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# On the Experimental Determination of Load Eccentricity and Initial Deviation in the Column Compressive Tests

*By*

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*Summary.* The separation of load eccentricity and initial deviation of a column, both inevitable for human production, is investigated. In the analysis the high applied-load vs. Euler-load ratio is used to simplify the theoretical approach, whose good agreement is supported by the experimental verification. The experimental measurement and determination are done for 7075-T6 and 2024-T4 aluminum alloy bars with rectangular solid section. The results show the minor magnitude of load eccentricity, say, less than 10% of initial deviation, in most cases.

## *Symbols*

$A$ = cross sectional area
$c_1, c_2$ = distance between the neutral axis and the exterior fiber
$e$ = load eccentricity
$EI$ = flexural rigidity of a column
$h$ = height or depth
$L$ = length
$n_E$ = applied load/Euler load
$P$ = applied compressive load
$P_E$ = Euler load
$w_o$ = initial deviation
$w_a$ = additional deflection
$w$ = total deflection
$\lambda$ = numerical constant
$\varepsilon$ = strain
$\delta_o$ = maximum initial deviation divided by the half-depth of column

## 1. INTRODUCTION

It is quite obvious that there exist the load eccentricity and the column initial deviation in the column compression, since the above-mentioned imperfections inherent to the human achievement are inevitable even in the elaborate experiment.

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Originally the load eccentricity is due to the loading mechanism and techniques, while the initial deviation of a column is inherent and constant. Therefore both of them are quite another things in nature, and should be discussed separately. Consequently their measurement and determination of individual magnitude are necessary to develop the theoretical approach in the solid mechanics such as the column creep buckling, the column compressive tests and the column instability phenomenon. Only the initial deviation is theoretically treated and determined in the experiment so far [1].

In the present paper, the authors achieve to develop the theoretical derivation for the separation of load eccentricity and the initial deviation, and determine both of them through the usual column compressive tests by use of strain gages and a dial gage. Generally the load eccentricity itself is dependent upon the load applied, however, in the analysis followed, we first establish the appropriate assumptions including the constant load eccentricity such that may not hurt the essential evaluation of the present problem, for the purpose of simplification. In fact, this simplification of constant load eccentricity is experimentally verified so far as the present test results are concerned as shown later. The more general case, including the different load eccentricities at both ends and the multi-term initial deviation, is also discussed for further reference.

## 2. Theoretical Determination of Load Eccentricity and Initial Deviation

For the simplification and the easier understanding of essential evaluation of the present problem, we take the following assumptions:

(1) The column under consideration has a constant uniform section and is simply supported at both ends.

(2) The column has the same amount of load eccentricity on one side of a column, and this load eccentricity is constant irrespective of applied load. We call this load eccentricity the equivalent one.

(3) The initial deviation of a column is a half sine wave*.

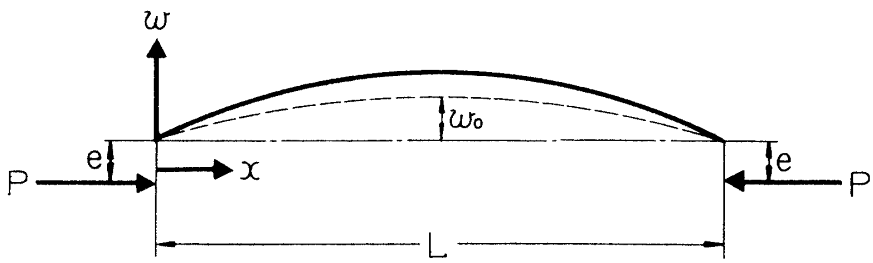


FIG. 1. The column with initial deviation under eccentric load.

$$w_0 = a_0 \sin \frac{\pi x}{L} \qquad (1)$$

* See Appendix 1 for details.

