Physics of Consolidation

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Abstract

During an attempt to systematize the theory of consolidation on the assumption that the deformation of clayey material obeys the law of isotropic elasticity, the physical background of Biot's theory of three-dimensional consolidation (J. Appl. Phys. 12, 155 (1941)) has become clear. The fundamental equations are constructed directly for the system of clay particles, whose pores are saturated with water. The external force which plays the essential part in the process of consolidation is the frictional traction between water and particles. The equation of excess-hydraulic pressure is similar to the equation of conduction of heat only in one-dimensional consolidation under steady surcharge.

The constancy of the *modulus of volume change*, introduced by Tschebotarioff (Soil Mechanics, Foundations and Earth Pressure (1952), p. 105), signifies the isotropic elasticity of the material. Its constancy, however, cannot be secured for clays, and the assumption of isotropic elasticity does not hold well for clayey foundations, though Terzaghi's theory of consolidation is based on this assumption. It is evident that systematic research is necessary to establish a theory of three-dimensional consolidation.

Introduction

It seems a common belief among researchers of soil mechanics that the same procedure as was employed by Terzaghi (17) in 1925 to deduce the equation of excess hydraulic pressure in one-dimensional consolidation may be extended without serious modifications to two- or three-dimensional cases, and that the equation of excess-hydraulic-pressure should be similar to that of heat-conduction also in this case. This procedure, however, involves unwarranted assumptions as will be evidenced in a subsequent section. The law of deformation of porous media should necessarily be taken into consideration in order to formulate exactly the process of consolidation.

To fulfill this want, Biot (1) presented in 1941 a general theory of three-dimensional consolidation. He introduced at first the stress-strain relation of the mixture of soil, water and air, following the usual method in the theory of elasticity. The moduli thus introduced, paralleled with the moduli of elasticity, are given later appropriate values so as to be applicable to the cases where the pores are saturated with water.

His theory, however, seems not to be widely accepted by researchers of soil mechanics. It is partly because his theory does not look to be based on sound physical concepts. The system composed of soil, water and air is more complicated than that of soil and water only. As an example, the transmission coefficient of water in the former case, that is, Darcy's coefficient in an unsaturated flow, varies with water

content and some other yet unknown factors (see refs. 2 & 3).

The clear-cut way of getting the general equations of three-dimensional consolidation may be the genuine extension of the physical meanings implied in Terzaghi's theory of one-dimensional consolidation to a three-dimensional case employing only the facts already established in soil mechanics.

The present notion of three-dimensional consolidation is somewhat confused. Terzaghi (4), Barron (5) and others (6, 7) claim that the equation of excess-pore-water-pressure should be similar to that of conduction of heat even in two- or three-dimensional cases. Terzaghi (4) and Rendulic (7), however, would not admit the modulus of consolidation in two- or three-dimensional cases as constant. Mandel (8) and Biot (1) state, on the other hand, that the equation of excess-pore-water-pressure differs from that of heat-conduction. Mikasa (10) and others (8 & 9) assume that the initial value of excess-pore-water-pressure should be the arithmetic mean of principal stresses. Biot (1), on the other hand, assumes $\varepsilon=0$ as the only initial condition, where ε means volume increase of the soil per unit of the initial volume. In one-dimensional consolidation with variable surcharge, Murayama et al. (11) assume the equation of excess-pore-water-pressure as similar to the equation of heat-conduction, but Akai (9) does not. Even in case of one-dimensional consolidation under steady surcharge, Aboshi (12) presented, in opposition to Terzaghi's theory, an experimental fact that the excess pore water pressure at a certain depth from the surface begins to increase with the application of surcharge and reaches its maximum after a certain lapse of time. It, therefore, is hoped to save the theory of consolidation from the present confused state.

The essential aim of the present research is placed in the systematization of the theory of consolidation along Biot's line, that is, the line that the law of deformation of porous media is assumed as that of an isotropic elastic body. But the present paper deals only with the clarification of the fact that the law of deformation of clay particles differs substantially from that of an isotropic elstic body. Another phase of research will be treated in future papers.

