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Linear Dynamic Analysis of the Spinning Axisymmetrical Rocket or Vehicle

By

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Summary. This report contains a dynamic analysis of the angular behavior of a spinning body flying either in vacuum or in atmosphere. The study is mainly concerned with the angular motion of a rigid axisymmetrical body having a thrust malalignment and another moment due to the deflection of control surfaces or control jets. In linear feedback system which is used in the control of spin-stabilized space vehicle, preferable combinations of the control parameters, such as feedback gain, spin rate, feedback phase difference and damping of the system, are obtained to optimize the stability and control of the vehicle.

In the case of asymmetrical feedback system the linear approximate solution has been obtained. The analysis of the feedback system will be applied to the motion of either launching or re-entry vehicle within atmosphere.

1. INTRODUCTION

Many reports have been treated about the dynamics of spinning axisymmetrical body, specifically for stability of such body flying either in vacuum or in atmosphere, with generally linearized theory [*1*~*17*]. In most of them, usually the main consideration is directed to the angular motion of the body flying with constant velocity and spin rate and therefore the characteristic equation is given as a quadratic equation with complex-constant coefficients. Hence, it is well known that angular motion of such spinning body is usually characterized as an epicycle in a complex-angular plane.

For the flight in atmosphere, the most important aerodynamic terms of the complex coefficient in the characteristic equation or in the transfer function are Magnus moment and pitching moment. It is generally known that the stability of the spinning rocket is strongly affected by the sign and amount of these coefficients. However, it is not easy to estimate the Magnus moment in wide circumferential range because of the non-linearity of the Magnus effects [*18*~*26*].

It has, furthermore, been recognized that other aerodynamic moments have also shown nonlinear characteristics due to the slender configuration, hence the spinning body shows undesirable motion [*27, 28, 49*], such as catastrophic yaw, lunar motion or others.

Recently, the spin stabilization technique has been applied to stabilize and sometimes to control the space vehicle during the ejection from the final stage of the launching, orbital flight and reentry phase [43~49]. Spin stabilization will be able to use as either open-loop system [29~34] or closed-loop system [35~42] to maintain a given vehicle attitude in space coordinates. Such a system suffers from the fact that attitude errors occur because of torques applied to the body by wind shear, tip off, thrust malalignment transients, solar radiation, gravity gradient, magnetic, and so forth. In the written reports but a few referring to the above described systems the stability criterion generally occupies the attention, while the effects of the system parameters upon the time response and the control parameters to optimize the response are almost out of consideration.

The analysis described herein for a spinning body provides estimation of performance and preferable control characteristic for a closed-loop linear-attitude-control system which will correct any attitude errors and damp out the coning produced by external moments.

In attitude control of the spin-stabilized space vehicle nonlinear control means, such as commutator which allocates the control moment corresponding with rolling angle and pulse jet with constant magnitude, are usually used for simplicity and reliability and for long life-time of the means [37~39, 42]. For estimation of the performance of such nonlinear system it must be very effective to understand the basic characteristics of linear system.

Since the dynamic characteristics of the attitude control for the spinning rocket in atmosphere may be expressed by same form as the one of closed loop of the attitude control system for the spinning space vehicle in vacuum, the detailed discussion for the time response of the system will have been directed to such closed system. Necessary optimum control parameters will be decided by minimizing two scalar quantities defined by the radii and sweep angles of two spirals composing an epicycle.

With good approximation the present analysis will be able to apply to the case of asymmetrical feedback system in which two reaction jets are usually pointed to opposite body-fixed directions for the control of transverse motion of the spinning body.

Symbols

A	quantity given by eqs. (2.40), (3.13) and (5.43)
B	quantity given by eqs. (2.41), (3.14) and (5.44)
C	complex coefficient
C_d	drag coefficient
C_L	lift coefficient
C_l	rolling moment coefficient
C_m	pitching or yawing moment coefficient
D	characteristic equation
e	exponential and quantity given in eqs. (2.38) and (5.41)
F	external force
f	arbitrary function and quantity given in eqs. (2.38) and (5.41)
g	gravity acceleration

h	altitude
I	moment of inertia
i	$\sqrt{-1}$
K	moment- of -inertia ratio I_X/I_Y or I_X/I_Z
K_b, K_s	rate and proportional feedback gains
k_0	quantity given by eq. (5.33)
$\mathcal{L}$, $\mathcal{L}^{-1}$	Laplace transform and inverse Laplace transform
l	referength length
M	moment
m	mass of rocket
p	rolling angular velocity or spin rate
q	pitching angular velocity for body axis
R, R'	scalar quantities defined by eqs. (4.9) and (4.12) respectively
r	yawing angular velocity for body axis and transverse distance
S	reference area, transfer function and sweep area defined by eq. (4.7)
S_s	trasfer function defined by eq. (3.32)
S'	sweep area defined by eq. (4.14)
s	flight distance
T	thrust
t	time
V	velocity of the body
V_E	re-entry velocity
v	transverse velocity of the body
W	weight of rocket
(X, Y, Z)	coordinates

Subscript

b	body axis
i	input
o	space fixed axis or initial value
X, Y, Z	components of (X, Y, Z) axes
X_b, Y_b, Z_b	components of (X_b, Y_b, Z_b) axes
p, α, β	derivatives e.g. $C_{L_\alpha}=\dfrac{\partial C_L}{\partial \alpha}$ $C_{Lp_\alpha}=\dfrac{\partial}{\partial \alpha}\dfrac{\partial C_L}{\partial\left(\dfrac{lp}{2V}\right)}$

Greeks

α	angle of attack
β	angle of side slip and exponential altitude parameter
δ	distance from the center of gravity to thrust line and fin-deflection angle
ε	complex attitude angle, $\psi+i\theta$
ε_t	complex thrust-malalignment angle, $\psi_t+i\theta_t$

$\eta, \eta_1, \eta_2,$	vectorial expressions of the response curve from the terminal point, see Fig. (4.9)
θ	pitching angle
θ_f, θ_e	quatities given in eqs. (2.38) and (5.41)
θ_t	thrust-malalignment angle in $(X_b\ Z_b)$ plane
θ_E	re-entry angle
λ_1, λ_2	Laplace transform parameters or characteristic roots
μ	Coriolis force parameter given in the eq. (2.22)
ν	p/V
ξ	complex angle of attack, $\beta-i\alpha$
ρ	air density
ϕ	rolling angle
ϕ_b, ϕ_s	phase differences of rate and proportional feedback
ψ	yawing angle
ψ_t	thrust-malalignment angle in (X_b, Y_b) plane
$\bar{\varepsilon}$	conjugate complex attitude angle, $\psi-i\theta$

2. Rotational Motion of Spinning Rocket

In this and next section, it is assumed that the rocket is flying at constant speed and constant rate of spin even in powered flight. It will be appreciated that from such audacious assumption many important principal results will be given for understanding of the behaviour of spinning body.

The attitude of a rocket can at instant be expressed by Eulerian angle (ψ, θ, ϕ) of body axes, (X_b, Y_b, Z_b), or (ψ, θ, o) of stability axes, (X, Y, Z), with respect to space fixed axes, (X_o, Y_o, Z_o), as shown in Fig. 2.1.

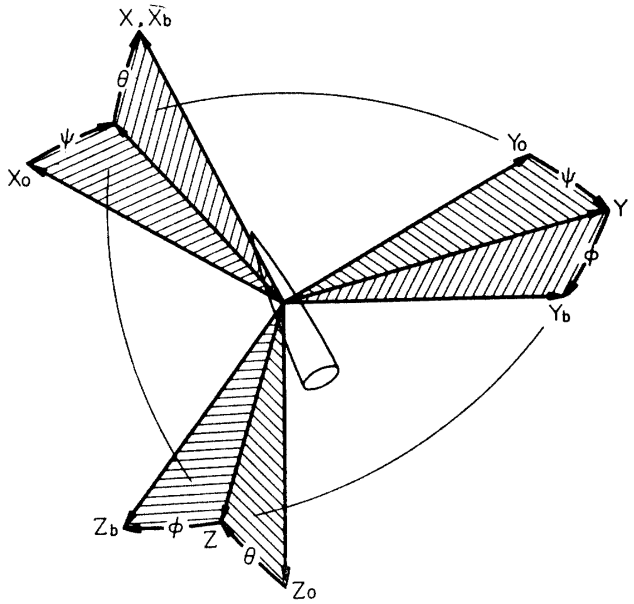

Figure 2.1. Orientation of body and stability axes with respect to space fixed axes.

The body axes, (X_b, Y_b, Z_b), remain axes fixed in the spinning body and therefore the X_b axis rotates with the body, while the X axis of stability axes does not rotate with the body but places along the axis of rotational symmetry.

Under the assumption that all Eulerian angles except rolling angle, ϕ, are small enough to neglect the higher order of Taylor expansion of cosine and sine terms, the transformation matrices established among those coordinate systems are given by

	X_o	Y_o	Z_o
X	1	ψ	$-\theta$
Y	$-\psi$	1	0
Z	θ	0	1
X_b	1	ψ	$-\theta$
Y_b	$\theta\sin\phi-\psi\cos\phi$	$\cos\phi$	$\sin\phi$
Z_b	$\theta\cos\phi+\psi\sin\phi$	$-\sin\phi$	$\cos\phi$

(2.1)

	X	Y	Z
X_b	1	0	0
Y_b	0	$\cos\phi$	$\sin\phi$
Z_b	0	$-\sin\phi$	$\cos\phi$

(2.2)

It is further assumed in this section that the center of gravity of the rocket is fixed at the stationary space and only angular motion of the body is allowable. Then the equations of rotational motion of the body are given as [*7, 8*]:

(i) for the body axes;

$$\left.\begin{aligned}\dot{q}-(1-K)pr&=M_{Yb}/I_Y\\ \dot{r}+(1-K)pq&=M_{Zb}/I_Z\end{aligned}\right\}\tag{2.3}$$

(ii) for the stability axes;

$$\left.\begin{aligned}\ddot{\theta}+Kp\dot{\psi}&=M_Y/I_Y\\ \ddot{\psi}-Kp\dot{\theta}&=M_Z/I_Z\,.\end{aligned}\right\}\tag{2.4}$$

Wherein p, q and r are the angular velocities of X_b, Y_b and Z_b axes respectively, and M_{Yb}, M_{Zb}, M_Y and M_Z are external moments along each axis designated by corresponding subscripts. K is a ratio of the longitudinal moment of inertia, I_X, to the lateral moment of inertia I_Y or I_Z and is less than 1 for rod-shaped body, equal to 1 for a sphere, and 1 to 2 for a disc-like configuration, *i.e.*

$$K=I_X/I_Y=I_X/I_Z\,. \tag{2.5}$$

There exist following relations among the moments and angular velocities in each coordinate system

$$\left.\begin{aligned} M_{Y_b}&=M_Y\cos\phi+M_Z\sin\phi \\ M_{Z_b}&=-M_Y\sin\phi+M_Z\cos\phi \end{aligned}\right\} \tag{2.6a}$$

$$\left.\begin{aligned} M_Y&=M_{Y_b}\cos\phi-M_{Z_b}\sin\phi \\ M_Z&=M_{Y_b}\sin\phi-M_{Z_b}\cos\phi \end{aligned}\right\} \tag{2.6b}$$

$$\left.\begin{aligned} \dot{\theta}&=q\cos\phi-r\sin\phi \\ \dot{\psi}&=q\sin\phi+r\cos\phi \end{aligned}\right\} \tag{2.7}$$

By introducing complex quantities such as

$$\dot{\varepsilon}_b=r+iq \tag{2.8}$$

$$\varepsilon=\psi+i\theta \tag{2.9}$$

in which $i=\sqrt{-1}$, the above equations may simply be expressed in complex form as follows:

(i) for the body axes;

$$\ddot{\varepsilon}_b-i(1-K)p\dot{\varepsilon}_b=(M_{Z_b}+iM_{Y_b})/I_Y \tag{2.10}$$

(ii) for the stability axes;

$$\ddot{\varepsilon}+iKp\dot{\varepsilon}=(M_Z+iM_Y)/I_Y \tag{2.11}$$

The angular motion of the spinning rocket is determined by the equations (2.10) or (2.11) jointly with the initial conditions.

Flight in Vacuum

In order to see the most simplified feature of spinning rockets, first of all let the rocket being flight in vacuum so that all aerodynamic quantities are diminished.

The pitching and yawing moments given in the right hand side of the equation (2.10) or (2.11) may consist of the thrust malalignment moment, control moment and Coriolis damping moment. In the above moments, only the thrust malalignment moment acts fixedly to the body-axes system in the planes of (X_b, Z_b) and $(X_b\, Y_b)$, as shown in Fig. 2.2.* Then, the moment, M_{Y_b} and M_{Z_b}, are given by

$$M_{Y_b}=T\delta_{Z_b}$$

$$M_{Z_b}=-T\delta_{Z_b}$$

* The control moment may, sometimes, be given as body-fixed-quantity whenever the momnt is not allocated sinusoidally to each control equipment.

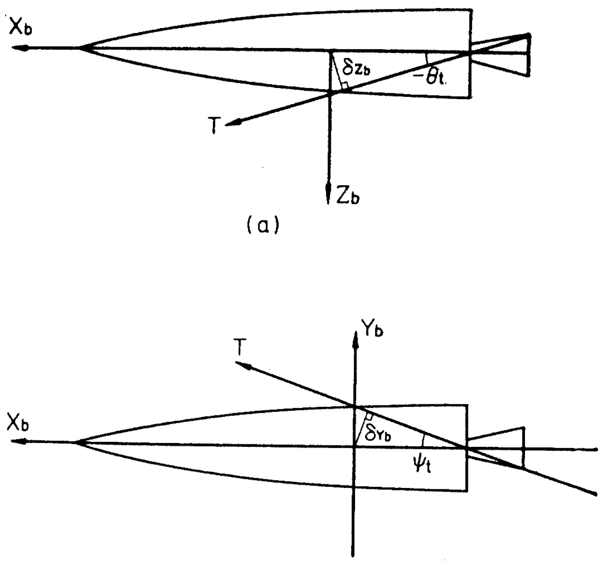

FIGURE 2.2. Thrust malalignment.

and therefore

$$(M_{Zb}+iM_{Yb})/I_Y=(-\delta_{Yb}+i\delta_{Zb})T/I_Y \tag{2.12}$$

By introducing the above moment into the equation (2.10), the following equation showing the thrust-malalignment effect is obtained as

$$\ddot{\varepsilon}_b-i(1-K)p\dot{\varepsilon}_b=(-\delta_{Yb}+i\delta_{Zb})T/I_Y \tag{2.13}$$

After applying Laplace transform to the above equation, such as

$$\mathcal{L}\{f(t)\}=\int_0^\infty e^{-\lambda t}f(t)dt=f(\lambda) \tag{2.14}$$

with the condition in which all initial conditions are zeroes, the complex angular velocity of the body is given in Laplace transformed expression as

$$\dot{\varepsilon}_b(\lambda)=\frac{\{-\delta_{Yb}(\lambda)+i\delta_{Zb}(\lambda)\}}{\lambda-i(1-K)p}\cdot\frac{T}{I_{Yb}} \tag{2.15}$$

By applying inverse Laplace transform to the above equation, such as

$$\mathcal{L}^{-1}\{f(\lambda)\}=\frac{1}{2\pi i}\int_{Br} e^{t\lambda}f(\lambda)d\lambda=f(t) \tag{2.16}$$

wherein $\int_{Br} d\lambda$ designates Bromwich integral, a time response of the complex angular velocity, $\dot{\varepsilon}_b$, for step inputs, δ_{Yb} and δ_{Zb}, is given as

$$\dot{\varepsilon}_b(t)=\frac{\sqrt{\delta_{Yb}^2+\delta_{Zb}^2}}{(1-K)p}\cdot\frac{T}{I_Y}\,e^{i\left(\tan^{-1}\frac{\delta_{Yb}}{\delta_{Zb}}+\pi\right)}\{1-e^{i(1-K)pt}\} \tag{2.17}$$

It is easy to trace a locus of the above response in the complex plane of angular velocity, $\dot{\varepsilon}_b$, given by the equation (2.17) as shown in Fig. 2.3. In the Fig. 2.3, it will be recognized that the locus of the $\dot{\varepsilon}_b(t)$ will make a circle whose radius is given as $\frac{\sqrt{\delta_{Yb}^2+\delta_{Zb}^2}}{(1-K)p}\frac{T}{I_Y}$ and its center, which is a mean value of $\dot{\varepsilon}_b$, is located at a point defined by radius being equal to the radius of the circle and angle of $\tan^{-1}\left(\frac{\delta_{Yb}}{\delta_{Zb}}\right)+\pi$.* Angular velocity of the locus on the circle is given by $(1-K)p$. It may be seen that in order to reduce the mean angular velocity let the spin rate, p, and/or the difference of moments of inertia, I_Y-I_X, preferably increase since the denominator of the mean value is $(1-K)pI_Y=(I_Y-I_X)p$.

In special case where the rocket has a shape of sphere *i.e.* $I_X=I_Y=I_Z$, the equation (2.13) will be reduced as

$$\ddot{\varepsilon}_b=(-\delta_{Yb}+i\delta_{Zb})T/I_Y \tag{2.18}$$

Thus, the response for step input of δ_{Yb} and δ_{Zb} is given by

$$\dot{\varepsilon}_b(t)=\sqrt{\delta_{Yb}+\delta_{Zb}^2}\,\frac{T}{I_Y}\,e^{i\tan^{-1}(\delta_{Zb}/-\delta_{Yb})}\cdot t \tag{2.19}$$

The complex angular velocity, $\dot{\varepsilon}_b$, will infinitely increase with time, t.

In the stability-axes system the thrust-malalignment moment is obtained from the equation (2.6) as

$$\left.\begin{aligned} M_Z+iM_Y&=(M_{Zb}+iM_{Yb})e^{-i\phi}\\ &=(-\delta_{Yb}+i\delta_{Zb})Te^{-i\phi}\end{aligned}\right\} \tag{2.20}$$

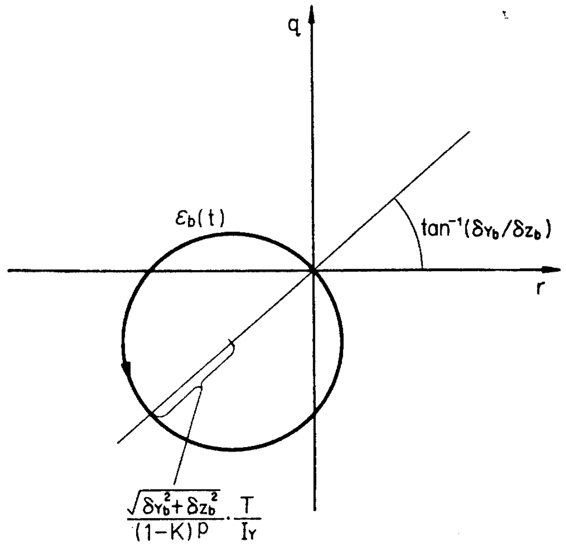

FIGURE 2.3. Locus of the complex angular velocity for step input of thrust malalignment.

* If the body is not symmetrical about X axis, *i.e.* $I_Y\neq I_Z$ then the system has an undamped oscillation similar to (2.17), for $I_X<I_Y$, I_Z or $I_X>I_Y$, I_Z, and has divergent motion for $I_Y<I_X<I_Z$ or $I_Z<I_X<I_Y$ [30].

Control moment is usually obtained by either fin deflection for fin-stabilized rockets, or control jet or thrust inclination for finned and unfinned rockets. For spinning rocket, such control moment must be allocated to each control means rotating with the body to create the moment for desired direction defined by stability axes, *i.e.*

$$\left.\begin{aligned} M_Z+iM_Y &= \delta M_Z+i\delta M_Y \\ &= (\delta M_{Zb}+i\delta M_{Yb})\, e^{-i\phi} \end{aligned}\right\} \tag{2.21a}$$

or

$$M_{Zb}+iM_{Yb}=(\delta M_Z+i\delta M_Y)\, e^{i\phi} \tag{2.21b}$$

where δM_Y, δM_Z, and δM_{Yb}, δM_{Zb} are control moments of stability axes and body axes respectively.

Coriolis damping moment is given during only burning flight and is obtained as [2]

$$M_Z+iM_Y=-\mu T\dot{\varepsilon} \tag{2.22}$$

wherein T is thrust of the rocket and μ is a parameter determined by the rocket configuration and the gas speed.

Total transverse moment is, thus, given by

$$\left.\begin{aligned} M_Z+iM_Y &= (-\delta_{Yb}+i\delta_{Zb})Te^{-i\phi} \\ &\quad +(\delta M_Z+i\delta M_Y)-\mu T\dot{\varepsilon} \end{aligned}\right\} \tag{2.23}$$

Substituting the above moment into the equation (2.11) the equation of motion for the stability-axes system is obtained as follows:

$$\ddot{\varepsilon}+iKp\dot{\varepsilon}=(-\delta_{Yb}+i\delta_{Zb})(T/I_Y)e^{-i\phi}+(\delta M_Z+i\delta M_Y)/I_Y-(\mu T/I_Y)\dot{\varepsilon} \tag{2.24}$$

By applying the Laplace transform with the following initial conditions

$$\text{at} \quad t=t_0: \quad \dot{\varepsilon}=\dot{\varepsilon}_0 \quad \text{and} \quad \varepsilon=\varepsilon_0\,. \tag{2.25}$$

The solution of the transformed complex angle, $\varepsilon(\lambda)$, is given by

$$\begin{aligned} \varepsilon(\lambda)=&\frac{1}{\lambda\left\{\lambda+\left(\dfrac{\mu T}{I_Y}+iK\right)\right\}}\Big[\{-\delta_{Yb}(\lambda+ip)+i\delta_{Zb}(\lambda+ip)\}T/I_Y \\ &+\{\delta M_Z(\lambda)+i\delta M_Y(\lambda)\}/I_Y+\varepsilon_0\cdot\lambda+\left\{\left(\frac{\mu T}{I_Y}+iKp\right)\varepsilon_0+\dot{\varepsilon}_0\right)\Big] \end{aligned} \tag{2.26}$$

The time response for the step input of the thrust malalignment is given by

$$\varepsilon(t)=\frac{\sqrt{\delta_{Yb}^2+\delta_{Zb}^2}}{\sqrt{\left(\dfrac{\mu T}{I_Y}\right)^2+(Kp)^2}}\,\frac{T}{I_Y p}\, e^{i\left\{\tan^{-1}\left(\frac{\delta Zb}{-\delta Yb}\right)+\tan^{-1}\left(\frac{Kp}{\mu T/IY}\right)+\frac{3}{2}\pi\right\}}$$

$$
\left.\cdot\left[1+\frac{p e^{-i\left\{\tan^{-1}\left(\frac{-(1-K)p}{\mu T/I_Y}\right)+\frac{\pi}{2}\right.}}{\sqrt{\left(\frac{\mu T}{I_Y}\right)^2+(1-K)^2p^2}}\cdot e^{-\left(\frac{\mu T}{I_Y}+iKp\right)t} + \frac{\sqrt{(\mu T/I_Y)^2+(KP)^2}\, e^{i\left\{\tan^{-1}\left(\frac{Kp}{\mu T/I_Y}\right)-\tan^{-1}\left(\frac{-(1-K)p}{\mu T/I_Y}\right)+\pi\right\}}}{\sqrt{\left(\frac{\mu T}{I_Y}\right)^2+(1-K)^2p^2}}\cdot e^{-ipt}\right]\right\} \quad (2.27)
$$

As described in the reference [2], since the Coriolis moment has very small magnitude the term $\mu T/I_Y$ is usually neglected comparing to the $(1-K)p$ except special case in which $K \doteqdot 1$, thus the equation (2.27) becomes

$$
\varepsilon(t) \doteqdot \frac{\sqrt{\delta_{Yb}^2+\delta_{Zb}^2}\,T}{I_X p^2}\, e^{i\left\{\tan^{-1}\left(\frac{\delta Zb}{-\delta Yb}\right)+\pi\right\}} \cdot \left\{1+\frac{e^{-\frac{\mu T}{IY}t-i(Kpt+\pi)}}{1-K}+\frac{K}{1-K}e^{-ipt}\right\} \quad (2.28a)
$$

The locus of the above complex angle, $\varepsilon(t)$, is shown in Fig. 2.4. From either equation (2.28a) or Fig. 2.4 it is appreciated that in the (ψ, θ) plane the complex angle is expressed by an epicycle consists of two circles one of which is undamped motion having the angular velocity p and the other is lightly damped motion, or spiral, having the angular velocity Kp since in conventional rocket system $\mu > O$. For special case in which $\mu < O$, the latter circle will lightly diverge. Ratio of the radii of the two circles is K.

