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Propagation of Spacially Non-uniform Shock Waves

By

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Summary. Effect of shear stress through a slip stream created behind a shock wave with non-uniform strength is discussed. An expression on the attenuation of the shock wave propagating through a channel with varying cross section and with momentum exchange between the neighbouring stream tubes was obtained. Using this relation and the ray-shock theory, diffraction of shock waves around a sharp convex corner was analysed. The result gives more reasonable shock pattern than those calculated by the ordinary analysis without shear effect.

1. INTRODUCTION

G. B. Whitham [1], [2] analysed diffraction of plane shock waves in his two papers. Introducing an orthogonal coordinate system constructed by a set of the successive positions of the shock wave-shocks- and their orthogonal trajectories-rays, and using a geometrical relation between the rays and the shocks and an approximate decay rule of the shock waves propagating through a channel with a varying cross section, he presented a theory, which can predict the shape of the diffracted shock wave. However, some, discrepancies between the theoretical prediction and the experimental data were reported in an author's paper [3].

In that paper, the decay rule was discussed on the more mathematically rigorous foundation, and a new theory was proposed, which takes an account of the shear stress between the neighbouring ray tubes using a hypothesis of turbulent mixing. However, as stated in that paper, there is no physically sound bases of this rather arbitrary assumption of the mixing mechanism. Furthermore, although the discrepancy between the theory and the experimental data was remarkably improved, some problems are still open to question; for example, we do not have any useful theory to obtain a shock pattern for the weak shock waves with Mach numbers near unity yet.

In this report, we started with the non-viscus two dimensional gasdynamic equations and derived an expression of the shear stress. Thus we can calculate the shear effect between the neighbouring ray tubes without any empirical factor (which is unavoidable in the previous calculation). From these equations, it is shown, the shear stress between the neighbouring ray tubes can not be neglected if the flow velocity is comparable to the velocity difference between the neighbouring ray tubes. This is certainly the case of the weak shock waves, in which the flow velocity of the initial undisturbed flow may decrease to zero (where the shock Mach number is 1). The equation obtained is a system of second order non-linear partial differential equations. Although further attempt to obtain a full solution of these equations has not been tried, it is seen that this solution may give a correct pattern of the diffracted shock wave even for the cases in which a weak shock wave diffracts around a sharp convex corner with a large turning angle. Then, based on the original solution without the shear stress given by Whitham, a correction to that solution is presented, which can give a flow pattern more agreeable with the experiments.

2. Quasi-one-dimensional Flow Analysis

The basic equations of gasdynamics of the two-dimensional non-viscus but compressible flow are written in a Cartesian coordinate (x, y) and time t as

$$\rho_t+\rho(u_x+u_y)+u\rho_x+v\rho_y=0 \tag{1}$$

$$u_t+uu_x+vu_y+\frac{1}{\rho}p_x=0 \tag{2}$$

$$v_t+uv_x+vv_y+\frac{1}{\rho}p_y=0 \tag{3}$$

$$s_t+us_x+vs_y=0 \tag{4}$$

where ρ, u, v, and s denote the density, the x- and y- components of the fluid velocity, the pressure and the specific entropy of the gas, respectively. The suffices x and y represent the partial derivatives with respect to the respective independent variables. The boundary conditions are given on the shock front as

$$\rho_2=\rho_0\frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} \tag{5}$$

$$u_2=\frac{2a_0}{(\gamma+1)}\left(M-\frac{1}{M}\right) \tag{6}$$

$$v_2=0 \tag{7}$$

$$p_2=\frac{p_0}{(\gamma+1)}(2\gamma M^2-\gamma+1) \tag{8}$$

$$a_2=\frac{a_0}{(\gamma+1)}\frac{\sqrt{(2\gamma M^2-\gamma+1)\{2+(\gamma-1)M^2\}}}{M} \tag{9}$$

where the suffices 2 and 0 denote the conditions just behind the shock wave and of the uniform undisturbed flow, respectively. γ is the ratio of the specific heats of the gas, a denotes the sound speed and M is the Mach number of the shock front. Here we assume that the shock wave is non-uniform but parallel to the y-axis in a moment considered. The position of the shock wave is not known a-priori, which is the main difficulty of this problem.