Deduction of the Fundamental Equations

The system we are now considering is composed of particles of clay and water filling the pores of accumulation. Four kinds of stresses are acting in this system: the hydraulic pressure in water; the frictional traction on the surface of grains caused by the motion of water; the stresses acting at the points in contact with the adjoining grains, which might be called here the intergranular stresses; and the stresses created inside the particles, putting them in mechanical equilibrium with the adjoining particles and water. These stresses have their own regions of action and they are, when expressed as functions of space, discontinuous functions being equal to zero outside the respective region of action.

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1. The relation between Biot's initial condition and Mikasa's and the point detected by Aboshi are not treated in this paper. Other points mentioned here are treated in the present paper.

The fact that the four kinds of stresses work in the system demand deliberate consideration so that the simplification adopted in soil mechanics might be clearly understood. The stresses inside the particles are brought into existence so that a particle inside which the stresses are working may be in mechanical equilibrium under the action of three other kinds of stresses working on its surface. In order, however, to make a rough estimation, we may assume that the four stresses may be divided into two groups, whose members are in equilibrium with each other, i.e., that the intergranular stresses are in equilibrium with the frictional traction, while the stresses inside the particles are in equilibrium with the hydraulic pressure. The hydraulic pressure may be assumed to be of the same value over the surface of a particle, since the size of a particle is very small as compared with the usual scale. It follows from these assumptions that the stresses inside the particles may be assumed to be equal to the hydraulic pressure, or, that the stresses inside the particles may be substituted by the hydraulic pressure.

There is another point to be considered. Intergranular stresses are discontinuous functions being equal to zero outside the contact points of particles. We may, however, assume the intergranular stresses to be continuous functions, because the size of particles are small. But, since we have assumed the intergranular stresses to be continuous functions, the porous media should be replaced by an ideal material which fills the region under consideration leaving no pores in it, but which can be percolated by water. The void ratio e should be considered in this case as a continuous function defined in this region, brought into existence concurrently with the introduction of the ideal material.

Let us now formulate the mechanical equilibrium of these stresses. We will choose arbitrarily any region of accumulation, and consider the stresses on the boundary surface of the region. The existence of bodily force is neglected, since this simplification is justified subsequently[2]. The boundary surface cuts through particles, and consists of sections of particles, sections of pores and boundary lines between sections of particles and pores. The stresses are as follows: the hydraulic pressure p on the sections of pores; the stresses inside the particles, of which the amount is expressed also by p, on the sections of particle; the intergranular stresses σ_x, σ_y, σ_z, τ_{xy}, τ_{yz} and τ_{zx}, the sign of which is taken as positive when normal stresses are pressure, all over the boundary surface; and the frictional traction along the boundary lines of the sections of particles and pores. The hydraulic pressure p may be regarded as working all over the boundary surface. The existence of frictional traction on the boundary surface may be neglected, since the resultants of stresses working on a line is very small in their magnitude as compared with those of stresses on a surface containing the line.

Another assumption that these stresses are in equilibrium is introduced, because the rate of consolidation is very small; or in the term of theoretical physics, the process of consolidation is assumed to be a quasi-static process. This assumption

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2. Cf. Footnote 4.

leads, as is evidenced in a subsequent section[3], to the one made by Terzaghi in one-dimensional consolidation that the sum of excess hydraulic pressure and intergranular pressure is always and everywhere constant.

Convert the surface integral of the stresses, which represents the mechanical equilibrium of porous media in the region under consideration, to the space integral by applying Gauss' formula in vector analysis, and we get the condition of equilibrium of stresses as follows;

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\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= -\frac{\partial p}{\partial x}, \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= -\frac{\partial p}{\partial y}, \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= -\frac{\partial p}{\partial z}.
\end{aligned}
\qquad (1)
$$

and

The set of equations (1) may be deduced in another way, as was shown in a past paper of the author (ref. 14). Take an infinitesimally small region in the system, and consider the stresses working on the region. If we may neglect the effect of the flow of pore water, we may assume that the stresses acting on this infinitesimally small region are only of two kinds, the hydraulic pressure and intergranular stresses. And the equilibrium of the stresses is formulated as the equations (1).