Then the lateral deflection $w(x)$ subject to a load $P$ applied axially satisfies the following equation,

$$EI\left(\frac{d^2w}{dx^2}-\frac{d^2w_o}{dx^2}\right)=-P(w+e) \tag{2}$$

The boundary conditions at both ends are given by

$$x=0\ \&\ x=L: \quad w=0, \quad \frac{d^2w}{dx^2}=-\frac{Pe}{EI} \tag{3}$$

Then we have

$$w(x)=\frac{a_0}{1-n_E}\sin\frac{\pi x}{L}+eF(x) \tag{4}$$

where

$$\left\{\begin{aligned} &P_E=\frac{\pi^2 EI}{L^2}=\text{Euler load}\\ &F(x)=\frac{\cos\left[\lambda L\left(1-\frac{2x}{L}\right)\right]}{\cos\lambda L}-1\\ &\lambda=\frac{\pi}{2L}\sqrt{n_E}\end{aligned}\right. \tag{5}$$

We see that in Eq. (4) the deflection $w(x)$ is a linear combination of $a_0$ and $e$. The function $F(x)$ takes the values shown below according to the various locations on the column length ;

$$\left\{\begin{aligned} &\text{at both ends, say, } x=0 \text{ and } x=L,\ F(o)=F(L)=0\\ &\text{at } x=\frac{L}{4}, \qquad F\left(\frac{L}{4}\right)=\frac{\cos\frac{\lambda L}{2}}{\cos\lambda L}-1\\ &\text{at } x=\frac{L}{2}, \qquad F\left(\frac{L}{2}\right)=\frac{1}{\cos\lambda L}-1\end{aligned}\right.$$

The additional deflection due to the applied load $P$ is

$$w_a=w(x)-w_o(x)=a_o\frac{n_E}{1-n_E}\sin\frac{\pi x}{L}+eF(x) \tag{6}$$

Since Eq. (6) includes the load $P$, the initial deviation $a_0$ and the load eccentricity $e$, the measurement of the additional deflection $w_a$ at various loading conditions implies the determination of $a_0$ and $e$ so far as the prescribed assumptions are satisfied [2].

For instance choosing the midpoint $x=L/2$ as the location where measurement is concerned, we have

$$w_a\left(\frac{L}{2}\right)=k_1a_0+k_2e \qquad (7)$$

where

$$k_1=\frac{n_E}{1-n_E}, \qquad k_2=\sec\left(\frac{\pi}{2}\sqrt{n_E}\right)-1 \qquad (8)$$

and the actual mathematical procedures such as the least square method can determine $a_o$ and $e$ provided $w_{ai}(L/2)$ is measured at the various corresponding load $P_i$:

$$\sum_{i=1}^{n}\left[k_1(P_i)\cdot a_o+k_2(P_i)\cdot e-w_{ai}\left(\frac{L}{2}\right)\right]^2=\text{Min}. \qquad (9)$$

TABLE 1. $k_1$, $k_2$ versus $n_E$

| $n_E$ | $k_1$ | $k_2$ |
|---|---|---|
| 0 | 0 | 0 |
| 0.25 | 0.333 | 0.414 |
| 0.50 | 1.000 | 1.260 |
| 0.75 | 3.000 | 3.810 |
| 1.00 | ∞ | ∞ |

However, as shown in Table 1, the difference $(k_2-k_1)$ does not become conspicuous with the $n_E$ increment, so it is required to measure $P$ and $w_a$ quite accurately in order to obtain $a_0$ and $e$ exactly in the experiment.

