

Influences of Temperature Change and Large Amplitude on Free Flexural Vibration of Rectangular Elastic Plates

By

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Summary. The fundamental equations of nonlinear flexural vibration for a rectangular elastic plate are solved approximately by employing a method of successive approximation, and the influences of the temperature change and large amplitude on the period of free vibration are established. Some numerical examples are given for a plate with hinged and immovable edges, and it is shown that the effects mentioned above are considerably large and cannot be ignored even when the temperature change is small.

Nomenclature.

$2a, 2b, d$	length, width and thickness of the rectangular plate, respectively.
c_b	speed of the longitudinal wave in the plate; $c_b^2 = E/(1-\nu^2)\rho$.
f, f', g	Eq. (3.3), Eqs. (2.27) and (2.28), Eqs. (2.31) and (2.32).
k	Eq. (3.23).
t	time.
u, v, w	displacement components in the middle plane in the x -, y - and z -directions, respectively.
$z(\theta, t)$	displacement at the center of the plate.
z_0, \bar{z}	Eq. (2.16).
z_1, z_2	absolute values of the maximum and minimum non-dimensional amplitudes, respectively.
C, I	integral constants; Eqs. (2.22) and (2.23), Eqs. (2.20) and (2.21).
D	flexural modulus of rigidity; $D = Eh^3/12(1-\nu^2)$.
E, G	moduli of elasticity and rigidity, respectively.
E	total energy of the vibrating system.
$K(k)$	complete elliptical integral of the first kind.
M_1, M_2, M_{12}	bending and twisting moments after the deformation; Eq. (2.4).
N_1, N_2, N_{12}	cross-sectional and shearing forces after the deformation; Eq. (2.4).
T, T^*	linear and nonlinear periods, respectively.
α, ρ	coefficient of linear thermal expansion and density of the plate material.

β, η	Eqs. (3.4) and (3.6), and Eq. (3.7).
τ, ξ, ζ	non-dimensional time; Eqs. (2.25), (3.1) and (3.4).
θ	temperature change from the initial state.
$\bar{\theta}, \tilde{\theta}$	mean temperature and temperature moment; Eqs. (2.6) and (2.7).
λ	aspect ratio of the rectangular plate.
ν	Poisson's ratio.
χ	Airy's stress function.
ω, ω^*	linear and nonlinear circular frequencies, respectively.
∇^2, ∇^4	operators,
	$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$

Subscripts "x" and "y" denote the partial differentiation with respect to x and y , respectively.

Subscripts "S" and "C" specify the quantity for the cases of the simply supported and clamped edge conditions, respectively.

1. INTRODUCTION

The vibration characteristics of thin plates and shells, subjected to the change in temperature, is one of the very important problems to be analyzed in checking the aeroelastic performance of high-speed flying vehicles [1], [2], and must be examined carefully in the design of structural components.

The period of the lateral vibration of rods and plates is influenced by many factors such as the internal friction of material, aerodynamic force, rotary inertia and amplitude [3]~[11]. The influences of these factors, except that of amplitude, can be analyzed in the problem of the small oscillation, that is in the scope of the linear theory. Many vibrations which we experience usually and deal with as infinitesimal ones are frequently accompanied by the finite amplitude, and it seems to be important to study the effect of the large amplitude on the vibration characteristics as precisely as possible.

The problems of thermal stress and vibration of structural components and structures are growing up to be of urgent importance in connection with the development of missiles and artificial satellites. The thermal deformation and stress of structural components under the transient heating and cooling conditions have been analyzed by the present author and others [12]~[18]. For the vibration phenomena due to the thermal shock, there exist solutions by Mura [19], Boley and others [20]~[23], but they are limited to the linear problems. In the problems of the nonlinear vibrations, it is difficult to separate the spatial variables from the time variable and the meaning of the normal mode which is powerful means in solving the linear vibration problems becomes ambiguous, and it seems, generally speaking, impossible to solve exactly the problems. Many methods of approximate solution such as a method of successive approximation, a perturbation method, the Galerkin method and an energy method have then been established.

In the present paper, the fundamental equations of nonlinear flexural vibration for a rectangular elastic plate are solved approximately by employing a method of successive approximation, and the influences of the temperature change and large amplitude on the period of free vibration are established. Some numerical examples are given for a plate with hinged and immovable edges, and it is shown that the effects mentioned above are considerably large and cannot be ignored even when the temperature change is small. The present work will form a link in the chain of the research of the above-mentioned problems and, on the other hand, aims at to make some preparations for analyses on the nonlinear transient phenomena of vibration of structural components subjected to the thermal shock.

2. FUNDAMENTAL EQUATIONS

The fundamental equations of motion of a rectangular plate, subjected to the change in temperature, are given as follows, where the effects of the internal friction, aerodynamic force and rotary inertia are neglected:

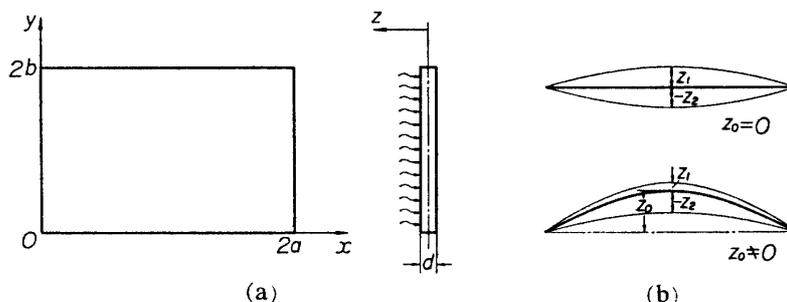


FIG. 1. Rectangular elastic plate.

