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An Investigation of Creep Bending Characteristics of Linear Polymer Column influenced by Non-Uniform Thermal Environments

By

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Summary. The creep bending characteristics such as the lateral creep deflection versus time relations are investigated in the non-uniform thermal environments for the linear polymer column having the idealized section. The linear polymer is described as a combination of Maxwell and Voigt elements. The non-uniform temperature distributions alongside the column length are assumed to be the idealized stepwise profile. The temperature dependence is expressed in terms of rate process appeared in the viscosity term. The numerical example for polymethylmethacrylate, whose physical properties are obtained through the usual tensile creep test, shows the conspicuous effects due to non-uniformity in the temperature distribution.

Symbols

A = cross-sectional area of idealized column
E = modulus of elasticity
H = activation energy
h = depth of idealized I section
k = constant in viscosity term
L = column length
$n_E = \dfrac{P}{P_{cr}}$
P = compressive load
R = universal gas constant
T = temperature
t = time
t_{cr} = critical time
σ = stress
λ = viscosity coefficient
ε = strain
δ_0 = initial deviation divided by one-half depth
δ_0^* = maximum initial deviation divided by one-half depth
δ = non-dimensional lateral deflection

$$\xi = \frac{x}{L}$$

$$\sigma_m = \frac{P}{A}$$

$$P_{cr} = \frac{\pi^2 E_1 A h^2}{4L^2}$$

1. Introduction

As is well known the thermoplastics are quite sensitive to thermal environments and therefore they might be well supposed to be much subjected to the effects of non-uniformity in temperature distribution alongside the column length in case of column creep bending initiated by the axial compressive load applied.

From the view-point of actual structural design of compressive members subjected to any thermal environments, no such linear polymers of thermoplastics may be allowed to be used. Although the present problem is rather far from the actual practice, however, for the fundamental understanding of viscoelastic behavior of linear polymers subjected to any thermal non-unifomity, the author believes the present work might be something.

For the analytical manipulation sake we adopt the four element model, that is, a combination of Maxwell and Voigt model as shown in Fig. 1, as a mechanical equivalent of fundamental linear polymer structure, which is typical of thermoplastics. The column, whose mechanical characteristics are of four element model, is assumed to have an idealized section [*1*] as shown in Fig. 3, and also to have an initial deviation δ_0 The non-uniform temperature profile alongside the column length is taken as an idealized asymmetric stepwise one, as shown in Fig. 2.

The temperature dependence is included in the viscosity term taking account of rate process expression, which is well recognized [*2*],[*3*]. The analytical approach for the time versus non-dimensional lateral creep deflection relations is done by use of Galerkin approximation. The numerical example for polymethylmethacrylate, which is supposed to be among the linear polymers, is examined to check the non-uniform temperature distribution effects.

2. Derivation of the Stress-Strain-Time Law for a Four Parameter Model

For the linear polymer the general stress-strain-time relationship governing the mechanical behavior of the four parameter model shown in Fig. 1 may be derived as follows [*4*].

If at any time t, the stress σ is applied to the model as shown in Fig. 1, where σ is positive in tension, the strain contribution ε_{1a} of the elastic element with Young's modulus E_1, is given by Hooke's law as