As for the strains on the column surface, the compressive strains $\varepsilon_1$ on the convex side and $\varepsilon_2$ on the concave side are given by

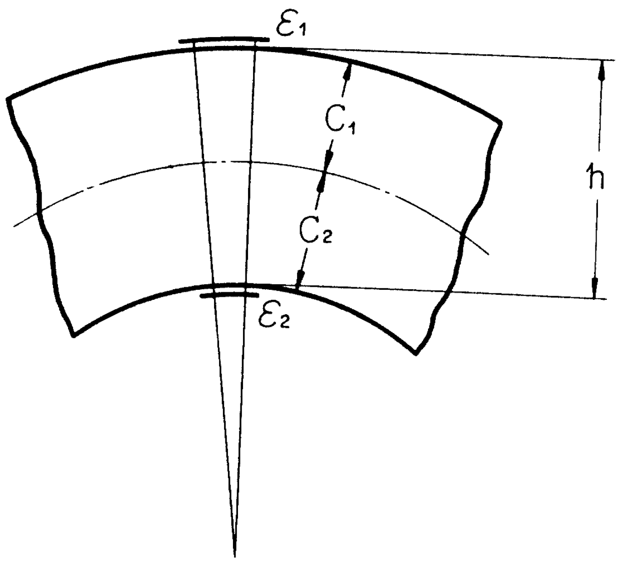


FIG. 2. Local surface strains produced on a column.

$$\begin{cases} \varepsilon_1 = \dfrac{P}{EA} - \dfrac{Pc_1}{EI}(k_3 a_0 + k_4 e) \\ \varepsilon_2 = \dfrac{P}{EA} + \dfrac{Pc_2}{EI}(k_3 a_0 + k_4 e) \end{cases} \tag{10}$$

where

$$k_3 = \frac{1}{1-n_E} \sin \frac{\pi x}{L}, \qquad k_4 = F(x) + 1 \tag{11}$$

It is understood that the experimental measurement of $\varepsilon_1$ and $\varepsilon_2$ at the various loading conditions may determine $a_0$ and $e$, quite similarly as previously described, in view of Eq. (10). [2] Again selecting $x = L/2$, then

$$k_3 = \frac{1}{1-n_E} \text{ and } k_4 = \sec\left(\frac{\pi}{2}\sqrt{n_E}\right) \tag{12}$$

Therefore Eq. (10) becomes

$$\begin{cases} \varepsilon_1 = \dfrac{\pi^2 I}{AL^2} n_E - \dfrac{\pi^2 c_1 h}{L^2} n_E \left[ \dfrac{\dfrac{a_0}{h}}{1-n_E} + \dfrac{e}{h} \sec\left(\dfrac{\pi}{2}\sqrt{n_E}\right) \right] \\ \varepsilon_2 = \dfrac{\pi^2 I}{AL^2} n_E + \dfrac{\pi^2 c_2 h}{L^2} n_E \left[ \dfrac{\dfrac{a_0}{h}}{1-n_E} + \dfrac{e}{h} \sec\left(\dfrac{\pi}{2}\sqrt{n_E}\right) \right] \end{cases} \tag{13}$$

Hence

$$\frac{a_0}{h} + \frac{e}{h}(1-n_E) \sec\left(\frac{\pi}{2}\sqrt{n_E}\right)$$

$$= \frac{I}{Ac_1 h}\left(1 - \frac{AL^2}{\pi^2 I}\frac{\varepsilon_1}{n_E}\right)(1-n_E) \tag{14}$$

$$= \frac{I}{Ac_2 h}\left(-1 + \frac{AL^2}{\pi^2 I}\frac{\varepsilon_2}{n_E}\right)(1-n_E) \tag{15}$$

Also rewriting

$$\frac{a_0}{h} + \frac{e}{h}(1-n_E) \sec\left(\frac{\pi}{2}\sqrt{n_E}\right)$$

$$= \frac{L^2}{\pi^2 h^2}\frac{1}{n_E}(1-n_E)(\varepsilon_2 - \varepsilon_1) \tag{16}$$

Consequently we can determine $a_0$ and $e$ experimentally by using Eq. (14) or Eq. (15) or Eq. (16), with the measurement of strains $\varepsilon_1$ and $\varepsilon_2$, and the corresponding loads.

## 3. Load Eccentricity and Initial Deviation in a More General Case

In this case the load eccentricity is different at both ends, but still independent of load applied, as an assumption.