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_{12}}{\partial y} = \rho d \frac{\partial^2 u}{\partial t^2}, \quad (2.1)$$

$$\frac{\partial N_{12}}{\partial x} + \frac{\partial N_2}{\partial y} = \rho d \frac{\partial^2 v}{\partial t^2}, \quad (2.2)$$

$$\begin{aligned} \frac{\partial^2 M_1}{\partial x^2} + 2 \frac{\partial^2 M_{12}}{\partial x \partial y} + \frac{\partial^2 M_2}{\partial y^2} + \frac{\partial}{\partial x} \left(N_1 \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left(N_2 \frac{\partial w}{\partial y} \right) \\ + \frac{\partial}{\partial x} \left(N_{12} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left(N_{12} \frac{\partial w}{\partial x} \right) = \rho d \frac{\partial^2 w}{\partial t^2}, \end{aligned} \quad (2.3)$$

where

$$\left. \begin{aligned} N_1 &= \frac{Ed}{(1-\nu^2)} \left[\left(u_x + \frac{1}{2} w_x^2 \right) + \nu \left(v_y + \frac{1}{2} w_y^2 \right) \right] - \frac{Ed\alpha}{(1-\nu)} \bar{\theta}, \\ N_2 &= \frac{Ed}{(1-\nu^2)} \left[\left(v_y + \frac{1}{2} w_y^2 \right) + \nu \left(u_x + \frac{1}{2} w_x^2 \right) \right] - \frac{Ed\alpha}{(1-\nu)} \bar{\theta}, \\ N_{12} &= Gd(u_y + v_x + w_x w_y) = \frac{Ed}{2(1+\nu)} (u_y + v_x + w_x w_y), \end{aligned} \right\} \quad (2.4)$$

$$\begin{aligned}
 M_1 &= -D(w_{xx} + \nu w_{yy}) - \frac{Ed^2\alpha}{(1-\nu)} \tilde{\theta}, \\
 M_2 &= -D(w_{yy} + \nu w_{xx}) - \frac{Ed^2\alpha}{(1-\nu)} \tilde{\theta}, \\
 M_{12} &= -D(1-\nu)w_{xy}, \\
 D &= \frac{Ed^3}{12(1-\nu^2)},
 \end{aligned} \tag{2.5}$$

$$\bar{\theta} = \frac{1}{d} \int_{-d/2}^{d/2} \theta(x, y, z) dz, \tag{2.6}$$

$$\tilde{\theta} = \frac{1}{d^2} \int_{-d/2}^{d/2} z\theta(x, y, z) dz. \tag{2.7}$$

Boundary conditions are given for the cases of simply supported and clamped edges, respectively, as

for the case of simply supported edge condition,

at $x=0, 2a$

$$\left. \begin{aligned}
 w &= 0, \\
 D(w_{xx} + \nu w_{yy}) + \frac{Ed^2\alpha}{(1-\nu)} \tilde{\theta} &= 0,
 \end{aligned} \right\} \tag{2.8.1}$$

at $y=0, 2b$

$$\left. \begin{aligned}
 w &= 0, \\
 D(w_{yy} + \nu w_{xx}) + \frac{Ed^2\alpha}{(1-\nu)} \tilde{\theta} &= 0.
 \end{aligned} \right\} \tag{2.8.2}$$

for the case of clamped edge condition,

at $x=0, 2a$

$$w = w_x = 0, \tag{2.9.1}$$

at $y=0, 2b$

$$w = w_y = 0. \tag{2.9.2}$$

Neglecting the inertia in the plate-plane in accordance with the assumption that the nonlinearity is not so large, Eqs. (2.1) and (2.2) are reduced to

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_{12}}{\partial y} = 0, \tag{2.1a}$$

$$\frac{\partial N_{12}}{\partial x} + \frac{\partial N_2}{\partial y} = 0, \tag{2.2a}$$

and Airy's stress function can be introduced as

$$\frac{N_1}{d} = \chi_{yy}, \quad \frac{N_2}{d} = \chi_{xx}, \quad \frac{N_{12}}{d} = -\chi_{xy}. \tag{2.10}$$

Using Eqs. (2.4) and (2.10), the equation of motion in the z -direction is reduced to

$$D\nabla^4 w = d(\chi_{yy}w_{xx} - 2\chi_{xy}w_{xy} + \chi_{xx}w_{yy}) - \frac{Ed^3\alpha}{(1-\nu)} \nabla^2 \tilde{\theta} - \rho d \frac{\partial^2 w}{\partial t^2}. \quad (2.11)$$

In this process, the so-called buoyancy terms in Eq. (2.3) vanish and it is due to the above assumption that the nonlinearity is not so large.

The equation of compatibility is given as

$$\nabla^4 \chi = E(w_{xy}^2 - w_{xx}w_{yy}) - E\alpha \nabla^2 \bar{\theta}. \quad (2.12)$$

Then, the problem is reduced to how to solve the simultaneous partial differential equations (2.11) and (2.12), and the normal displacement at the center of plate and the period or circular frequency of vibration of the plate under the change in temperature can be obtained.

The temperature distribution over the plate is assumed to be symmetrical with respect to the center of plate and is given as

$$\theta = \sum_i \sum_j \bar{\theta}_{ij} \cos \frac{i\pi x}{2a} \cos \frac{j\pi y}{2b}, \quad (i, j = 0, 2, 4, \dots \text{even}), \quad (2.13)$$

$$\tilde{\theta} = \sum_p \sum_q \tilde{\theta}_{pq} \sin \frac{p\pi x}{2a} \sin \frac{q\pi y}{2b}, \quad (p, q = 1, 3, 5, \dots \text{odd}). \quad (2.14)$$

Eq. (2.14) will become more general expression if the term of $\tilde{\theta}_e$ (temperature moment at the edges) is added in the right-hand side, and the following process of solution can be applied for that case in the same way as in the present paper. However, the addition of $\tilde{\theta}_e$ does not introduce the new phenomena in the present problem, but only results in complicating the calculation, and so the term of $\tilde{\theta}_e$ is neglected here, assuming that there exists no temperature gradient through the thickness at the edges.

In the present paper, the undamped free vibration of the rectangular plate subjected to the change in temperature is analyzed. The nonlinear terms in the fundamental equations express the effects of the finite displacement due to the temperature change and of the finite vibration. Since it seems to be natural to expect that there exists no remarkable difference between the wave form of the present nonlinear vibration and that of the small vibration [3], the lowest mode of vibration is assumed to be the same as the deflection form due to the temperature change only, and the normal displacement of the plate at any time is assumed as Eq. (2.15) in the sum of the displacement due to the temperature change and the amplitude of vibration.

$$w(x, y; \theta, t) = z(\theta, t)w_s(x, y), \quad (2.15)$$

where

$$\frac{z(\theta, t)}{d} = z_0(\theta) + [\bar{z}(t)]_{\theta=\text{const}}, \quad (2.16)$$

$$w_s(x, y) = \sin \frac{\pi x}{2a} \sin \frac{\pi y}{2b}, \quad (2.17)$$

$$w_c(x, y) = \frac{1}{4} \left(1 - \cos \frac{\pi x}{a} \right) \left(1 - \cos \frac{\pi y}{b} \right). \quad (2.18)$$

Equations (2.17) and (2.18) satisfy the boundary conditions, Eqs. (2.8) and (2.9), respectively. If $\tilde{\theta}_e$ which has been neglected in Eq. (2.14) is taken into consideration, Eq. (2.17) does not satisfy completely Eq. (2.8) and it is necessary to add the another terms which come from the complementary solution of $\nabla^4 w_s = 0$ accompanied by the integral constants determined so as to satisfy the boundary conditions, Eq. (2.8) [13]. For the case of clamped edges, the solution is not related to $\tilde{\theta}_e$ [14].