$$\varepsilon_{1a} = \frac{\sigma}{E_1} \tag{1}$$

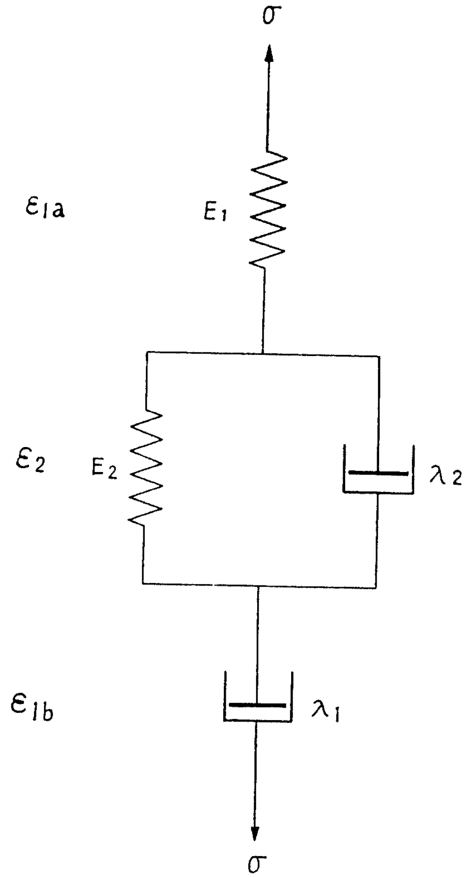

FIG. 1. Four Parameter Model Representation

For the viscous element with the viscosity coefficient λ_1 Newton's law gives

$$\dot{\varepsilon}_{1b}=\frac{\sigma}{\lambda_1} \qquad (2)$$

in which the dot above the strain indicates the differentiation with respect to time t.

From Eqs. (1) and (2) we have the combined strain contribution related to the applied stress, i.e.,

$$\dot{\varepsilon}_1=\frac{\dot{\sigma}}{E_1}+\frac{\sigma}{\lambda_1} \qquad (3)$$

This shows the stress-strain-time law for a Maxwell model.

Next, another strain contribution due to a Voigt model is to be determined. The strain relating to the Voigt model is ε_2, which is common to both a spring and a dashpot, but the stresses induced must be different, say, σ_{2a} and σ_{2b}, where the sum of these stresses is the applied stress σ. Thus we have

$$\varepsilon_2=\frac{\sigma_{2a}}{E_2} \qquad (4)$$

$$\dot{\varepsilon}_2=\frac{\sigma_{2b}}{\lambda_2} \qquad (5)$$

where E_2 is the modulus of elasticity and λ_2 the viscosity coefficient as shown in Fig. 1. Hence the strain ε_2 is related to the applied stress σ by the equation

$$\lambda_2\dot{\varepsilon}_2+E_2\varepsilon_2=\sigma \tag{6}$$

Eq. (6) is the defining equation of a Kelvin or Voigt model.

The total strain ε for a system shown in Fig. 1 is

$$\varepsilon=\varepsilon_1+\varepsilon_2 \tag{7}$$

where ε_1 is given by Eq. (3) and ε_2 by Eq. (6).

Elimination of ε_1 and ε_2 from Eqs. (3), (6) and (7) results in the following stress-strain-time relation governing the mechanical behavior of four parameter model shown in Fig. 1:

$$\ddot{\sigma}+\frac{E_2}{\lambda_2}\left(1+\frac{E_1\lambda_2}{E_2\lambda_1}+\frac{E_1}{E_2}\right)\dot{\sigma}+\frac{E_1E_2}{\lambda_1\lambda_2}\sigma=E_1\left(\ddot{\varepsilon}+\frac{E_2}{\lambda_2}\dot{\varepsilon}\right) \tag{8}$$

3. Temperature Dependence and Non-Uniformity in the Temperature Profile

The creep deformations are likely to be controlled by some process involving the activation energy, therefore, we may represent the viscosity term in terms of rate process resulting in the following temperature dependence [2],[3].

$$\lambda=ke^{\frac{H}{RT}} \tag{9}$$

As to the non-uniformity in the temperature profile we just assume the asymmetric stepwise one as shown in Fig. 2. The uniform temperature considered in the present work is indicated by the dotted line.

4. Solution for a Creep Bending Problem by the Galerkin Approach

For the essential evaluation of the non-uniform temperature distribution effects the idealized column proposed by Hoff [1] is conceived so as to avoid any detour in the theoretical treatment followed.

As shown in Fig. 3, the axial compressive load P is applied to the idealized column of four element model, causing the lateral creep deflection due to the initial imperfection of $\delta_0=\delta_0^*\sin\pi\xi$. The produced stresses for the concave and the convex side of a column are

$$\sigma_i=\frac{P}{A}\left(1+\frac{w}{\left(\frac{h}{2}\right)}\right) \tag{10}$$