Another interpretation of the equations (1) is possible, and it is the most pertinent to the process of consolidation. This interpretation is based on the view that the frictional traction averaged in unit volume of accumulation may be formulated as the negative of the gradient of hydraulic pressure p. This view was once derived by the present author (ref. 2 & 13), when he constructed the equation of motion of underground water and soil moisture from Navier-Stokes equation which was assumed to represent the microscopic motion of the fluid flowing through the accumulation of soil grains and air bubbles. This derivation may not be repeated here, since the physical meaning of the above-mentioned view can be known easily. The loss of energy of the flow of pore water, which may be assumed in this case to fill the pores, is caused by the frictional traction on the surface of grains. Therefore, the loss of energy per unit volume of accumulation, which is the same as the negative of the gradient of hydraulic pressure, may be assumed to be equal to the frictional traction averaged in the unit volume of the accumulation.

In case of the ideal material introduced above as a mathematical substitute of the real porous media, the frictional traction averaged in a unit volume of real porous media should be regarded as a bodily force acting on the unit volume of the ideal material.

Take an infinitesimally small region in the accumulation of ideal material introduced above as a mathematical substitute of the real porous media, and consider the stresses working on this region. The hydraulic pressure is disregarded in this case,

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3. Cf. Footnote 5.

since it is in equilibrium itself. The frictional traction averaged in a unit volume of porous media should be regarded in this case as a bodily force acting on the unit volume of ideal material. Other stresses on the region are the intergranular stresses. And the equilibrium of these forces are formulated as the equations in (1). This view was once mentioned by de Josselin (16), though not clearly.

As the next step of our deduction, we assume that the volume change of accumulation is caused solely by the outlet of pore water. This assumption is formulated as

$$\dot{\varepsilon} = \mathrm{div}\ V, \tag{2}$$

where $\dot{\varepsilon} = \partial \varepsilon / \partial t$ means the rate of volume decrease of the accumulation per unit of the initial volume, and V the flux of pore water through a section of unit area under consideration. Combining this equation with Darcy's law

$$V = -k\ \mathrm{grad}\ p, \tag{3}$$

we get

$$\dot{\varepsilon} = -\mathrm{div}\ (k\ \mathrm{grad}\ p), \tag{4}$$

where k is Darcy's constant. If k is constant with regard to space coordinates, Eq. (4) becomes to

$$\dot{\varepsilon} = -k\Delta p, \tag{4'}$$

where Δ is the Laplacian operator standing for $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

When only the one-dimensional consolidation is dealt with, $\dot{\varepsilon}$ means simply the rate of decrease of the height per unit length. But the deformation in a three-dimensional consolidation is very complicated, and the flow of material of accumulation must be traced in detail.

To this end, two assumptions shall be introduced. Firstly, we assume that the deformation of the accumulation of particles is caused only by intergranular stresses, the amount of which is regulated by the flow of pore water. In this connection, however, it should be noticed that the *excess* intergranular stresses, or, the differences computed of every intergranular stress by subtracting the initial value from it, must be used to express the stress‐strain relation of the accumulation of particles, since the system is in equilibrium before the initial time. Subtracting the equation which expresses the state of equilibrium at the initial time from the respective equation of (1)[4], we get the differential equations which express the relation between the *excess* intergranular stresses and the excess hydraulic pressure. Changing the meaning in the equations in (1), and letting the same notations as used there to represent the *excess* intergranular stresses and the excess hydraulic pressure, the equations in (1) remain unchanged in their appearance.

As the second assumption, we introduce the relation between the *excess* intergranular stresses and the strains. The law of elasticity in isotropic material is assumed to be valid here for simplicity, though any other of the alternatives may

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4. This is the reason why bodily forces may not be considered in the equations in (1).

be employed. And the components of strain are connected with the components of stress by the formulas:

$$\sigma_x=2G\Big(\varepsilon_x+\frac{\nu}{1-2\nu}\varepsilon\Big), \quad \sigma_y=2G\Big(\varepsilon_y+\frac{\nu}{1-2\nu}\varepsilon\Big), \quad \sigma_z=2G\Big(\varepsilon_z+\frac{\nu}{1-2\nu}\varepsilon\Big), \Big\}$$

$$\tau_{xy}=G\gamma_{xy}, \quad \tau_{yz}=G\gamma_{yz}, \quad \tau_{zx}=G\gamma_{yz}, \tag{5}$$

the direction of compression being taken as positive both for strain and stress. In these equations, ε_x, ε_y, ε_z, γ_{xy}, γ_{yz}, and γ_{zx} are expressed, in terms of the components of displacement u, v and w, as

$$\varepsilon_x=\frac{\partial u}{\partial x}, \quad \varepsilon_y=\frac{\partial v}{\partial y}, \quad \varepsilon_z=\frac{\partial w}{\partial z},$$

$$\gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}, \quad \gamma_{yz}=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}, \quad \gamma_{zx}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x},$$

and ε stands for $\varepsilon_x+\varepsilon_y+\varepsilon_z$, G for the rigidity modulus, and ν for Poisson's ratio.