Substituting Eqs. (2.13) and (2.15) into Eq. (2.12) and integrating the resulting equation, the stress function is given as

$$\chi = \frac{1}{2} C_1 x^2 + \frac{1}{2} C_2 y^2 + I, \quad (2.19)$$

where

$$\begin{aligned} I_s = & \frac{Ez^2}{32} \left[\left(\frac{a}{b} \right)^2 \cos \frac{\pi x}{a} + \left(\frac{b}{a} \right)^2 \cos \frac{\pi y}{b} \right] \\ & + E\alpha \left\{ \sum_{i=2}^{\infty} \frac{\bar{\theta}_{i0}}{\left(\frac{i\pi}{2a} \right)^2} \cos \frac{i\pi x}{2a} + \sum_{j=2}^{\infty} \frac{\bar{\theta}_{0j}}{\left(\frac{j\pi}{2b} \right)^2} \cos \frac{j\pi y}{2b} \right. \\ & \left. + \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} \frac{\bar{\theta}_{ij}}{\left[\left(\frac{i\pi}{2a} \right)^2 + \left(\frac{j\pi}{2b} \right)^2 \right]} \cos \frac{i\pi x}{2a} \cos \frac{j\pi y}{2b} \right\}, \quad (2.20) \end{aligned}$$

$$\begin{aligned} I_c = & \frac{Ez^2}{32} \left\{ \left(\frac{a}{b} \right)^2 \cos \frac{\pi x}{a} + \left(\frac{b}{a} \right)^2 \cos \frac{\pi y}{b} - \frac{1}{16} \left[\left(\frac{a}{b} \right)^2 \cos \frac{2\pi x}{a} \right. \right. \\ & \left. \left. + \left(\frac{b}{a} \right)^2 \cos \frac{2\pi y}{b} \right] - \frac{2 \left(\frac{a}{b} \right)^2}{\left[1 + \left(\frac{a}{b} \right)^2 \right]^2} \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \right. \\ & \left. + \frac{\left(\frac{a}{b} \right)^2}{\left[1 + 4 \left(\frac{a}{b} \right)^2 \right]^2} \cos \frac{\pi x}{a} \cos \frac{2\pi y}{b} \right. \\ & \left. + \frac{\left(\frac{a}{b} \right)^2}{\left[4 + \left(\frac{a}{b} \right)^2 \right]^2} \cos \frac{2\pi x}{a} \cos \frac{\pi y}{b} \right\} \\ & + E\alpha \left\{ \sum_{i=2}^{\infty} \frac{\bar{\theta}_{i0}}{\left(\frac{i\pi}{2a} \right)^2} \cos \frac{i\pi x}{2a} + \sum_{j=2}^{\infty} \frac{\bar{\theta}_{0j}}{\left(\frac{j\pi}{2b} \right)^2} \cos \frac{j\pi y}{2b} \right. \\ & \left. + \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} \frac{\bar{\theta}_{ij}}{\left[\left(\frac{i\pi}{2a} \right)^2 + \left(\frac{j\pi}{2b} \right)^2 \right]} \cos \frac{i\pi x}{2a} \cos \frac{j\pi y}{2b} \right\}. \quad (2.21) \end{aligned}$$

Integral constants, C_1 and C_2 , are determined using the conditions on displacements in the plate plane. Where $u=0$ at $x=0, 2a$ and $v=0$ at $y=0, 2b$, C_1 and C_2 are determined as

$$\left. \begin{aligned} C_{1,s} &= -\frac{E\alpha\bar{\theta}_{00}}{(1-\nu)} + \frac{\pi^2 E z^2}{32(1-\nu^2)} \left(\frac{\nu}{a^2} + \frac{1}{b^2} \right), \\ C_{2,s} &= -\frac{E\alpha\bar{\theta}_{00}}{(1-\nu)} + \frac{\pi^2 E z^2}{32(1-\nu^2)} \left(\frac{1}{a^2} + \frac{\nu}{b^2} \right), \end{aligned} \right\} \quad (2.22)$$

$$\left. \begin{aligned} C_{1,c} &= -\frac{E\alpha\bar{\theta}_{00}}{(1-\nu)} + \frac{3\pi^2 E z^2}{128(1-\nu^2)} \left(\frac{\nu}{a^2} + \frac{1}{b^2} \right), \\ C_{2,c} &= -\frac{E\alpha\bar{\theta}_{00}}{(1-\nu)} + \frac{3\pi^2 E z^2}{128(1-\nu^2)} \left(\frac{1}{a^2} + \frac{\nu}{b^2} \right). \end{aligned} \right\} \quad (2.23)$$

C_1 and C_2 can be easily obtained approximately even if displacements in the plate plane at edges are permitted [24].

The equation of vibration is then derived. By expressing each of the variables in Eq. (2.11) by the sum of two components, that is, the one corresponding to the deflection state due to the temperature change only and the other corresponding to the vibration state, and applying the Galerkin method to Eq. (2.11) after eliminating the former part the following equation is finally obtained.