It is interesting to notify that after damped out the second term of the

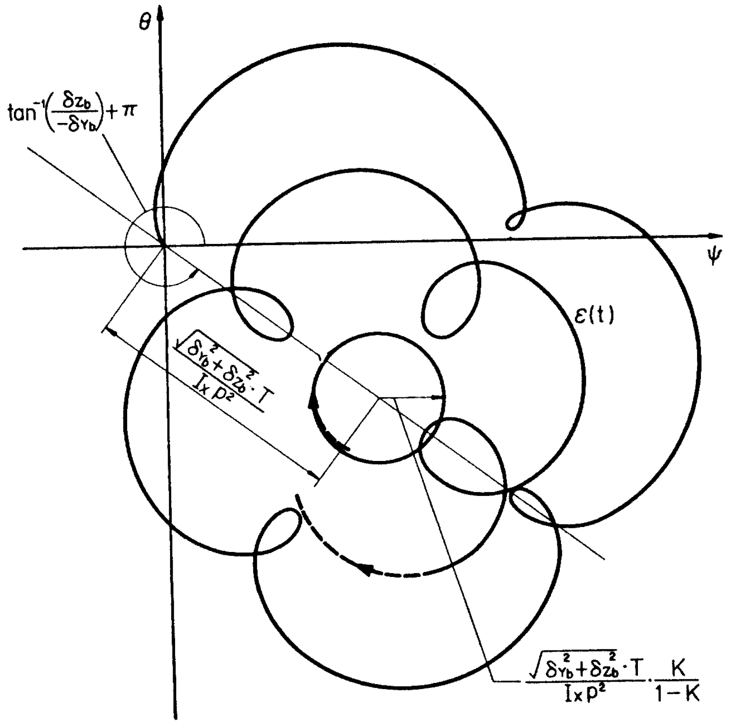

FIGURE 2.4. Locus of the complex angle for step input of thrust malalignment (for $\mu > 0$).

equation (2.28) the mean value of $\varepsilon(t)$ is given by $\sqrt{\delta_{Yb}^2+\delta_{Zb}^2}\,T/I_Xp^2$, hence the angular deviation caused by thrust malalignment will be reduced with inversely proportional to the product of moment of inertia along the X axis by square of spin rate, I_Xp^2.

For a given mass of rocket, given spin rate and given thrust malalignment there is an optimum inertia ratio to minimize the angular deviation of the rocket.

If the rocket damping, μ, is almost zero, the above optimum condition is obtained by the following equation:

$$\frac{\partial}{\partial K}\left\{\frac{1}{I_X}\left(\frac{1+K}{1-K}\right)\right\}=0 \tag{2.29}$$

and if the rocket damping, μ, is large positive value,

$$\frac{\partial}{\partial K}\left\{\frac{1}{I_X}\cdot\frac{1}{1-K}\right\}=0 \tag{2.30}$$

For a general body of revolution I_X may be written as a function of the inertia ratio alone so that for a homogeneous circular cylinder the former equation, (2.29), gives the optimum inertia ratio [*14*], $K \doteqdot 0.177$ and the latter equation (2.29b) gives the optimum value, $K \doteqdot 0.279$.

As K approaches 1 the radii of the circles increase and the equation (2.27) will again be rewritten by replacing the equation (2.28a) with the following equation:

$$\left.\begin{aligned}\varepsilon(t)\doteqdot \frac{\sqrt{\delta_{Yb}^2+\delta_{Zb}^2}\,T}{I_Yp^2}\,e^{i\left\{\tan^{-1}\left(\frac{\delta Zb}{-\delta Yb}\right)+\pi\right\}}\cdot\Bigg\{1&+\frac{pe^{i\left\{\frac{\pi}{2}+(o\text{ or }\pi)\right\}}}{\mu T/I_Y}\cdot e^{-\left(\frac{\mu T}{IY}+ip\right)t}\\ &+\frac{pe^{i\left\{\frac{3}{2}\pi-(o\text{ or }\pi)\right\}}}{\mu T/I_Y}\cdot e^{-ipt}\Bigg\}\end{aligned}\right\} \tag{2.28b*}$$

It will be recognized from the above equation that the mean value of the complex angle is equal to the one of former case but the radii of the circles are inversely proportional to the spin rate p insted of p^2 in the former case.

The time response for the step input of the control moments will be written as

$$\left.\begin{aligned}\varepsilon(t)=\frac{\sqrt{\delta M_Y^2+\delta M_Z^2}}{I_Y}\,e^{i\tan^{-1}\left(\frac{\delta MY}{\delta MZ}\right)}\cdot\frac{e^{-i\tan^{-1}\left(\frac{Kp}{\mu T/IY}\right)}}{\sqrt{\left(\frac{\mu T}{I_Y}\right)^2+(Kp)^2}}\\ \cdot\left[t-\frac{e^{-i\tan^{-1}\left(\frac{Kp}{\mu T/IY}\right)}}{\sqrt{\left(\frac{\mu T}{I_Y}\right)^2+(Kp)^2}}\left\{1-e^{-\left(\frac{\mu T}{IY}+iKp\right)t}\right\}\right]\end{aligned}\right\} \tag{2.31}$$

For large Kp the above equation will be reduced as

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* In the terms of $e^{i\left\{\frac{\pi}{2}+(o\text{ or }\pi)\right\}}$ and $e^{i\left\{\frac{3}{2}\pi-(o\text{ or }\pi)\right\}}$ and susequent similar expression, o corresponds to $\mu>o$ and π corresponds to $\mu<o$ respectively.

$$\varepsilon(t) \doteqdot \frac{\sqrt{\delta M_Y^2+\delta M_Z^2}}{I_X p} e^{i\left\{\tan^{-1}\left(\frac{\delta M_Y}{\delta M_Z}\right)-\frac{\pi}{2}\right\}} \cdot \left\{t-\frac{e^{-i\frac{\pi}{2}}}{Kp}\left(1-e^{-\left(\frac{\mu T}{I_Y}+iKp\right)t}\right)\right\} \tag{2.31a}$$

and hence the locus of $\varepsilon(t)$ in the (ψ, θ) plane is as shown in Fig. 2.5.

From the above equation (2.31a) or Fig. 2.5 it will be seen that the angular deviation caused by control moment is perpendicular to the direction defined by control moment, *i.e.* it is corresponding to phase shift of $-\pi/2$, and is increasing with time. A circle carried on a straight line which is shifted by $1/Kp$ from the origin in Fig. 2.5. shows lightly damping motion in usuall case. The mean complex angular velocity, $\dot{\varepsilon}(t)$, is inversely proportional to the product of the moment of inertia by rate of spin, $I_X p$.

For small Kp, *i.e.* $Kp<(\mu T/I_Y)^2$, the equation (2.31) will be simplfied as

$$\varepsilon(t)=\frac{\sqrt{\delta M_Y^2+\delta M_Z^2}}{\mu T} e^{i\left\{\tan^{-1}\left(\frac{\delta M_Y}{\delta M_Z}\right)-(0 \text{ or } \pi)\right\}} \cdot \left\{t-\frac{e^{-i(0 \text{ or } \pi)}}{\mu T/I_Y}\left(1-e^{-\left(\frac{\mu T}{I_Y}+iKp\right)t}\right)\right\} \tag{2.31b}$$

Thus, the direction of the complex angle, in this case, will coincide with the direction defined by the control moments for lightly damped motion.

The time response of the initial conditions is given by

$$\varepsilon(t)=\varepsilon_0+\frac{\dot{\varepsilon}_0 e^{-i\tan^{-1}\left(\frac{Kp}{\mu T/I_Y}\right)}}{\sqrt{\left(\frac{\mu T}{I_Y}\right)^2+(Kp)^2}} \cdot \left\{1-e^{-\left(\frac{\mu T}{I_Y}+iKp\right)t}\right\} \tag{2.32}$$

The above equation is simply rewritten for small $\mu T/I_Y$

$$\varepsilon(t)=\varepsilon_0+\frac{\dot{\varepsilon}_0 e^{-i\frac{\pi}{2}}}{Kp}\left\{1-e^{-\left(\frac{\mu T}{I_Y}+iKp\right)t}\right\} \tag{2.32a}$$

and for small Kp

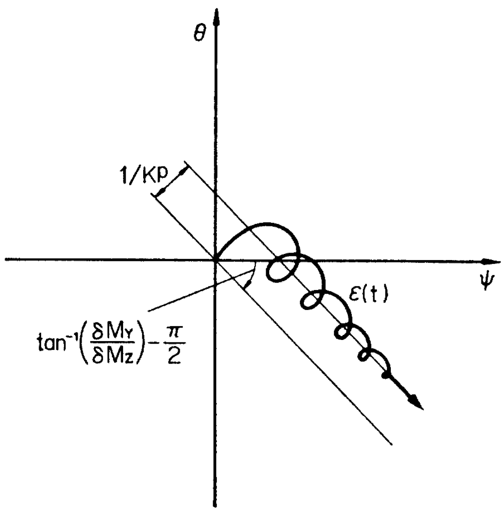


FIGURE 2.5. Locus of the complex angle for the step input of control movements (for $\mu>0$).

$$\varepsilon(t)=\varepsilon_0+\frac{\dot{\varepsilon}_0 e^{-i(0\ \mathrm{or}\ \pi)}}{\mu T/I_Y}\cdot\left\{1-e^{-\left(\frac{\mu T}{I_Y}+iK_p\right)t}\right\} \tag{2.32b}$$

In both cases the locus of $\varepsilon(t)$ is shown in Fig. 2.6.

It will be appreciated that although the effect of the initial angular velocity of the deviation of complex attitude angle will be reduced with increasing Kp but still remains the initial angle ε_0 unaffected as well known in gyroscopic art.

Further, it is natural result that in coasting flight without power if the spin rate decreases to zero $\varepsilon(t)$ of the equation (2.32a) becomes

$$\varepsilon(t) \doteqdot \varepsilon_0+\dot{\varepsilon}_0 t \tag{2.32c}$$

consequently complex attitude angle will increase with time, t, due to the initial angular velocity given at the end of powered flight. Hence, it must be careful to make despin in vacuum if the rocket has no automatic attitude-control system.

In considering a rectangular-pulse-pitching moment as control input, it will be found that the motion of the rocket will be obtained by superposition of the above results, (2.27) or (2.31) and (2.32), with appropriate time shift corresponding to the pulse duration. For example, by replacing the thrust-malalignment moment, $T\delta_{Yb}$ or $T\delta_{Zb}$, with appropriate control moment produced by pulse

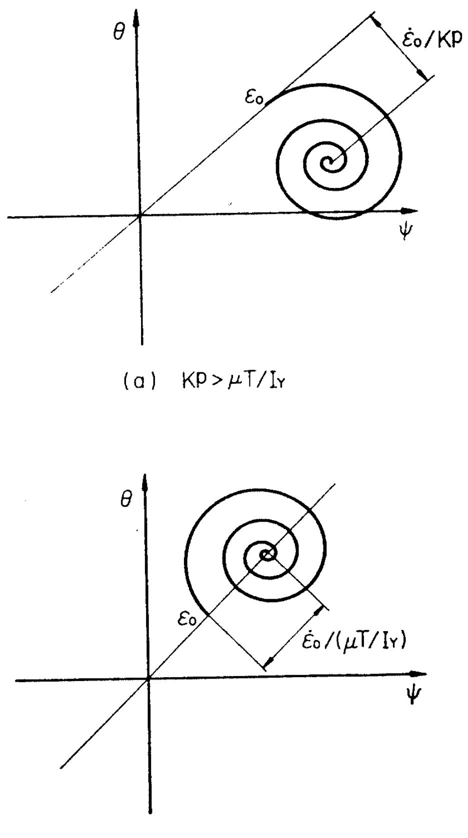


FIGURE 2.6. Locus of the complex angle for initial conditions (for $\mu>0$).

jet the response for two-pulse control inputs will be obtained from the combinations of the equations (2.27) and (2.32) in more practical form than the results given by references [*30*], [*37*] and [*39*].

Flight in Atmosphere

During flight in atmosphere additional external moments caused by aerodynamic forces have to be considered. Having neglected the nonlinear term the most pronounced terms in these moments are given by the following linear expansions:

$$\left.\begin{aligned} M_Y &= \frac{1}{2}\rho V^2 Sl\left\{C_{m\alpha}\theta - C_{mp\beta}\frac{lp}{2V}\phi + C_{mq}\frac{l\dot{\theta}}{2V}\right\} \\ M_Z &= \frac{1}{2}\rho V^2 Sl\left\{C_{m\alpha}\phi + C_{mp\beta}\frac{lp}{2V}\theta + C_{mq}\frac{l\dot{\phi}}{2V}\right\} \end{aligned}\right\} \tag{2.33}$$

By using complex variables, the above equations become

$$M_Z + iM_Y = \frac{1}{2}\rho V^2 Sl\left\{\left(C_{m\alpha} - iC_{mp\beta}\frac{lp}{2V}\right)\varepsilon + C_{mq}\frac{l}{2V}\dot{\varepsilon}\right\} \tag{2.34}$$

Substituting the above additional moments with the moment expressed by the equation (2.23) into the equation (2.11), the following equation for angular deviation, ε, can be derived:

$$\left.\begin{aligned} &\ddot{\varepsilon} + \left\{\left(\frac{\mu T}{I_Y} - \frac{\rho V^2 Sl}{2I_Y}\cdot\frac{l}{2V}C_{mq}\right) + iKp\right\}\dot{\varepsilon} + \frac{\rho V^2 Sl}{2I_Y}\left(-C_{m\alpha} + iC_{mp\beta}\frac{lp}{2V}\right)\varepsilon \\ &= (-\delta_{Yb} + i\delta_{Zb})(T/I_Y)e^{-ipt} + (\delta M_Z + i\delta M_Y)/I_Y \end{aligned}\right\} \tag{2.36}$$

The solution of this complex angle in Laplace transformed form is

$$\left.\begin{aligned} \varepsilon(\lambda) = \frac{1}{D(\lambda)}\Big[&\{-\delta_{Yb}(\lambda+ip) + i\delta_{Zb}(\lambda+ip)\}\frac{T}{I_Y} + \{\delta M_Z(\lambda) + i\delta M_Y(\lambda)\}/I_Y \\ &+ \varepsilon_0\cdot\lambda + \left(\frac{\mu T}{I_Y} - \frac{\rho V^2 Sl}{2I_Y}\cdot\frac{l}{2V}C_{mq} + iKp\right)\varepsilon_0 + \dot{\varepsilon}_0\Big] \end{aligned}\right\} \tag{2.36}$$

where $D(\lambda)$ is the left hand side of the characteristic equation, such as

$$\left.\begin{aligned} D(\lambda) = \lambda^2 &+ \left\{\left(\frac{\mu T}{I_Y} - \frac{\rho V^2 Sl}{2I_Y}\cdot\frac{l}{2V}C_{mq}\right) + iKp\right\}\lambda \\ &+ \frac{\rho V^2 Sl}{2I_Y}\left(-C_{m\alpha} + iC_{mp\beta}\frac{lp}{2V}\right) = 0 \end{aligned}\right\} \tag{2.37}$$

The stability criterion of the above system characterized by the second-order characteristic equation with complex coefficients is, for example, given in the reference [*7*] and, as the most convenient form to be simply applicable to stability estimation, in the reference [*29*]. According to the latter criterion, it is necessary to calculate the following quantities for stability decision:

$$\left.\begin{aligned} f&=\frac{\rho V^2 Sl}{2I_Y}\sqrt{C_{m\alpha}^2+C_{mp\beta}^2\left(\frac{lp}{2V}\right)^2} \\ \theta_f&=\tan^{-1}\left(\frac{C_{mp\beta}\dfrac{lp}{2V}}{-C_{m\alpha}}\right) \\ e&=\frac{1}{2\sqrt{f}}\sqrt{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\cdot\frac{l}{2V}C_{mq}\right)^2+(Kp)^2} \\ \theta_e&=\tan^{-1}\left(\frac{Kp}{\dfrac{\mu T}{I_Y}-\dfrac{\rho V^2 Sl}{2I_Y}\dfrac{l}{2V}C_{mq}}\right) \end{aligned}\right\} \tag{2.38}$$

If a point defined by θ_f and θ_e of the above quantities in a plane of (θ_f, θ_e) is located inside region limited by a given e as shown in Fig. 2.7. the system will be within the stability region.

Two roots of this characteristic equation are written as

$$\left.\begin{aligned} \begin{matrix}\lambda_1\\ \lambda_2\end{matrix}&=-\frac{1}{2}\left\{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2I_Y}C_{mq}\right)+iKp\right\}\pm\frac{1}{2}Ae^{i\frac{B}{2}} \\ &=-\frac{1}{2}\left\{\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}\pm A\cos\left(\frac{B}{2}\right)\right\}-\frac{i}{2}\left\{Kp\mp A\sin\left(\frac{B}{2}\right)\right\}, \end{aligned}\right\} \tag{2.39}$$

where

$$\left.\begin{aligned} A=\Bigg[&\left\{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\cdot\frac{l}{2V}C_{mq}\right)^2+4\cdot\frac{\rho V^2 Sl}{2I_Y}C_{m\alpha}-(Kp)^2\right\}^2 \\ &+4\left\{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}\right)K-2\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mp\beta}\right\}^2p^2\Bigg]^{\frac{1}{4}} \end{aligned}\right\} \tag{2.40}$$

$$B=\tan^{-1}\left\{\frac{2\left\{\left(\dfrac{\mu T}{I_Y}-\dfrac{\rho V^2 Sl}{2I_Y}\dfrac{l}{2V}C_{mq}\right)K-2\dfrac{\rho V^2 Sl}{2I_Y}\dfrac{l}{2V}C_{mp\beta}\right\}p}{\left(\dfrac{\mu T}{I_Y}-\dfrac{\rho V^2 Sl}{2I_Y}\dfrac{l}{2V}C_{mq}\right)^2+4\dfrac{\rho V^2 Sl}{2I_Y}C_{m\beta}-(Kp)^2}\right\} \tag{2.41}$$

It can be seen immediately from the equation (2.39) that the term of $\frac{1}{2}A\cos\left(\frac{B}{2}\right)$ makes to reduce the damping of one of two oscillations of the system and to increase the damping of the other one, as well as term $\frac{1}{2}A\sin\left(\frac{B}{2}\right)$ makes to reduce or increase the angular velocity of the system. To be stable, the system must satisfy the following condition:

$$\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}>\left|A\cos\left(\frac{B}{2}\right)\right|. \tag{2.42}$$

In order to keep the system to be optimum damping condition, *i.e.* to keep two roots, or the two oscillations, of the system to be equal damping it has to

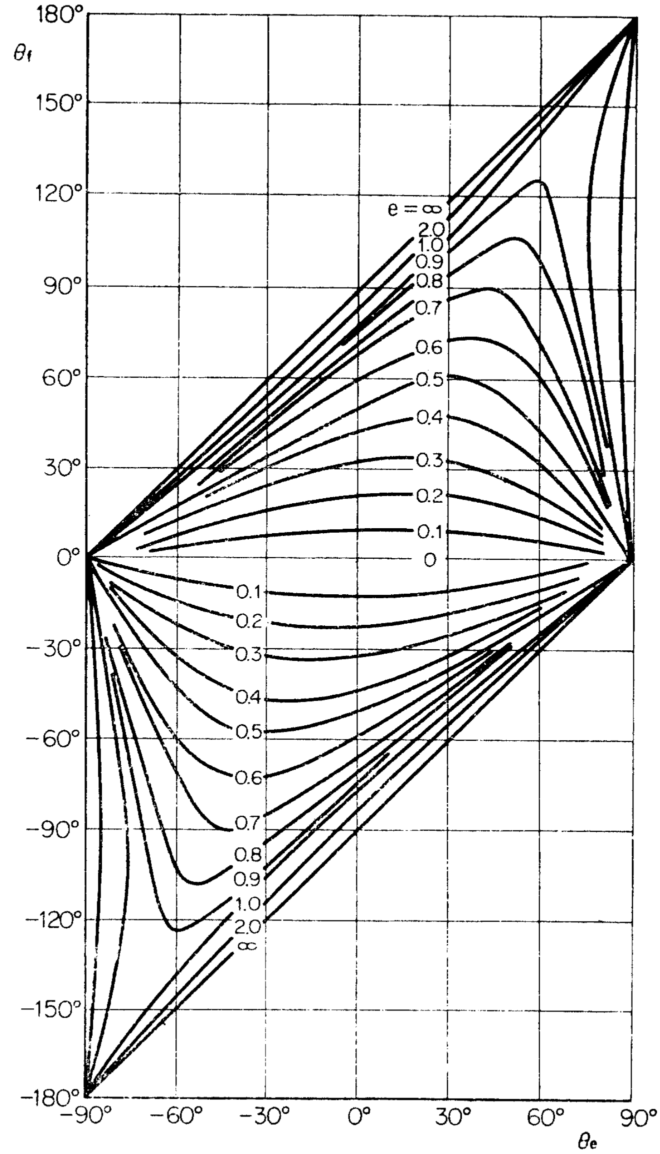


FIGURE 2.7. Stability region for the second order system with complex coefficients.

be satisfied that the term of $A\cos\left(\frac{B}{2}\right)$ is zero *i.e.* either $A=O$ or $B=\pi$. The condition in which $A=O$ will be obtained from the equation (2.40) as

$$\left.\begin{aligned}&\left(\frac{\mu T}{I_Y}-\frac{\rho V^2Sl}{2I_Y}\frac{l}{2V}C_{mq}\right)^2+4\frac{\rho V^2Sl}{2I_Y}C_{m_\alpha}-(Kp)^2=0\\ \text{and}\\ &\left(\frac{\mu T}{I_Y}-\frac{\rho V^2Sl}{2I_Y}\frac{l}{2V}C_{mq}\right)K-2\frac{\rho V^2Sl}{2I_Y}\frac{l}{2V}C_{mp\beta}=0\end{aligned}\right\} \quad (2.43)$$

The condition in which $B=\pi$ will be obtained if the following equations are satisfied:

$$\left.\begin{aligned}&\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}\right)^2+4\frac{\rho V^2 Sl}{2I_Y}C_{m_\alpha}-(Kp)^2<0\\ \text{and}\quad&\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}\right)K=2\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mp\beta}.\end{aligned}\right\}\tag{2.44}$$

The latter one in equation (2.44) is equivalent to the condition:

$$\sin\theta_f=e^2\sin 2\theta_e \tag{2.45}$$

which has been given in the reference [29]. Anyhow it is usually difficult to keep the above condition because this condition does not include the term of rate of spin, p, but depends only on the rocket configuration and the speed.