and

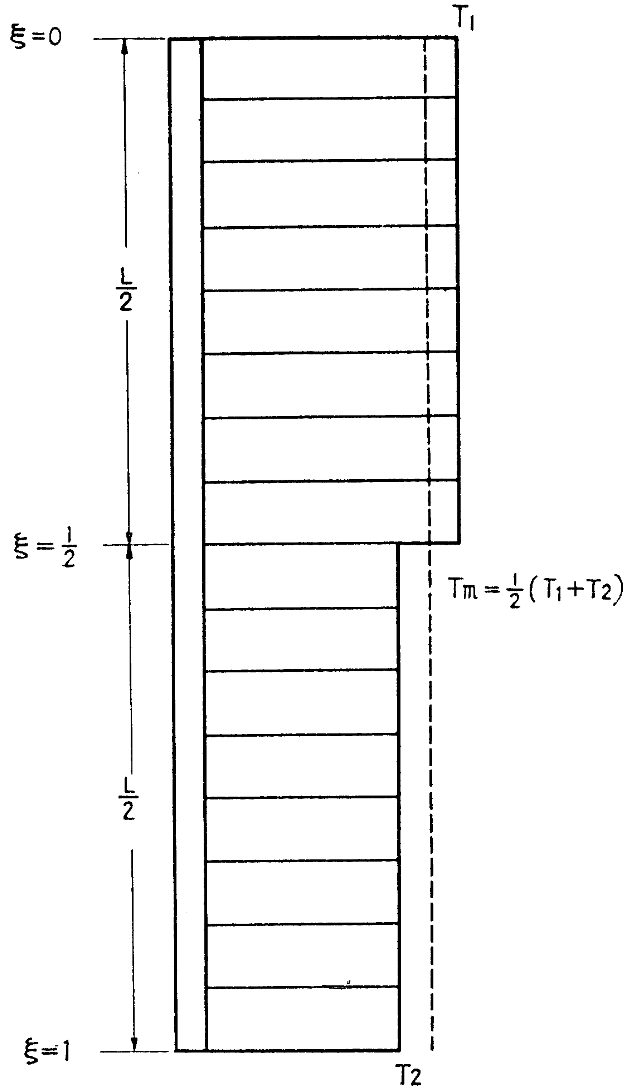

FIG. 2. Idealized Asymmetric Stepwise Temperature Distribution assumed

$$\sigma_a=\frac{P}{A}\left(1-\frac{w}{\left(\frac{h}{2}\right)}\right) \tag{11}$$

Denoting $P/A=\sigma_m$ and $w/(h/2)=\delta$, for convenience, we can rewrite Eqs. (10) and (11) as

$$\sigma_i=\sigma_m(1+\delta) \tag{12}$$

$$\sigma_a=\sigma_m(1-\delta) \tag{13}$$

$$(\delta<1)$$

respectively.