Substituting the equations in (5) into the equations in (1), the equations

$$G\Big(\Delta u+\frac{1}{1-2\nu}\frac{\partial\varepsilon}{\partial x}\Big)=-\frac{\partial p}{\partial x},$$
$$G\Big(\Delta v+\frac{1}{1-2\nu}\frac{\partial\varepsilon}{\partial y}\Big)=-\frac{\partial p}{\partial y}, \Big\} \tag{6}$$

and
$$G\Big(\Delta w+\frac{1}{1-2\nu}\frac{\partial\varepsilon}{\partial z}\Big)=-\frac{\partial p}{\partial z},$$

are reached. Eq. (4) and the set of equations (6) form the fundamental equations of three-dimensional consolidation. This set of equations is the same as Biot's equation (6.5) (ref. 1).

In this connection we will formulate the relation between ε and the void ratio e. Void ratio means the volume of void per unit volume of soil particles only. Suppose a unit volume of accumulation composed of soil particles and pore water containing soil particles of volume $\chi_s(t)$ at time t. After an infinitesimally small interval dt, its volume is reduced to $1-\dot\varepsilon dt$, but the total of volumes of soil particles remains constant. Therefore,

$$\chi_s(t+dt)=\frac{\chi_s(t)}{1-\dot\varepsilon dt}.$$

Rewriting this, we get the differential equation which represents the change of $\chi_s(t)$ with respect to time,

$$\dot\chi_s=\dot\varepsilon\chi_s, \tag{7}$$

where $\dot\chi_s$ stands for $\partial\chi_s/\partial t$. Integrating this with the initial condition that $\varepsilon=0$ at $t=0$, we get

$$\ln(\chi_s/\chi_s^{(0)})=\varepsilon, \tag{8}$$

where $\chi_s^{(0)}$ is the initial value of χ_s. Or, using the relation

$$\chi_s=1/(1+e), \tag{9}$$

Eq. (8) is changed to

$$\varepsilon = \ln \frac{1+e_0}{1+e}, \tag{8'}$$

where e_0 is the initial value of e.

Modifications of the Fundamental Equations

Differentiating the first, the second and the third equation of (6) with respect to x, y and z respectively and summing them up, we obtain the relation

$$\kappa \Delta \varepsilon = -\Delta p, \tag{10}$$

where

$$\kappa = 2G \frac{1-\nu}{1-2\nu}. \tag{11}$$

Eq. (10) gives rise to the relation

$$\kappa \varepsilon = -p + D, \tag{12}$$

where D is a harmonic function, that is, a function which satisfies $\Delta D = 0$. D is equal to the initial value of p at $t=0$, since $\varepsilon = 0$ at $t=0$; and D tends to $\kappa \varepsilon_\infty$ when $t \to \infty$, since $p \to 0$ when $t \to \infty$, where ε_∞ means the final value of ε. Combining Eq. (4') and Eq. (12), an equation of excess pore water pressure

$$\dot{p} = k\kappa \Delta p + \dot{D} \tag{13}$$

is reached, where $\dot{p}$ and $\dot{D}$ stand for $\partial p / \partial t$ and $\partial D / \partial t$ respectively. If D vanishes, the equation is reduced to the equation of a heat-conduction type with $k\kappa$ corresponding to C_v, the coefficient of consolidation in Terzaghi's theory.

To examine the nature of D, a set of equations (6) is expressed in the vector form;

$$\kappa \operatorname{grad} \varepsilon - G \operatorname{rot} \Omega = -\operatorname{grad} p, \tag{14}$$

where Ω is the rotation of displacement vector (u, v, w). Comparing Eq. (13) with Eq. (11), it is obvious that D is connected with Ω by the relation

$$\operatorname{grad} D = G \operatorname{rot} \Omega. \tag{15}$$

The rotation of displacement vector is not necessarily zero in three-dimensional case, and Eq. (13) cannot usually be the starting point of the mathematical treatment of three-dimensional consolidation.