Time response for step input of the thrust malalignment is given by

$$\left.\begin{aligned}\varepsilon(t)=&\frac{\sqrt{\delta_{Y_b}^2+\delta_{Z_b}^2}\,T}{I_Y}e^{i\tan^{-1}\left(\frac{\delta Zb}{-\delta Yb}\right)}\\ &\cdot\sqrt{\left\{\frac{\rho V^2 Sl}{2I_Y}C_{m_\alpha}+p^2(1-K)\right\}^2+p^2\left\{\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}(C_{mq}+C_{mp\beta})-\frac{\mu T}{I_Y}\right\}^2}\\ &\cdot e^{i\tan^{-1}\frac{\left\{\frac{\rho V S^2 l}{2IY}\frac{l}{2V}(Cmq+Cmp\beta)-\frac{\mu T}{IY}\right\}p}{-\left\{\frac{\rho V^2 Sl}{2IY}Cm\alpha+(1-K)p^2\right\}}}\cdot\left[e^{-ipt}-\frac{1}{2}(e^{\lambda_1 t}+e^{\lambda_2 t})\right.\\ &-\frac{1}{2A}\sqrt{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}\right)^2+(2-K)^2p^2}\\ &\cdot e^{i\left\{\tan^{-1}\frac{-(2-K)p}{\frac{\mu T}{IY}-\frac{\rho V^2 Sl}{2IY}\frac{l}{2V}Cmq}-\frac{B}{2}\right\}}\cdot(e^{\lambda_1 t}-e^{\lambda_2 t})\end{aligned}\right\}\tag{2.46}$$

The first term in bracket, e^{-ipt}, shows a circle having angular velocity $-p$. Two terms, $(e^{\lambda_1 t}\pm e^{\lambda_2 t})$, presented in second and third terms in the bracket show an epicycle composed on two circles. It is important to say that the effect of the third term on the angular deviation of the system depends on the magnitude of $\sqrt{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}\right)^2+(2-K)^2p^2}\,/\,2A$.

Time response for the step inputs of the control moments is similarly given by

$$\left.\begin{aligned}\varepsilon(t)=&\frac{\sqrt{\delta M_Y^2+\delta M_Z^2}}{I_Z}e^{i\tan^{-1}\left(\frac{\delta MY}{\delta MZ}\right)}\cdot\frac{e^{-i\tan^{-1}\left(\frac{Cmp\beta\frac{lp}{2V}}{-Cm\alpha}\right)}}{\frac{\rho V^2 Sl}{2I_Y}\sqrt{C_{m\alpha}^2+\left(C_{mp\beta}\frac{lp}{2V}\right)^2}}\\ &\cdot\left[1-\frac{1}{2}(e^{\lambda_1 t}+e^{\lambda_2 t})-\frac{1}{2A}\sqrt{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}\right)^2+(Kp)^2}\right.\\ &\left.\cdot e^{i\left\{\tan^{-1}\frac{Kp}{\frac{\mu T}{IY}-\frac{\rho V^2 Sl}{2IY}\frac{l}{2V}Cmq}\right\}}\cdot(e^{\lambda_1 t}-e^{\lambda_2 t})\right].\end{aligned}\right\}\tag{2.47}$$

Since the terms in bracket of the equation (2.47) are similar expression to those of the equation (2.46) except the coefficient of the third term, it is not necessary to add further words.

The effect of the initial conditions is given by

$$
\left.
\begin{aligned}
\varepsilon(t)=&\varepsilon_0\left[\frac{1}{2}(e^{\lambda_1 t}+e^{\lambda_2 t})-\frac{1}{2A}\sqrt{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}\right)^2+(Kp)^2}\right.\\
&\left.\cdot e^{i\left\{\tan^{-1}\frac{Kp}{\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}}-\frac{B}{2}\right\}}\cdot(e^{\lambda_1 t}-e^{\lambda_2 t})\right]\\
&+\frac{e^{-i\frac{B}{2}}}{A}\left\{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}+iKp\right)\varepsilon_0+\dot{\varepsilon}_0\right\}(e^{\lambda_1 t}-e^{\lambda_2 t})
\end{aligned}
\right\}\quad(2.48)
$$

The above those equations (2.46~48) were derived with understanding that $A\neq O$. While $A=O$, λ_1 exactly coincids with λ_2 and therefore total response for the above step inputs is shown as

$$
\left.
\begin{aligned}
\varepsilon(t)=&\frac{\sqrt{\delta_{Yb}^{\ 2}+\delta_{Zb}^{\ 2}}\,T}{I_Y}e^{i\tan\left(\frac{\delta Zb}{-\delta Yb}\right)}\cdot\frac{4e^{-2i\tan^{-1}\frac{Kp}{\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}}}}{\left\{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}\right)^2+(2-K^2)p^2\right\}}\\
&\cdot\left[(e^{ipt}-e^{\lambda_1 t})+\frac{1}{2}\sqrt{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}\right)^2+(2-K)^2p^2}\right.\\
&\left.\cdot e^{i\left\{\tan\frac{-(2-K)p}{\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}}+\pi\right\}}\cdot te^{\lambda_1 t}\right]+\frac{\sqrt{\delta M_Y^2+\delta M_Z^2}}{I_Y}e^{i\tan\left(\frac{\delta M_Y}{\delta M_Z}\right)}\\
&\cdot\frac{4e^{-2i\tan^{-1}\frac{Kp}{\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}}}}{\left\{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}\right)^2+(Kp)^2\right\}}\left[1-e^{\lambda_1 t}\right.\\
&+\frac{1}{2}\sqrt{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}\right)^2+(Kp)^2}\\
&\left.\cdot e^{i\left\{\tan^{-1}\frac{Kp}{\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}}+\pi\right\}}\cdot te^{\lambda_1 t}\right]\\
&+\varepsilon_0\left\{1-\frac{t}{2}\sqrt{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}\right)^2+(Kp)^2}\right.\\
&\left.\cdot e^{i\tan^{-1}\frac{Kp}{\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}}}\right\}e^{\lambda_1 t}\\
&+\left\{\left(\frac{\mu T}{I_Y}-\frac{\rho V^2 Sl}{2I_Y}\frac{l}{2V}C_{mq}+iKp\right)\varepsilon_0+\dot{\varepsilon}_0\right\}t\cdot e^{\lambda_1 t}.
\end{aligned}
\right\}\quad(2.49)
$$

In the results given by equations (2.46~48) the most typical terms are

$$
-\frac{1}{2}(e^{\lambda_1 t}+e^{\lambda_2 t})-\frac{C}{2A}(e^{\lambda_1 t}-e^{\lambda_2 t})=-\frac{1}{2}\left\{\left(1+\frac{C}{A}\right)e^{\lambda_1 t}+\left(1-\frac{C}{A}\right)e^{\lambda_2 t}\right\}
$$

where C is a representation of the complex coefficients shown through those equations. As will be described in subsequent section, since this expression will correspond to the second and third terms of the equation (3.19) the detailed discussion of these quantities will further be notified in later.

3. SIMPLE FEEDBACK SYSTEM

Spin stabilization has been used as an open-loop system to maintain a constant rocket attitude in space coordinate. The system described herein for a spinning body probides a closed-loop attitude-control system or feedback system which will correct any attitude errors and damp out the rotational motion. In order to perform the above stabilizing action the system must be provided with not only sensing devices, such as two rate gyros which detect two orthogonal transverse angular velocities of the rocket and absolute attitude sensors, but also control systems, such as body mounted reaction-control jets which actuate with properly phased input signals derived from the above sensing devices.

The rates of transverse rotational angles are fedback to the reaction jets after modifying them with gain change, K_b and phase shift, ϕ_b, as shown in Fig. 3.1.

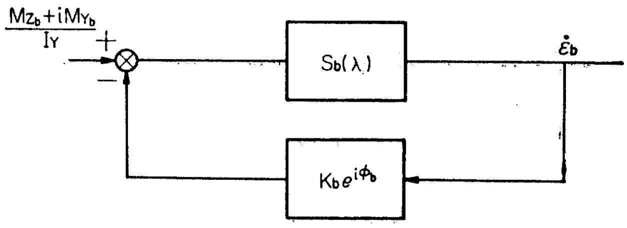


FIGURE 3.1. Augular rate feedback system.

For simplicity the following analyses performed in this section are restricted only for rotational motion of the spinning rocket flying in vacuum.

A transfer function of the rotational motion of spinning rocket flying in vacuum, $S_b(\lambda)$, is given from the equation (2.10) as

$$S_b(\lambda)=\frac{\dot{\varepsilon}_b(\lambda)}{\{M_{Zb}(\lambda)-iM_{Yb}(\lambda)\}/I_Y}=\frac{1}{\lambda-i(1-K)p} \tag{3.1}$$

In closed loop, such as Fig. 3.1, complex angular velocity about the body axes is given by

$$\dot{\varepsilon}_b(\lambda)=\frac{1}{\lambda-i(1-K)p+K_b e^{i\phi_b}}\cdot\frac{M_{Zb}(\lambda)+iM_{Yb}(\lambda)}{I_{Yb}}. \tag{3.2}$$

From the equations (2.7~9), on the other hand, the following relations are established:

$$\dot{\varepsilon}=\dot{\varepsilon}_b e^{-ipt}. \tag{3.3}$$

By applying the Laplace transform, the above equation becomes

$$\varepsilon(\lambda)=\frac{1}{\lambda}\dot{\varepsilon}_b(\lambda+ip), \tag{3.4}$$

and combining with the equation (3.2) this will again be written as

$$\varepsilon(\lambda)=\frac{1}{\lambda(\lambda+iKp+K_b e^{i\phi_b})}\cdot\frac{M_{Zb}(\lambda+ip)+iM_{Yb}(\lambda+ip)}{I_Y}. \tag{3.5}$$

If the Coriolis damping is neglected then the moment given in the above equation for the step input of the thrust malalignment is

$$\frac{M_{Zb}(\lambda+ip)+iM_{Zb}(\lambda+ip)}{I_{Yb}}=\frac{T(-\delta_{Yb}+i\delta_{Zb})}{I_{Yb}}\cdot\frac{1}{\lambda+ip}. \tag{3.6}$$

Thus, the time response for the thrust malalignment is given by

$$\left.\begin{aligned}\varepsilon(t)=&\frac{\sqrt{\delta_{Yb}^2+\delta_{Zb}^2}\,T}{I_{Yb}}\cdot e^{i\tan^{-1}\left(\frac{\delta Zb}{-\delta Yb}\right)}\cdot\frac{e^{i\left\{-\tan^{-1}\left(\frac{K_b\sin\phi_b+Kp}{K_b\cos\phi_b}\right)+\frac{3}{2}\pi\right\}}}{p\sqrt{(K_b\cos\phi_b)^2+(K_b\sin\phi_b+Kp)^2}}\\&\cdot\left[1-\frac{e^{-i\tan^{-1}\left(\frac{K_b\sin\phi_b-(1-K)p}{K_b\cos\phi_b}\right)}}{\sqrt{(K_b\cos\phi_b)^2+\{K_b\sin\phi_b-(1-K)p\}^2}}\right.\\&\cdot\left\{\sqrt{(K_b\cos\phi_b)^2+(K_b\sin\phi_b+Kp)^2}\cdot e^{i\left\{-pt+\tan^{-1}\left(\frac{K_b\sin\phi_b+Kp}{K_b\cos\phi_b}\right)\right\}}\right.\\&\left.\left.-ipe^{-(K_b\cos\phi_b)t-i(K_b\sin\phi_b+Kp)t}\right\}\right].\end{aligned}\right\} \tag{3.7}$$

The locus of fhe above response in (ψ,θ) plane is shown in Fig. 3.2. Although the final term in the bracket of the equation (3.7) will be damped out as long as $\phi_b \neq \pi/2$, the condition required for the most stable operation of the

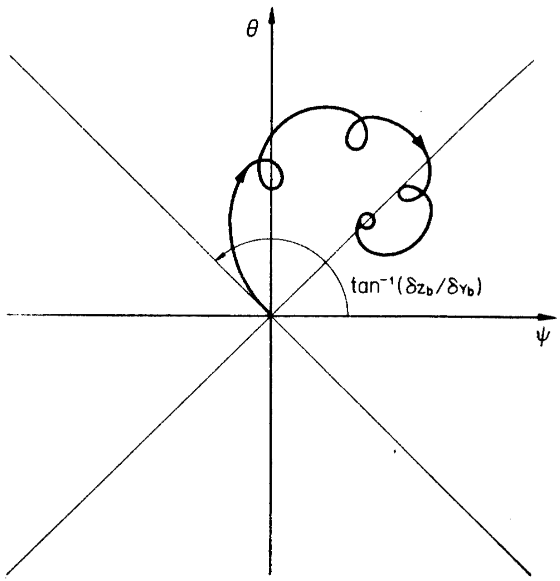


FIGURE 3.2. Locus of the complex angle for the thrust malalignment about the feedback system.

artificial system damping is $\phi_b = O$.

By using the relations of (2.6) and $\phi_b = O$, the equation (3.5) will be re-written as

$$\left.\begin{aligned} \varepsilon(\lambda) &= \frac{1}{\lambda(\lambda+K_b+iKp)} \cdot \frac{M_Z(\lambda)+iM_Y(\lambda)}{I_Y} \\ &= S(\lambda) \cdot \frac{M_Z(\lambda)+iM_Y(\lambda)}{I_Y} \end{aligned}\right\} \tag{3.8}$$

wherein

$$S(\lambda) = \frac{1}{\lambda(\lambda+K_b+iKp)} \tag{3.9}$$

On the other hand, it will be sure that the attitude rate on the stability axes, $\dot{\varepsilon}$, is also detected from a roll-stabilized platform as will be described hereinafter.

It will be able to find the attitude of the spinning body with respect to stationary space by using either a pair of two degree-of-freedom gyros mounted on a roll-stable platform [*42*] or electro-optical sensors [*38*]. The error outputs from the sensors provide signals which, when processed through the suitable electronic amplifiers, produce the corrective control moment after allocating the command signals through a roll-axis commutator or other resolving devices to each reaction jet. Usually, not only the reaction jet provides constant-level thrust pulses but also the roll commutator is only devided into finite segments, *e.g.* four segments, so that the system will not be able to describe as a linear feedback system. However, it will be assumed here that the control moment on the stability axes is given by equivalent linear expression in order to get easily the mathematical prospect for the system motion.

Then, a closed loop of the system will be shown as Fig. 3.3 and the complex angle for the closed loop will be given by

$$\varepsilon(\lambda) = \frac{K_s e^{i\phi_s}}{\lambda^2+\lambda(K_b+iKp)+K_s e^{i\phi_s}}\, \varepsilon_i(\lambda)\,. \tag{3.10}$$

Wherein K_s and ϕ_s show gain change and phase shift of the feedback system respectively.

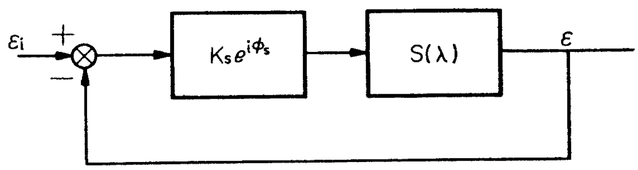


FIGURE 3.3. Attitude-angle feedback system.

The characteristic equation of the system and its roots are given by

$$D(\lambda) = \lambda^2+\lambda(K_b+iKp)+K_s e^{i\phi_s} = 0 \tag{3.11}$$

and

$$\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}=-\frac{1}{2}(K_b+iKp)\pm\frac{1}{2}Ae^{iB/2} \tag{3.12}$$

respectively, where

$$A=\{(K_b^2-K^2p^2-4K_s\cos\phi_s)^2+4(K_bKp-2K_s\sin\phi_s)^2\}^{1/4} \tag{3.13}$$

$$B=\tan^{-1}\left\{\frac{2(K_bKp-2K_s\sin\phi_s)}{K_b^2-K^2p^2-4K_s\cos\phi_s}\right\} \tag{3.14}$$

Comparing the above equations (3.12~14) with the equations (2.39~41) the following corresponding relations are obtained.

$$\left.\begin{array}{ll}\dfrac{\mu T}{I_Y}-\dfrac{\rho V^2Sl}{2I_Y}\dfrac{l}{2V}C_{mq} & \longrightarrow K_b\\ -\dfrac{\rho V^2Sl}{2I_Y}C_{m_\alpha} & \longrightarrow K_s\cos\phi_s\\ \dfrac{\rho V^2Sl}{2I_Y}\dfrac{lp}{2V}C_{mp\beta} & \longrightarrow K_s\sin\phi_s\end{array}\right\} \tag{3.15}$$

It is easy to understand that damping, K_b, parallel feedback gain, $K_s\cos\phi_s$, and orthogonal feedback gain, $K_s\sin\phi_s$, are equivalent to the aerodynamic damping moment, aerodynamic restoring moment and Magnus moment respectively. The most strong damping for both roots is obtained in such cases

$A=0$: i.e.

$$\left.\begin{array}{l}K_s\cos\phi_s=(K_b^2-K^2p^2)/4\\ K_s\sin\phi_s=K_bKp/2\end{array}\right\} \tag{3.16}$$

or $B=\pi$: i.e.

$$\left.\begin{array}{l}K_s\cos\phi_s>(K_b^2-K^2p^2)/4\\ K_s\sin\phi_s=K_bKp/2\end{array}\right\} \tag{3.17}$$

In the former case, in which the equation (3.16) is established, time response of the step control input, ε_i, is specifically given by

$$\varepsilon(t)=\varepsilon_i\left[1-\left\{1+\frac{1}{2}(K_b+iKp)\right\}e^{-\frac{1}{2}(K_b+iKp)t}\right] \tag{3.18}$$

Generally, time response of the step input of ε_i is given by

$$\left.\begin{array}{l}\varepsilon(t)=\varepsilon_i\left[1-\dfrac{1}{2}\left\{1+\dfrac{1}{A}\sqrt{K_b^2+(Kp)^2}\,e^{i\left(\tan^{-1}\frac{Kp}{K_b}-\frac{B}{2}\right)}\right\}e^{\lambda_1 t}\right.\\ \qquad\left.-\dfrac{1}{2}\left\{1-\dfrac{1}{A}\sqrt{K_b^2+(Kp)^2}\,e^{i\left(\tan^{-1}\frac{Kp}{K_b}-\frac{B}{2}\right)}\right\}e^{\lambda_2 t}\right]\end{array}\right\} \tag{3.19}$$

As previously mentioned this equation is equivalent to the equation (2.47) under the relation of (3.15).

It is very interesting to say that in order to get quick response of the closed-loop system the feedback-phase difference, ϕ_s, should preferably satisfy the relations (3.16) or (3.17), in other words, for a given K_b and Kp either $K_s \cos\phi_s$ and $K_s \sin\phi_s$ will be decided from the relation (3.16) or only $K_s \sin\phi_s$ will be decided from the relation (3.17). In actual system, however, K_s will be given as a fixed value or some fixed values because of, *e.g.*, capacity of the reaction jet so that $K_s \cos\phi_s$ and therefore ϕ_s will also be decided from the relation (3.17). In the latter case, as spin rate, p, increases from zero to infinity the phase difference, ϕ_s, may increase from zero to $\pi/2$. In the case of $\phi_s=\pi/2$, for an example, it will be apparent that yawing angular deviation causes the actuation for the reaction jet to create pitching moment. Other system parameters except $K_s \sin\phi_s$ (and $K_s \cos\phi_s$ for (3.16)), such as Kp and $K_s \cos\phi_s$, will be decided for a some optimum control as described in subsepuent section.

In a system employing only a single rate gyro and two oppositely directed control jets for stabilizing the lateral motion of a rocket, it will be impossible to treat the motion as a symmetrical system such as has been done previously. Let the system be expressed the following equation of motion:

$$\left.\begin{aligned} &\dot{q}-(1-K)pr=0 \\ &\dot{r}+K_b r+(1-K)pq=M_{Zb}/I_Y \end{aligned}\right\} \tag{3.20}$$

That is to say, the system comprises a yaw-rate gyro and two yaw-control jets. After taking the Laplace transform to each equation of motion and then combining the above obtained equations with i, the complex angular velocity in the body axis is given as

$$\dot{\varepsilon}_b(\lambda)=\frac{\lambda+i(1-K)p}{\lambda^2+K_b\lambda+(1-K)^2p^2}\cdot\frac{M_{Zb}(\lambda)}{I_Y} \tag{3.21}$$

By using the equation (3.4)

$$\left.\begin{aligned} \varepsilon(\lambda)&=\frac{\lambda+i(2-K)p}{\lambda\{\lambda^2+\lambda(K_b+2ip)+(K^2-2K)p^2+iK_b p\}}\cdot\frac{M_{Zb}(\lambda+ip)}{I_Y} \\ &\equiv S_s(\lambda)\frac{M_{Zb}(\lambda+ip)}{I_Y} \end{aligned}\right\} \tag{3.22}$$

Two roots of the quadratic equation of the denominator of (3.22) are

$$\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}=-\left(\frac{1}{2}K_b+ip\right)\pm\frac{1}{2}\sqrt{K_b^2-4(1-K)^2p^2}\,. \tag{3.23}$$

Comparing the above equation (3.22) and the roots of the equation (3.23) with the equation (3.5), it will be appreciated that if $K_b^2<4(1-K)^2p^2$, the damping of the equation (3.22) is the half of that given by the equation (3.5) when $\phi_b=O$, because of the single rate gyro, and if $K_b^2>4(1-K)^2p^2$, then the damping will increase for one root and decrease for the other.