In order to obtain an order estimation of the each term, we assume for a while an incompressible stationary flow, then from the equation (1), we have

$$v=\int u_x dy \approx u_y \Delta y\,. \tag{10}$$

Then the equation (2) is modefied into

$$u_t+(u+u_y\Delta y)u_x+\frac{1}{\rho}p_x=0\,. \tag{11}$$

Thus, if the velocity difference between the neighbouring ray tubes, $u_y\Delta y$, is comparable with the velocity, u, then the term vu_y cannot be neglected. This is the term omitted in the ordinary quasi-one-dimensional analysis, and expresses the effect of the shear stress between the neighbouring ray tubes.

Furthermore, it is worthwhile to note here, if we introduce the vorticity $\omega(=u_y-v_x)$, this term can be approximately expressed as $v\omega$, that is, this term corresponds to the convective transport of the vorticity.

Thus, we find a quasi-one-dimensional flow equation, which includes this vorticity transport effect, and with which we are going to start,

$$\rho_t+u\rho_x+\rho u_x+\frac{\rho u}{A}A_x=0 \tag{12}$$

$$u_t+(u+v)u_x+\frac{1}{\rho}p_x=0 \tag{13}$$

$$p_t+up_x+a^2\rho u_x+\frac{\rho u a^2}{A}A_x=0 \tag{14}$$

where $v=u_yA$ and A is the width of the channel. Note that v is a function of x and t, and has no explicit relation with u.

3. Generalized Decay Rule

Following the process adopted in the previous paper, the generalized decay rule is obtained as follows; The characteristics of these equations are

$$\frac{dx}{dt}=u+\frac{1}{2}v+a\sqrt{1+\frac{v^2}{4a^2}} \quad \text{on C+characteristics} \tag{15}$$

$$\frac{dx}{dt}=u+\frac{1}{2}v-a\sqrt{1+\frac{v^2}{4a^2}} \quad \text{on C−characteristics} \tag{16}$$

$$\frac{dx}{dt}=u \quad \text{on P characteristics} \tag{17}$$

Comparing with the ordinary case treated in the previous paper, we find that the vorticity transport effect has no influence on the P characteristics but changes the gradients of the C+ and C−characteristics. The characteristic form for C+characteristics is

$$\left\{\sqrt{1+\frac{v^2}{4a^2}}+\frac{1}{2}\frac{v}{a}\right\}\left[u_t+\left\{u+\frac{1}{2}v+a\sqrt{1+\frac{v^2}{4a^2}}\right\}u_x\right]$$
$$+\frac{1}{a\rho}\left[p_t+\left(u+\frac{1}{2}v+a\sqrt{1+\frac{v^2}{4a^2}}\right)p_x\right]+\frac{au}{A}A_x=0\,. \tag{18}$$

The further procedure is also quite similar to the one used before. The equation (18) is integrated along the C+characteristics (15) from the initial uniform flow region (denoted by the suffix 1) to the region just behind the shock wave (denoted by the suffix 2),

$$\int_{u_1}^{u_2}\left\{\sqrt{1+\frac{v^2}{4a^2}}+\frac{1}{2}\frac{v}{a}\right\}du+\int_{p_1}^{p_2}\frac{1}{a\rho}\,d\rho$$
$$+\int_{\ln A_1}^{\ln A_2}\frac{au}{u+\frac{1}{2}v+a\sqrt{1+\frac{v^2}{4a^2}}}\,d(\ln A)=0 \tag{19}$$

and then partially differentiate with respect to x. Now if we neglect the terms from the region except the neighbouring zone to the shock front, we obtain

$$\left\{\sqrt{1+\frac{v_2^2}{4a_2^2}}+\frac{1}{2}\frac{v_2}{a_2}\right\}\frac{\partial u_2}{\partial x}+\frac{1}{\rho_2 a_2}\frac{\partial p_2}{\partial x}$$
$$+\frac{a_2 u_2}{u_2+\frac{1}{2}v_2+a_2\sqrt{1+\frac{v_2^2}{4a_2^2}}}\frac{\partial(\ln A_2)}{\partial x}=0\,. \tag{20}$$