The strain difference is described by

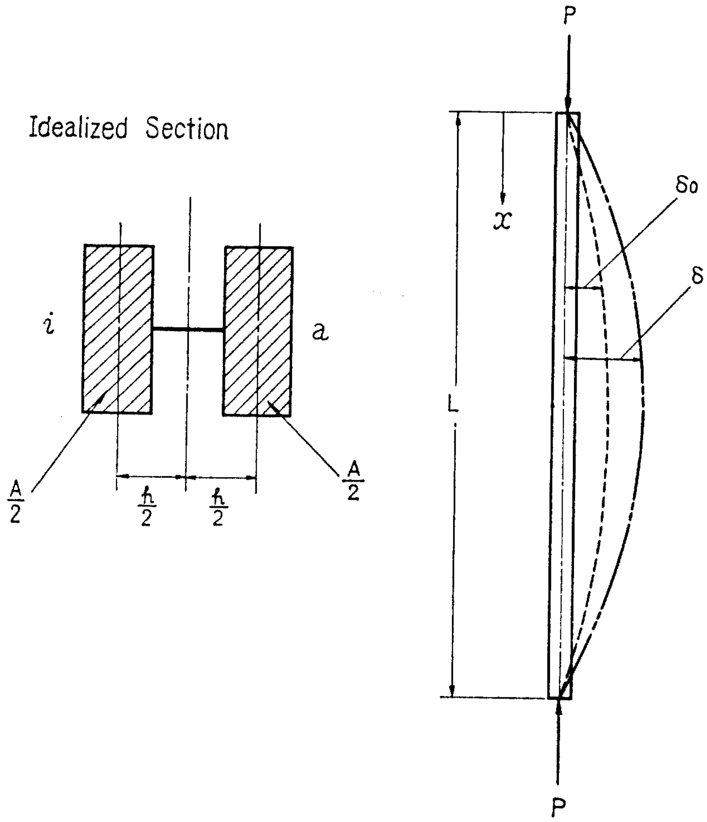

FIG. 3. Idealized Column with Initial Deviation

$$\varepsilon_a - \varepsilon_i = h\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w_0}{\partial x^2}\right) = \frac{h^2}{2L^2}\left(\frac{\partial^2 \delta}{\partial \xi^2} - \frac{\partial^2 \delta_0}{\partial \xi^2}\right) \tag{14}$$

The boundary conditions for pinned ends, shown in Fig. 3, are

$$\delta = \frac{\partial^2 \delta}{\partial \xi^2} = 0 \quad \text{at} \quad \xi = 0 \quad \text{and} \quad \xi = 1 \tag{15}$$

The initial deviation of a column centerline is assumed to be*

$$w_0(\xi) = a_0 \sin \pi\xi \tag{16}$$
$$(0 \leqq \xi \leqq 1)$$

and also

$$\delta_0 = \delta_0^* \sin \pi\xi \tag{17}$$

for the non-dimensional form, which already appears in the previous page, where $\delta_0 = w_0/(h/2)$ and $\delta_0^* = a_0/(h/2)$.

Assuming the non-dimensional deflection**

* In the present analysis it is assumed that the n_E value is near unity, so that the first harmonic sine is predominant.

** It is assumed that the first harmonic sine is predominant, and Eq. (18) is to satisfy the previous boundary conditions given by Eq. (15).

$$\delta(\xi, t) = f(t) \cdot \sin \pi\xi \tag{18}$$

Then we have, denoting the differentiation with respect to time t by the dot,

$$\left.\begin{aligned} \dot{\delta} &= \dot{f} \sin \pi\xi \\ \ddot{\delta} &= \ddot{f} \sin \pi\xi \\ \dot{\delta}_{\xi\xi} &= -\dot{f}\pi^2 \sin \pi\xi \\ \ddot{\delta}_{\xi\xi} &= -\ddot{f}\pi^2 \sin \pi\xi \end{aligned}\right\} \tag{19}$$

Combining Eps. (8), (12), (13), (14), (18) and (19), we have the fundamental equation,

$$A_1\ddot{f} + B_1\dot{f} + C_1 f = 0 \tag{20}$$

where

$$\left.\begin{aligned} A_1 &= -2\sigma_m \sin \pi\xi + \frac{E_1 h^2}{2L^2}\pi^2 \sin \pi\xi \\ B_1 &= -2\sigma_m \frac{E_2}{\lambda_2}\left(1 + \frac{\lambda_2 E_1}{\lambda_1 E_2} + \frac{E_1}{E_2}\right)\sin \pi\xi + \frac{E_1 E_2 h^2}{2\lambda_2 L^2}\pi^2 \sin \pi\xi \\ C_1 &= -2\sigma_m \frac{E_1 E_2}{\lambda_1 \lambda_2} \sin \pi\xi \end{aligned}\right\} \tag{21}$$

Now we are going to apply the Galerkin approximation to Eq. (20) to have the relationship between $f(t)$ and time t.*

Application of the one-term Galerkin approximation to Eq. (20) yields

$$\int_0^1 [A_1\ddot{f} + B_1\dot{f} + C_1 f] \sin \pi\xi d\xi = 0 \tag{22}$$

Considering both the non-uniform temperature profile prescribed in Fig. 2 and the temperature dependence given by Eq. (9), we lead to the following equation after mathematical manipulation,

$$A_2\ddot{f} + B_2\dot{f} + C_2 f = 0 \tag{23}$$

where

$$\left.\begin{aligned} A_2 &= \frac{1}{2}\left(-2\sigma_m + \frac{E_1 h^2 \pi^2}{2L^2}\right) \\ B_2 &= \frac{1}{4}\left(e^{-\frac{H}{RT_1}} + e^{-\frac{H}{RT_2}}\right)\left[\left(\frac{-2\sigma_m E_2}{k_2}\right)\left(1 + \frac{k_2 E_1}{k_1 E_2} + \frac{E_1}{E_2}\right) + \frac{E_1 E_2 h^2 \pi^2}{2L^2 k_2}\right] \\ C_2 &= -\frac{2\sigma_m E_1 E_2}{4k_1 k_2}\left(e^{-\frac{2H}{RT_1}} + e^{-\frac{2H}{RT_2}}\right) \end{aligned}\right\} \tag{24}$$