In attitude control system of the spin-stabilized vehicle, it is mostly common to use either a single or two body-mounted reaction jet in order to control the attitude error which is detected by body-mounted attitude-error sensor or sensors. In such system, the attitude errors and their rates are fedback to the reaction jet after modifying to the control input for the body-axes system. Then the equations of motion for such system are, for example, given by

$$\left.\begin{aligned} &\dot{q}-(1-K)pr=0 \\ &\dot{r}+K_b r+(1-K)pq+K_s\cos(\phi_s+pt)\phi-K_s\sin(\phi_s+pt)\theta \\ &\quad =K_s\cos(\phi_s+pt)\phi_i-K_s\sin(\phi_s+pt)\theta_i \end{aligned}\right\} \tag{3.24}$$

Combining the above equations with $i=\sqrt{-1}$ and using the relation of (2.7), the above equations become

$$\left.\begin{aligned} \ddot{\varepsilon}+\left(\frac{K_b}{2}+iKp\right)\dot{\varepsilon}+\frac{K_s}{2}e^{i\phi_s}\varepsilon=\frac{K_s}{2}e^{i\phi_s}\varepsilon_i+\frac{K_s}{2}e^{-i(\phi_s+2pt)}\cdot\bar{\varepsilon}_i \\ -\frac{K_b}{2}e^{-i2pt}\dot{\bar{\varepsilon}}-\frac{K_s}{2}e^{-i(\phi_s+2pt)}\cdot\bar{\varepsilon} \end{aligned}\right\} \tag{3.25}$$

Applying the Laplace transform to the equation (3.25) with initial conditions being zeroes,

$$\left.\begin{aligned} \left\{\lambda^2+\left(\frac{K_b}{2}+iKp\right)\lambda+\frac{K_s}{2}e^{i\phi_s}\right\}\varepsilon(\lambda)=\frac{K_s}{2}e^{i\phi_s}\varepsilon_i(\lambda) \\ +\frac{K_s}{2}e^{-i\phi_s}\bar{\varepsilon}_i(\lambda+i2p)-\left\{\frac{K_b}{2}(\lambda+i2p)+\frac{K_s}{2}e^{-i\phi_s}\right\}\bar{\varepsilon}(\lambda+i2p) \end{aligned}\right\} \tag{3.26}$$

It will be sure that since the second and third terms of the right-hand side of the above equations consist of conjugate-complex input and feedback terms shifted to higher frequency respectively, these terms will be regarded to be zero as the first approximation within a normal flight condition or except small spin rate. For such case, the solution of $\varepsilon(\lambda)$ is approximated by

$$\varepsilon(\lambda)\doteqdot\frac{\dfrac{K_s}{2}e^{i\phi_s}}{\lambda^2+\lambda\left(\dfrac{K_b}{2}+iKp\right)+\dfrac{K_s}{2}e^{i\phi_s}}\cdot\varepsilon_i(\lambda) \tag{3.27}$$

It must be again notified that both the damping, K_b, and the feedback gain, K_s, are reduced as half values of those of the symmetric system given by the equation (3.10).

By introducing the above solution into the right-hand side of the equation (3.26), the second approximation for a solution of the equation (3.26) is given as

$$\left.\begin{aligned}\varepsilon(\lambda)&=\frac{\frac{K_s}{2}e^{i\phi_s}}{\lambda^2+\lambda\left(\frac{K_b}{2}+iKp\right)+\frac{K_s}{2}e^{i\phi_s}}\varepsilon_i(\lambda)\\&+\frac{\frac{K_s}{2}e^{-i\phi_s}(\lambda+i2p)\{\lambda+i(2-K)p\}\bar{\varepsilon}_i(\lambda+i2p)}{\left\{\lambda^2+\lambda\left(\frac{K_b}{2}+iKp\right)+\frac{K_s}{2}e^{i\phi_s}\right\}\left\{(\lambda+i2p)^2+\left(\frac{K_b}{2}-iKp\right)\times(\lambda+i2p)+\frac{K_s}{2}e^{-i\phi_s}\right\}}\end{aligned}\right\}\quad(3.28)$$

Thus, for the step input of the input, ε_i, the time response of the attitude angle, $\varepsilon(t)$, is given by

$$\left.\begin{aligned}\varepsilon(t)&=\varepsilon_i\left[1+\frac{\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_1 t}-\frac{\lambda_1}{\lambda_1-\lambda_2}e^{\lambda_2 t}\right]\\&+\bar{\varepsilon}_i\left(\frac{\frac{K_s}{2}e^{-i\phi_s}\{\lambda_1+i(2-K)p\}}{(\lambda_1-\lambda_2)(\lambda_1-\bar{\lambda}_1+i2p)(\lambda_1-\bar{\lambda}_2+i2p)}e^{\lambda_1 t}\right.\\&-\frac{\frac{K_s}{2}e^{-i\phi_s}\{\lambda_2+i(2-K)p\}}{(\lambda_1-\lambda_2)(\lambda_2-\bar{\lambda}_1+iKp)(\lambda_2-\bar{\lambda}_2+i2p)}e^{\lambda_2 t}\\&+\frac{\frac{K_s}{2}e^{-i\phi_s}(\bar{\lambda}_1-iKp)}{(\bar{\lambda}_1-\lambda_1-i2p)(\bar{\lambda}_1-\lambda_2-i2p)(\bar{\lambda}_1-\bar{\lambda}_2)}e^{(\bar{\lambda}-i2p)t}\\&\left.-\frac{\frac{K_s}{2}e^{-i\phi_s}(\bar{\lambda}_2-iKp)}{(\bar{\lambda}_2-\lambda_1-i2p)(\bar{\lambda}_2-\lambda_2-i2p)(\bar{\lambda}_1-\bar{\lambda}_2)}e^{(\bar{\lambda}_2-i2p)t}\right)+\cdots\end{aligned}\right\}\quad(3.29)$$

Where $\lambda_i(i=1,2)$ are two roots of the characteristic equation of the equation (3.27), $\bar{\lambda}_i$ are conjugate-complex values of λ_i.

In actual system it will be able to use the equation (3.27) instead of the equation (3.29) with satisfactory approximation. By the way of illustration, if the system parameters are such that $K_b=2$, $K_s\cos\phi_s=0.5$, $K_s\sin\phi_s=1.22$ and $Kp=1.225$, which will correspond to the one of optimum condition given in Table 4.1 in Section 4, the magnitude of the correction terms in the equation (3.29) are the order of 10^{-2}.

In order to apply the above analysis to the actual control system of the spinning vehicle it will be necessary to introduce suitable compensation network and the nonlinear systems, such as discontinuous commutator or resolver, proper thresholds and on-off reaction jet, instead of linear means assumed in the above analysis.

It is most convenient to estimate the dynamics of such nonlinear systems from simulator studies, but it may be quite sure that a fundamental understandings about general nature of the motion will be given from the above consideration on the linear systems.

4. Time Response and Optimum Control

For step input a typical time response of the spinning space vehicle having automatic control system, as well as of the spinning rocket flying in atmosphere without feedback system, is characterized by

$$\frac{\varepsilon(t)}{\varepsilon_i}=1+\frac{\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_1 t}-\frac{\lambda_1}{\lambda_1-\lambda_2}e^{\lambda_2 t}\ (\text{for } \lambda_1 \neq \lambda_2), \tag{4.1a}$$

$$=1-\left\{1-\frac{t}{2}\sqrt{K_b^2+(Kp)^2}\,e^{i\tan^{-1}\left(\frac{Kp}{Kb}\right)}\right\}e^{\lambda_1 t}\ (\text{for } \lambda_1=\lambda_2) \tag{4.1b}$$

The equation (4.1a) is rewritten by using the equation (3.19) as

$$\left.\begin{aligned}\frac{\varepsilon(t)}{\varepsilon_i}&=1-\frac{1}{2}\left\{1+\frac{1}{A}\sqrt{K_b^2+(Kp)^2}\,e^{i\left(\tan^{-1}\frac{Kp}{Kb}-\frac{B}{2}\right)}\right\}e^{\lambda_1 t}\\&\quad-\frac{1}{2}\left\{1-\frac{1}{A}\sqrt{K_b^2+(Kp)^2}\,e^{i\left(\tan^{-1}\frac{Kp}{Kb}-\frac{B}{2}\right)}\right\}e^{\lambda_2 t}\end{aligned}\right\} \tag{4.2}$$

It is apparent that the system parameters characterizing the dynamics of the spinning body are damping, K_b, product of inertia-moments ratio by spin rate, Kp, feedback gain, K_s, and its phase difference, ϕ_s. The particular quantities defining the time response of the system, such as A, B, λ_1, and λ_2 in the equation (4.2) are the functions of the above described quantities.

Now, in order to see the change of time response for the change of various values of system parameters let K_b keep constant and Kp increase for preselected values of K_s and ϕ_s. Fig. 4.1a, b, and c present the cases in which $K_b=1$, $K_s\cos\phi_s=2$, and $K_s\sin\phi_s=0$, 2, -2, for various Kp. The A and B are shown in Fig. 4a where each A increases from some value to infinity and each B aproches to π. The loci of characteristic roots for increasing Kp are shown in Fig. 4.1b. In any case λ_1 tends to approch to the origin and λ_2 increases negative frequency. When $K_s\sin\phi_s=0$, two roots are conjugate complex numbers each other at $Kp=0$, then λ_1 becomes more light damping and low frequency and λ_2 becomes more heavy damping and high frequency as Kp increases. While $K_s\sin\phi_s=2$, one root λ_1 is in negative real domain but the other root λ_2 is in positive real domain at $Kp=0$, *i.e.* the system in unstable at $Kp=0$. As Kp increases λ_2 enters into the negative real domain and hence the system becomes stable after λ_2 crosses the imaginary axis. In the case of

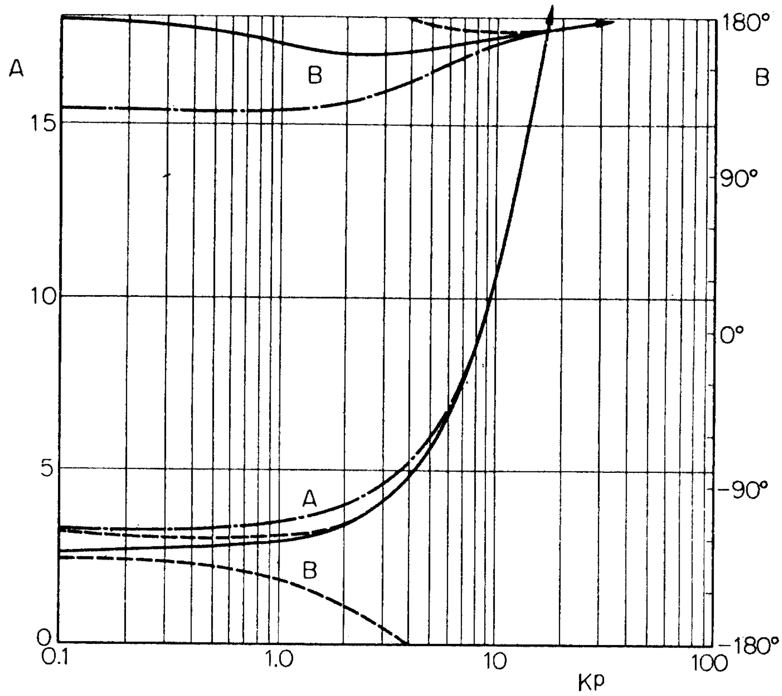


FIGURE 4.1. (a) A and B for $K_b=1$, $K_s \cos\phi_s=2$ $K_s \sin\phi_s=0$, 2, −2

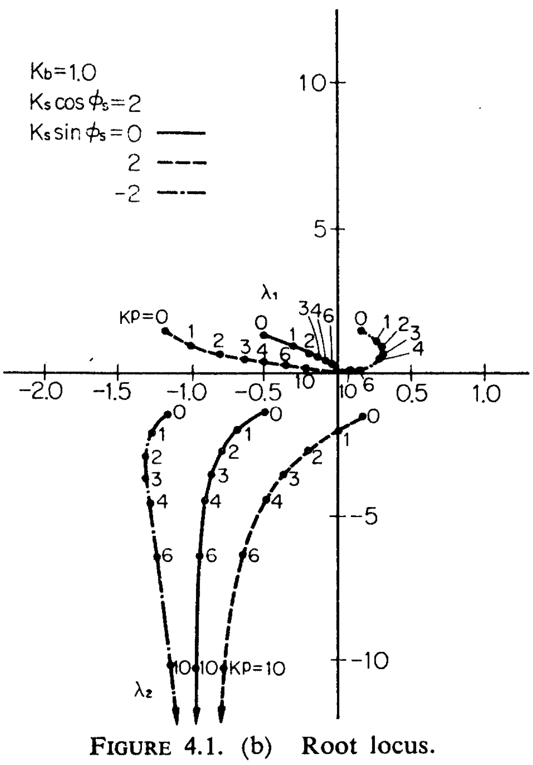


FIGURE 4.1. (b) Root locus.

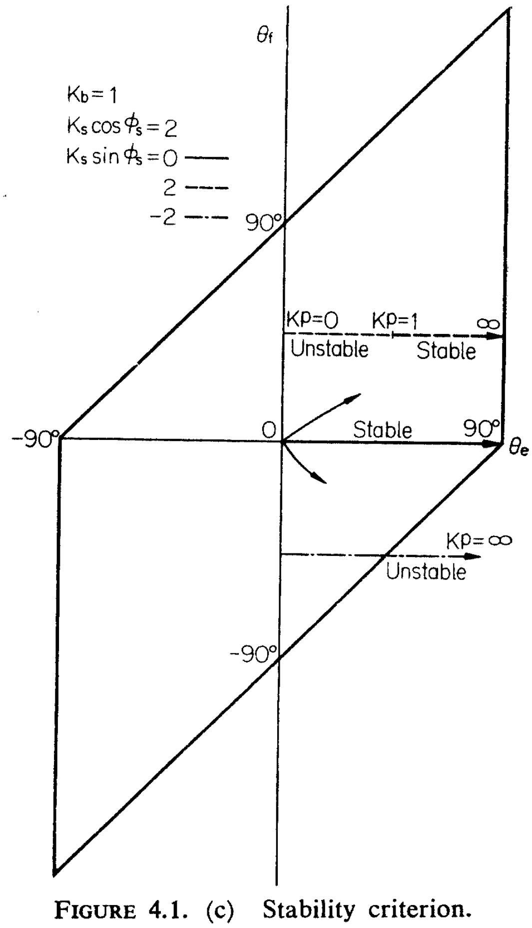


FIGURE 4.1. (c) Stability criterion.

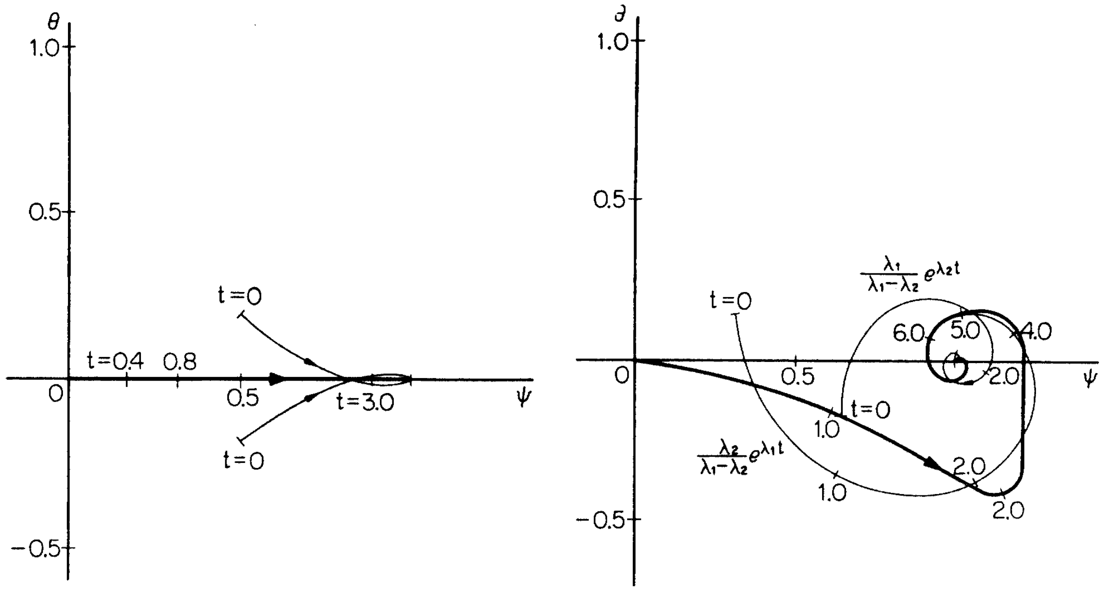
$K_s \sin\phi_s = -2$, one root λ_1 is always in positive real domain and never becomes stable for any Kp.

Fig. 4.1.c shows the change of θ_e and θ_f with increasing Kp. Thick lines show that the system is in stable. As Kp increases θ_e takes zero to $\pi/2$, so that in the case of $K_s \sin\phi_s = -2$, the system does not enter into the stable region for $Kp > O$.

For spinning rocket flying in atmosphere, Magnus moment will change from zero to some positive or negative value as spin rate increases, hence $K_s \sin\phi_s = 0$ at $Kp = 0$ and $K_s \sin\phi_s$ takes positive or negative value for positive or negative Magnus moment respectively. These loci are shown in Fig. 4.1c by two lines with arrows leaving from the origin.

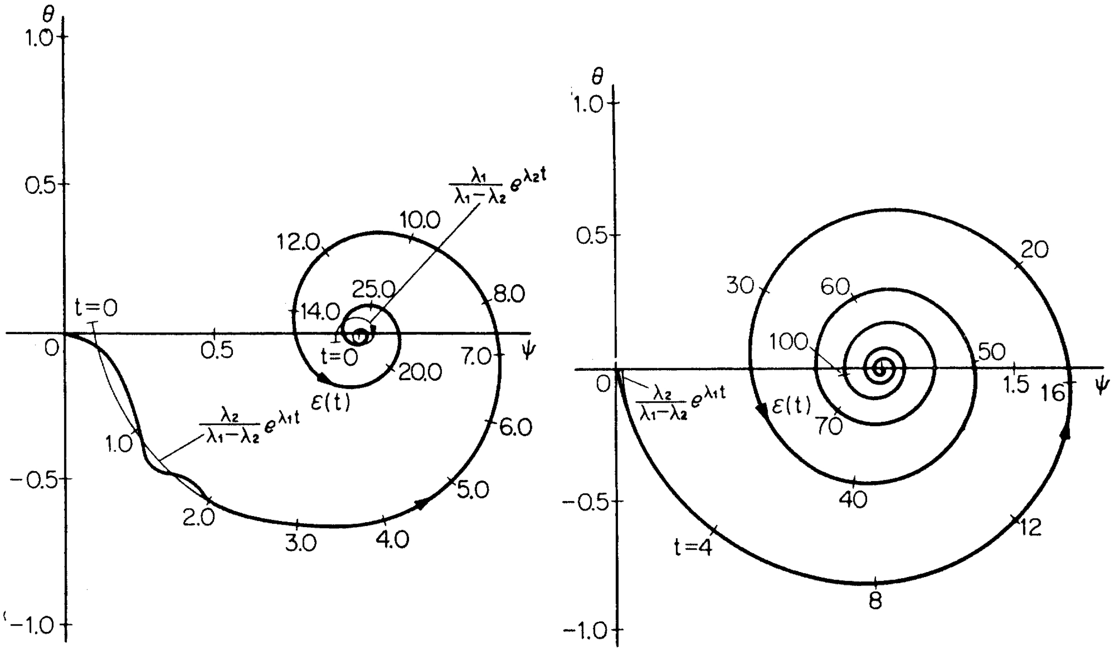
Fig. 4,2a, b, c and d show time response of step input, $\varepsilon_i = 1$, for the cases in which $K_b = 1$, $K_s \cos\phi_s = 2$, and $K_s \sin\phi_s = 0$. When $Kp = O$ the system shows the time response of usual nonspinning axisymmetrical rocket having positive restoring moment as Fig. 4.2a. It must be careful that in the figures of time response described hereinafter, the second and third terms of the equation (4.2) are drawn in the same figures as $\varepsilon(t)$ but their origins are located at 1.

The effect of spin rate is shown in Fig. 4.2b, c and d corresponding to $Kp = 1$,4 and 10 respectively. The third term of the equation (4.2), which is designated by $\frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2 t}$ in those figures, becomes smaller as Kp increases, and therefore the first term, 1, and the second term, which is designated by $\frac{\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1 t}$ are predominant for large Kp. The reason for this will be apparent



(a) $\begin{cases} K_b = 1 \\ K_s \cos\phi_s = 2 \\ K_s \sin\phi_s = 0 \\ Kp = 0 \end{cases}$

(b) $\begin{cases} K_b = 1 \\ K_s \cos\phi_s = 2 \\ K_s \sin\phi_s = 0 \\ Kp = 1 \end{cases}$



(c) $\begin{cases} K_b=1 \\ K_s\cos\phi_s=2 \\ K_s\sin\phi_s=0 \\ Kp=4 \end{cases}$ (d) $\begin{cases} K_b=1 \\ K_s\cos\phi_s=2 \\ K_s\sin\phi_s=0 \\ Kp=10 \end{cases}$

FIGURE 4.2. Time response for step input, $\varepsilon_i=1$, for $K_b=1$, $K_s\cos\phi_s=2$, $K_s\sin\phi_s=0$

from the equation (4.1a) and the Fig. 4.1b since absolute value of λ_2 increases with increasing Kp so that for large Kp the equation (4.1a) will be expressed as the following approximation:

$$\frac{\varepsilon(t)}{\varepsilon_i}=1-e^{\lambda_1 t} \quad (\text{for } \lambda_1 \fallingdotseq \lambda_2) \tag{4.3}$$

As the spin rate increases the system also reduces the damping or increases time required for approaching to a terminal point. This will again be described later.

Fig. 4.3a, b and c show the cases in which $K_b=1$, $K_s\cos\phi_s=2$, $K_s\sin\phi_s=2$, and $Kp=0$, 1 and 10 respectively. According to $Kp=0$, 1 and 10, the system is unstable, neutral and stable respectively.

It is interesting to say that in the case of $Kp=0$, the second term of $e^{\lambda_1 t}$ is smaller than the third term of $e^{\lambda_2 t}$, but the second term is predominant at higher Kp.

Fig. 4.4a and b show the cases in which $K_b=1$, $K_s\cos\phi_s=2$, $K_s\sin\phi_s=-2$ and $Kp=0$ and 10 respectively. In either case the system is unstable.

Fig. 4.5a, b and c present the cases in which $K_b=1$, $K_s\cos\phi_s=-2$ and $K_s\sin\phi_s=0$, 2, -2 for various Kp. From Fig. 4.5a, it will be apparent that as Kp increases A, for each case, increases to innnity after some of then once decreases. Herein, each B also approches to π.