Since the flow quantities with the suffix 2 are determined by the relations (5) to (9) as the functions of M, this equation immediately gives the decay rule, which is

$$D=-\frac{A}{M}\frac{dM}{dA}=\frac{1-\frac{1}{M^2}}{\left[\left\{\sqrt{1+\frac{v_2^2}{a_2^2}}+\frac{1}{2}\frac{v_2}{a_2}\right\}\left(1+\frac{1}{M^2}\right)+2\mu\right]}$$
$$\times\frac{1}{\left[\sqrt{1+\frac{v_2^2}{4a_2^2}}+\frac{2(1-\mu^2)}{(\gamma+1)\mu}\left\{1+\frac{(\gamma+1)Mv_2}{4a_2(M^2-1)}\right\}\right]} \tag{21}$$

where

$$\mu^2=\frac{M(1-\mu^2)v_2}{a_0(M^2-1)\mu}$$

$$\frac{v_2}{a_2}=\frac{M(1-\mu^2)v_2}{a_0(M^2-1)\mu}\,.$$

For practical cases, the higher order terms of v_2/a_0 can be neglected, then we have

$$D=D_0\left[1-\frac{M(1-\mu^2)\left[1+\frac{1}{M^2}+\frac{1}{(\gamma+1)\mu}\left\{1+\gamma\mu^2+\frac{1-\mu^2}{M}\right\}\right]}{\mu(M^2-1)\left[1+\frac{1}{M^2}+2\mu\right]\left[1+\frac{2(1-\mu^2)}{(\gamma+1)\mu}\right]}\frac{v_2}{a_0}\right] \tag{22}$$

where

$$D_0=\frac{1-\frac{1}{M^2}}{\left(1+\frac{1}{M^2}+2\mu\right)\left(1+\frac{2(1-\mu^2)}{(\gamma+1)\mu}\right)} \tag{23}$$

and D_0 is the decay coefficient for the case without shear effect.

Since the experimental results inevitably require the lower decay coefficients than D_0, as discussed in the previous paper, this form is very interesting.

4. Generalized Ray Shock Theory

The ray-shock theory discussed in the previous paper gives a relation which is derived from a geometrical consideration. It is written as

$$M_\xi+\frac{M^2D}{A}\theta_\eta=0 \tag{24}$$

$$\theta_\xi+\frac{1}{A}M_\eta=0 \tag{25}$$

where ξ=constant represents the shock positions, and η=constant is the rays, and we take $\xi=a_0t$. The suffices ξ and η represent the partial derivatives with respect to ξ and η, respectively. Since $\frac{\partial}{\partial y}=\frac{1}{A}\frac{\partial}{\partial \eta}$ on the shocks, we have from the equation (22)

$$D=D_0[1-BM_\eta] \tag{26}$$

$$B=\frac{M(1-\mu^2)\left[1+\frac{1}{M^2}+\frac{1}{(\gamma+1)\mu}\left\{1+\gamma\mu^2+\frac{1-\mu^2}{M^2}\right\}\right]}{\mu(M^2-1)\left[1+\frac{1}{M^2}+2\mu\right]\left[1+\frac{2(1-\mu^2)}{(\gamma+1)\mu}\right]}.$$

This is a system of non-linear second order equations with respect to the highest order terms. However, the above process readily suggests an approximate method, that is, we can start with the solution of the equations for the ordinary case (without shear effect) as the first order approximation, the equations of which are

$$M_\xi+\frac{M^2D_0}{A}\theta_\eta=0 \tag{27}$$

$$\theta_\xi+\frac{1}{A}M_\eta=0 \tag{28}$$

where

$$D_0=\frac{K}{2}\left(1-\frac{1}{M^2}\right) \tag{29}$$

$$K^{-1}=\frac{1}{2}\left(1+\frac{2}{\gamma+1}\frac{1-\mu^2}{\mu}\right)\left(1+2\mu+\frac{1}{M^2}\right)$$

and for practical purposes, we can assume K=constant ($K=\frac{1}{2}$ at $M_1=1$) $K=0.394$ for $M_1=\infty$, $\gamma=1.4$).