$$\frac{d^2 \bar{z}}{d\tau^2} + (f'_1 + 3f'_3 z_0^2) \bar{z} + 3f'_3 z_0 \bar{z}^2 + f'_3 \bar{z}^3 = 0, \quad (2.24)$$

where

$$\tau = \frac{c_b}{\sqrt{12}} \frac{d}{(2b)^2} t = \frac{1}{(2b)^2} \sqrt{\frac{D}{\rho d}} t, \quad (2.25)$$

$$c_b^2 = \frac{E}{(1-\nu^2)\rho}, \quad (2.26)$$

$$\left. \begin{aligned} f'_{1,s} &= \pi^4 \left(1 + \frac{1}{\lambda^2} \right)^2 - 12(1+\nu)\pi^2 \left(\frac{2b}{d} \right)^2 \alpha \left[\left(1 + \frac{1}{\lambda^2} \right) \bar{\theta}_{00} \right. \\ &\quad \left. - \frac{(1-\nu)}{2} \left(\frac{\bar{\theta}_{02}}{\lambda^2} + \bar{\theta}_{20} \right) \right], \\ f'_{3,s} &= 3\pi^4 \left[\frac{(3-\nu^2)}{4} \left(1 + \frac{1}{\lambda^4} \right) + \frac{\nu}{\lambda^2} \right], \end{aligned} \right\} \quad (2.27)$$

$$\left. \begin{aligned} f'_{1,c} &= \frac{16\pi^4}{9} \left(3 + \frac{2}{\lambda^2} + \frac{3}{\lambda^4} \right) \\ &\quad - \frac{8}{3} (1+\nu)\pi^2 \left(\frac{2b}{d} \right)^2 \alpha \left[6 \left(1 + \frac{1}{\lambda^2} \right) \bar{\theta}_{00} \right. \\ &\quad \left. - (1-\nu) \left\{ \left[\frac{1}{\lambda^2} (4\bar{\theta}_{02} - \bar{\theta}_{04}) + (4\bar{\theta}_{20} - \bar{\theta}_{40}) \right] \right\} \right] \end{aligned} \right\}$$

$$\left. \begin{aligned}
 & -\frac{4\bar{\theta}_{22}}{(1+\lambda^2)} + 2\left[\frac{\bar{\theta}_{24}}{(1+4\lambda^2)} + \frac{\bar{\theta}_{42}}{(4+\lambda^2)}\right] \Bigg\}, \\
 f'_{3,c} = & 3\pi^4 \left\{ \frac{(35-17\nu^2)}{36} \left(1 + \frac{1}{\lambda^4}\right) + \frac{\nu}{\lambda^2} \right. \\
 & \left. + \frac{2(1-\nu^2)}{9} \left[\frac{4}{(1+\lambda^2)^2} + \frac{1}{(1+4\lambda^2)^2} + \frac{1}{(4+\lambda^2)^2} \right] \right\},
 \end{aligned} \right\} \quad (2.28)$$

$$\lambda = \frac{a}{b}. \quad (2.29)$$

$z_0(\theta)$ is the so-called "static" solution and can be obtained from Eq. (2.30).

$$f'_1 z_0 + f'_3 z_0^3 = g, \quad (2.30)$$

where

$$g_s = 12(1+\nu)\pi^2 \left(1 + \frac{1}{\lambda^2}\right) \left(\frac{2b}{d}\right)^2 \alpha \tilde{\theta}_{11}, \quad (2.31)$$

$$g_c = \frac{4096}{3} (1+\nu) \left(\frac{2b}{d}\right)^2 \sum_{p,q:\text{odd}} \sum \frac{(p^2 + \lambda^2 q^2)}{\lambda^2 p q (p^2 - 4)(q^2 - 4)} \alpha \tilde{\theta}_{pq}. \quad (2.32)$$

The solutions of Eq. (2.30) have already been given by the present author [13], [14], and Eq. (2.24) is solved in the following chapter.

3. ANALYTICAL SOLUTIONS

By the use of the following transformation,

$$\xi = \sqrt{f'_1 + 3f'_3 z_0^2} \tau, \quad (3.1)$$

Eq. (2.24) is reduced to

$$\frac{d^2 \bar{z}}{d\xi^2} + \bar{z} + f_2 \bar{z}^2 + f_3 \bar{z}^3 = 0, \quad (3.2)$$

where

$$\left. \begin{aligned}
 f_2 &= \frac{3f'_3 z_0}{f'_1 + 3f'_3 z_0^2}, \\
 f_3 &= \frac{f'_3}{f'_1 + 3f'_3 z_0^2}.
 \end{aligned} \right\} \quad (3.3)$$

The solution of Eq. (3.2) can be obtained using the elliptical function, but it is too complicated to be used in examining physically the problem even though it is complete in the mathematical sense, so a method of successive approximation [25] is used to solve Eq. (3.2) in the present analysis.

Introducing the new independent variable ζ as

$$\zeta = \sqrt{1+\beta} \xi, \quad (3.4)$$

Eq. (3.2) is reduced to

$$(1 + \beta)\ddot{\bar{z}} + \bar{z} = -f_2\bar{z}^2 - f_3\bar{z}^3, \tag{3.5}$$

where “.” means the differentiation with respect to ζ . Eq. (3.5) is the equation of motion with respect to the vertical displacement, \bar{z} , which has the maximum amplitude, z_1 and the minimum amplitude, $-z_2$. Then, β and \bar{z} are expanded in the power series of z_2 , that is,

$$\beta = -\beta_1 z_2 + \beta_2 z_2^2 - \beta_3 z_2^3 + \beta_4 z_2^4 - \dots, \tag{3.6}$$

$$\bar{z} = -\eta_1(\zeta)z_2 + \eta_2(\zeta)z_2^2 - \eta_3(\zeta)z_2^3 + \eta_4(\zeta)z_2^4 - \dots \tag{3.7}$$

Substituting Eqs. (3.6) and (3.7) into Eq. (3.5), and putting the terms of the same order with respect to z_2 in the both sides to be equal with each other, the following equations are obtained.

$$\ddot{\eta}_1 + \eta_1 = 0, \tag{3.8.1}$$

$$\ddot{\eta}_2 + \eta_2 = -\beta_1 \ddot{\eta}_1 - f_2 \eta_1^2, \tag{3.8.2}$$

$$\ddot{\eta}_3 + \eta_3 = -\beta_2 \ddot{\eta}_1 - \beta_1 \ddot{\eta}_2 - 2f_2 \eta_1 \eta_2 - f_3 \eta_1^3, \tag{3.8.3}$$

$$\ddot{\eta}_4 + \eta_4 = -\beta_3 \ddot{\eta}_1 - \beta_2 \ddot{\eta}_2 - \beta_1 \ddot{\eta}_3 - f_2(2\eta_1 \eta_3 + \eta_2^2) - 3f_3 \eta_1^2 \eta_2, \tag{3.8.4}$$

$$\begin{aligned} \ddot{\eta}_5 + \eta_5 = & -\beta_4 \ddot{\eta}_1 - \beta_3 \ddot{\eta}_2 - \beta_2 \ddot{\eta}_3 - \beta_1 \ddot{\eta}_4 \\ & - 2f_2(\eta_1 \eta_4 + \eta_2 \eta_3) - 3f_3(\eta_1^2 \eta_3 + \eta_1 \eta_2^2), \end{aligned} \tag{3.8.5}$$