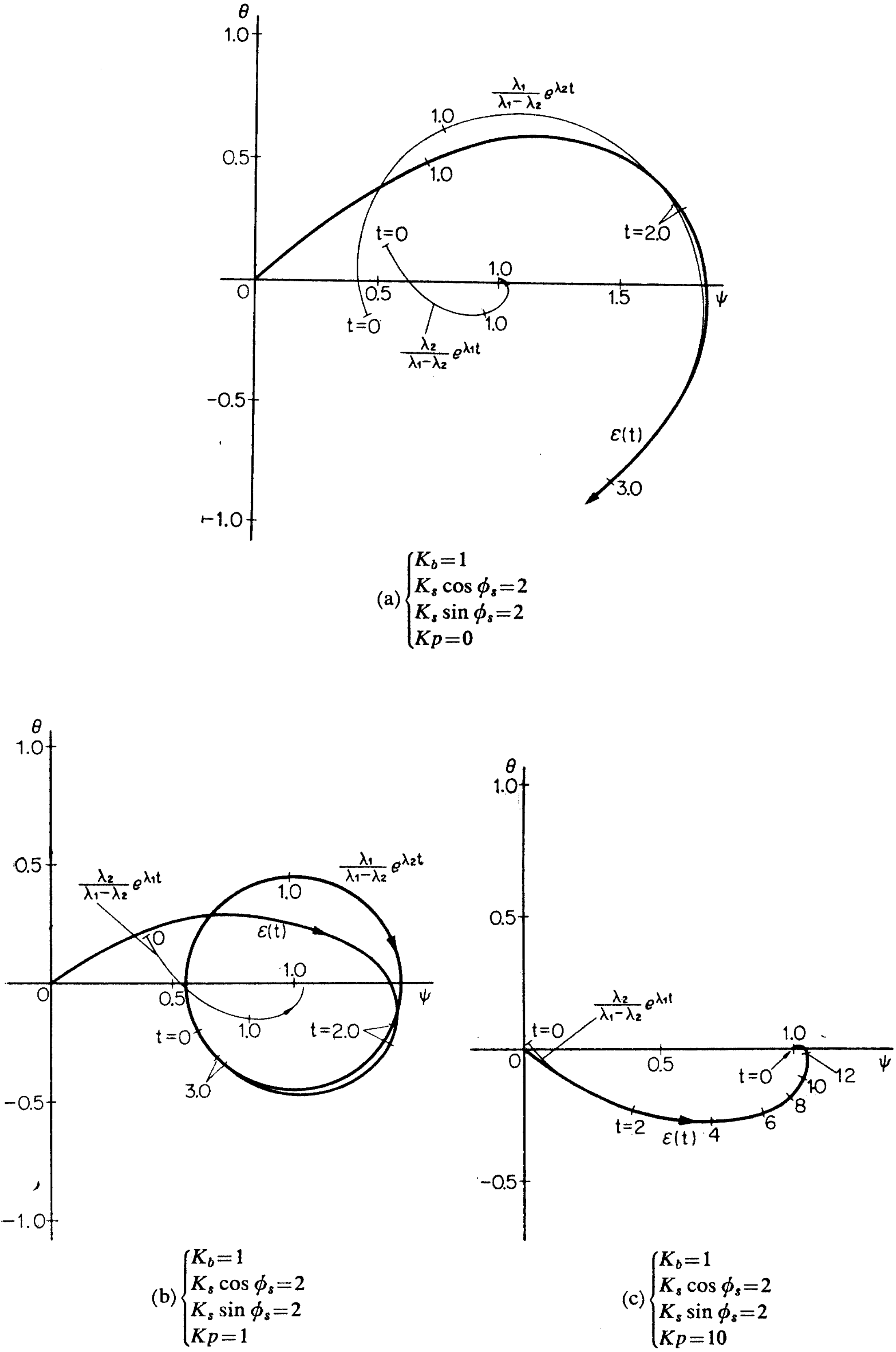
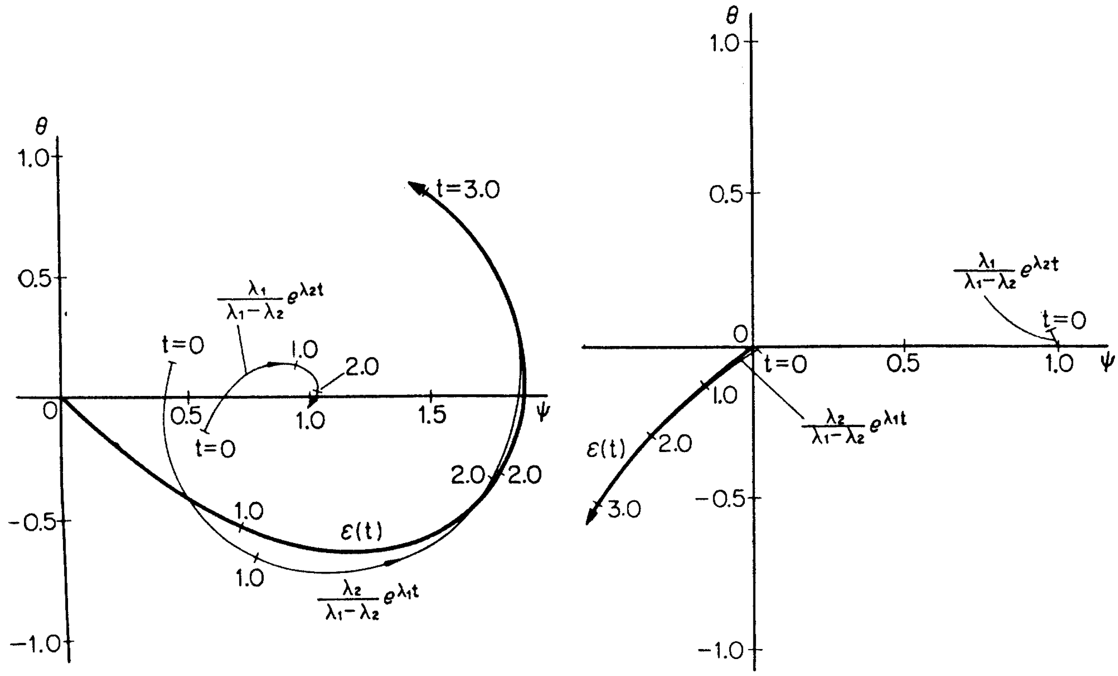


FIGURE 4.3. Time response for step input, $\varepsilon_i=1$, for $K_b=1$, $K_s\cos\phi_s=2$, $\sin\phi_s=2$



(a) $\begin{cases} K_b=1 \\ K_s\cos\phi_s=2 \\ K\ \sin\phi\ =-2 \\ Kp=0 \end{cases}$ (b) $\begin{cases} K_b=1 \\ K_s\cos\phi_s=2 \\ K\ \sin\phi\ =-2 \\ Kp=10 \end{cases}$

FIGURE 4.4. Time response for step inpct, $\varepsilon_i=1$, for $K_b=1$, $K_s\cos\phi_s=2$, $K\ \sin\phi\ =-2$

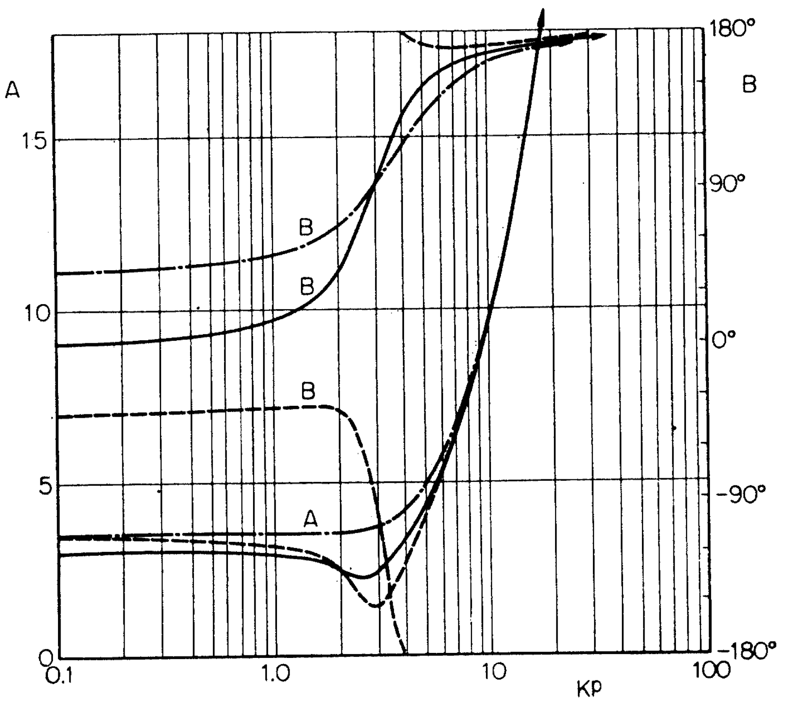


FIGURE 4.5 (a) A and B for $K_b=1$, $K_s\cos\phi_s=-2$, $K\ \sin\phi\ =0,\ 2,\ -2$

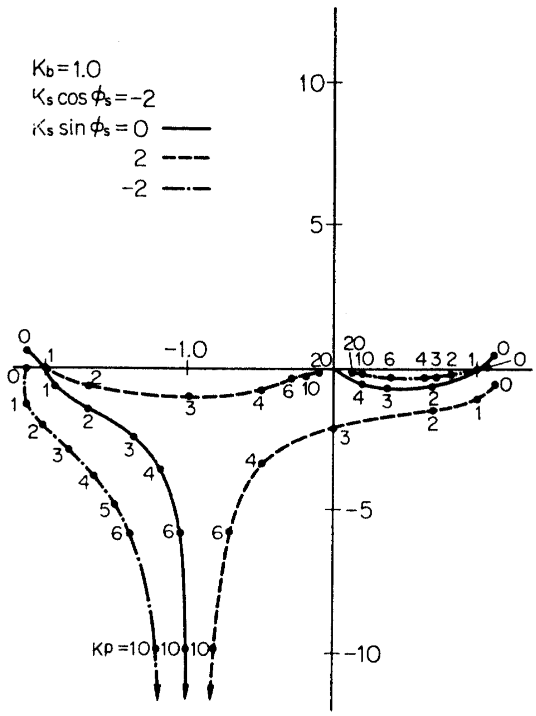


FIGURE 4.5 (b) Root locus

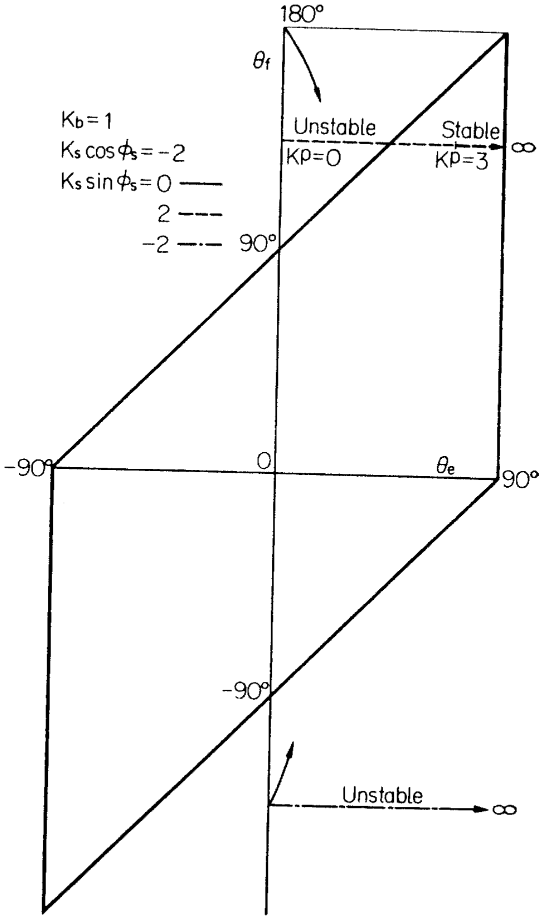


FIGURE 4.5 (c) Stability criterion

The root loci of the characteristic equation are shown in Fig. 4.5b, from which it will be observed that in only one case, $K_s \sin\phi_s = 2$, the system has stable region for $Kp > 3$ and in the remaining cases the system is always unstable. At $Kp = 4$, which satisfies the conditions (3.17) for the stable case of $K_s \sin\phi_s = 2$, negative real components of both roots λ_1 and λ_2 are equal, and hence the system will be in one of optimum condition.

The Fig. 4.5c shows the change of θ_f and θ_e with increasing Kp. For spinning rocket flying in atmosphere θ_f and θ_e will change along a line of two curves with arrow leaving from two points defined by $\theta_e = 0$ and $\theta_f = \pm\pi$, corresponding to positive and negative Magnus moment respectively.

Fig. 4.6a, b and c show the time response for step input, $\varepsilon_i = 1$, corresponding to the cases in which $K_b = 1$, $K_s \cos\phi_s = -2$, $K_s \sin\phi_s = 2$ and $Kp = 0$, 3 and 10 respectively. Fig. 4.6a, b and c show unstable motion for nonspinning case, and almost neutral motion and stable mntion for $Kp = 3$ and 10 respectively.

Fig. 4.7a, b and c show A and B, root loci of λ_1 and λ_2 and the change of θ_e and θ_f respectively for each case in which $K_b = 1$, $K_s \cos\phi_s = 0$ $K_s \sin\phi_s = 2$

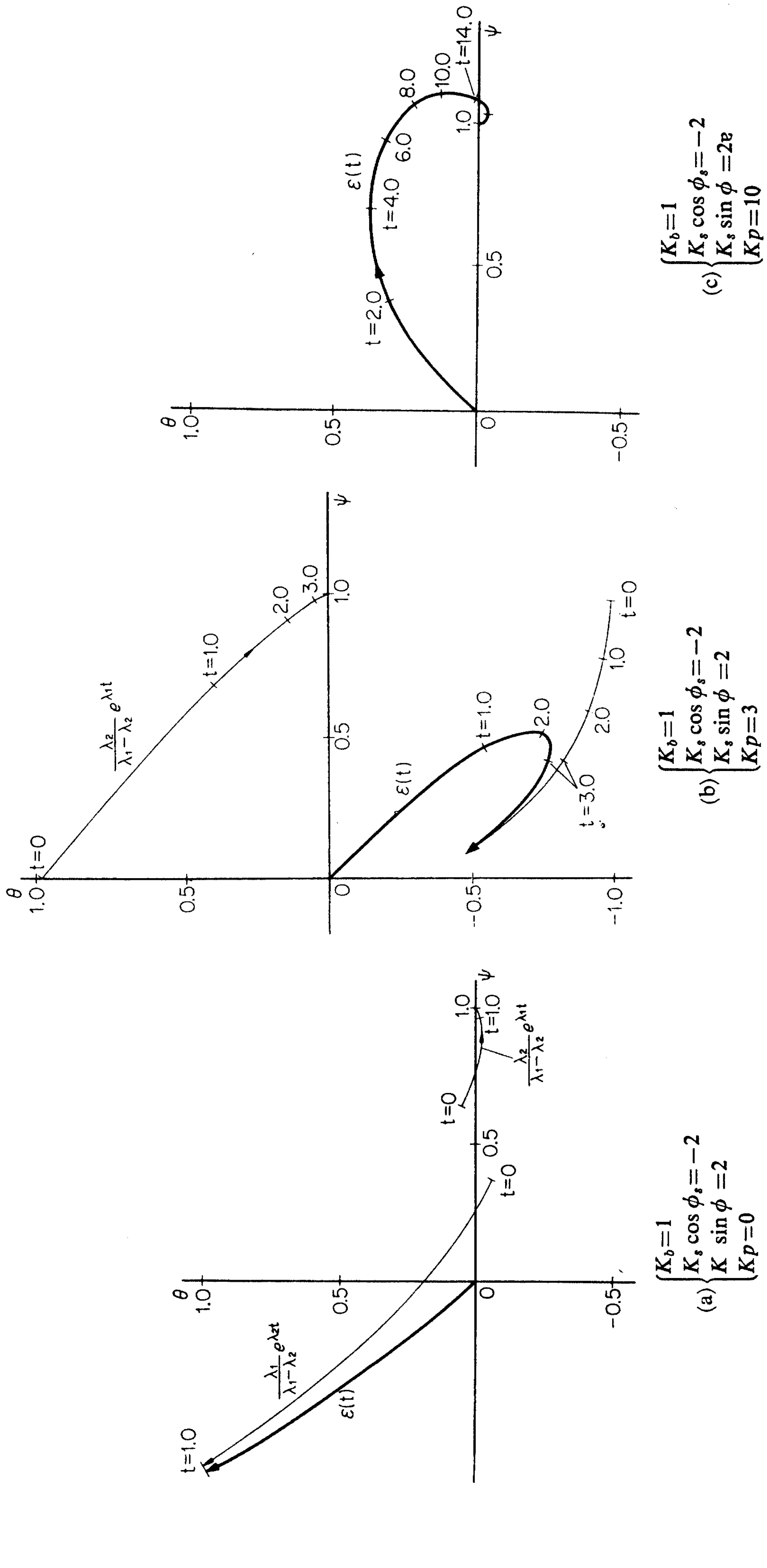


FIGURE 4.6. Time response for step input, $\varepsilon_i=1$, for $K_b=1$, $K_s\cos\phi_s=-2$, $K_s\sin\phi=2$

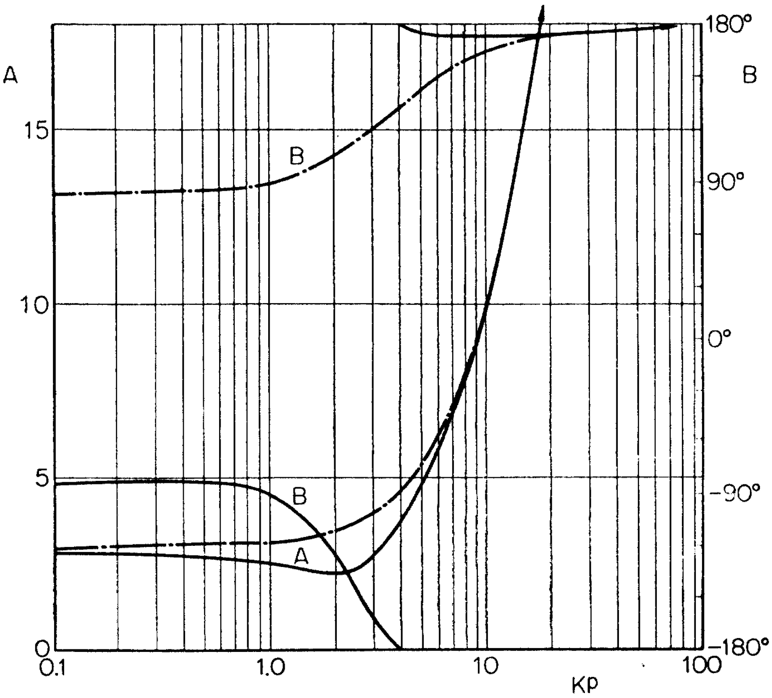


FIGURE 4.7 (a) A and B
for $K_b=1$, $K_s\cos\phi_s=0$, $K_s\sin\phi=2, -2$

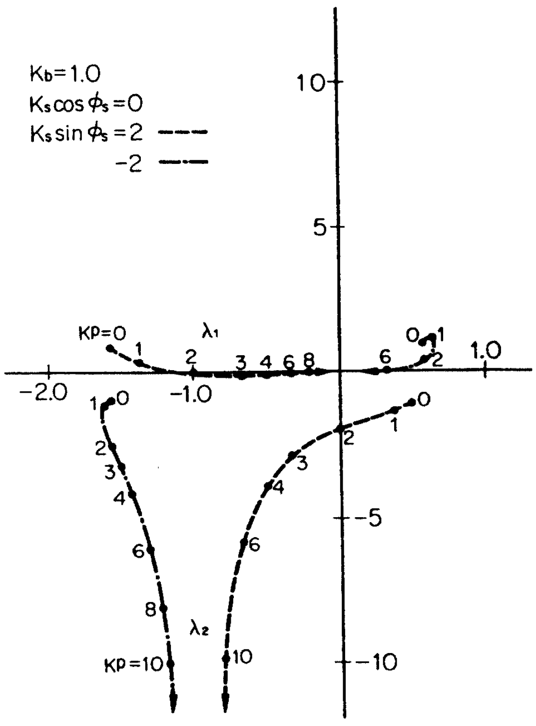


FIGURE 4.7 (b) Root locus

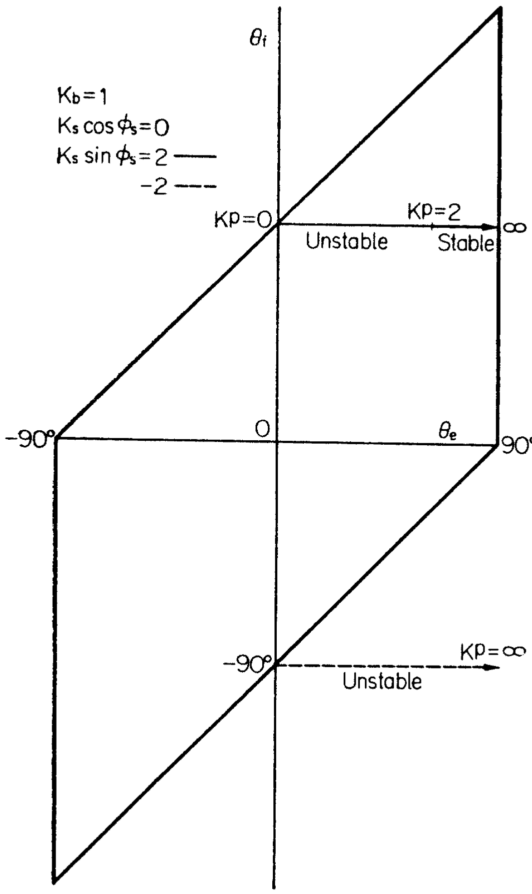


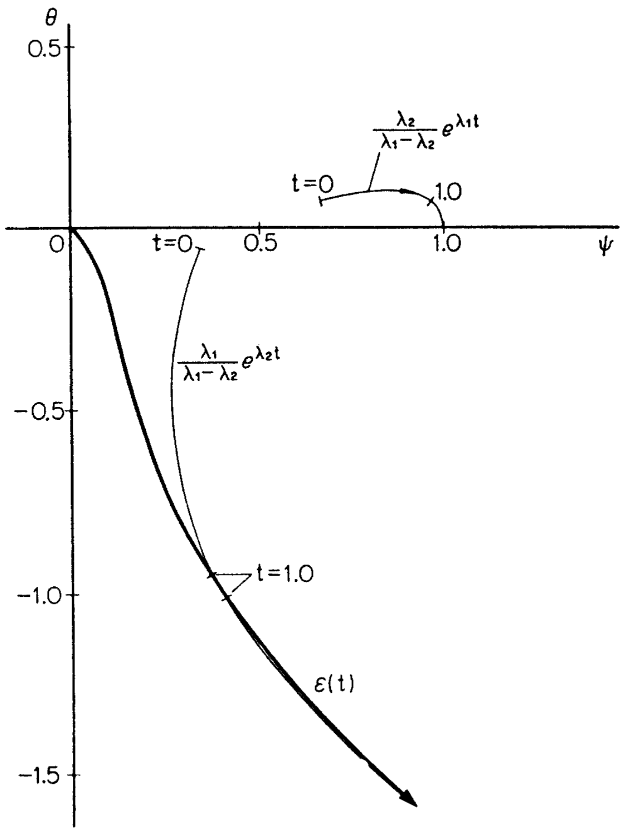
FIGURE 4.7 (c) Stability criterion

and -2. As saying before, the system can not be stabilized by positive spinning motion for negative value of $K_s \sin\phi_s$. For $K_s \sin\phi_s = 2$, $Kp = 4$ is also the spin rate at which optimum damping is obtained.