Eq. (12) and Eq. (15) form the intermediate integrals of the set of differential equations in (6). And we may take the set of equations (4'), (12) and (15) as the fundamental equations of three-dimensional consolidation instead of those of (4') and (6). The components of Eq. (15) are

$$\left. \begin{array}{l} \dfrac{1}{G} \dfrac{\partial D}{\partial x} = \dfrac{\partial \Omega_z}{\partial y} - \dfrac{\partial \Omega_y}{\partial z}, \\[2mm] \dfrac{1}{G} \dfrac{\partial D}{\partial y} = \dfrac{\partial \Omega_x}{\partial z} - \dfrac{\partial \Omega_z}{\partial x}, \\[2mm] \dfrac{1}{G} \dfrac{\partial D}{\partial z} = \dfrac{\partial \Omega_y}{\partial x} - \dfrac{\partial \Omega_x}{\partial y}, \end{array} \right\} \tag{15'}$$

where

$$\left.\begin{aligned}
\Omega_x &= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \\
\Omega_y &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \\
\Omega_z &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.
\end{aligned}\right\} \tag{16}$$

If we take the case of two-dimensional consolidation, where the system is independent of z direction, the equations in (15′) become

$$\left.\begin{aligned}
\frac{1}{G}\frac{\partial D}{\partial x} &= \frac{\partial \Omega_z}{\partial y}, \\
\frac{1}{G}\frac{\partial D}{\partial y} &= -\frac{\partial \Omega_z}{\partial x}.
\end{aligned}\right\} \tag{15″}$$

These equations constitute the so-called Cauchy-Riemann's equation in the theory of functions of complex variables, whereby D and Ω_z are

$$\frac{1}{G}D = \text{real part of } f(x+iy),$$

$$\Omega_z = \text{imaginary part of } f(x+iy),$$

where $f(x+iy)$ is an arbitrary regular function of complex variable $x+iy$.

One-Dimensional Consolidation

Let us suppose the case where the system is consolidated only along the z direction. In this case, u and v are equal to zero, and all the quantities that are not zero are functions of z and t. Under these assumptions we get from the equations in (16) that $\Omega_x = \Omega_y = \Omega_z = 0$; from the equations in (15′), we get D as a function of t only; and from the equations in (5),

$$\sigma_z = \kappa \mathcal{E} \tag{17}$$

and

$$\sigma_x = \sigma_y = \frac{2G\nu}{1-2\nu}\mathcal{E}, \tag{18}$$

where

$$\mathcal{E} = \frac{\partial w}{\partial z}.$$

Combining Eq. (17) with Eq. (12), we get

$$\sigma_z + p = D.$$

If p is constantly zero at the surface, D should be equal, as shown by this equation, to the surcharge, say q, because $\sigma_z = q$ at $z=0$. Therefore,

$$\sigma_z + p = q. \tag{19}$$

Eq. (19) verifies Terzaghi's assumption that the sum of intergranular pressure and excess pore water pressure should be equal to the surcharge[5].

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5. This verifies the point at Footnote 3.

Surcharge q can be constant, but may vary with time according to circumstances. In the case where q is steady, the equation of excess hydraulic pressure is a heat-conduction type, as seen in Eq. (13). In the case, however, where q varies with time, it is not so simple, but rather similar to the equation of heat-conduction with continuous sources of heat in the system.

Empirical Value of κ

Taking the case of one-dimensional consolidation, and combining Eq. (8) with Eq. (17), we get

$$\ln(1+e_0) - \ln(1+e) = (1/\kappa)\,\sigma_z.$$

As the consolidation progresses, p becomes smaller, and finally it vanishes at every point in the system. σ_z is equal at this stage to the surcharge, say q. And the above equation becomes

$$\ln(1+e_0) - \ln(1+e_\infty) = (1/\kappa)\,q,$$

where e_∞ is the final void ratio.