The solution for Eq. (23) is given by

$$f(t) = K_1 e^{\mu_1 t} + K_2 e^{\mu_2 t} \tag{25}$$

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* It is assumed that Eq. (18) holds at any time, and $f(t)$ is to be determined so that Eq. (20) may be satisfied in the region $0 \leqq \xi \leqq 1$ by the Galerkin method.

where μ_1 and μ_2 are as follows.

$$\mu_{\frac{1}{2}}=\frac{-B_2\pm\sqrt{B_2^2-4A_2C_2}}{2A_2} \tag{26}$$

The constants K_1 and K_2 appeared in Eq. (25) are to be determined by the initial conditions described in the followings.

$$\text{At } t=0 \qquad \varepsilon_2=0 \tag{27}$$

which is characteristic of a Voigt model. Moreover, due to a spring in a Maxwell model, there occurs the instantaneous elastic deflection, i.e.,

$$\text{At } t=0 \qquad \delta(\xi, 0)=f(0)\sin\pi\xi=\frac{\delta_0^*\sin\pi\xi}{1-n_E} \tag{28}$$

with the knowledge of elasticity.

The above two relations described in Eqs. (27) and (28) can determine K_1 and K_2, whose details are shown in what follows.

The viscosity term can be represented in terms of displacement, say,

$$\frac{\sigma}{\lambda}=\dot{\varepsilon}=\frac{1}{l}\frac{dx}{dt} \tag{29}$$

from which we obtain

$$\frac{dx}{dt}=\frac{\sigma}{\lambda}l \tag{30}$$

Now for the asymmetric stepwise temperature distribution shown in Fig. 2, the viscosity term can be expressed as the sum of individual temperature effect. Then we have

$$\frac{dx}{dt}=\frac{\sigma}{\lambda_{T1}}\frac{l}{2}+\frac{\sigma}{\lambda_{T2}}\frac{l}{2} \tag{31}$$

Substitution of $l=1$ into Eq. (31) leads to

$$\frac{d\varepsilon}{dt}=\frac{\sigma}{2}\left(\frac{1}{\lambda_{T1}}+\frac{1}{\lambda_{T2}}\right) \tag{32}$$

With Eq. (9), Eq. (32) is rewritten as

$$\dot{\varepsilon}=\frac{\sigma}{2k}\left(e^{-\frac{H}{RT_1}}+e^{-\frac{H}{RT_2}}\right) \tag{33}$$

The whole strain in the present model is

$$\varepsilon=\varepsilon_{1a}+\varepsilon_{1b}+\varepsilon_2=\varepsilon_1+\varepsilon_2 \tag{34}$$

and the differentiation of Eq. (34) with respect to time yields

$$\dot{\varepsilon}=\dot{\varepsilon}_1+\dot{\varepsilon}_2 \tag{35}$$

With reference to Eq. (33) we have

$$\dot{\varepsilon}_1=\frac{\dot{\sigma}}{E_1}+\frac{\sigma}{2k_1}\left(e^{-\frac{H}{RT_1}}+e^{-\frac{H}{RT_2}}\right) \tag{36}$$

which is for a Maxwell model.

As to a Voigt model the induced strain must be equal, but the stress is different, say, σ_{2a} and σ_{2b}. Consequently,

$$\varepsilon_2=\frac{\sigma_{2a}}{E_2} \tag{37}$$

$$\dot{\varepsilon}_2=\frac{\sigma_{2b}}{2k_2}\left(e^{-\frac{H}{RT_1}}+e^{-\frac{H}{RT_2}}\right) \tag{38}$$

Substitution of Eqs. (37) and (38) into Eq. (6) derives

$$\frac{2k_2\dot{\varepsilon}_2}{e^{-\frac{H}{RT_1}}+e^{-\frac{H}{RT_2}}}+E_2\varepsilon_2=\sigma \tag{39}$$