This system of the equations is a quasi-linar hyperbolic partial differential equation, which is reducible. Under the boundary conditions adjacent to the constant condition (initial uniform shock wave), the solution forms a simple wave region, as shown in Fig. 4 of the previous paper, bounded by the first L+charactestic, $\xi=\frac{A_1}{M_1\sqrt{D_{01}}}\eta$, and the last L+characteristic $\xi=\frac{A_w}{M_w\sqrt{D_{0w}}}\eta$, where the suffices 1 and w denote the conditions in the initial uniform flow and on the wall after diffraction, respectively. All the L+characteristics are straight lines started at the origin, which is

$$\frac{\xi}{\eta}=\text{constant}\left(=\frac{A}{M\sqrt{D_0}}\right). \tag{30}$$

The L−characteristics are straight on the left side of the first L+characteristic, $\xi/\eta=\xi_1/\eta_1$, and on the right side of the last L+characteristic, $\xi/\eta=\xi_w/\eta_w$. Between them they are hyperbolas,

$$\xi\eta=\xi_1\eta_1(=\xi_w\eta_w)\ . \tag{31}$$

The values of θ, M, etc. on the simple wave region are functions of ξ/η only, which are

$$\frac{\eta}{\xi}=\sqrt{\frac{K}{2}}\frac{(M^2-1)^{\frac{1}{K}+\frac{1}{2}}}{(M_1^2-1)^{\frac{1}{K}}} \tag{32}$$

$$\frac{M}{M_1}=\cosh\sqrt{\frac{K}{2}}\,\theta+\sqrt{1+\frac{1}{M_1^2}}\sinh\sqrt{\frac{K}{2}}\,\theta\,. \tag{33}$$

Using these relations (30) (32) (33), we have

$$M_\eta=\frac{(M_1^2-1)^{\frac{1}{2K}}}{\sqrt{2K}\left(\frac{1}{K}+\frac{1}{2}\right)(M^2-1)^{\frac{1}{2K}-\frac{1}{2}}M}\cdot\frac{1}{\xi_1} \tag{34}$$

Substituting this relation (34) into the original equation, we have a system of the approximate equations,

$$M_\xi+\frac{M^2D_0}{A}\left(1-B_0\frac{1}{\xi_1}\right)\theta_\eta=0 \tag{35}$$

$$\theta_\xi + \frac{1}{A} M_\eta = 0 \tag{36}$$

where

$$B_0 = B \frac{\sqrt{2K}\,(M_1^2-1)^{\frac{1}{2K}}}{(K+2)\,(M^2-1)^{\frac{1}{2K}-\frac{1}{2}}\,M} \cdot \frac{1}{\xi_1}\,. \tag{37}$$

This is still a system of the quasi-linear hyperbolic partial differential equations, but not reducible and does not form centered simple wave any more.

The characteristics of the equations (35) and (36) are

$$\frac{d\eta}{d\xi} = \frac{M}{A}\sqrt{D_0\left(1-B_0\frac{1}{\xi_1}\right)} \quad \text{on L+characteristics} \tag{38}$$

$$\frac{d\eta}{d\xi} = -\frac{M}{A}\sqrt{D_0\left(1-B_0\frac{1}{\xi_1}\right)} \quad \text{on L}-\text{characteristics.} \tag{39}$$

This relation shows that the characteristics slowly change their gradient for the smaller region of ξ but take the same form as the ordinary case for $\xi=\infty$. For the region with very small value of $\xi(\xi \leq B_0)$, this approximation fails to be useful.

The characteristic forms of the original equations are

$$\int d\theta + \int \frac{dM}{M\sqrt{D_0\left(1-B_0\frac{1}{\xi_1}\right)}} = \text{const.} \quad \text{along L+characteristics} \tag{40}$$

$$\int d\theta - \int \frac{dM}{M\sqrt{D_0\left(1-B_0\frac{1}{\xi_1}\right)}} = \text{const.} \quad \text{along L}-\text{characteristics.} \tag{41}$$

Using the original characteristics (30) (31) as the approximate characteristics of these equations, we can integrate the equations (40) (41). From the boundary condition, the integral constant (the Rieman invariant) of the equation (41) is found to be zero, then we have

$$\begin{aligned}\theta &= \int \frac{dM}{M\sqrt{D_0\left(1-B_0\frac{1}{\xi_1}\right)}} \\ &\doteqdot \int_{M_1}^{M} \frac{dM}{M\sqrt{D_0}} + \frac{1}{2\xi_1}\int_{M_1}^{M} \frac{B_0 dM}{M\sqrt{D_0}}\end{aligned} \tag{42}$$