In the routine test of consolidation, the surcharge is successively increased at interval of 24 hours. Assuming that the value of void ratio after 24 hours may be substituted for e_∞, and signifying the final void ratio at the ith stage by e_{i+1}, we get for the first stage

$$\ln(1+e_0) - \ln(1+e_1) = (1/\kappa)\,q_0.$$

Similarly, for the second stage

$$\ln(1+e_1) - \ln(1+e_2) = (1/\kappa)\,\Delta q_1,$$

and so on, and finally for the nth stage

$$\ln(1+e_{n-1}) - \ln(1+e_n) = (1/\kappa)\,\Delta q_{n-1},$$

where Δq_{i-1} means the increase of surcharge at the ith stage. Summing up each side of the above equations from the first to the ith stage, we get

$$\ln(1+e_0) - \ln(1+e_i) = (1/\kappa)\,q_i, \qquad\qquad (20)$$

where q_i is the surcharge at the ith stage, or $q_i = q_0 + \Delta q_1 + \cdots\cdots + \Delta q_{i-1}$. And it is concluded that, if the assumptions adopted as the bases of this treatise are valid in the case of clay, the relation between $\ln(1+e)$ and q should be a straight line, and, therefore, that its grade should be the reciprocal of κ.

This proposition was tested with twenty eight data obtained from four different localities. The conclusion is that, although the linear dependence of $\ln(1+e)$ on q may be regarded as tolerably satisfied in a few cases, it cannot be true in a larger part of the cases- One of the affirmative examples is shown in Fig. 1. In this figure, the surcharge is increased successively from point 1 to point 6, and decreased from point 6 to point 12, and increased again from point 12 to point 19. Linearity holds well from point 1 to point 6, and from point 16 to point 19. In other stages of this figure, the linearity does not hold well, but the departure from linearity

seems to be caused more or less by the fact that the setup is so constructed that the water once expelled out of the system can hardly be taken up completely, admitting air into the system in the process of decreasing surcharge. One of the negative examples is shown in Fig, 2.
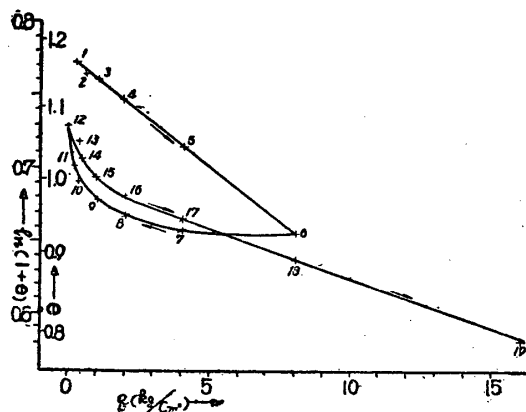

Fig. 1

In (1+e) versus q relation of a Boston Blue clay.
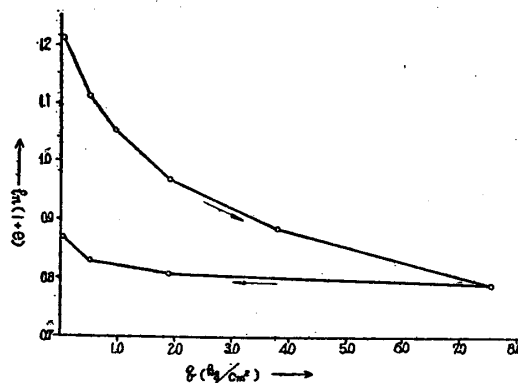
Prepared from Taylor (18), Fig. 10.5

Fig. 2

In (1+e) versus q relation of a clay, undisturbed sample from Amagasaki, Japan.

The relation between κ and the modulus of consolidation familiar in soil mechanics can be known easily. Regarding e_i as a continuous function of q_i in Eq. (19), we get

$$\frac{1}{1+e}\frac{de}{dq} = -\frac{1}{\kappa}.$$

(21)

Therefore, a_v, introduced by Terzaghi (4) and termed by him as the *coefficient of compressibility*, and m_v, introduced by Tschebotarioff (15) and termed by him as the *modulus of volume change*, may be put as

$$a_v = (1+e)/\kappa$$

(22)

and

$$m_v = 1/\kappa$$

(23)

respectively.

Criticism of the Present Theory of Three-Dimensional Consolidation

At present, there are two ways of formulating the process of consolidation: the one is the theory along Biot's line, and the other is the equation of excess-pore-water-pressure of the heat-conduction type. We have seen in previous sections that the two ways of formulation cannot give identical results except in one-dimensional consolidation with steady surcharge. In this section we will examine in the latter formulation the validity of three-dimensional consolidation.