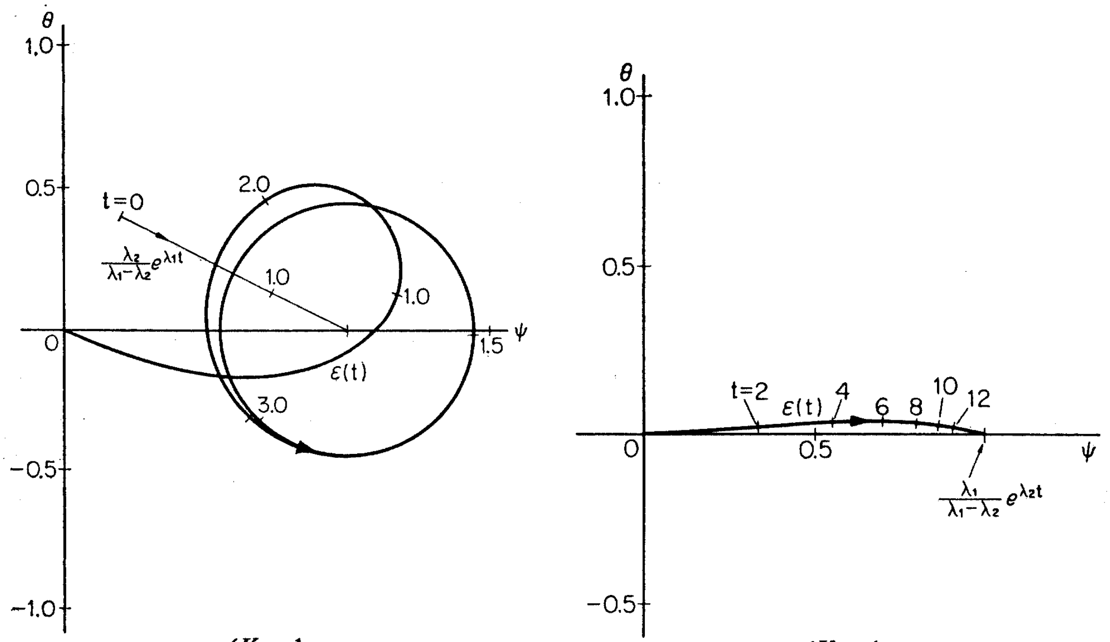
Fig. 4.8a, b and c show time response for each case in which $K_b = 1$, $K_s \cos\phi_s = 0$ and $K_s \sin\phi_s = 2$ and $Kp = 0$, 2, and 10 respectively. At $Kp = 0$, the system is unstable, while at $Kp = 2$ and 10 the system is neutral and stable respectively. Herein λ_1 also becomes predominant as Kp increases.

It will be interesting to compare, for instance, the Fig. 4.8c with the other cases having same values of $K_b = 1$, $K_s \sin\phi_s = 2$ and $Kp = 10$, such as Fig. 4.3c and Fig. 4.6c. From this comparion it will be sure that the $K_s \cos\phi_s$ tries to bend the path of time response approaching to final or terminal point, in other word, the $K_s \cos\phi_s$ seems to obstract to go straight. It must also be notified that the total time to reach the final point is almost unaffected by $K_s \cos\phi_s$, for large Kp.

Now, for the purpose to point the space vehicle to a predetermined direction in the space there may be some "optimum" way to determine the feedback parameters, such as feedback gain, necessary phase difference, damping and preferable spin rate, by which the vehicle can quickly respond to the control



$$(a)\begin{cases} K_b = 1 \\ K_s \cos\phi_s = 0 \\ K\ \sin\phi = 2 \\ Kp = 0 \end{cases}$$



(b) $\begin{cases} K_b=1 \\ K_s \cos\phi_s=0 \\ K_s \sin\phi=2 \\ Kp=2 \end{cases}$ (c) $\begin{cases} K_b=1 \\ K_s \cos\phi_s=0 \\ K_s \sin\phi=2 \\ Kp=10 \end{cases}$

FIGURE 4.8. Time response for step input, $\varepsilon_i=1$, for $K_b=1$, $K_s \cos\phi_s=0$ $K_s \sin\phi=2$

moment which is created by reaction jets corresponding to the error signal detected by sensors on board and modified by suitable compensation network.

Here, some of them will be discussed.

At first, it will be considered that a sweep area of a vector η, which is a radius vector starting from the terminal or destination point and ending on the time response curve as shown in Fig. 4.9, will be minimized for particular combi-

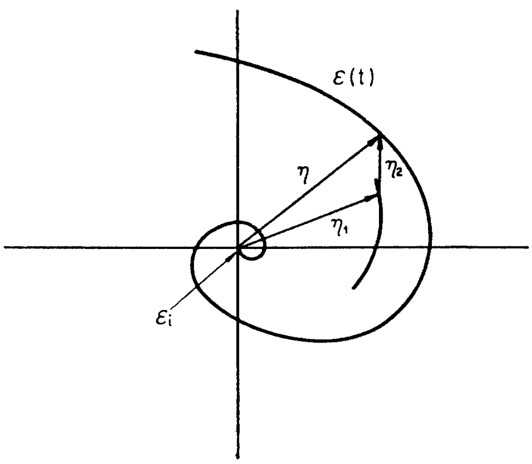


FIGURE 4.9. Vectorial expression of the time response from the terminal point.

nation of the system parameters. That is to say, let find the condition that the sweep area S defined by*

$$S=\left|\frac{1}{8i}\int_0^{\infty}\{(\dot{\eta}-\dot{\bar{\eta}})(\eta+\bar{\eta})-(\dot{\eta}+\dot{\bar{\eta}})(\eta-\bar{\eta})\}dt\right| \tag{4.4}$$

is minimum. Wherein $\dot{\eta}$ designates the time derivative of the vector η, and $\bar{\eta}$ shows conjugate complex vector of η. By substituting η, which is given by

$$\left.\begin{aligned}\eta&=\eta_1+\eta_2\\&=\frac{\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_1 t}-\frac{\lambda_1}{\lambda_1-\lambda_2}e^{\lambda_2 t},\end{aligned}\right\} \tag{4.5}$$

into (4.4), and then completing the integration

$$\left.\begin{aligned}S=&\left|\frac{1}{4i(\lambda_1-\lambda_2)(\bar{\lambda}_1-\bar{\lambda}_2)}\left\{\frac{-\lambda_2\bar{\lambda}_2(\lambda_1-\bar{\lambda}_1)}{\lambda_1+\bar{\lambda}_1}+\frac{\lambda_2\bar{\lambda}_1(\lambda_1-\bar{\lambda}_2)}{\lambda_1+\bar{\lambda}_2}\right.\right.\\&\left.\left.+\frac{\lambda_1\bar{\lambda}_2(\lambda_2-\bar{\lambda}_1)}{\lambda_2+\bar{\lambda}_1}-\frac{\lambda_1\bar{\lambda}_1(\lambda_2-\bar{\lambda}_2)}{\lambda_2+\bar{\lambda}_2}\right\}\right|\end{aligned}\right\} \tag{4.6}$$

Although it is not necessary to use the condition of (3.17) to find the minimum sweep area, but if this condition is preselected the above equation will be expressed more simply as

$$S=\frac{1}{4}\left|\frac{K_s\sin\phi_s(K_s\cos\phi_s-K_b^2)}{K_s^2\sin^2\phi_s+K_b^2K_s\cos\phi_s}\right| \tag{4.7}$$

Thus, when

$$K_s\cos\phi_s=K_b^2 \tag{4.8}$$

the sweep area becomes zero, *i.e.* the vector η loiters around a line passing through the origin and the final point of the motion and its total sweep area is zero.

Second, a scalar quantity, R, defined by square of the vector η, such as

$$\left.\begin{aligned}R&=\int_0^{\infty}\eta\bar{\eta}dt\\&=\frac{1}{(\lambda_1-\lambda_2)(\bar{\lambda}_1-\bar{\lambda}_2)}\left\{\frac{-\lambda_2\bar{\lambda}_2}{\lambda_1+\bar{\lambda}_1}+\frac{\lambda_1\bar{\lambda}_2}{\bar{\lambda}_1+\lambda_2}+\frac{\lambda_2\bar{\lambda}_1}{\lambda_1+\bar{\lambda}_2}+\frac{-\lambda_1\bar{\lambda}_1}{\lambda_2+\bar{\lambda}_2}\right\}\end{aligned}\right\} \tag{4.9}$$

will be minimized. Under the both condition (3.17) and (4.8) this becomes

$$R=\frac{1}{K_b} \tag{4.10}$$

---

* Although there is another scalar quatity relating to the sweep area, such as

$$S'=\int_0^{\infty}\left|\frac{1}{4i}(\dot{\eta}-\dot{\bar{\eta}})(\eta+\bar{\eta})-(\dot{\eta}+\dot{\bar{\eta}})(\eta-\bar{\eta})\right|^2dt,$$

the scalar quantities given by (4.4) or (4.14) will be treated here for simplicity.

This relation only imposes that the R can be reduced as K_b increases. In this case, therefore, there is some arbitrary choice for the system parameters. All decided relations from the above minimum conditions are as follows:

$$\left.\begin{aligned} K_b \sin\phi_s &= (Kp/2)K_b \\ K_s \cos\phi_s &= K_b^2 > \frac{K_b^2-(Kp)^2}{4} \end{aligned}\right\} \tag{4.11}$$

Instead of the R, if a new scalar quantity, R', such as

$$\left.\begin{aligned} R' &= \int_0^\infty (\eta_1\bar{\eta}_1+\eta_2\bar{\eta}_2)\,dt\left(>\int_0^\infty \eta\bar{\eta}\,dt\right) \\ &= \frac{-1}{(\lambda_1-\lambda_2)(\bar{\lambda}_1-\bar{\lambda}_2)}\left\{\frac{\lambda_2\bar{\lambda}_2}{\lambda_1+\bar{\lambda}_1}+\frac{\lambda_1\bar{\lambda}_1}{\lambda_2+\bar{\lambda}_2}\right\} \end{aligned}\right\} \tag{4.12}$$

is introduced, then, under the condition of (3.17), this will be

$$R' = \frac{2K_s\cos\phi_s+(Kp)^2}{K_b\{(Kp)^2+4K_s\cos\phi-K_b^2\}} = \frac{2(K_b^2K_s\cos\phi_s+2K_s^2\sin^2\phi_s)}{K_b\{4K_s^2\sin^2\phi_s+4K_b^2K_s\cos\phi_s-K_b^4\}} \tag{4.13}$$

Further more, it will be possible to introduce a sum of sweep area, S', of each vector, η_1 or η_2, corresponding to each spiral instead of the resultant vector, η, *i.e.*

$$S' = \frac{1}{2(\lambda_1-\lambda_2)(\bar{\lambda}_1-\bar{\lambda}_2)}\left\{\frac{-\lambda_2\bar{\lambda}_2}{\lambda_1+\bar{\lambda}_1}\left|\frac{\lambda_1-\bar{\lambda}_1}{2i}\right| - \frac{\lambda_1\bar{\lambda}_1}{\lambda_2+\bar{\lambda}_2}\left|\frac{\lambda_2-\bar{\lambda}_2}{2i}\right|\right\} \tag{4.14}$$

Under the condition of (3.17), this will be

$$\left.\begin{aligned} S' = \frac{1}{8A^2K_b}\, &\{K_b^2(|Kp-A|+|Kp+A|) \\ &+(Kp+A)^2|Kp-A|+(Kp-A)^2|Kp+A|\} \end{aligned}\right\} \tag{4.15}$$

When the following condition (a) or (b) is satisfied:

(a) $K_s\cos\phi_s = (K_b^2-K^2p^2)/2$ for $4K_s\cos\phi_s > K_b^2 > 2(Kp)^2$ (4.16a)

(b) $K_s\mathrm{cor}\,\phi_s = K_b^2/4$ or $Kp=A$ for otherwise (4.16b)

the S' takes a minimum value. By substituting this relation, (a) or (b) into the equation (4.13), the R' will be given as

$$R' = \frac{K_b^3}{K_b^4-K_s^2\sin^2\phi_s} \quad \text{for } 4K_s\cos\phi_s > K_b^2 > 2(Kp)^2 \tag{4.17a}$$

$$R' = \frac{K_b^4+8(K_s\sin\phi_s)^2}{8(K_s\sin\phi_s)^2K_b} \quad \text{for otherwise} \tag{4.17b}$$

The R' of (4.17a) will be small for either large K_b or small $K_s\sin\phi_s$. This fact will correspond to the characteristics of non-spinning body. While the

R' of (4.17b) will be minimum at

$$K_s \sin\phi_s = (\sqrt{3}/2\sqrt{2})K_b^2 \tag{4.18}$$

for a given K_b.

By combining the minimum conditions (3.17), (4.16a, b) and (4.18) the following relations are obtained:

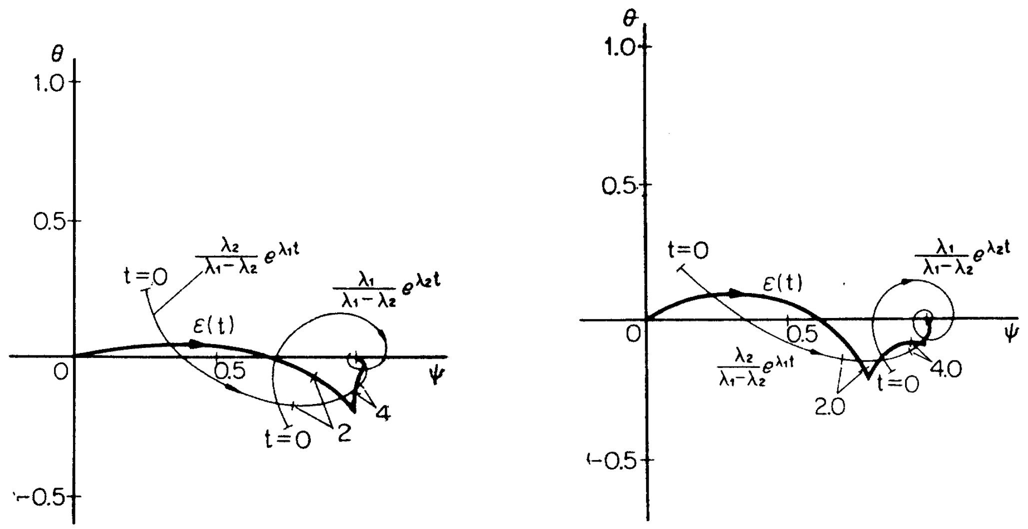
$$\left.\begin{aligned} K_s\sin\phi_s &= K_b Kp/2 \\ K_s\cos\phi_s &= (K_b^2 - K^2p^2)/2 \end{aligned}\right\} \quad \text{for} \quad 4K_s\cos\phi_s > K_b^2 > 2K^2p^2 \tag{4.19a}$$

$$\left.\begin{aligned} K_s\sin\phi_s &= (\sqrt{3}/2\sqrt{2})K_b^2 \\ K_s\cos\phi_s &= K_b^2/4 \\ Kp &= \sqrt{3/2}\,K_b \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} K_s &= (\sqrt{7}/4)K_b^2 \\ \phi_s &= \tan^{-1}\sqrt{6} = 67°48' \\ Kp &= \sqrt{3/2}K_b \end{aligned}\right\} \ \text{for otherwise} \tag{4.19b}$$

Either relation (4.11) or (4.19a, b) may not be always absolute optimum condition because of the condition (3.17), however it seems to be some measure for "optimum" control. It is still interesting to say that for non-spinning body, $Kp=0$, the condition (4.19a) gives $K_s\cos\phi_s = K_b^2/2$ and $K_s\sin\phi_s=0$, thereupon the system will be optimum when the damping coefficient, $K_b/2\sqrt{K_s\cos\phi_s}$, is equal to $1/\sqrt{2}$.

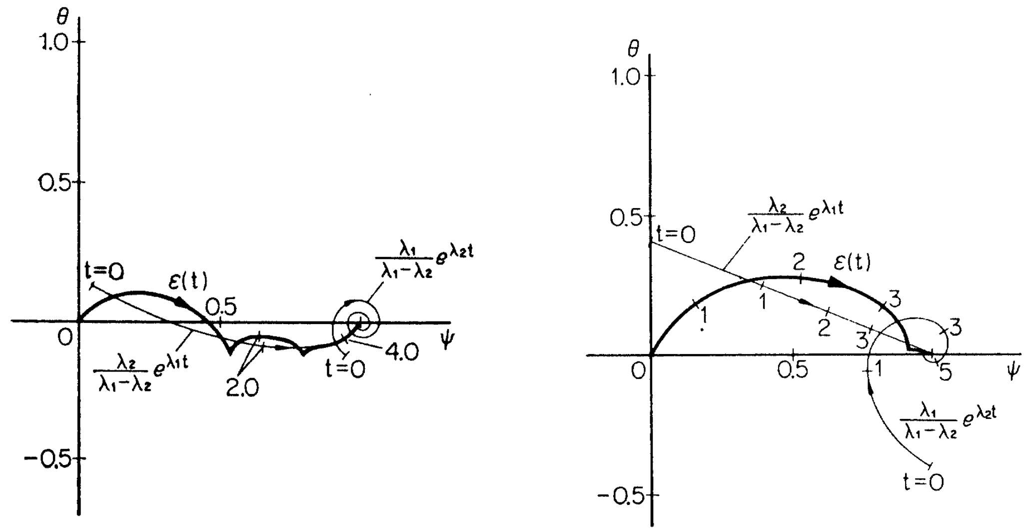
Fig. 4.10a, b, c, and d, show time response of th esystem having such optimum system parameters which are given in Table 4.1.

In the case of the condition (4.11), as shown in Fig. 4.10a, b, and c the total sweep area seems to be certainly zero. It is, however, interesting to notify that each locus showing the time response has some cusp or node and the number of cusp increases as Kp increases. As previously mentioned, the term of $e^{\lambda_2 t}$ also



(a) $\begin{cases} K_b=1 \\ K_s\cos\phi_s=1 \\ K_s\sin\phi={}_2/^1 \\ Kp=1 \end{cases}$

(b) $K_b=1$, $K_s\cos\phi_s=1$, $K_s\sin\phi=1$, $Kp=2$



(c) $K_b=1$
$K_s\cos\phi_s=1$
$K_s\sin\phi\ =2$
$Kp=4$

(d) $K_b=1$
$K_s\cos\phi_s=0.25$
$K\ \sin\phi\ =0.612$
$Kp=1.225$

FIGURE 4.10. Time response for step input, $\varepsilon_i=1$ for optimum condition (4.11) or (4.19).

TABLE 4.1 Optimum system parameter

condition \ parameters	K_b	$K_s\cos\phi_s$	$K_s\sin\phi_s$	Kp	K_s	ϕ_s
(4.11)	1	1	0.5	1	1.12	26°34′
	1	1	1	2	1.41	45°
	1	1	2	4	2.24	63°26′
(4.19b)	1	0.25	0.6124	1.225	0.661	67°48′

becomes small with increasing Kp.

For the condition (4.19b), the time response is shown in Fig. 4.10d. In this case the term of $e^{\lambda_1 t}$ presents a straight line so that the locus is a little more smooth than the case of (4.11). The sweep area is bigger than the one of (4.11).

The time required to reach around the final point is almost same since K_b is kept constant for all cases.

In the preceding discussion it is assumed that the condition (3.17) is always held, but if the condition (3.16) is established instead of (3.17), the system has also equal damping for both roots. Then the R will be given by

$$R=\frac{1}{2K_b^3}\{5K_b^2+(Kp)^2\}. \tag{4.20}$$

This relation points to be better for non-spinning body.

It has been shown that for large spin rate the equation (4.3) is effective as an approximation of the equation (4.2). Generally, as Kp increases the following approximations may be established:

$$\left.\begin{aligned}
A &\fallingdotseq (Kp)\left\{1+\frac{2K_s\cos\phi_s}{(Kp)^2}+\frac{1}{2}\left(\frac{K_b}{Kp}\right)^2\right\} \\
B &\fallingdotseq \tan^{-1}\left\{\frac{2\left(\dfrac{K_b}{Kp}-\dfrac{2K_s\sin\phi_s)}{(Kp)^2}\right)}{-1}\right\} \\
\lambda_1 &\fallingdotseq -\frac{K_s\sin\phi_s}{Kp}+i\,\frac{4K_s\cos\phi_s+K_b^2}{4Kp} \\
\lambda_2 &\fallingdotseq -\left(K_b-\frac{K_s\sin\phi_s}{Kp}\right)-i\left(Kp+\frac{K_s\cos\phi_s}{Kp}+\frac{K_b^2}{4Kp}\right)
\end{aligned}\right\} \tag{4.21}$$

$$\frac{\varepsilon(t)}{\varepsilon_i}=1-e^{(-K_s\sin\phi_s/Kp+i(4K_s\cos\phi_s+K_b^2)/4Kp)t} \tag{4.22}$$

Many important matters will be obtained from the above approximation for large Kp.

First, it is apparent that $K_s\sin\phi_s$ has to be positive but to be $K_s\sin\phi_s<Kp\times K_b$ for the stability of the system. Damping of the system will be reduced as Kp increases.

Second, either by increasing Kp or by making

$$K_s\cos\phi_s=-K_b^2/4 \tag{4.23}$$

the locus of the time response curve will approach to a straight line. Therefore, this relation also seems to be one of optimum condition. In the plane of complex attitude angle the response curve is almost straight line. Unfortunately, in this case, however, the time required to reach the final point is not so small, *i.e.* the system has slow response characteristic as shown, for an example, in Fig. 4.8c, comparing with other optimum cases.

Having decided the feedback terms, K_b and $K_se^{i\phi_s}$, the practical control moments for body-axes system shown in the equation (2.10) or (2.21b) will be given by

$$\left.\begin{aligned}
(M_{Z_b}+iM_{Y_b})/I_Y &= -K_b\dot{\varepsilon}e^{i\phi}-K_s(\varepsilon-\varepsilon_i)e^{i(\phi_s+\phi)} \\
\text{or}\qquad &= -K_b\dot{\varepsilon}_b-K_s(\varepsilon-\varepsilon_i)e^{i(\phi_s+\phi)}.
\end{aligned}\right\} \tag{4.24}$$

5. Motion of the Spinning Rocket Including Translational Motion

It is an object of this section to investigate some generalized motions of the spinning rocket during powered or unpowered flight sustaining rotational motion as well as translational motion which was out of consideration in the previous sections. It is, however, assumed for simplicity of the analysis that the gravitational force affecting the translational motion of the rocket is still neglected and the center of gravity is fixed in the body even in powered flight.