Since the free vibration is considered here, the initial conditions to be applied to η 's are set as

$$\left. \begin{aligned} \eta_1(0) = 1, \quad \eta_2(0) = \eta_3(0) = \eta_4(0) = \dots = 0, \\ \dot{\eta}_1(0) = \dot{\eta}_2(0) = \dot{\eta}_3(0) = \dots = 0. \end{aligned} \right\} \tag{3.9}$$

Considering Eqs. (3.9), Eq. (3.8.1) is solved as

$$\eta_1 = \cos \zeta. \tag{3.10}$$

Using Eq. (3.10), Eq. (3.8.2) is reduced to

$$\ddot{\eta}_2 + \eta_2 = \beta_1 \cos \zeta - \frac{1}{2} f_2 (1 + \cos 2\zeta). \tag{3.11}$$

To eliminate the secular term from the solution of Eq. (3.11), β_1 must be

$$\beta_1 = 0. \tag{3.12}$$

Substituting Eq. (3.12) into Eq. (3.11) and satisfying the conditions, Eqs. (3.9), the following solution for η_2 is obtained.

$$\eta_2 = -\frac{1}{2} f_2 + \frac{1}{3} f_2 \cos \zeta + \frac{1}{6} f_2 \cos 2\zeta. \tag{3.13}$$

Solving successively Eqs. (3.8) in the same way, the following equations are obtained.

$$\begin{aligned}
\eta_1 &= \cos \zeta, \\
\eta_2 &= -\frac{1}{2} f_2 + \frac{1}{3} f_2 \cos \zeta + \frac{1}{6} f_2 \cos 2\zeta, \\
\eta_3 &= -\frac{1}{3} f_2^2 + \left(\frac{29}{144} f_2^2 - \frac{1}{32} f_3 \right) \cos \zeta + \frac{1}{9} f_2^2 \cos 2\zeta \\
&\quad + \left(\frac{1}{48} f_2^2 + \frac{1}{32} f_3 \right) \cos 3\zeta, \\
\eta_4 &= \left(-\frac{25}{48} f_2^3 + \frac{21}{32} f_2 f_3 \right) + \left(\frac{119}{432} f_2^3 - \frac{35}{96} f_2 f_3 \right) \cos \zeta \\
&\quad + \left(\frac{2}{9} f_2^3 - \frac{1}{3} f_2 f_3 \right) \cos 2\zeta + \left(\frac{1}{48} f_2^3 + \frac{1}{32} f_2 f_3 \right) \cos 3\zeta \\
&\quad + \left(\frac{1}{432} f_2^3 + \frac{1}{96} f_2 f_3 \right) \cos 4\zeta, \\
\eta_5 &= \left(-\frac{25}{36} f_2^4 + \frac{29}{24} f_2^2 f_3 \right) + \left(\frac{7103}{20736} f_2^4 - \frac{1607}{2304} f_2^2 f_3 \right) \\
&\quad + \frac{23}{1024} f_2^3 \cos \zeta + \left(\frac{8}{27} f_2^4 - \frac{5}{9} f_2^2 f_3 \right) \cos 2\zeta + \left(\frac{31}{576} f_2^4 \right. \\
&\quad + \frac{11}{384} f_2^2 f_3 - \frac{3}{128} f_3^2 \left. \right) \cos 3\zeta + \left(\frac{1}{648} f_2^4 + \frac{1}{72} f_2^2 f_3 \right) \cos 4\zeta \\
&\quad + \left(\frac{5}{20736} f_2^4 + \frac{5}{2304} f_2^2 f_3 + \frac{1}{1024} f_3^2 \right) \cos 5\zeta, \\
&\dots\dots\dots
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\beta_1 &= 0, \\
\beta_2 &= -\frac{5}{6} f_2^2 + \frac{3}{4} f_3, \\
\beta_3 &= -\frac{5}{9} f_2^3 + \frac{1}{2} f_2 f_3, \\
\beta_4 &= -\frac{235}{288} f_2^4 + \frac{185}{96} f_2^2 f_3 - \frac{3}{128} f_3^2, \\
&\dots\dots\dots
\end{aligned} \tag{3.15}$$

Using Eqs. (3.14), Eq. (3.7) is reduced to

$$\begin{aligned}
\bar{z} &= \left[-\frac{1}{2} f_2 z_2^2 + \frac{1}{3} f_2^2 z_2^3 - \left(\frac{25}{48} f_2^3 - \frac{21}{32} f_2 f_3 \right) z_2^4 \right. \\
&\quad \left. + \left(\frac{25}{36} f_2^4 - \frac{29}{24} f_2^2 f_3 \right) z_2^5 - \dots \right] \\
&\quad + \left[-z_2 + \frac{1}{3} f_2 z_2^2 - \left(\frac{29}{144} f_2^2 - \frac{1}{32} f_3 \right) z_2^3 + \left(\frac{119}{432} f_2^3 - \frac{35}{96} f_2 f_3 \right) z_2^4 \right.
\end{aligned}$$

amplitude, which is the characteristics of the nonlinear vibration, and changes with the thermal stress and deflection due to the temperature change.

Next, the relation between the maximum and minimum values of amplitude, z_1 and $-z_2$, is given.

Applying the so-called energy integral to Eq. (3.2), the following equation is obtained.

$$\left(\frac{d\bar{z}}{d\xi}\right)^2 + \bar{z}^2 + \frac{2}{3}f_2\bar{z}^3 + \frac{1}{2}f_3\bar{z}^4 = 2E = \text{const}, \quad (3.19)$$

where, E is the total energy of the vibrating system (conservative system). Using the condition, $d\bar{z}/d\xi = 0$ at $\bar{z} = z_1, -z_2$, Eq. (3.19) is reduced to

$$z_1^2 \left(1 + \frac{2}{3}f_2z_1 + \frac{1}{2}f_3z_1^2\right) = z_2^2 \left(1 - \frac{2}{3}f_2z_2 + \frac{1}{2}f_3z_2^2\right) = 2E. \quad (3.20)$$

z_1 and z_2 can be determined independently whenever E is given in accordance with the special initial conditions. For the case of arbitrary free vibration such as one presented in the present paper, if one of z_1 or z_2 is given, the other can be determined through Eqs. (3.20).