The equation of excess-pore-water-pressure in three-dimensional consolidation is deduced as follows (Terzaghi 17). The quantity of water expelled out of an infinitesimally small volume dv during an interval dt is div $V \cdot dv \cdot dt$. Or, when combined with Darcy's formula $V = -\text{grad} p$, this becomes $-\text{div}(k \text{ grad} p) \cdot dv \cdot dt$. If, moreover, k is constant with regard to the space-coordinate, the above formula is transformed

to

$$-k\Delta p. \tag{24}$$

The quantity of water contained in dv is $(e/1+e)dv$, and its decrease in an interval dt is $-(\partial/\partial t)(edv/(1+e))dt$. In this quantity $dv/(1+e)$ does not change, because this is equal to the total of volume of soil particles. Therefore the above quantity is transformed to

$$-\frac{dv}{1+e}\frac{\partial e}{\partial t}dt. \tag{25}$$

Equating (25) to (24), we get

$$\frac{1}{1+e}\frac{\partial e}{\partial t}=k\Delta p. \tag{26}$$

Terzaghi assumes that

$$-\frac{de}{d\sigma_z}=a_v=\text{const.}, \tag{27}$$

where σ_z is the normal intergranular stress in z direction. If, however, the deformation of porous media obeys the law of isotropic elasticity, Eq. (27) should be replaced by Eq. (21), which shall be expressed here as

$$-\frac{1}{1+e}\frac{de}{d\sigma_z}=\frac{1}{\kappa}. \tag{28}$$

On integration, this equation becomes

$$\frac{1}{\kappa}\sigma_z=\ln\frac{1+e_0}{1+e}, \tag{28'}$$

where the initial condition that $\sigma_z=0$ when $e=e_0$ is employed. Eq. (28′) may be obtained in another way, that is, by combining Eq. (17) with Eq. (8′), to show that Eq. (28) should be used here in place of (27).

Another assumption employed by him is that

$$p+\sigma_z=p_0. \tag{29}$$

Combining (28) with (29), we get

$$\frac{1}{1+e}\frac{de}{dp}=\frac{1}{\kappa}. \tag{30}$$

Introducing (29) into (28), the equation of excess-pore-water-pressure is obtained as

$$\frac{\partial p}{\partial t}=k\kappa\Delta p. \tag{31}$$

This procedure of deducing Eq. (31) involves unwarranted assumptions; namely, Eq. (17) and Eq. (28) hold true only in one-dimensional consolidation. They should be replaced in case of three-dimensional consolidation by the equations in (5) and Eq. (12) respectively.

It should be recognized further that, as can be easily seen by differentiating Eq. (8′), the left-hand-side-member of Eq. (26) is identical with ε, and therefore that Eq. (26) is the same as Eq. (4′).

When, therefore, all the invalidities in Terzaghi's theory are corrected, the equation of excess-pore-water-pressure is obtained as Eq. (13), which contains other variables than p such as u, v, w, and their derivatives $\dot{u}$, $\dot{v}$, $\dot{w}$, and it cannot be solved by itself.

It is logical to infer that, if the isotropic elasticity of clayey foundation may be assumed, the consolidation theory along Biot's line is the only right one. But this inference is also not perfect, because an actual clay does not necessarily satisfy the assumption of isotropic elasticity as shown in Fig. 2.

On the contrary, in one-dimensional consolidation the constancy of $k\kappa$ seems to hold fairly well, though neither k nor κ is constant. The present author is now examining this point from theoretical point of view in order to clarify the nature of secondary consolidation and, though the research is not yet completed, he thinks he may say that the range of variation of $k\kappa$ is far smaller than that of κ. In some cases among those he examined, κ reached at the final stage of consolidation 7 or 10 times its initial value, and $k\kappa$, two or three times the initial value of itself. And, moreover, it seems even possible to reduce further the range of variation of $k\kappa$ by selecting a more appropriate functional form of k on e. The secondary consolidation seems to be caused by the motion of adsorbed moisture (heterogeneous phase composed of water and other substances absorbed around particles), which cannot so easily set in motion as the capillary moisture (homogeneous phase composed of water and solutes in pores). This research will be published in due time.