The relation between velocity vector of the rocket and the stability axes, (X, Y, Z), is shown in Fig. 5.1.

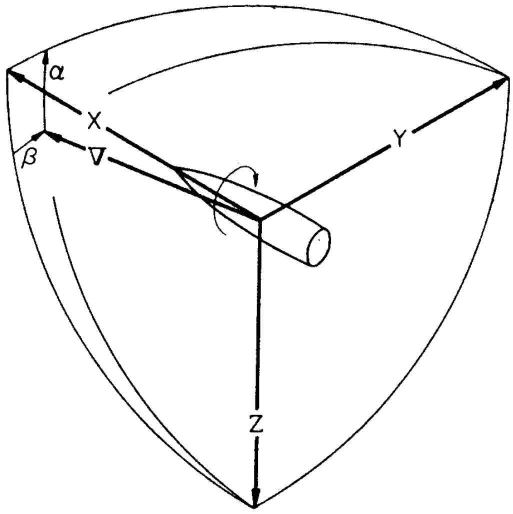


FIGURE 5.1. Orientation of velocity vector with respect to stability axes.

The equations of translational motion of a rocket are given by

$$\left.\begin{aligned} \dot{V} &= F_X/m \\ V(\dot{\beta}+\dot{\psi}) &= F_Y/m \\ V(\dot{\alpha}-\dot{\theta}) &= F_Z/m \end{aligned}\right\} \tag{5.1}$$

Wherein m is a mass of the rocket, *i.e.* $m=W/g$. Neglecting the gravitational force, the external forces given in the above equations are written as

$$\left.\begin{aligned} F_X/m &= T/m - \frac{1}{2}(\rho V^2 S/m)C_d \\ F_Y/m &= (T/m)(\psi_t \cos\phi + \theta_t \sin\phi) \\ &\quad - \frac{1}{2}(\rho V^2 S/m)\left(C_{L_\alpha}\beta + C_{L_{p\beta}}\frac{lp}{2V}\alpha + C_{L_{\dot{\alpha}}}\frac{l}{2V}\dot{\beta}\right) \\ F_Z/m &= (T/m)(\psi_t \sin\phi - \theta_t \cos\phi) \\ &\quad - \frac{1}{2}(\rho V^2 S/m)\left(C_{L_\alpha}\alpha + C_{L_{p\beta}}\frac{lp}{2V}\beta + C_{L_{\dot{\alpha}}}\frac{l}{2V}\dot{\alpha}\right) \end{aligned}\right\} \tag{5.2}$$

Wherein ψ_t and θ_t are the thrust-malalignment angles as shown in Fig. 2.2. The aerodynamic terms considered here are restricted only the first Taylor expansions of the main aerodynamic forces.

Substituting the equation (5.2) into the equation (5.1) and then combining these equations with i as previously having done, the following two equations are derived.

$$\dot{V}=T/m-\frac{1}{2}(\rho V^2S/m)C_d \tag{5.3}$$

and

$$V(\dot{\xi}+\dot{\varepsilon})=\frac{T}{m}\varepsilon_t e^{-i\phi}-\frac{1}{2}\frac{\rho V^2S}{m}\left(C_{L_\alpha}\xi+iC_{L_{p\beta}}\frac{\rho l}{2V}\xi+C_{L_{\dot\alpha}}\frac{l}{2V}\dot{\xi}\right) \tag{5.4}$$

where

$$\xi=\beta-i\alpha \tag{5.5}$$

and

$$\varepsilon_t=\psi_t+i\theta_t \tag{5.6}$$

Rotational motion of the rocket along the three axes is expressed by the following three equations of motion:

$$\left.\begin{aligned}\dot{p}&=M_X/I_X\\ \ddot{\theta}+\dot{\psi}Kp&=M_Y/I_Y\\ \ddot{\psi}-\dot{\theta}Kp&=M_Z/I_Y\end{aligned}\right\} \tag{5.7}$$

The external moments given in the above equations are similarly obtained from the control jet, thrust malalignment, the Coriolis damping and the aerodynamic forces, and they are consequently appeared as

$$\left.\begin{aligned}M_X/I_X&=\delta M_X/I_X+\frac{1}{2}(\rho V^2Sl/I_X)\left(C_{l_\delta}\delta+C_{l_p}\frac{l}{2V}p\right)\\ M_Y/I_Y&=(T/I_Y)(\delta_{Z_b}\cos\phi+\delta_{Y_b}\sin\phi)+\delta M_Y/I_Y-(\mu T/I_Y)\dot{\theta}\\ &\quad+\frac{1}{2}(\rho V^2Sl/I_Y)\left(C_{m_\alpha}\alpha+C_{m_{p\beta}}\frac{lp}{2V}\beta+C_{mq}\frac{l}{2V}\dot{\theta}+C_{m_{\dot\alpha}}\frac{l}{2V}\dot{\alpha}\right)\\ M_Z/I_Y&=(T/I_Y)(\delta_{Z_b}\sin\phi-\delta_{Y_b}\cos\phi)+\delta M_Z/I_Y-(\mu T/I_Y)\dot{\psi}\\ &\quad+\frac{1}{2}(\rho V^2Sl/I_Y)\left(-C_{m_\alpha}\beta+C_{m_{p\beta}}\frac{lp}{2V}\alpha+C_{mq}\frac{l}{2V}\dot{\psi}-C_{m_{\dot\alpha}}\frac{l}{2V}\dot{\beta}\right)\end{aligned}\right\} \tag{5.8}$$

Substituting the equation (5.8) into the equation (5.7) and then combining these equations with $\dot{\varepsilon}$

$$\dot{p}=\delta M_X/I_X+\frac{1}{2}(\rho V^2Sl/I_X)\left(C_{l_\delta}\delta+C_{l_p}\frac{l}{2V}p\right) \tag{5.9}$$

and

$$\left.\begin{aligned}&\ddot{\varepsilon}+iKp\dot{\varepsilon}=(T/I_Y)(-\delta_{Y_b}+i\delta_{Z_b})e^{-i\phi}+(\delta_{MZ}+i\delta_{MY})/I_Y-(\mu T/I_Y)\dot{\varepsilon}\\ &+\frac{1}{2}(\rho V^2Sl/I_Y)\left(-C_{m_\alpha}\xi+iC_{m_{p\beta}}\frac{lp}{2V}\xi+C_{mq}\frac{l}{2V}\dot{\varepsilon}-C_{m_{\dot\alpha}}\frac{l}{2V}\dot{\xi}\right)\end{aligned}\right\} \tag{5.10}$$

By combining the above equations (5.3), (5.4), (5.9) and (5.10) unknown variables, such as velocity V, spin rate p, complex attitude angle ε, and complex angle of attack ξ, will be obtained as functions of either time t or flight distance s with appropriate initial conditions.

Flight in Vacuum

It is important to consider the dynamics of the spinning rocket flying in vacuum as a fundamental treatment of the present analysis. By neglecting the aerodynamic term and using initial conditions such as at $t=t_0$, $V=V_0$ and $s=s_0$, the equation (5.3) will be integrated as

$$V(t)=V_0+\int_{t_0}^{t}(T/m)dt \tag{5.11}$$

$$s(t)=s_0+V_0(t-t_0)+\int_{t_0}^{t}\int_{t_0}^{t}(T/m)dt^2 \tag{5.12}$$

In special case of $T/m=$constant,

$$V=V_0+(T/m)(t-t_0) \tag{5.11$'$}$$

$$s=s_0+V_0(t-t_0)+\frac{1}{2}(T/m)(t-t_0)^2 \tag{5.12$'$}$$

Similarly, with initial conditions such as $t=t_0$, $p=p_0$ and $\phi=\phi_0$ the equation (5.9) becomes

$$p(t)=p_0+\int_{t_0}^{t}(\delta M_X/I_X)dt \tag{5.13}$$

$$\phi(t)=\phi_0+p(t-t_0)+\int_{t_0}^{t}\int_{t_0}^{t}(\delta M_X/I_X)dt^2 \tag{5.14}$$

If $\delta M_X/I_X=$constant, the above equations will again be

$$p(t)=p_0+(\delta M_X/I_X)(t-t_0) \tag{5.13$'$}$$

$$\phi(t)=\phi_0+p_0(t-t_0)+\frac{1}{2}(\delta M_X/I_X)(t-t_0)^2 \tag{5.14$'$}$$

From the equation (5.10) the complex angular velocity, $\dot{\varepsilon}$, will similarly be given by

$$\left.\begin{aligned}&\left[\frac{d}{dt}+\left\{\frac{\mu T}{I_Y}+iK\left(p_0+\int_{t_0}^{t}(\delta M_X/I_X)dt\right)\right\}\right]\dot{\varepsilon}\\&\quad=\left(\frac{T}{I_Y}\right)(-\delta_{Y_b}+i\delta_{Z_b})e^{-i\{\phi_0+p_0(t-t_0)+\int_{t_0}^{t}\int_{t_0}^{t}(\delta M_X/I_X)dt^2\}}.\\&\quad+(\delta M_Z+i\delta M_Y)/I_Y\end{aligned}\right\} \tag{5.15}$$

and the time response of the equation is given by

$$\dot{\varepsilon}(t)=\left[\dot{\varepsilon}_0+\int_{t_0}^{t}\left\{\frac{T}{I_Y}(-\delta_{Y_b}+i\delta_{Z_b})e^{-i\phi}+\frac{\delta M_Z+i\delta M_Y}{I_Y}\right\}e^{\int_{t_0}^{t}\left(\frac{\mu T}{IY}+iKp\right)dt}\cdot dt\right]e^{-\int_{t_0}^{t}\left(\frac{\mu T}{IY}+iKp\right)dt} \tag{5.16}$$

and

$$\varepsilon(t)=\varepsilon_0+\int_{t_0}^{t}\left\{\left[\dot{\varepsilon}_0+\int_{t_0}^{t}\left\{\frac{T}{I_Y}(-\delta_{Y_b}+i\delta_{Z_b})e^{-i\phi}+\frac{\delta M_Z+i\delta M_Y}{I_Y}\right\}e^{\int_{t_0}^{t}\left(\frac{\mu T}{IY}+iKp\right)dt}\cdot dt\right]e^{-\int_{t_0}^{t}\left(\frac{\mu T}{IY}+iKp\right)dt}\right\}dt \tag{5.17}$$

When both spin rate and rolling angle are constants the equation (5.15) coincides with the equation (2.24).

Therefore the equation (5.17) seems to be an extending solution of the equations (2.27) and (2.31) for variable spin rate. Let $\mu T/I_Y$ and p approach to zero then the equation (5.17) will become

$$\varepsilon(t)\doteqdot\varepsilon_0+\int_{t_0}^{t}\left[\dot{\varepsilon}_0+\int_{t_0}^{t}\left\{\frac{T}{I_Y}(-\delta_{Y_b}+i\delta_{Z_b})e^{-i\phi}+\frac{\delta M_Z+i\delta M_Y}{I_Y}\right\}dt\right]dt \tag{5.18}$$

This result shows the angular behaviour of the spinning rocket when the spin is almost stopped, and while no external moments exist this will be same result of the equation (2.32c).

Under the same assumption that T/m=constant, the equation (5.4) becomes

$$\dot{\xi}=\frac{(T/m)\,\varepsilon_t e^{-i\phi}}{V_0+(T/m)(t-t_0)}-\dot{\varepsilon} \tag{5.18}$$

Sabstituting the equation (5.6) into the above equation and integrating with t, the complex angle of attack is given by

$$\left.\begin{aligned}\xi(t)=\xi_0-\varepsilon_0+&\int_{t_0}^{t}\frac{(T/m)\varepsilon_t e^{-i\phi}}{V_0+(T/m)(t-t_0)}dt\\&-\int_{t_0}^{t}\left\{\left[\dot{\varepsilon}_0+\int_{t_0}^{t}\frac{T}{I_Y}(-\delta_{Y_b}+i\delta_{Z_b})e^{-i\phi}+\frac{\delta M_Z+i\delta M_Y}{I_Y}\right\}\right.\\&\left.\qquad e^{\int_{t_0}^{t}\left(\frac{\mu T}{IY}+iKp\right)dt}\cdot dt\right]e^{-\int_{t_0}^{t}\left(\frac{\mu T}{IY}+iKp\right)dt}\Big\}dt\end{aligned}\right\} \tag{5.19}$$

Transverse velocity and distance of the center of gravity, v and r, are

$$\left.\begin{aligned}v(t)&=v_0+\int_{t_0}^{t}V(\dot{\xi}+\dot{\varepsilon})dt\\&=v_0+\int_{t_0}^{t}(T/m)\varepsilon_t e^{-i\phi}dt\end{aligned}\right\} \tag{5.20}$$

and

$$r(t)=r_0+v_0(t-t_0)+\int_{t_0}^{t}\int_{t_0}^{t}(T/m)\,\varepsilon_t e^{-i\phi}dt^2 \tag{5.21}$$

Where v_0 and r_0 are initial transverse velocity and shift at $t=t_0$ respectively. If T/m=constant and $\delta M_x=0$, then

$$v(t)=v_0+(T/m)\varepsilon_t\frac{1}{p_0}e^{-i\{\phi_0+\frac{\pi}{2}+p(t-t_0)\}} \tag{5.20}$$

$$r(t)=r_0+v_0(t-t_0)-(T/m)\varepsilon_t\frac{1}{p_0^2}e^{-i\{\phi_0+p_0(t-t_0)\}} \tag{5.21}$$

In vacuum, as be considered herein, it is impossible to control the initial transverse velocity, v_0, of the rocket, but possible to reduce the effect of thrust malalignment of spinning motion.

Flight in Atmosphere

As usually done let an independent variable change from time, t, to flight distance, s, by using the following relation:

$$\begin{aligned}\frac{d}{dt}&=V\frac{d}{ds}\\ \frac{d^2}{dt^2}&=\frac{1}{2}\frac{dV^2}{ds}\frac{d}{ds}+V^2\frac{d^2}{ds^2}\end{aligned} \tag{5.22}$$

The velocity given by the equation (5.3) is given as

$$\frac{d}{ds}(V^2)=2T/m-(\rho V^2S/m)C_d \tag{5.23}$$

In the above equation, even if the thrust-mass ration, T/m, is a constant, air density, ρ, and drag coefficient, C_d, are given as a function of flight altitude h, mainly a function of the flight path. It is not a present object to calculate precise flight path or altitude of the rocket but to know the effect of the spinning motion of the rocket, so that the both quantities ρ and C_d are also assumed as constant values hereinafters except in the case of re-entry problem. Thus, for initial condition such as $V=V_0$ at $s=s_0$, the equation (5.23) can be integrated as

$$V(s)^2=2\frac{T}{m}\frac{m}{\rho SC_d}+\left(V_0^2-\frac{2T}{m}\frac{m}{\rho SC_d}\right)e^{-\frac{\rho SC_d}{m}(s-s_0)} \tag{5.24}$$

Similarly, the spin rate given by the equation (5.9) will be integrated with similar assumption and initial condition $p=p_0$ at $s=s_0$ as follows:

$$\left.\begin{aligned} p(s)=&\left[p_0+\int_{s_0}^{s}\left(\frac{\delta M_X}{I_X}\frac{1}{V}+\frac{1}{2}\frac{\rho Sl}{I_X}C_{l\delta}\delta V\right)e^{-\frac{1}{4}\frac{\rho Sl^2}{IX}C_{lp}(s-s_0)}ds\right]\\ &\cdot e^{\frac{1}{4}\frac{\rho Sl^2}{IX}C_{lp}(s-s_0)}\end{aligned}\right\} \tag{5.25}$$

Generally, during powered flight the thrust-mass ratio is predominant rather than aerodynamic drag, therefore the equation (5.24) will be able to reduce as

$$V^2=V_0^2+\frac{2T}{m}(s-s_0) \tag{5.26b}$$

On the other hand, for coasting or umpowered flight the equation (5.24) will be

$$V^2=V_0^2e^{-\frac{\rho Sd}{m}(s-s_0)} \tag{5.26b}$$

wherein the initial values V_0 and s_0 are not same to those of the equation (5.26a).

Substituting the above results into the equation (5.25) and assuming that $\delta M_X/I_X$ and $\rho SC_{l\delta}\delta/I_X$ are constant, then the spin rate, p, is given by,

(a) for powerd flight:

$$\left.\begin{aligned} p(s)=&\left[p_0+\frac{\delta M_X}{I_X}\int_{s_0}^{s}\frac{e^{-\frac{1}{4}\frac{\rho Sl^2}{IX}C_{lp}(s-s_0)}}{\sqrt{V_0^2+\frac{2T}{m}(s-s_0)}}ds\right.\\ &+\frac{1}{2}\frac{\rho Sl}{I_X}C_{l\delta}\int_{s_0}^{s}\sqrt{V_0^2+\frac{2T}{m}(s-s_0)}\\ &\left.\cdot e^{-\frac{1}{4}\frac{\rho Sl^2}{IX}C_{lp}(s-s_0)}ds\right]e^{\frac{1}{4}\frac{\rho Sl^2}{IX}(s-s_0)}\end{aligned}\right\} \tag{5.27a}$$

(b) for unpowered flight:

$$\left.\begin{aligned} p(s)=&p_0e^{\frac{1}{4}\frac{\rho Sl^2}{IX}C_{lp}(s-s_0)}+\frac{\delta M_X}{I_X}\frac{e^{\frac{\rho Sd}{2m}(s-s_0)}-e^{\frac{1}{4}\frac{\rho Sl^2}{IX}C_{lp}(s-s_0)}}{V_0\left(\frac{\rho SC_d}{2m}-\frac{1}{4}\frac{\rho Sl^2}{I_X}C_{lp}\right)}\\ &-\frac{1}{2}\frac{\rho Sl}{I_X}C_{l\delta}\delta V_0\frac{e^{-\frac{\rho Sd}{2n}(s-s_0)}-e^{\frac{1}{4}\frac{\rho Sl^2}{IX}C_{lp}(s-s_0)}}{\frac{\rho SC_d}{2m}+\frac{1}{4}\frac{\rho Sl^2}{I_X}C_{lp}}\end{aligned}\right\} \tag{5.27b}$$

Similarly rolling angle, ϕ, is given from the equation (5.25) as

$$\left.\begin{aligned} \phi(s)=\phi_0+\int_{s_0}^{s}\Bigg\{p_0+\int_{s_0}^{s}&\frac{\delta M_X}{I_Y}\frac{1}{V}\\ &+\frac{1}{2}\frac{\rho Sl}{I_Y}C_{l\delta}\delta V\Bigg)e^{-\frac{1}{4}\frac{\rho Sl^2}{IX}C_{lp}(s-s_0)}ds\Bigg\}\\ &\cdot e^{\frac{1}{4}\frac{\rho Sl^2}{IX}C_{lp}(s-s_0)}ds\end{aligned}\right\} \tag{5.28}$$

and as approximation this will be

(a) for powered flight:

$$
\left.\begin{aligned}
\phi(s)=\phi_0+&\frac{p_0}{\frac{1}{4}\cdot\frac{\rho Sl^2}{I_X}C_{lp}}\left(e^{\frac{1}{4}\cdot\frac{\rho Sl^2}{I_X}C_{lp}(s-s_0)}-1\right)\\
&+\frac{\delta M_X}{I_X}\int_{s_0}^{s}\int_{s_0}^{s}\frac{e^{-\frac{1}{4}\frac{\rho Sl^2}{I_X}C_{lp}(s-s_0)}}{\sqrt{V_0^2+\frac{2T}{m}(s-s_0)}}ds\cdot e^{\frac{1}{4}\frac{\rho Sl^2}{I_X}C_{lp}(s-s_0)}ds\\
&+\frac{\rho Sl}{2I_X}C_{l\delta}\delta\int_{s_0}^{s}\int_{s_0}^{s}\sqrt{V_0^2+\frac{2T}{m}(s-s_0)}\\
&\cdot e^{-\frac{1}{4}\frac{\rho Sl^2}{I_X}C_{lp}(s-s_0)}ds\cdot e^{\frac{1}{4}\frac{\rho Sl^2}{I_X}C_{lp}(s-s_0)}\cdot ds
\end{aligned}\right\} \tag{5.29}
$$

(b) for unpowered flight:

$$
\left.\begin{aligned}
\phi(s)=\phi_0+&\frac{p_0}{\frac{1}{4}\frac{\rho Sl^2}{I_X}C_{lp}}\left(e^{\frac{1}{4}\frac{\rho Sl^2}{I_X}(s-s_0)}-1\right)\\
&+\frac{\delta M_X}{I_X}\frac{1}{V_0\left(\frac{\rho SC_d}{2m}-\frac{1}{4}\frac{\rho Sl^2}{I_X}C_{lp}\right)}\left\{\frac{e^{\frac{\rho SC_d}{2m}(s-s_0)}-1}{\rho SC_d/2m}\right.\\
&\left.-\frac{e^{\frac{\rho Sl^2}{4I_X}C_{lp}(s-s_0)}-1}{\rho Sl^2C_{lp}/4I_X}\right\}+\frac{\rho Sl}{2I_X}C_{l\delta}\delta\frac{V_0}{\frac{\rho SC_d}{2m}+\frac{1}{4}\frac{\rho Sl^2}{I_X}C_{lp}}\\
&\left\{\frac{e^{-\frac{\rho SC_d}{2m}(s-s_0)}-1}{\rho SC_d/2m}-\frac{e^{\frac{\rho Sl^2}{4I_X}C_{lp}(s-s_0)}-1}{\rho Sl^2C_{lp}/4I_X}\right\}
\end{aligned}\right\} \tag{5.29b}
$$

For ballistic re-entry bodies, on the other hand, the rate of change of altitude with time can be approximated by [43]

$$\frac{dh}{dt}=-V\sin\theta_E \tag{5.30}$$

where θ_E is an angle between horizontal and re-entry velocity, positive downward. The variation of atmospheric density with altitude can be approximately by

$$\rho=\rho_0 e^{-\beta h} \tag{5.31}$$

where ρ_0 is a nominal low-altitude density and β is an exponential altitude parameter. Additionally, the velocity is given by

$$V=V_E e^{-\frac{k_0}{2}e^{-\beta h}} \tag{5.32}$$

where V_E is a velocity at start of re-entry and k_0 is given by

$$k_0 = \frac{\rho_0}{\beta \sin\theta_E} \frac{C_d S}{m} \tag{5.33}$$

and $C_d S$ shows flat area of the re-entry body. If $\delta M_X / I_X = 0$, the equation (5.9) can then be integrated to give [48]

$$p(h) = V_E \frac{ml}{I_X C_d} C_{l\delta} \delta \frac{1}{1 - \frac{1}{2} \frac{C_{lp}}{C_d} \frac{ml^2}{I_X}} \left(e^{-\frac{C_{lp}}{4C_d} \frac{ml^2}{I_X} k_0 e^{-\beta h}} - e^{-\frac{k_0}{2} e^{-\beta h}}\right) \tag{5.34}$$