From Eqs. (35), (36) and (39),

$$\varepsilon_2=\frac{\dot{\sigma}}{E_1E_2}\frac{2k_2}{\left(e^{-\frac{H}{RT_1}}+e^{-\frac{H}{RT_2}}\right)}+\frac{k_2\sigma}{E_2}\left(\frac{1}{k_1}+\frac{1}{k_2}\right)-\frac{2k_2\dot{\varepsilon}}{E_2\left(e^{-\frac{H}{RT_1}}+e^{-\frac{H}{RT_2}}\right)} \tag{40}$$

With Eqs. (12), (13) and (14), Eq. (40) reduces to

$$\left.\begin{aligned}\varepsilon_2=&\frac{-4\sigma_m k_2\dot{\delta}}{E_1E_2\left(e^{-\frac{H}{RT_1}}+e^{-\frac{H}{RT_2}}\right)}-\frac{2\sigma_m k_2\delta}{E_2}\left(\frac{1}{k_1}+\frac{1}{k_2}\right)\\&-\frac{k_2h^2}{E_2L^2}\frac{\dot{\delta}_{\xi\xi}}{\left(e^{-\frac{H}{RT_1}}+e^{-\frac{H}{RT_2}}\right)}\end{aligned}\right\} \tag{41}$$

Combination of Eqs. (18), (19), (25), (27) and (41), at time $t=0$, gives

$$\left.\begin{aligned}&\frac{4\sigma_m k_2(K_1\mu_1+K_2\mu_2)}{E_1E_2\left(e^{-\frac{H}{RT_1}}+e^{-\frac{H}{RT_2}}\right)}+\frac{2\sigma_m k_2}{E_2}\left(\frac{1}{k_1}+\frac{1}{k_2}\right)(K_1+K_2)\\&\qquad-\frac{k_2h^2\pi^2(K_1\mu_1+K_2\mu_2)}{E_2L^2\left(e^{-\frac{H}{RT_1}}+e^{-\frac{H}{RT_2}}\right)}=0\end{aligned}\right\} \tag{42}$$

since $\sin\pi\xi\neq 0$.

For the initially deviated column the instantaneous lateral deflection shown in Eq. (28) is observed upon application of axial compressive load P, and the combination of Eqs. (25) and (28) yields the following equation at time $t=0$,

$$K_1+K_2=\frac{\delta_0^*}{1-n_E} \tag{43}$$

Thus we can determine K_1 and K_2 from Eqs. (42) and (43), as shown below.

$$K_1=\frac{(\alpha_1\mu_2+\alpha_2-\alpha_3\mu_2)\delta_0^*}{(1-n_E)(\mu_2-\mu_1)(\alpha_1-\alpha_3)} \tag{44}$$

$$K_2=\frac{(\alpha_3\mu_1-\alpha_2-\alpha_1\mu_1)\delta_0^*}{(1-n_E)(\mu_2-\mu_1)(\alpha_1-\alpha_3)} \tag{45}$$

where

$$\left.\begin{aligned}
\alpha_1&=\frac{4\sigma_m k_2}{E_1E_2\left(e^{-\frac{H}{RT_1}}+e^{-\frac{H}{RT_2}}\right)}\\
\alpha_2&=\frac{2\sigma_m k_2}{E_2}\left(\frac{1}{k_1}+\frac{1}{k_2}\right)\\
\alpha_3&=\frac{k_2h^2\pi^2}{E_2L^2\left(e^{-\frac{H}{RT_1}}+e^{-\frac{H}{RT_2}}\right)}
\end{aligned}\right\} \tag{46}$$

Finally we can recognize the creep bending behavior of a four parameter model, as a mechanically equivalent representation of linear polymer, through Eq. (25) with the constants K_1 and K_2 described by Eqs. (44) and (45).

5. Numerical Examples and Discussions

Polymethylmethacrylate might be regarded as a kind of linear polymer in the form of four parameter model under the glass transition temperature, and its physical properties are obtained through the usual tensile creep test, of which details will be roughly described in what follows.