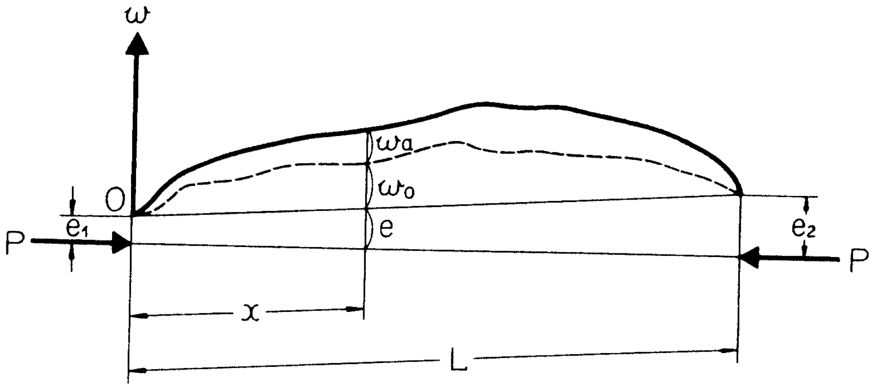


Fig. 3. The column having the multi-term initial deviation with different load eccentricities at both ends.

The initial column deviation is assumed as

$$w_0=\sum_{n=1}^{m}a_n \sin \frac{n\pi x}{L} \tag{17}$$

and the load eccentricity at $x$ is given by

$$e(x)=e_1+\frac{e_2-e_1}{L}x \tag{18}$$

As previously treated we derive the following differential equations:

$$EI\left(\frac{d^2w}{dx^2}-\frac{d^2w_0}{dx^2}\right)=-P(w+e) \tag{19}$$

The boundary conditions are

$$\begin{cases} \text{at } x=0: & w=0 \text{ and } \dfrac{d^2w}{dx_2}=-\dfrac{Pe_1}{EI} \\ \text{at } x=L: & w=0 \text{ and } \dfrac{d^2w}{dx^2}=-\dfrac{Pe_2}{EI} \end{cases} \tag{20}$$

Then the additional deflection $w_a$ is expressed as

$$w_a(x)=w(x)-w_0(x)=\left(\frac{e_2}{\sin 2\lambda L}-e_1 \cot 2\lambda L\right)\sin 2\lambda x$$
$$+e_1 \cos 2\lambda x-\left(e_1+\frac{e_2-e_1}{L}x\right)+\sum_{n=1}^{m}a_n\frac{P/P_{En}}{1-P/P_{En}}\sin\frac{n\pi x}{L} \tag{21}$$

where

$$P_{En}=\frac{n^2\pi^2 EI}{L^2}=n^2 P_E \text{ and } \lambda L=\frac{\pi}{2}\sqrt{n_E} \tag{22}$$

For example, at $x=L/2$, the additional deflection $w_a(L/2)$ is written as

$$\begin{aligned} w_a(L/2) &= e_1(\cos\lambda L-\cot 2\lambda L \sin\lambda L-1/2) \\ &+e_2\left(\frac{\sin\lambda L}{\sin 2\lambda L}-\frac{1}{2}\right) \\ &+a_1\frac{n_E}{1-n_E}-a_3\frac{n_E}{9-n_E}+a_5\frac{n_E}{25-n_E} \\ &+\dots \end{aligned} \tag{23}$$

Similarly the additional deflections for other locations along the column length can be described, so that we can determine $e_1$, $e_2$, $a_1$, $a_2$, $a_3$, ... etc. by measuring the additional deflections at various locations at each corresponding loading condition.

As to the strains on the column surface the same theoretical treatment may be proceeded as similarly as previously described and thus we can also determine $e_1$, $e_2$, $a_1$, $a_2$, $a_3$, ... etc.

The combined measurement of strains and additional deflections, by using Eqs. (7) and (16), of course, leads to the determination of load eccentricity and initial deviation, quite the same procedure as developed before.