For the case where there exists no vertical displacement due to the temperature change, Eq. (3.2) is reduced to Eq. (3.21) because of $f_2 = 0$ according to $z_0 = 0$ in Eq. (2.24).

$$\frac{d^2\bar{z}}{d\tau^2} + f_1'\bar{z} + f_3'\bar{z}^3 = 0. \quad (3.21)$$

Through the energy integral, Eq. (3.21) is reduced to

$$\begin{aligned} \tau &= \int_0^{\bar{z}} \frac{d\bar{z}}{\sqrt{2E - f_1'\bar{z}^2 - (f_3'\bar{z}^4)/2}} \\ &= \int_0^{\varphi} \frac{d\varphi}{\sqrt{f_1' + f_3'z_2^2(1 - \cos^2\varphi/2)}} \\ &= \frac{1}{\sqrt{f_1' + f_3'z_2^2}} \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \cos^2\varphi}}, \end{aligned} \quad (3.22)$$

where

$$k^2 = \frac{f_3'z_2^2}{2(f_1' + f_3'z_2^2)} = \frac{1}{2} \left(1 + \frac{f_1'}{f_3'z_2^2}\right)^{-1}. \quad (3.23)$$

Then, the period of vibration is given as

$$\begin{aligned} T^*(\tau) &= \frac{4}{\sqrt{f_1' + f_3'z_2^2}} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \cos^2\varphi}} \\ &= \frac{4}{\sqrt{f_1' + f_3'z_2^2}} K(k), \end{aligned} \quad (3.24)$$

or

$$T^*(t) = \frac{4(2b)^2}{\sqrt{f_1' + f_3' z_0^2}} \sqrt{\frac{\rho d}{D}} K(k), \tag{3.25}$$

where, $K(k)$ is the complete elliptical integral of the first kind. It is natural that the asymptotic expansion of Eq. (3.25) coincides with Eq. (3.18) where z_0 is put to be equal to zero.

4. NUMERICAL EXAMPLES

Influences of the temperature change and large amplitude on the free vibration of rectangular elastic plates have been given analytically in the preceding chapter, and they are explained concretely by some numerical examples in this chapter.

It is assumed that the rectangular plate is simply supported at four edges and subjected to the following change in temperature.

$$\left. \begin{aligned} \theta = \bar{\theta} = \Theta \sin \frac{\pi x}{2a} \sin \frac{\pi y}{2b}, \\ \tilde{\theta} = 0, \quad \text{i.e., } g = 0. \end{aligned} \right\} \tag{4.1}$$

The plate buckles at the critical temperature, Θ_{cr} , and so its vibration behavior is studied separately for each of the cases of before and after the thermal buckling. The plate will start to deflect from the beginning of heating when $\tilde{\theta} \neq 0$, and its behavior can be dealt with in the same way as for the case of 4.1 or 4.2 after the temperature settles down.

Using Eq. (2.30) with Eqs. (4.1), the critical temperature, Θ_{cr} , and the deflection at the center of the plate after buckling are obtained and shown in Fig. 2. The change of the critical temperature, Θ_{cr} with aspect ratio, λ is shown in Fig. 3, where Poisson's ratio is assumed to be $\nu = 1/3$.

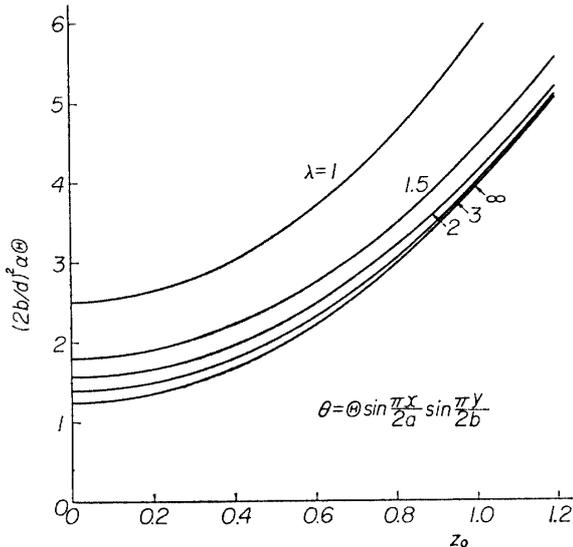


FIG. 2. Relation between temperature rise and deflection at the center of plate, simply supported edges.

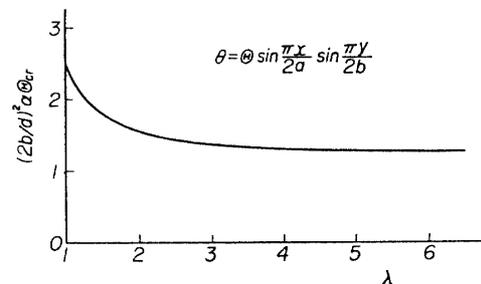


FIG. 3. Variation of thermal buckling coefficient with aspect ratio, simply supported edges.

4.1. THE CASE OF $\theta < \theta_{cr}(z_0=0)$.

The change of the period with the amplitude and temperature is given by Eq. (3.18) (putting $z_0=0$) or Eq. (3.25) and shown in Figs. 4 a, b and c, inclusively. It can be seen from these figures that the period changes remarkably with the increase of amplitude and the change in temperature. It is natural that Fig. 4a