For the moduli of elasticity, E_1 and E_2, and the viscosity coefficient λ_1, all these values are found in the creep strain versus elapsed time curve under the constant load as shown in Fig. 4. As for another viscosity coefficient λ_2, the creep strain versus logarithmic time plotting, shown in Fig. 5, is used to determine the retardation time τ, from which λ_2 is calculated since $\tau=\lambda_2/E_2$. The retardation time position on the curve is the location where the gradient is maximum. The physical properties thus obtained are as follows:

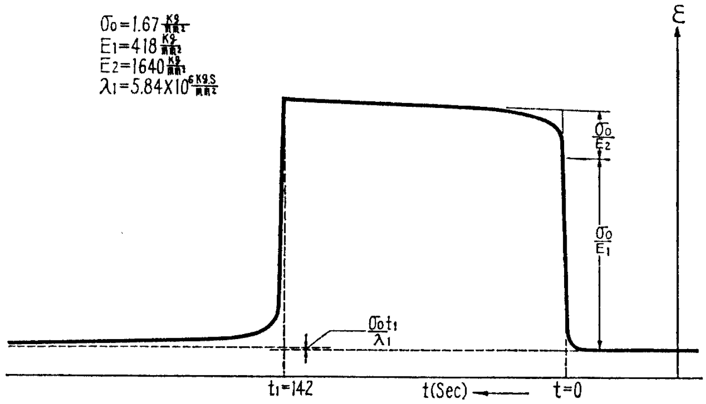

FIG. 4. Tensile Creep Test Curve obtained

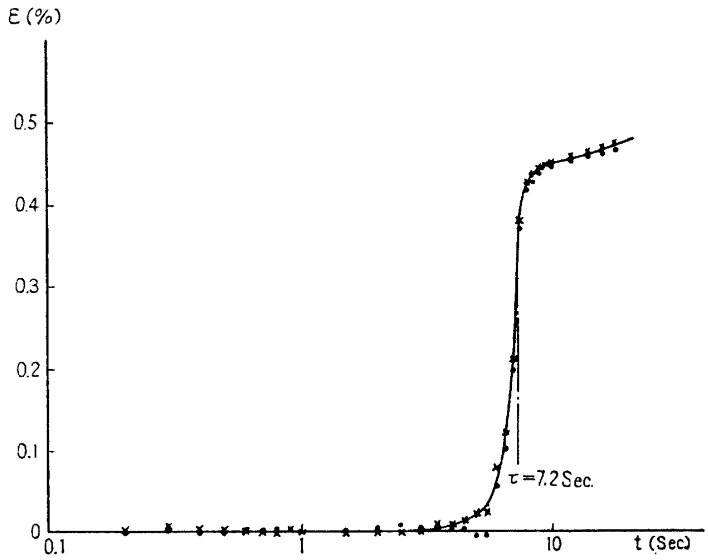

FIG. 5. Determination of Retardation Time

$$E_1 = 418 \text{ Kg/mm}^2$$
$$E_2 = 1640 \text{ Kg/mm}^2$$
$$\lambda_1 = 5.84 \times 10^6 \text{Kg} \cdot \text{s/mm}^2$$
$$\lambda_2 = 1.18 \times 10^4 \text{Kg} \cdot \text{s/mm}^2$$
$$\tau = 7.2 \text{ sec.}$$

The above values are derived from the test results at 12°C (=285°K) with the tensile stress $\sigma_0 = 1.67$ Kg/mm^2.

For easier understanding of non-uniform temperature distribution effects the following numerical examples are tried. In the examples, we assume

$$T_m = 285°\text{K}$$
$$H = 29000 \text{ cal/mole}$$
$$R = 2 \text{ cal/°K mole}$$
$$n_E = 0.9 \quad (\sigma_m = 2.32 \text{ Kg/mm}^2)$$
$$h/L = 1/20$$
$$\delta_0^* = 1/100 \qquad (f(o) = 0.1)$$

in addition to the above physical properties. The numerical results obtained are shown in Table 1 and Figs. 6 and 7.