## 4. Experimental Results and Discussions

By using a dead-weight lever type loading apparatus 7075-T6 and 2024-T4 aluminum alloy bars with rectangular solid section are tested to be measured the surface strains and the additional deflection, both at midpoint. The rigid knife-edge caps attached to both ends of a column specimen give the negligible effects on the column length correction*. The strains are measured by the strain gages and the deflection by the dial gage. The results using Eqs. (7) and (16) are shown in Tables 2 and 3. Since we only take a half-sine wave for the deflection pattern in the simplified analysis the $P/P_E$ ratio must be close to unity in order to represent the actual deflection in good agreement. Therefore we use $P/P_E=0.903$ and 0.898 for 7075-T6 and 0.891 for 2024-T4. The review of the results obtained tells that most of cases show the minor magnitude of equivalent load eccentricity, say, less than 10% of initial deviation. This confirms the neglect of load eccentricity as treated heretofore as seen in the precedents. However, the load eccentricity and the initial deviation are quite another things in nature, therefore they should be discussed separately in evaluating the effects of imperfection in cases of creep buckling, compressive tests and column instability phenomenon.

* See Appendix 2 for details.

## 5. Conclusions

The experimental determination of load eccentricity and initial column deviation is achieved. The results show the minor magnitude of load eccentricity.

TABLE 2. $a_0$ and $e$ experimentally determined

| 7075–T6 | | | | | | |
|---|---|---|---|---|---|---|
| No. | $\varepsilon_2-\varepsilon_1$ | $w_a\left(\frac{L}{2}\right)$(mm) | $a_0$ (mm) | $e$ (mm) | $\delta_0$ | |
| 1 | $2418\times10^{-6}$ | 1.09 | 0.1120 | 0.00446 | 0.0319 | |
| 2 | 360 | 0.17 | 0.0070 | 0.00850 | 0.0020 | |
| 3 | 964 | 0.44 | 0.0378 | 0.00737 | 0.0108 | $n_E=0.903$ |
| 4 | 1840 | 0.84 | 0.0825 | 0.00582 | 0.0236 | $P=590$ kg |
| 5 | 3660 | 1.66 | 0.1690 | 0.00737 | 0.0483 | $L=177$ mm |
| 6 | 2290 | 1.04 | 0.1050 | 0.00524 | 0.0300 | $h=7$ mm |
| 7 | 1188 | 0.54 | 0.0452 | 0.01018 | 0.0129 | |
| 8 | 2555 | 1.16 | 0.1177 | 0.00524 | 0.0336 | |
| 101 | 2505 | 1.18 | 0.1308 | 0.00259 | 0.0384 | |
| 102 | 2140 | 1.01 | 0.1130 | 0.00120 | 0.0332 | |
| 103 | 1818 | 0.86 | 0.0942 | 0.00239 | 0.0277 | |
| 104 | 1610 | 0.76 | 0.0748 | 0.00915 | 0.0220 | |
| 105 | 1274 | 0.60 | 0.0575 | 0.00868 | 0.0169 | |
| 106 | 3600 | 1.70 | 0.1870 | 0.00458 | 0.0550 | |
| 107 | 1091 | 0.52 | 0.0486 | 0.00812 | 0.0143 | |
| 108 | 820 | 0.39 | 0.0334 | 0.00860 | 0.0098 | |
| 109 | 1111 | 0.53 | 0.0486 | 0.00891 | 0.0143 | $n_E=0.898$ |
| 110 | 2663 | 1.26 | 0.1348 | 0.00637 | 0.0396 | $P=530$ kg |
| 111 | 1252 | 0.59 | 0.0551 | 0.00975 | 0.0162 | $L=177$ mm |
| 112 | 1472 | 0.69 | 0.0700 | 0.00716 | 0.0206 | $h=6.8$ mm |
| 113 | 454 | 0.22 | 0.0140 | 0.00847 | 0.0041 | |
| 114 | 618 | 0.29 | 0.0326 | 0.00032 | 0.0096 | |
| 115 | 1430 | 0.67 | 0.0721 | 0.00358 | 0.0212 | |
| 116 | 1682 | 0.79 | 0.0823 | 0.00633 | 0.0242 | |
| 117 | 1383 | 0.65 | 0.0653 | 0.00712 | 0.0192 | |
| 118 | 1591 | 0.75 | 0.0823 | 0.00231 | 0.0242 | |
| 119 | 1615 | 0.76 | 0.0785 | 0.00645 | 0.0231 | |
| 120 | 1660 | 0.78 | 0.0790 | 0.00785 | 0.0232 | |