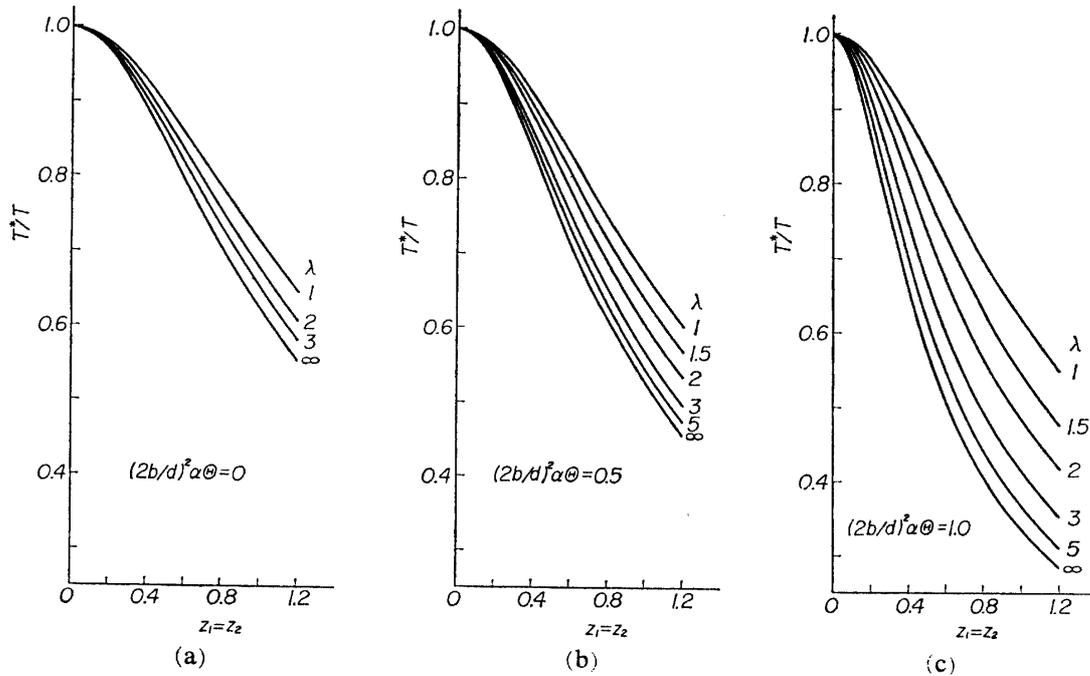


FIG. 4. Influence of large amplitude on period of vibration of rectangular plates, simply supported edges.

($\theta=0$) coincides with the result by Chu and Herrmann [9]. The change of the period with aspect ratio of rectangular plates, λ is shown in Fig. 5. This figure indicates that the larger the amplitude and the temperature change become, the larger the difference between the solutions by the linear and the nonlinear theory will be. For the case of $\lambda=1$, the change of the circular frequency with the temperature is shown in Fig. 6 using the amplitude as a parameter. Figure 6 shows the characteristics similar to that of the vibration of the bar subjected to the axial force, and the temperature where the restoring force vanishes, that is $\omega=0$ in the linear solution, coincides with the thermal buckling temperature of the plate.

4.2. THE CASE OF $\theta > \theta_{cr}(z_0 \neq 0)$.

Using the values of z_0 , which can be obtained from Eq. (2.30), in Eq. (3.18), the change of the circular frequency with the amplitude and temperature is obtained and shown in Fig. 6 for the case of $\lambda=1$. The circular frequency increases with the increase of temperature and decreases with the increase of amplitude where the temperature is constant, and this characteristics is opposed to that of the prebuckling state.

The relation between the maximum amplitude, z_1 and the minimum amplitude,

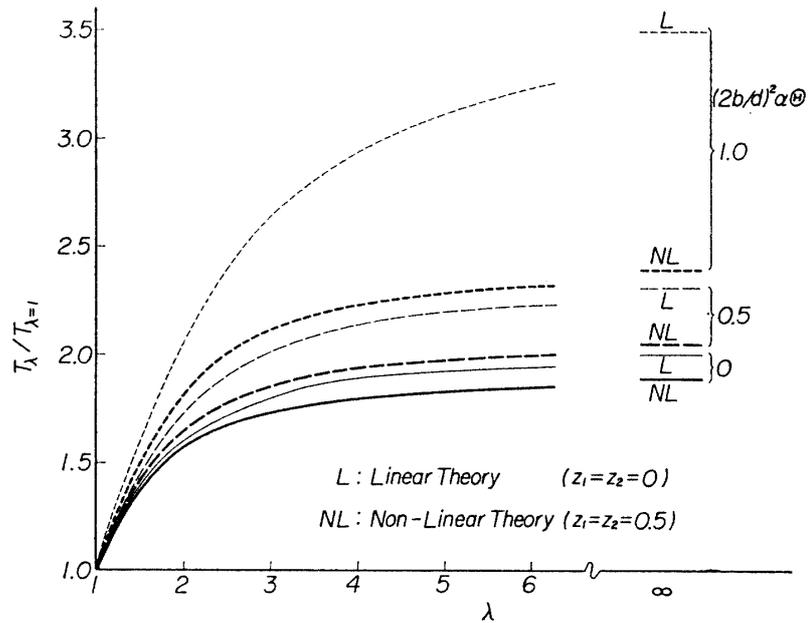


FIG. 5. Variation of period of vibration of heated rectangular plates with aspect ratio, simply supported edges.

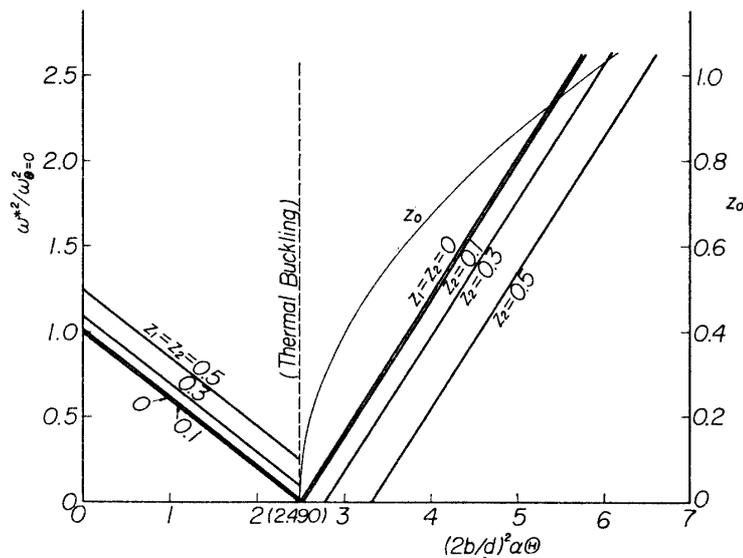


FIG. 6. Variation of frequency of vibration of plate with temperature rise and large amplitude, simply supported edges, $\lambda=1$.

$-z_2$ is calculated through Eq. (3.20) and shown in Fig. 7. This curve is symmetrical with respect to $-z_2/z_0 = -1$ which is the initial unstrained state, and this is natural because the direction of the deflection due to buckling is not restricted toward any of upper or lower side. The half of the curve shown by broken line is used for the case where the plate buckles toward the lower side.