Fig. 6 shows the conspicuous reduction in elapsed time up to the prescribed deflection in case of non-uniform temperature distribution. For instance, in case of only ±1.75% temperature variation compared with the uniform temperature of 285°K, i.e., T_1=290°K and T_2=280°K, the calculated elapsed time to reach f=1 [5] reduces to 69.5% compared with the uniform temperature case. If it has ±3.51% temperature deviation, namely T_1=295°K and T_2=275°K, the situation becomes worse and shows about one-third of elapsed time degradation. Therefore, we have to be careful in case we work the present column as a compression member under uniform temperature profile alongside the column length, so that we

TABLE 1. Thermal Non-Uniformity Effects on Creep Bending Characteristics

T_1	°K	305	300	295	290	285
T_2	°K	265	270	275	280	285
$\pm\Delta T$	°C	20	15	10	5	0
$\frac{\pm\Delta T}{T_m}$	%	7.02	5.27	3.51	1.75	0
t (Sec.)	$f=0.5$	0.44	0.93	2.10	4.54	6.54
	$f=1.0$	0.66	1.42	3.20	6.91	9.95
	$f=1.5$	0.80	1.71	3.87	8.37	12.00
	$f=2.0$	0.90	1.93	4.36	9.40	13.45
$\frac{t}{t_m}$	$f=1.0$	0.066	0.143	0.322	0.695	1

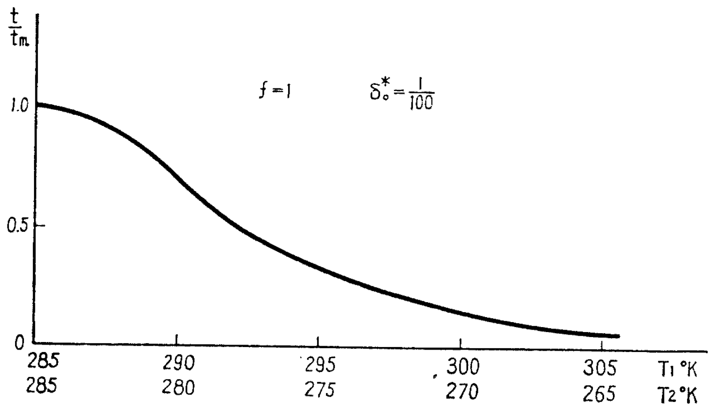

FIG. 6. Elapsed Time versus Thermal Non-Uniformity Relation under the Prescribed Condition

investigate the actual temperature profile to check any non-uniformity in temperature profile to avoid such catastrophic degradation of elapsed time as shown in the results. The previous results already pointed out by Hayashi et al. [6] and after-

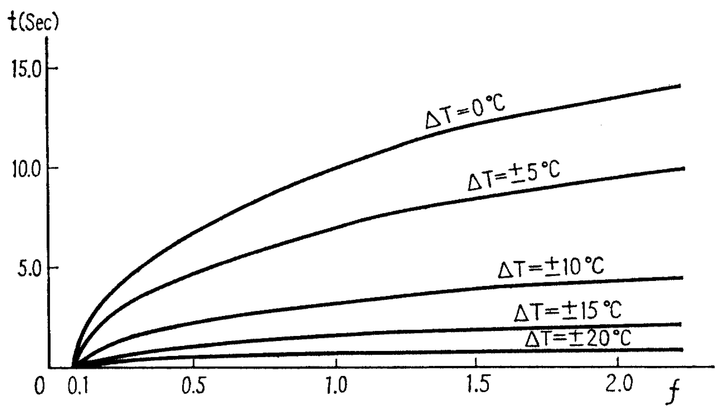

FIG. 7. Elapsed Time versus Non-Dimensional Lateral Creep Deflection Curves influenced by Thermal Non-Uniformity

wards by Kobayashi [7], both for metal, also agree with the present tendency.

In Fig. 7, we can perceive that the lateral creep deflection rate is accelerated as the magnitude of non-uniformity in temperature distribution increases. Consequently, for large temperature difference case the lateral deflection versus time curves soon become asymptotic.

It is easily recognized that T_1 and T_2 are interchangeable in this case. Therefore, the above results seen in Figs. 6 and 7 are also available for estimating the lower elapsed time limit for the corresponding temperature fluctuation.

6. Conclusions

The non-uniform thermal environment effects are found to be conspicuous in case of column creep bending of four parameter model. Since T_1 and T_2 are interchangeable in the present problem, the obtained results also show the lower elapsed time limit for the corresponding temperature fluctuation.

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