TABLE 3. $a_0$ and $e$ experimentally determined

| 2024–T4 | | | | | | |
|---|---|---|---|---|---|---|
| No. | $\varepsilon_2-\varepsilon_1$ | $w_a\left(\frac{L}{2}\right)$(mm) | $a_0$ (mm) | $e$ (mm) | $\delta_0$ | |
| 51 | $724\times10^{-6}$ | 0.33 | 0.0345 | 0.00444 | 0.0099 | |
| 52 | 264 | 0.12 | 0.0143 | 0.00027 | 0.0041 | |
| 53 | 501 | 0.23 | 0.0209 | 0.00550 | 0.0060 | |
| 54 | 1160 | 0.52 | 0.0621 | 0.00169 | 0.0178 | |
| 55 | 201 | 0.09 | 0.0047 | 0.00517 | 0.0013 | |
| 56 | 1370 | 0.62 | 0.0728 | 0.00244 | 0.0208 | $n_E=0.891$ |
| 57 | 1382 | 0.62 | 0.0545 | 0.01760 | 0.0156 | $P=590$ kg |
| 58 | 592 | 0.27 | 0.0183 | 0.01150 | 0.0052 | $L=177$ mm |
| 59 | 1910 | 0.86 | 0.0725 | 0.02660 | 0.0207 | $h=7$ mm |
| 60 | 1540 | 0.70 | 0.0811 | 0.00324 | 0.0232 | |
| 61 | 1540 | 0.69 | 0.0680 | 0.01370 | 0.0195 | |
| 62 | 422 | 0.19 | 0.0167 | 0.00533 | 0.0048 | |
| 63 | 2030 | 0.91 | 0.0833 | 0.02310 | 0.0238 | |

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## APPENDIX 1

Consider the deflection curve of an initially deviated column, whose both ends are simply supported, as shown in Fig. 4. No load eccentricity is assumed in this case.

Suppose the initial deviation

$$w_0 = \sum_{k=1}^{n} b_k \sin \frac{k\pi x}{L} \tag{1a}$$

then the equilibrium equation for the column with the prescribed initial deviation $w_0$ under the constant axial compressive load $P$ is

$$EI\left(\frac{d^2w}{dx^2} - \frac{d^2w_0}{dx^2}\right) = -Pw \tag{2a}$$

Combining Eq. (2a) with the following boundary conditions :