Therefore, rolling angle ϕ is given by

$$\left.\begin{aligned} \phi &= \phi_0 + \int_\infty^h p(h) dh \\ &= \phi_0 + V_E \frac{ml}{I_X C_d} C_{l\delta} \delta \frac{1}{1 - \frac{1}{2} \frac{C_{lp}}{C_d} \frac{ml^2}{I_X}} \\ &\quad \left\{ \int_\infty^h e^{-\frac{C_{lp}}{4C_d} \frac{ml^2}{I_X} k_0 e^{-\beta h}} dh - \int_\infty^h e^{-\frac{k_0}{2} e^{-\beta h}} dh \right\} \end{aligned}\right\} \tag{5.35}$$

Furthermore, the equilibrium roll rate can be obtained as

$$p = -\left(\frac{2}{l}\right)\left(\frac{C_{l\delta}}{C_{lp}}\right) \delta V_E e^{-\frac{k_0}{2} e^{-\beta h}} \tag{5.36}$$

The equations (5.4) and (5.10) will similarly be transformed under the assumption that the Coriolis damping and the lift damping, $C_{L\dot{\alpha}}$, are negligiblly small, such as

$$\left.\begin{aligned} &\left\{\frac{d}{ds} + \frac{\rho S}{2m}\left(C_{L\alpha} + i C_{Lp_\alpha} \frac{l\nu}{2}\right)\right\}\xi + \frac{d\varepsilon}{ds} = \frac{T}{m} \varepsilon_t \frac{e^{-i\phi}}{V^2} \\ &\frac{\rho S l}{2 I_Y}\left\{C_{m\dot{\alpha}} \frac{l}{2} \frac{d}{ds} + C_{m_\alpha} - i C_{mp_\alpha} \frac{l\nu}{2}\right\}\xi + \left\{\frac{d^2}{ds^2}\right. \\ &\quad \left. + \left(-\frac{\rho S l^2}{4 I_Y} C_{mq} + i K \nu\right)\frac{d}{ds}\right\}\varepsilon = \frac{T}{I_Y}(-\delta_{Y_b} + i\delta_{Z_b}) \frac{e^{-i\phi}}{V^2} \\ &\quad + \frac{1}{I_Y}(\delta M_Z + i \delta M_Y) \frac{1}{V^2} \end{aligned}\right\} \tag{5.37}$$

where ν is a spin-velocity ratio and is given by

$$\nu = p/V \tag{5.38}$$

If the spin-velocity ratio, ν, and air density, ρ, are constant, the equations (5.37) are linerized constant coefficient differential equations and therefore their characteristic equation is given by

$$\left.\begin{aligned}D(\lambda)=\lambda\Big[\lambda^2+\lambda\Big\{\Big(\frac{\rho S}{2m}C_{L\alpha}-\frac{\rho Sl^2}{4I_Y}(C_{mq}+C_{m\dot{\alpha}})\Big)\\+i\Big(\frac{\rho Sl}{4m}C_{Lp_\alpha}+K\Big)\nu\Big\}+\Big\{-\Big(\frac{\rho S}{2m}C_{L\alpha}\frac{\rho Sl^2}{4I_Y}C_{mq}+\frac{\rho Sl}{4m}C_{Lp_\alpha}K\nu^2\\+\frac{\rho Sl}{2I_Y}C_{m\alpha}\Big)+i\Big(-\frac{\rho Sl}{4m}C_{Lp_\alpha}\frac{\rho Sl^2}{4I_Y}C_{mq}+\frac{\rho S}{2m}C_{L\alpha}K\\+\frac{\rho Sl^2}{4I_Y}C_{mp_\alpha}\Big)\nu\Big\}\Big]=0\end{aligned}\right\}\quad(5.39)$$

The assumption $\nu=$constant is sustained under such condition that the aerodynamic damping of the rolling motion is not so predominant. In other words, for flight in vacuum ν is given as

(a) for powered flight:

$$\nu=\frac{\delta M_X/I_X}{T/m}\tag{5.40a}$$

(b) for unpowered flight:

$$\nu=p_0/V_0\tag{5.40b}$$

For flight in atmosphere if the spinning motion is created by the deflected fins and $p=0$ then

(c) for launching vehicle:

$$\nu=-\Big(\frac{ml}{I_X}\Big)\Big(\frac{C_{l\delta}}{C_\delta}\Big)\delta\tag{5.40c}$$

(d) for re-entry vehicle after equilibrium-roll rate attained:

$$\nu=-\Big(\frac{4}{l}\Big)\Big(\frac{C_{l\delta}}{C_{lp}}\Big)\delta\tag{5.40d}$$

For constant air density, the stability of the system will again be decided from the Fig. 2.7 with the necessary coefficients such as

$$\left.\begin{aligned}f=\Big\{\Big(\frac{\rho S}{2m}C_{L\alpha}\frac{\rho Sl^2}{4I_Y}C_{mq}+\frac{\rho Sl}{4m}C_{Lp_\alpha}K\nu^2+\frac{\rho Sl}{2I_Y}C_{m\alpha}\Big)^2\\+\Big(-\frac{\rho Sl}{4m}C_{Lp_\alpha}\frac{\rho Sl^2}{4I_Y}C_{mq}+\frac{\rho S}{2m}C_{L\alpha}K+\frac{\rho Sl^2}{4I_Y}C_{mp_\alpha}\Big)^2\nu^2\Big\}^{\frac{1}{2}}\\\theta_f=\tan^{-1}\left\{\frac{\Big(-\frac{\rho Sl}{4m}C_{Lp_\alpha}\frac{\rho Sl^2}{4I_Y}C_{mq}+\frac{\rho S}{2m}C_{L\alpha}K+\frac{\rho Sl^2}{4I_Y}C_{mp_\alpha}\Big)\nu}{-\Big(\frac{\rho S}{2m}C_{L\alpha}\frac{\rho Sl^2}{4I_Y}C_{mq}+\frac{\rho Sl}{4m}C_{Lp_\alpha}K\nu^2+\frac{\rho Sl}{2I_Y}C_{m\alpha}\Big)}\right\}\\e=\frac{1}{2\sqrt{f}}\Big\{\Big(\frac{\rho S}{2m}C_{L\alpha}-\frac{\rho Sl^2}{4I_Y}(C_{mq}+C_{m\dot{\alpha}})\Big)^2+\Big(\frac{\rho Sl}{4m}C_{Lp_\alpha}+K\Big)^2\nu^2\Big\}^{\frac{1}{2}}\end{aligned}\right\}\quad(5.41)$$

$$\theta_e=\tan^{-1}\left\{\frac{\left(\frac{\rho Sl}{4m}C_{Lp_\alpha}+K\right)\nu}{\left(\frac{\rho S}{2m}C_{L_\alpha}-\frac{\rho Sl^2}{4I_Y}(C_{mq}+C_{m\dot{\alpha}})\right)}\right\}$$

The characteristic roots are given by

$$\begin{aligned}
\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}=&\frac{1}{2}\left\{-\left(\frac{\rho S}{2m}C_{L_\alpha}-\frac{\rho Sl^2}{4I_Y}(C_{mq}+C_{m\dot{\alpha}})\right)-i\left(\frac{\rho Sl}{4m}C_{Lp_\alpha}+K\right)\nu\right\}\\
&\pm\frac{1}{2}\left[\left\{\left(\frac{\rho S}{2m}C_{L_\alpha}\right)^2+\left(\frac{\rho Sl^2}{4I_Y}\right)(C_{mq}+C_{m\dot{\alpha}})^2\right.\right.\\
&+2\frac{\rho S}{2m}C_{L_\alpha}\frac{\rho Sl^2}{4I_Y}(C_{mq}+C_{m\dot{\alpha}})+4\frac{\rho Sl}{2I_Y}C_{m_\alpha}-\nu^2\left.\left(\frac{\rho Sl}{4m}C_{Lp_\alpha}-K\right)^2\right\}\\
&+2i\nu\left\{\frac{\rho S}{2m}C_{L_\alpha}\frac{\rho Sl}{4m}C_{L_\alpha}+\frac{\rho Sl^2}{4I_Y}(C_{mq}+C_{m\dot{\alpha}})\frac{\rho Sl}{4m}C_{Lp_\alpha}-\frac{\rho SC_{L_\alpha}}{2m}K\right.\\
&\left.\left.-\frac{\rho Sl^2}{4I_Y}(C_{mq}+C_{m\dot{\alpha}})K-\frac{\rho Sl^2}{2I_Y}C_{mp_\alpha}\right\}\right]^{\frac{1}{2}}\\
=&\frac{1}{2}\left\{-\left(\frac{\rho S}{2m}C_{L_\alpha}-\frac{\rho Sl^2}{4I_Y}(C_{mq}+C_{m\dot{\alpha}})\right)-i\left(\frac{\rho Sl}{4m}C_{Lp_\alpha}+K\right)\nu\right\}\\
&\pm\frac{1}{2}Ae^{i\frac{B}{2}}
\end{aligned}\tag{5.42}$$

$$\begin{aligned}
A=&\left[\left\{\left(\frac{\rho S}{2m}C_{L_\alpha}\right)^2+\left(\frac{\rho Sl^2}{4I_Y}\right)^2(C_{mq}+C_{m\dot{\alpha}})^2+2\frac{\rho S}{2m}C_{L_\alpha}\frac{\rho Sl^2}{4I_Y}(C_{mq}+{}_{m\dot{\alpha}})\right.\right.\\
&\left.+4\frac{\rho Sl}{2I_Y}C_{m_\alpha}-\nu^2\left(\frac{\rho Sl}{4m}C_{Lp_\alpha}-K\right)^2\right\}^2\\
&+4\nu^2\left\{\frac{\rho S}{2m}C_{L_\alpha}\frac{\rho Sl}{4m}C_{Lp_\alpha}+\frac{\rho Sl^2}{4I_Y}(C_{mq}+C_{m\dot{\alpha}})\cdot\frac{\rho Sl}{4m}C_{Lp_\alpha}\right.\\
&\left.\left.-\frac{\rho S}{2m}C_{L_\alpha}K-\frac{\rho Sl^2}{4I_Y}(C_{mq}+C_{m\dot{\alpha}})K-\frac{\rho Sl^2}{2I_Y}C_{mp_\alpha}\right\}^2\right]^{1/4}
\end{aligned}\tag{5.43}$$

$$B=\tan^{-1}\left[\frac{2\nu\left\{\frac{\rho S}{2m}C_{L_\alpha}\frac{\rho Sl}{4m}C_{Lp_\alpha}+\frac{\rho Sl^2}{4I_Y}(C_{mq}+C_{m\dot{\alpha}})\frac{\rho Sl}{4m}C_{Lp_\alpha}-\frac{\rho S}{2m}C_{L_\alpha}K-\frac{\rho Sl^2}{4I_Y}(C_{mq}+C_{m\dot{\alpha}})K-\frac{\rho Sl^2}{2I_Y}C_{mp_\alpha}\right\}}{\left\{\left(\frac{\rho S}{2m}C_{L_\alpha}\right)^2+\left(\frac{\rho Sl^2}{4I_Y}\right)^2(C_{mq}+C_{m\dot{\alpha}})^2+2\frac{\rho S}{2m}\frac{\rho Sl^2}{4I_Y}(C_{mq}+C_{m\dot{\alpha}})+4\frac{\rho Sl}{2I_Y}C_{m_\alpha}-\nu^2\left(\frac{\rho Sl}{4m}C_{Lp_\alpha}-K\right)^2\right\}}\right]\tag{5.44}$$

Now, the particular solution for the forcing functions of the equation (5.37) are given by

$$\left.\begin{aligned}\xi=\sum_{j=1}^{2}\Bigg\{&\frac{e^{-i\frac{B}{2}}}{A}(-1)^{j+1}\int_{s_0}^{s}\Bigg[\frac{T}{m}\varepsilon_t\frac{e^{-i\phi}}{V^2}\Bigg\{\frac{\rho Sl^2}{4I_Y}C_{m\dot{\alpha}}\\&-\frac{\rho S}{2m}\Big(C_{L_\alpha}+iC_{Lp_\alpha}\frac{l\nu}{2}\Big)-\lambda_{3-j}\Bigg\}-\Bigg\{\frac{T}{I_Y}(-\delta_{Y_b}+i\delta_{Z_b})\frac{e^{-i\phi}}{V^2}\\&+\frac{1}{I_Y}(\delta M_Z+i\delta M_Y)\frac{1}{V^2}\Bigg\}\Bigg]e^{-\lambda_j s}ds\cdot e^{\lambda_j s}\Bigg\}\end{aligned}\right\}\tag{5.45}$$

$$\left.\begin{aligned}\frac{d\varepsilon}{ds}=\sum_{j=1}^{2}\Bigg\{&\frac{e^{-i\frac{B}{2}}}{A}(-1)^{j}\Bigg\{\frac{\rho S}{2m}\Big(C_{L_\alpha}+iC_{Lp_\alpha}\frac{l\nu}{2}\Big)+\lambda_j\Bigg\}\\&\int_{s_0}^{s}\Bigg[\frac{T}{m}\varepsilon_t\frac{e^{-i\phi}}{V^2}\Bigg\{\frac{\rho Sl}{2I_Y}C_{m\dot{\alpha}}\frac{l}{2}-\frac{\rho S}{2m}\Big(C_{L_\alpha}+iC_{Lp_\alpha}\frac{l\nu}{2}\Big)-\lambda_{3-j}\Bigg\}\\&-\Bigg\{\frac{T}{I_Y}(-\delta_{Y_b}+i\delta_{Z_b})\frac{e^{-i\phi}}{V^2}+\frac{1}{I_Y}(\delta M_Z+iM_Y)\frac{1}{V^2}\Bigg\}\Bigg]e^{-\lambda_j s}ds\cdot e^{\lambda_j s}\Bigg\}\end{aligned}\right\}\tag{5.46}$$

For step input of the forcing functions, it is necessary to calculate

$$\int_{s_0}^{s}\frac{1}{V^2}e^{-\lambda_i s}ds \quad \text{and} \quad \int_{s_0}^{s}\frac{e^{-i\phi}}{V^2}e^{-\lambda_i s}ds \qquad (i=1,2)$$

for either powered or unpowered flight.

When air density, ρ, is not constant, *i.e.* ρ is expressed by the equation (5.31), the altitude will be approximated by the following equation:

$$h=h_0-\sin\theta_E(s-s_0)\tag{5.47}$$

wherein θ_E is considered as a constant value and is positive for re-entry case and negative for launching case respectively. Thus, from (5.31)

$$\rho=\rho_0e^{-\beta h}=\rho_0e^{-\beta\{h_0-\sin\theta_E(s-s_0)\}}=\rho_0'e^{(\beta\sin\theta_E)s}\tag{5.48}$$

where

$$\rho_0'=\rho_0e^{-\beta(h_0+s_0\sin\theta_E)}\tag{5.49}$$

By similar way as reference [45], assuming a solution of the equation (5.37) as

$$\left.\begin{aligned}\xi&=A_1e^{\int\lambda_1 ds}+A_2e^{\int\lambda_2 ds}\\\frac{d\varepsilon}{ds}&=B_1e^{\int\lambda_1 ds}+B_2e^{\int\lambda_2 ds}\end{aligned}\right\}\tag{5.50}$$

the following relations are obtained from the equation (5.37)

$$\left.\begin{aligned}
&\left\{\lambda_i+\frac{S}{2m}\rho_0' e^{(\beta\sin\theta_E)s}\left(C_{L_\alpha}+iC_{Lp_\alpha}\frac{l\nu}{2}\right)\right\}A_i+B_i=0\\
&\left\{\left(\frac{Sl^2}{4I_Y}\rho_0' e^{(\beta\sin\theta_E)s}C_{m\dot{\alpha}}\right)\lambda_i+\frac{Sl}{2I_Y}\rho_0' e^{(\beta\sin\theta_E)}\left(C_{m_\alpha}-iC_{mp_\alpha}\frac{l\nu}{2}\right)\right\}A_i\\
&\quad+\left\{\lambda_i+\left(-\frac{Sl^2}{4I_Y}\rho_0' e^{(\beta\sin\theta_E)s}C_{mq}+i\nu K\right)\right\}B_i=0 \qquad i=1,2
\end{aligned}\right\} \tag{5.51}$$

Since A_i and B_i are constants, a nontrivial solution exists only if the determinant of the coefficients of the equation (5.51) equals zero.

$$\left.\begin{aligned}
&\lambda_i^2+\lambda_i\left\{-\frac{Sl}{4I_Y}\rho_0' e^{(\beta\sin\theta_E)s}(C_{mq}+C_{m\dot{\alpha}})+\frac{S}{2m}l_0' e^{(\beta\sin\theta_E)s}C_{L_\alpha}\right.\\
&\quad\left.+i\left(\frac{Sl}{4m}\rho_0' e^{(\beta\sin\theta_E)s}C_{Lp_\alpha}+K\right)\nu\right\}\\
&\quad+\left\{-\left(\frac{S}{2m}\rho_0' e^{(\beta\sin\theta_E)s}C_{L_\alpha}\frac{Sl^2}{4I_Y}\rho_0' e^{(\beta\sin\theta_E)s}C_{mq}\right.\right.\\
&\quad\left.+\frac{Sl}{4m}\rho_0' e^{(\beta\sin\theta_E)s}C_{Lp_\alpha}K\nu^2+\frac{Sl}{2I_Y}\rho_0' e^{(\beta\sin\theta_E)s}C_{m_\alpha}\right)\\
&\quad+i\left(-\frac{Sl}{4m}\rho_0' e^{(\beta\sin\theta_E)s}C_{Lp_\alpha}\cdot\frac{Sl^2}{4I_Y}\rho_0' e^{(\beta\sin\theta_E)s}C_{mq}\right.\\
&\quad\left.\left.+\frac{S}{2m}\rho_0' e^{(\beta\sin\theta_E)s}C_{L_\alpha}K+\frac{Sl^2}{4I_Y}\rho_0' e^{(\beta\sin\theta_E)s}C_{mp_\alpha}\right)\nu\right\}=0
\end{aligned}\right\} \tag{5.52}$$

When air density, ρ, is constant, the above equation coincides with the qudratic form of the characteristic equation (5.39). Thus roots λ_i are also given by the equation (5.42) wherein air density ρ is a function of distance, s. While ρ is small, the effective spin rate $K\nu$ may be regarded as large value so that λ_i will be approximated by the equation (4.21). It will be apparent that the homogeneous solutions are given by substituting the above λ_i into the equation (5.50).

Then a particular solution for the forcing functions of the eqution (5.37) is given, instead of the equations (5.45~46), as

$$\left.\begin{aligned}
&\xi=\sum_{j=1}^{2}\left\{\frac{e^{-i\frac{B}{2}}}{A}(-1)^{j+1}\int_{s_0}^{s}\frac{T}{m}\varepsilon_t\frac{e^{-i\phi}}{V^2}\left\{\frac{Sl^2}{4I_Y}\rho_0' e^{(\beta\sin\theta_E)s}C_{m\dot{\alpha}}\right.\right.\\
&\quad\left.-\frac{S}{2m}\rho_0' e^{(\beta\sin\theta_E)s}\left(C_{L_\alpha}+iC_{Lp_\alpha}\frac{l\nu}{2}\right)-\lambda_{3-j}\right.\\
&\quad-\left\{\frac{T}{I_Y}(\delta_{Y_b}+i\delta_{Z_b})\frac{e^{-i\phi}}{V^2}\right.\\
&\quad\left.\left.+\frac{1}{I_Y}(\delta M_Z+iM_Y)\frac{1}{V^2}\right\}\right]e^{-\int\lambda_j ds}ds\cdot e^{\int_{s_0}^{s}\lambda_j ds}\Bigg\}
\end{aligned}\right\} \tag{5.53}$$

$$
\left.
\begin{aligned}
\frac{d\varepsilon}{ds} = \sum_{j=1}^{2} \Bigg\{ & \frac{e^{-i\frac{B}{2}}}{A} (-1)^j \left(\frac{S}{2m} \rho_0' e^{(\beta \sin\theta E)s} \left(C_{L_\alpha} + i C_{Lp_\alpha} \right) \right. \\
& \left. + \lambda_i \right) \int_{s_0}^{s} \Bigg[\frac{T}{m} \varepsilon_t \frac{e^{-i\phi}}{V^2} \Bigg\{ \frac{Sl}{4I_Y} \rho_0' e^{(\beta \sin\theta E)s} C_{m\dot{\alpha}} \\
& - \frac{S}{2m} \rho_0' e^{(\beta \sin\theta E)s} \left(C_{L_\alpha} + i C_{Lp_\alpha} \frac{l\nu}{2} \right) - \lambda_{3-j} \Bigg\} \\
& - \Bigg\{ \frac{T}{I_Y} (-\delta_{Y_b} + i \delta_{Z_b}) \frac{e^{-i\phi}}{V^2} \\
& + \frac{1}{I_Y} (\delta_{MZ} + i \delta_{MY}) \frac{1}{V^2} \Bigg\} \Bigg] e^{-\int \lambda_j ds} ds \cdot e^{\int_{s_0}^{s} \lambda_j ds} \Bigg\} .
\end{aligned}
\right\} \quad (5.54)
$$

6. Conclusion

General equations of motion of the spinning rocket or space vehicle flying either in vacuum or atmosphere and their simplified solutions have been obtained under suitable assumptions.

The effect of system parameters on the characteristics of the time response curve has been precisely analyzed for both open-loop and closed-loop systems.

For feedback system such as space vehicle having specifically automatic attitude control means, some optimum combinations of parameters, which will minimize the time required to reach about a destination point or the sweep area swinging around the point, have been obtained like the equation (4.11), (4.19a, b) or (4.23). From these equations the following statements have been recognized.

The system damping is directly related to the positive K_b so that K_b will be well to be large. Orthogonal or indirect feedback gain, $K_s \sin\phi_s$, should satisfy the relation, $K_s \sin\phi_s = K_b Kp/2$, for quick response, but parallel or direct feedback gain, $K_s \cos\phi_s$, may somewhat be selected within positive value arbitrarily. However, it is not preferable to increase $K_s \cos\phi_s$ too much because the spiral motion will be stressed. For large spin rate, more precisely for large Kp, the time response characterized by an epicycle in the attitude-angle will be approximated by a spiral. The damping of this spiral is decided by $K_s \sin\phi_s / Kp$ instead of K_b and the locus of response curve will be approximated as a straight line if $K_s \cos\phi_s$ is equal to $-K_b^2/4$. In this case, therefore, the system has light damping but a straight or smooth time-response curve.

For a single feedback system having two reaction jets pointed oppositely as transverse control moment, a corresponding symmetrical system has been obtained, whose damping, K_b, and feedback gain, K_s, should be twice, otherwise same as those of symmetrical case. The approximation is fairly good for normal operating range.

For re-entry flight of spinning axisymmetrical body analytical solutions of the motion have been obtained as well as either powered or coasting flight in atmosphere.

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