \left(\frac{1}{T_s^0} - \frac{1}{T_s} \right). \end{aligned} \quad (\text{C.4})$$

The first derivatives of these expression with respect to the gradient f are

$$\begin{aligned} \frac{1}{u} \left(\frac{\partial u}{\partial f} \right)_p &= \beta \left[\left(\frac{\partial T_s}{\partial f} \right)_p - \frac{\kappa}{u} + \frac{\kappa f}{u} \left(\frac{\partial u}{\partial f} \right)_p \right] \\ \frac{1}{u} \left(\frac{\partial u}{\partial f} \right)_p &= \frac{E}{RT_s^2} \left(\frac{\partial T_s}{\partial f} \right)_p. \end{aligned} \quad (\text{C.5})$$

From Eqs. (C.5) it follows

$$\left(\frac{\partial u}{\partial f} \right)_p = \frac{\kappa}{\Delta} \frac{k}{D}, \quad \left(\frac{\partial T_s}{\partial f} \right)_p = \frac{\kappa}{u^0} \frac{r}{D}. \quad (\text{C.6})$$

The first derivatives of Eqs. (C.4) with respect to $l = \ln p$ are

$$\begin{aligned} \frac{1}{u} \left(\frac{\partial u}{\partial l} \right)_f &= n + \beta \left[\left(\frac{\partial T_s}{\partial l} \right)_f + \frac{\kappa f}{u^2} \left(\frac{\partial u}{\partial l} \right)_f \right] \\ \frac{1}{u} \left(\frac{\partial u}{\partial l} \right)_f &= \frac{E}{RT_s^2} \left(\frac{\partial T_s}{\partial l} \right)_f. \end{aligned} \quad (\text{C.7})$$

From Eqs. (C.7) we have

$$\left(\frac{\partial u}{\partial l} \right)_f = -u^0 \frac{\nu}{D}, \quad \left(\frac{\partial T_s}{\partial l} \right)_f = -\Delta \frac{\mu}{D}. \quad (\text{C.8})$$

To obtain the second derivatives we rewrite Eqs. (C.5) in the form

$$\begin{aligned}\frac{1}{\beta} \left(\frac{\partial u}{\partial f} \right)_p &= u \left(\frac{\partial T_s}{\partial f} \right)_p - \kappa - \frac{\kappa}{u} f \left(\frac{\partial u}{\partial f} \right)_p \\ \left(\frac{\partial T_s}{\partial f} \right)_p &= \frac{RT_s^2}{E} \frac{1}{u} \left(\frac{\partial u}{\partial f} \right)_p.\end{aligned}\tag{C.9}$$

The second derivatives with respect to the gradient may be obtained from Eqs. (C.9)

$$\begin{aligned}\frac{1}{\beta} \left(\frac{\partial^2 u}{\partial f^2} \right)_p &= \left(\frac{\partial u}{\partial f} \right)_p \left(\frac{\partial T_s}{\partial f} \right)_p + \frac{\kappa}{u} \left(\frac{\partial u}{\partial f} \right)_p - \frac{\kappa}{u^2} f \left(\frac{\partial u}{\partial f} \right)_p^2 + \frac{\kappa}{u} f \left(\frac{\partial^2 u}{\partial f^2} \right)_p + u \left(\frac{\partial^2 T_s}{\partial f^2} \right)_p \\ \left(\frac{\partial^2 T_s}{\partial f^2} \right)_p &= \frac{2T_s R}{E} \left(\frac{\partial T_s}{\partial f} \right)_p \frac{1}{u} \left(\frac{\partial u}{\partial f} \right)_p - \frac{RT_s^2}{E} \frac{1}{u^2} \left(\frac{\partial u}{\partial f} \right)_p^2 + R \frac{T_s^2}{E} \frac{1}{u} \left(\frac{\partial^2 u}{\partial f^2} \right)_p.\end{aligned}$$

That leads

$$\begin{aligned}\left(\frac{\partial^2 u}{\partial f^2} \right)_p &= \frac{u^0}{(f^0)^2} \frac{k^2 [1 - r(\epsilon + 1)]}{D^3} \\ \left(\frac{\partial^2 T_s}{\partial f^2} \right)_p &= \frac{\Delta}{(f^0)^2} \frac{kr[2(1-r) - k + \epsilon(k-1)]}{D^3}.\end{aligned}\tag{C.10}$$

Denoting

$$X = \frac{\partial^2 u}{\partial l \partial f}, \quad Y = \frac{\partial^2 T_s}{\partial l \partial f}$$

we have from Eqs. (C.9)

$$\begin{aligned}X &= \beta \left(\frac{\partial u}{\partial l} \frac{\partial T_s}{\partial f} + uY - \frac{\kappa}{u^2} f \frac{\partial u}{\partial l} \frac{\partial u}{\partial f} + \frac{\kappa}{u} fX \right) \\ Y &= \frac{2RT_s}{E} \frac{\partial T_s}{\partial l} \frac{1}{u} \frac{\partial u}{\partial f} - \frac{RT_s^2}{E} \frac{1}{u^2} \frac{\partial u}{\partial l} \frac{\partial u}{\partial f} + \frac{RT_s^2}{E} \frac{1}{u} X.\end{aligned}$$

That leads

$$\begin{aligned}\frac{\partial^2 u}{\partial f \partial l} &= \frac{u^0}{f^0} \frac{k\nu}{D^3} (\epsilon r - k) \\ \frac{\partial^2 T_s}{\partial f \partial l} &= \frac{\Delta}{f^0} \frac{r\nu}{D^3} [\epsilon(1-k) + r - 1].\end{aligned}\tag{C.11}$$