Further calculations are made for the cases of $M_1 \to 1$ and $M_1 \to \infty$. For $M_1 \to 1$, we have

$$\mu^2 = 1, \qquad K = \frac{1}{2}, \qquad D_0 = \frac{1}{2}\frac{M-1}{M^2}$$

$$B=\frac{3}{4}M \qquad B_0=\frac{3(M_1-1)}{5\sqrt{2}\,(M-1)^{\frac{1}{2}}}\cdot\frac{1}{\xi_1}$$

$$\frac{B_0}{M\sqrt{D_0}}=\frac{3}{5}\frac{M_1-1}{M-1}\frac{1}{\xi_1}$$

$$\theta=2^{3/2}(\sqrt{M-1}-\sqrt{M_1-1})+\frac{3}{5}(M_1-1)\ln\frac{M-1}{M_1-1}\frac{1}{\xi_1}\,. \tag{43}$$

After a simple calculation we obtain

$$M=1+\left[\sqrt{M_1-1}+2^{-3/2}\theta-\frac{6}{5}(M_1-1)\ln\left(1+\frac{2^{-3/2}\theta}{\sqrt{M_1-1}}\right)\cdot\frac{1}{\xi_1}\right]^2. \tag{44}$$

Especially the Mach number along the wall is

$$M_w=1+\left[\sqrt{M_1-1}+2^{-3/2}\theta-\frac{6}{5}(M_1-1)\ln\left(1+\frac{2^{-3/2}\theta}{\sqrt{M_1-1}}\right)\cdot\frac{1}{\xi_1}\right]^2. \tag{45}$$

Thus, it is found that the shock wave diffracted around a corner is given less attenuation (higher M_w) on an account of this correction, which is experimentally confirmed (note the sign of θ_w is negative).

For $M_1\to\infty$, we have

$$\mu^2=\frac{2\gamma}{\gamma-1},\qquad K^{-1}=\frac{1}{2}\left(1+\sqrt{\frac{2}{\gamma(\gamma-1)}}\right)\left(1+\sqrt{\frac{2(\gamma-1)}{\gamma}}\right),\qquad D_0=\frac{K}{2}$$

$$B=\frac{(1-\mu^2)\left[1+\frac{1}{(\gamma+1)\mu}\{1+\gamma\mu^2\}\right]}{M\mu(1+2\mu)\left[1+\frac{2(1-\mu^2)}{(\gamma+1)\mu}\right]}=\frac{1}{M}\times(\text{function of }\gamma)=\frac{1}{M}B'(\gamma)$$

$$B_0=\frac{M_1^{\frac{1}{K}}}{M^{\frac{1}{K}+1}}\cdot B_0'(\gamma)\cdot\frac{1}{\xi_1}\left(=\frac{1}{M}B'(\gamma)\frac{\sqrt{2K}}{(K+2)}\left(\frac{M_1}{M}\right)^{\frac{1}{K}}\frac{1}{\xi_1}\right)$$

$$\theta=\sqrt{\frac{2}{K}}\ln\frac{M}{M_1}+\frac{2}{K}\left(1+\frac{1}{K}\right)B_0'(\gamma)M_1^{\frac{1}{K}}\left\{M^{-\left(\frac{1}{K}+1\right)}-M_1^{-\left(\frac{1}{K}+1\right)}\right\}\frac{1}{\xi_1} \tag{46}$$

and

$$\frac{M}{M_1}=e^{\sqrt{\frac{K}{2}}\theta}\,e^{-\sqrt{\frac{2}{K}}\left(1+\frac{1}{K}\right)B_0'(\gamma)M_1^{-1}\left(e^{\sqrt{\frac{K}{2}}\theta\left(1+\frac{1}{K}\right)}-1\right)\frac{1}{\xi_1}} \tag{47}$$

$$\frac{M_w}{M_1}=e^{-\sqrt{\frac{K}{2}}\theta_w}\,e^{-\sqrt{\frac{2}{K}}\left(1+\frac{1}{K}\right)B_0'(\gamma)M_1^{-1}\left(e^{\sqrt{\frac{K}{2}}\theta\left(1+\frac{1}{K}\right)}-1\right)\frac{1}{\xi_1}}\,. \tag{48}$$

These relations (44) and (47) should be compared with the corresponding relations given in the previous paper (the equations (6.14) and (6.15)). Although a straightforward derivation between these relations is not available, the vorticity transfer concept introduced in this paper gives the same trend of the correction term as the one given by the turbulent mixing concept which was developed in the previous paper and was given an experimental support.

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