In two- or three-dimensional consolidation, however, the constancy of $k\kappa(=c_v)$ does not seem to hold so well. Terzaghi (20, p. 295) remarks on this point, "For processes of consolidation involving linear flow this assumption (constancy of c_v) is known to be reasonably accurate. However, in connection with two- and three-dimensional consolidation, the same assumption should be regarded as a potential source of errors whose importance is not yet known." It is clarified in this paper that two- or three-dimensional consolidation is substantially different from one-dimensional consolidation under steady surcharge. In the former, the rotational displacement of soil particles can take place and the equation of excess-pore-water-pressure is not necessarily of the type of heat-conduction. In one-dimensional consolidation, on the other hand, a plane which lies in parallel with the surface of surcharge at the initial time moves in parallel with the initial plane throughout the process of consolidation. The coefficient κ, therefore, may have substantially different values in the two forms of motion of particles. It is because the non-linearity of κ, probably caused by other properties than the elasticity of soil structure, have possibilities of being exaggerated in rotational displacements. The irreversibility of κ in increasing and decreasing processes of surcharge may add to this tendency.

The mathematical treatment of three-dimensional consolidation can be carried out rigorously, if the isotropic elasticity may be assumed, and if k may be assumed to be kept constant with regard to its space coordinate throughout the process of consolidation, by solving the set of differential equations (4'), (12) and (15) simul-

taneously. The initial condition in this treatment should be $\varepsilon=0$, as Biot put it. Strictly speaking, the equation of excess pore water pressure in the form of heat conduction should not be adopted as a fundamental equation.

It should be remarked, however, that the calculation along Biot's line may not necessarily give better results than the equation in the form of heat conduction, since the constancy of coefficients of consolidation do not hold well. The latter, though illogical, may give in some cases results which can fit better to the empirical data than the former.

Therefore, it seems indispensable, in order to establish a theory of three-dimensional consolidation, to employ for the first step such ideal materials as sponge, gum or vinyl to test Biot's theory of three-dimensional consolidation. Various properties other than isotropic elasticity, such as anisotropy, viscoelasticity, and so on, should be added to Biot's theory after it has been verified by those ideal materials.

The Meaning of Dott-Differentiation

There are two methods in hydrodynamics of formulating the motion of liquid, the Eulerian and the Lagrangian. In the former the change of quantities at any point fixed in space is followed, whereas in the latter the motion of particles contained in the system is in question. The formulas containing the differentials with respect to time are utterly different in the two methods. Denoting the differential of F with respect to time in the Eulerian method by $\partial F/\partial t$, and that in the Lagrangian method by $\delta F/\delta t$, they are connected, as is well known in hydrodynamics, by the relation

$$\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + \dot{u}\frac{\partial F}{\partial x} + \dot{v}\frac{\partial F}{\partial y} + \dot{w}\frac{\partial F}{\partial z},$$

where $\dot{u}$, $\dot{v}$ and $\dot{w}$ are x, y and z component of velocity respectively. In order to develop the present theory in the future, it seems important to recognize that also in the process of consolidation there are two ways of differentiation, even though the discrepancy between them may not be so large in usual cases.

In the fundamental formulas deduced in previous sections, the differentials with respect to space make no difference between the two ways of formulation. The differential with respect to time, however, should be considered in detail. Let us consider a differentially small region of accumulation located at point $(x,\ y,\ z)$ at time t. The excess intergranular stresses and the excess pore water pressure in this small region satisfies the equations in (1). The region is displaced after an interval δt to the point $(x+\dot{u}\delta t,\ y+\dot{v}\delta t,\ z+\dot{w}\delta t)$ and its volume is decreased by the amount of water expelled out of the region. The amount of decrease per unit initial volume is formulated as

$$\varepsilon(t+\delta t) - \varepsilon(t) = \int_t^{t+\delta t} \operatorname{div} V \cdot dt.$$

For $\delta t \to 0$, this equation becomes

$$\frac{\delta \varepsilon}{\delta t} = \operatorname{div} V.$$

Therefore, it is obvious that the differential with respect to time in Eq. (2) is the one in the Lagrangian type.

Since the equations in (6) hold without any modification, so does Eq. (12), too. The dott-differentiation in Eq. (13), therefore, is of the Lagrangian type.

The dott-differentiation in Eq. (7) is also of the Lagrangian type. The integration of Eq. (7), therefore, does not necessarily lead to Eq. (8) in this case.

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