$$w = \frac{d^2w}{dx^2} = 0 \qquad \text{at } x=0 \text{ and } x=L \tag{3a}$$

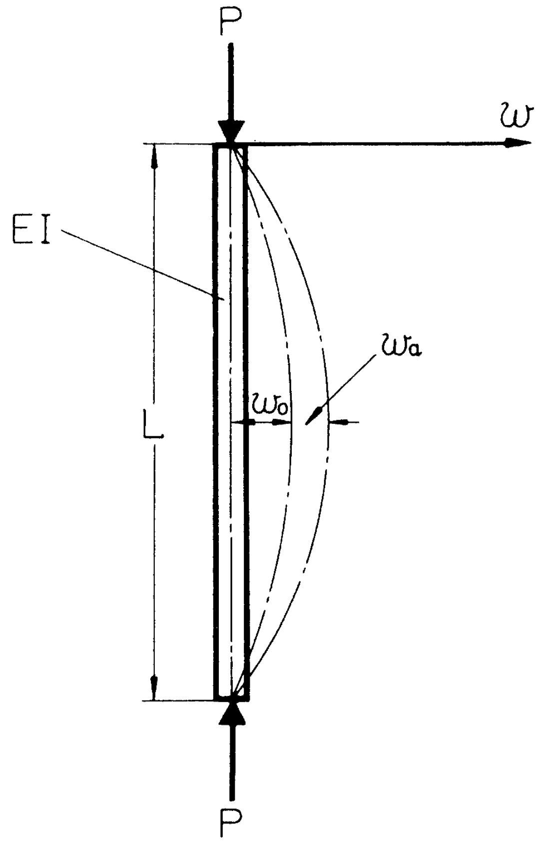


FIG. 4. The column under compressive load (with initial deviation only).

leads to the deflection $w$, say,

$$w=\sum_{k=1}^{n}\frac{k^2}{k^2-n_E}b_k\sin\frac{k\pi x}{L}=\frac{b_1}{1-n_E}\sin\frac{\pi x}{L}+\frac{4b_2}{4-n_E}\sin\frac{2\pi x}{L}+\frac{9b_3}{9-n_E}\sin\frac{3\pi x}{L}+\cdots\cdot \tag{4a}$$

The additional deflection $w_a$ is expressed as

$$w_a=w-w_0=\frac{n_Eb_1}{1-n_E}\sin\frac{\pi x}{L}+\frac{n_Eb_2}{4-n_E}\sin\frac{2\pi x}{L}+\frac{n_Eb_3}{9-n_E}\sin\frac{3\pi x}{L}+\cdots\cdot+\frac{n_Eb_n}{n^2-n_E}\sin\frac{n\pi x}{L} \tag{5a}$$

where $n_E=P/P_E$ and $P_E=\pi^2EI/L^2$.

In case the $n_E$ value is near unity, i.e., the applied load approaches the Euler load, the total deflection $w$ and the additional deflection $w_a$, shown by Eqs. (4a) and (5a), indicate the first single term predominance. For example, $n_E=0.9$,

$$w=10b_1\sin\frac{\pi x}{L}+1.29b_2\sin\frac{2\pi x}{L}+1.11b_3\sin\frac{3\pi x}{L}+\cdots\cdot$$

$$w_a=9b_1\sin\frac{\pi x}{L}+0.29b_2\sin\frac{2\pi x}{L}+0.11b_3\sin\frac{3\pi x}{L}+\cdots\cdot$$

Consequently, the predominant first term can be easily recognized as seen above [*3*].

## APPENDIX 2

In the present experiment, the SK-2 hardened tool steel knife-edge caps are fitted at both ends of a test column in order to keep the simply supported end condition as exact as possible. Here we are going to investigate the additional effects