In Fig. 6, there exists the temperature range where the nonlinear free vibration cannot occur depending on the value of amplitude, and it seems that this range

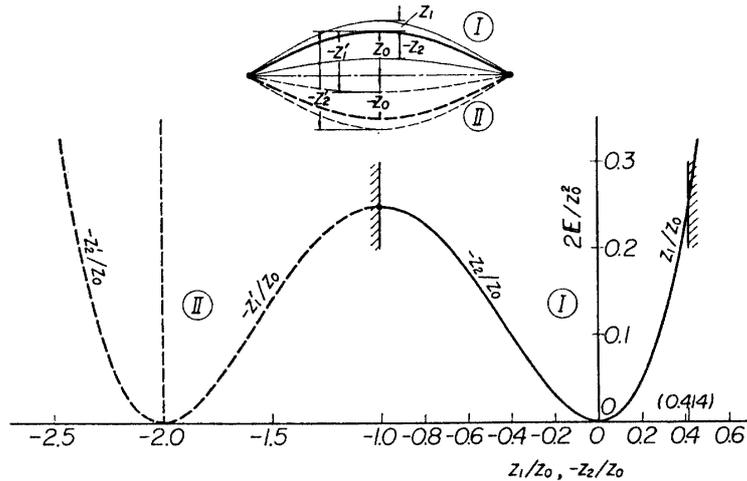


FIG. 7. Relation between z_1 and z_2 after buckling, simply supported edges, $\lambda=1$.

corresponds to the amplitude range, $-z_2/z_0 < -1$ in Fig. 7. That is to say, $-z_2/z_0 < -1$ means that the absolute value of the minimum amplitude is larger than the deflection caused by the heating (buckling), and the snap-through takes place if the assumed initial deflection ($-z_2$, the minimum amplitude) is tried to be set. The maximum absolute value of the minimum amplitude, that is, $z_2 = z_0$ corresponds to the maximum value of amplitude, $z_1 = 0.414z_0$.

By the aid of Fig. 7, Fig. 6 is transformed into Fig. 8. It also is shown in Fig. 8 that the circular frequency of the vibration decreases with the temperature rise and reaches the minimum at the buckling temperature and thereafter it increases with the increase of temperature, and that the effect of amplitude on the vibration is

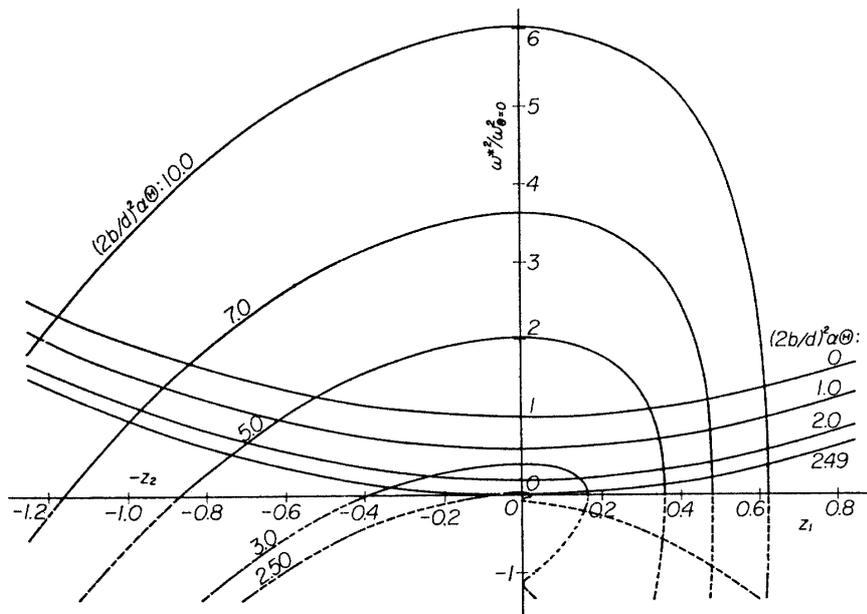


FIG. 8. Variation of frequency of vibration of plate with temperature rise and large amplitude, simply supported edges, $\lambda=1$.

very important. The points z_2 's, where the curves for $\theta > \theta_{cr}$ intersect the abscissa, coincide with the respective z_0 's corresponding to the postbuckling displacements at each of the temperatures.

The above explanation can be given for the case where the initial condition of the free vibration of plate is given by the minimum value of amplitude, $-z_2$. If the initial condition is given by the maximum value of amplitude, z_1 , the another type of vibration accompanied by the snap-through phenomenon can also exist mathematically. That is to say, if the initial condition of the free vibration, z_1 larger than $z_1 = 0.414z_0$ where the total energy of the system is equal to the energy corresponding to $-z_2/z_0 = -1$ (Fig. 7) is set, the free vibration where the neutral point of the upper and lower amplitudes is $-z_0$ (initial unstrained state) could be taken place, and it can also be shown by the use of the phase plane.

In the present paper, the governing equation of the free vibration, Eq. (2.24) has been derived based on the assumption that the nonlinearity is not so large and so the inertia in the plate plane can be neglected and hence buoyancy terms in the equilibrium equation in the z -direction have been eliminated. Moreover, the stress state in the latter case mentioned above is not necessarily assured to be in the elastic range of the material. So, the latter case with larger amplitude has to be analyzed more carefully than the case studied in the present paper and its analysis will be given separately later.

5. CONCLUSIONS

The influences of the temperature change and large amplitude on the free flexural vibration of rectangular plates have been studied. It has been shown that the vibration characteristics of plates are affected remarkably by the in-plane stress and deflection of plates induced by the temperature change and that this tendency is conspicuous where the finite amplitude is taken into consideration especially for the case of larger change in temperature.

It is well known that many vibration problems are dealt with as small oscillations for convenience' sake of analysis and frequently it is sufficiently suitable to the practical understanding of the problems. However, in some special cases such as problems in the aerothermoelasticity, it seems that the linear theory is not any more sufficient tool to explain the phenomena and the problems have to be analyzed as "in the large", that is the nonlinear ones.

In the present paper, the vibration of plates at the equilibrium state of temperature after being changed has been analyzed, and so the temperature change during the vibration has not been considered. For the vibration induced by the thermal shock it is impossible to assume the deflection as in Eq. (2.16) and Eq. (2.11) has to be solved directly. And, the problem has been analyzed only for the case of the lowest order of vibration mode, but the vibrations of higher order of mode can be examined in the same way.

Some experiments [26] had been carried out to check the present analysis and a good agreement between the theoretical and experimental results was seen, and the present analysis has been developed to cover the cases with initial imperfections.

They will be published later on.

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