To obtain the second derivatives with respect to the pressure we rewrite Eqs. (C.7) in the form

$$\frac{\partial u}{\partial l} = nu + \beta \left(u \frac{\partial T_s}{\partial l} + \frac{\kappa}{u} f \frac{\partial u}{\partial l} \right), \quad \frac{\partial T_s}{\partial l} = \frac{RT_s^2}{E} \frac{1}{u} \frac{\partial u}{\partial l}.$$

Denoting

$$X = \frac{\partial^2 u}{\partial l^2}, \quad Y = \frac{\partial^2 T_s}{\partial l^2}$$

we have

$$X = n \frac{\partial u}{\partial l} + \beta \left[\frac{\partial u}{\partial l} \frac{\partial T_s}{\partial l} + uY - \frac{\kappa}{u^2} f \left(\frac{\partial u}{\partial l} \right)^2 + \frac{\kappa}{u} fX \right]$$

$$Y = \frac{2RT_s}{E} \frac{1}{u} \frac{\partial T_s}{\partial l} \frac{\partial u}{\partial l} - \frac{RT_s^2}{E} \frac{1}{u^2} \left(\frac{\partial u}{\partial l} \right)^2 + \frac{RT_s^2}{E} \frac{1}{u} X.$$

That gives

$$\left(\frac{\partial^2 u}{\partial l^2} \right)_f = \frac{u^0}{D^3} \nu^2 (2k + r - 1 - r\epsilon)$$

$$\left(\frac{\partial^2 T_s}{\partial l^2} \right)_f = \frac{\Delta}{D^3} \nu \mu [k + \epsilon(k - 1)]. \quad (\text{C.12})$$

Here is the full list of the derivatives

$$\left(\frac{\partial u}{\partial f} \right)_p = \frac{\kappa}{\Delta} \frac{k}{D}, \quad \left(\frac{\partial T_s}{\partial f} \right)_p = \frac{\kappa}{u^0} \frac{r}{D}$$

$$\left(\frac{\partial u}{\partial l} \right)_f = -u^0 \frac{\nu}{D}, \quad \left(\frac{\partial T_s}{\partial l} \right)_f = -\Delta \frac{\mu}{D}$$

$$\left(\frac{\partial^2 u}{\partial f^2} \right)_p = \frac{u^0}{(f^0)^2} \frac{k^2 [1 - r(\epsilon + 1)]}{D^3}$$

$$\left(\frac{\partial^2 T_s}{\partial f^2} \right)_p = \frac{\Delta}{(f^0)^2} \frac{kr[2(1 - r) - k + \epsilon(k - 1)]}{D^3} \quad (\text{C.13})$$

$$\frac{\partial^2 u}{\partial f \partial l} = \frac{u^0}{f^0} \frac{k\nu(\epsilon r - k)}{D^3}, \quad \frac{\partial^2 T_s}{\partial f \partial l} = \frac{\Delta}{f^0} \frac{r\nu[\epsilon(1 - k) + r - 1]}{D^3}$$

$$\left(\frac{\partial^2 u}{\partial l^2} \right)_f = u^0 \frac{\nu^2 (2k + r - 1 - r\epsilon)}{D^3}, \quad \left(\frac{\partial^2 T_s}{\partial l^2} \right)_f = \Delta \frac{\mu\nu [k + \epsilon(k - 1)]}{D^3}.$$

In nondimensional form we have

$$v_\varphi = \frac{k}{D}, \quad \vartheta_\varphi = \frac{r}{D}$$

$$v_\eta = -\frac{\nu}{D}, \quad \vartheta_\eta = -\frac{\mu}{D}$$

$$v_{\varphi\varphi} = \frac{k^2 [1 - r(\epsilon + 1)]}{D^3}, \quad \vartheta_{\varphi\varphi} = \frac{kr[2(1 - r) - k + \epsilon(k - 1)]}{D^3} \quad (\text{C.14})$$

$$v_{\varphi\eta} = \frac{k\nu(\epsilon r - k)}{D^3}, \quad \vartheta_{\varphi\eta} = \frac{r\nu[\epsilon(1 - k) + r - 1]}{D^3}$$

$$v_{\eta\eta} = \frac{\nu^2 (2k + r - 1 - r\epsilon)}{D^3}, \quad \vartheta_{\eta\eta} = \frac{\mu\nu [k + \epsilon(k - 1)]}{D^3}.$$

Let us introduce the complexes

$$\begin{aligned}
L_{\varphi\varphi} &= \vartheta_{\varphi} v_{\varphi\varphi} - v_{\varphi} \vartheta_{\varphi\varphi} \\
L_{\varphi\eta} &= \vartheta_{\varphi} v_{\varphi\eta} - v_{\varphi} \vartheta_{\varphi\eta} \\
L_{\eta\eta} &= \vartheta_{\varphi} v_{\eta\eta} - v_{\varphi} \vartheta_{\eta\eta}.
\end{aligned}
\tag{C.15}$$

It is easy to calculate them using Eqs. (C.14). They are

$$\begin{aligned}
L_{\varphi\varphi} &= \frac{k^2 r}{D^3} (1 - \epsilon) \\
L_{\varphi\eta} &= \frac{kr\nu}{D^3} (\epsilon - 1) \\
L_{\eta\eta} &= \frac{r\nu^2}{D^3} (1 - \epsilon).
\end{aligned}
\tag{C.16}$$

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