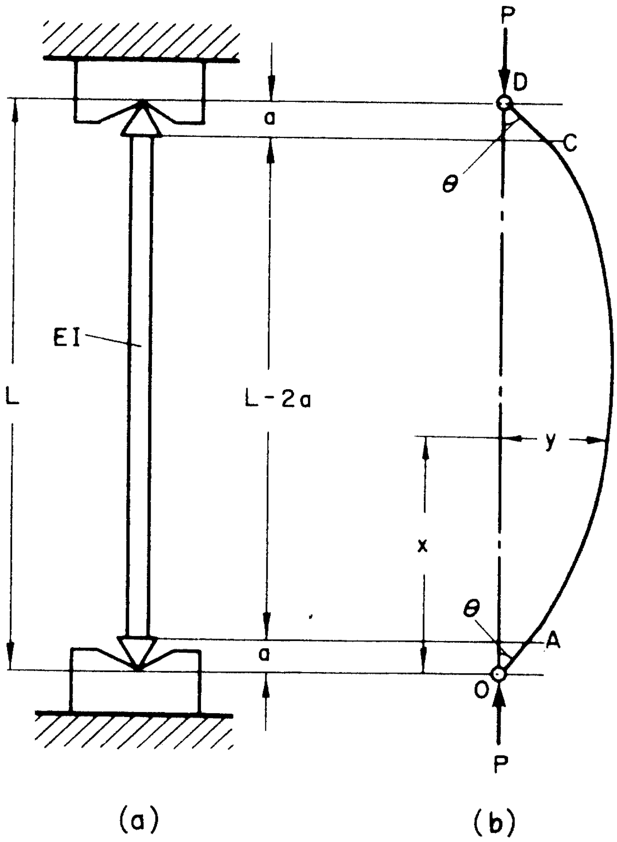


FIG. 5. The column specimen with rigid knife-edge caps.

due to end caps to find the correction, if any; since the end fixtures are stiffer than the test column itself, the equivalent uniform column length is slightly smaller than the actual distance from knife-edge to knife-edge.
The very effects are already discussed in References [*4*] and [*5*], however, the essential points are briefed for reference in what follows. The column in question is assumed to be without any initial deviation and without any load eccentricity.

Since the sectional dimensions of knife-edge caps fitted are taken so large compared with those of a test column as to be able to regard the flexural rigidity infinite, OA and CD shown in Fig. 5(b) could be treated to be straight. Suppose the symmetric deflection pattern with the gradient $\theta$ at the ends as shown in Fig. 5.

Then we have

$$EIy''+Py=0 \tag{1b}$$

for $a \leqq x \leqq L/2$,
and the boundary conditions are

$$y(a)=a\theta \text{ and } y'(L/2)=0 \tag{2b}$$

The rigid OA part is expressed as

$$\theta=\frac{dy}{dx}=\frac{y}{a} \qquad \text{at } x=a \tag{3b}$$

Combination of Eqs. (1b), (2b) and (3b) yields

$$\left.\begin{aligned} &y(x)=A \sin \lambda x+B \cos \lambda x \\ &\lambda^2=\frac{P}{EI} \end{aligned}\right\} \tag{4b}$$

Substitution of Eq. (4b) into Eqs. (2b) and (3b), and elimination of $\theta$ lead to

$$\cot\frac{\lambda}{2}(L-2a)=a\lambda \tag{5b}$$

Then we have

$$\tan\left[\frac{\pi}{2}-\frac{\lambda L}{2}\left(1-\frac{2a}{L}\right)\right]=\frac{\lambda L}{2}\frac{2a}{L} \tag{6b}$$

Putting

$$\frac{\lambda L}{2}=\frac{L}{2}\sqrt{\frac{P}{EI}}\equiv\xi \text{ and } \frac{2a}{L}\equiv\eta \tag{7b}$$

Eq. (6b) can be written as

$$\tan\left[\frac{\pi}{2}-\xi(1-\eta)\right]=\xi\eta \tag{8b}$$

Suppose $\xi=(\pi/2)+\varepsilon$, and both $\varepsilon$ and $\eta$ are enough small as to have the following critical load when Eq. (8b) is expanded in series.

$$P=\frac{\pi^2 EI}{L^2}\left[1+\frac{\pi^2}{6}\left(\frac{2a}{L}\right)^3\right] \tag{9b}$$

The term including $a$ gives the correction due to the added knife-edge caps.

In the present experimental achievement we have $a=8.5$ mm. and $L=177$ mm., therefore the correction term seen in Eq. (9b) is

$$\frac{\pi^2}{6}\left(\frac{2a}{L}\right)^3=\frac{\pi^2}{6}\left(\frac{17}{177}\right)^3=\frac{1.46}{1000}$$

and shows only 0.15% correction compared with that of the non-capped specimen, indicating the negligible effects attributed to the knife-edge caps.