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Body Divergence of a Multi-Stage Rocket

By

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Summary: The body of a rocket is designed to satisfy the static stability criteria. The body should not be considered rigid but elastic, especially when the body is slender. Even if the body is statically stable for a rigid one, the critical speed exists for an elastic body. This speed is known as the divergence speed.

For a single-stage rocket, this divergence speed is already obtained by one of the authors [1]. The method of deriving this speed is given in the following. The lift of the single-stage rocket operates mainly on around the nose and the fin. It dose not seem much errorneous to consider the total lift concentrated on the nose and the fin [2]. So the deformation of the body can be determined in such a way that the inertia force of the body corresponding to some flight speed is supported at the nose and the fin. The angle of attack of the nose and of the fin are now obtained and the lift of each point can be calculated. The divergence speed is derived by equating the restoring moment due to the lift zero.

In calculating the divergence speed for an actual rocket, the values for the lift coefficient can be obtained by the wind-tunnel test and the values for the deformation can be obtained by the bending test of the actual rocket or the simple calculation. This method is very clear and simple in calculating the divergence speed. In 1965 National Aerospace Laboratory fired NAL-7 rockets. The area of the fin and the center of the gravity of these rockets were varied, and the stability of the flight was examined. From these experimental results the above mentioned analysis has been found very useful.

In this paper, this method of analysis is extended to the multi-stage rocket. It does not seem much errorneous to consider the lift of a multi-stage rocket concentrated on the several points, and so the analysis for a single-stage rocket can be extended.

Nomenclature

- W total weight of the rocket.
- L total lift of the rocket.
- EI bending stiffness of the body.
- A reference area of lift coefficient.
- v velocity of the rocket.
- q dynamic pressure; $q=\rho v^2/2$.
- L_i lift on i-point.
- $C_{L\alpha i}$ lift-curve slope for i-point.
- α_i angle of attack of i-point.
- n number of loading points.

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1. Introduction

It is not so important to take into account of the bending stiffness of the body in deriving the static stability criteria of a rocket, if the slenderness ratio of the rocket or its velocity is not large. It is because that the bending deflection of the body is small. But for the slender body the bending deformation should be considered. For example, the angles of attack of the nose and the fins have different values because of its flexibility.

It is very convenient for the analysis, if the aerodynamic loads can be treated as the concentrated loads on several points. For a single-stage rocket it is not much errorneous to consider the aerodynamic loads operate on the nose and the fin. For a multi-stage rocket, it is also not much errorneous to consider the aerodynamic load operate on the nose and the fin of each booster, even if the main rocket-boosters interference exists. So the aerodynamic loads could be treated as the concentrated loads. For example, the lift of a two-stage rocket is assumed to operate on around the nose and the fin of the main rocket and the fins of the boosters.

The bending stiffness of the body is necessary to derive the divergence speed. It is desirable that the values concerning the bending stiffness can be obtained by simple experiments or calculations.

In derving the divergence speed of a multi-stage rocket, the most important problem is that the angle of attack of the each center of the pressure is interconnected with the way of the distribution of the stiffness and the lift. It is also to be noted that the lift-curve slope and the position of the center of the pressure are variable with Mach number.

In the following analysis, the divergence speed of a multi-stage rocket is derived by extending the analysis for a single-stage rocket.

2. Body Divergence of a Multi-Stage Rocket

The total lift of a N-stage rocket could be assumed to work on the $(N+1)$ points with sufficient accuracy. The points on which the aerodynamic load operates are written by the symbol i in the following analysis. The lift force on the i-point is

$$L_i=\frac{1}{2}\rho v^2 A C_{L\alpha_i}\alpha_i \tag{1}$$

The angle of attack of the each point is not the same because of the flexibility. The total lift force can be written in the following form.

$$L=\sum L_i=\frac{1}{2}\rho v^2 A \sum C_{L\alpha_i}\alpha_i \tag{2}$$

The ratio of the lift force on the i-point to the total lift force is

$$\frac{L_i}{L}=\frac{C_{L\alpha_i}\alpha_i}{\sum C_{L\alpha_i}\alpha_i} \tag{3}$$

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Let's assume that the weight of the body is supported at all points of the each center of pressure by the ratio given by Eq. (3). The body deflects and we want to know the slope of its deflection at all the points. The difference in angle between the slope at i-point and some reference point is denoted by Θ_i. We can obtain Θ_i in the following way. The addition of the $(n-1)$ loading conditions as shown in Fig. 1 is equivalent to the supporting condition now considered. In Fig. 1 Θ_{0i} is

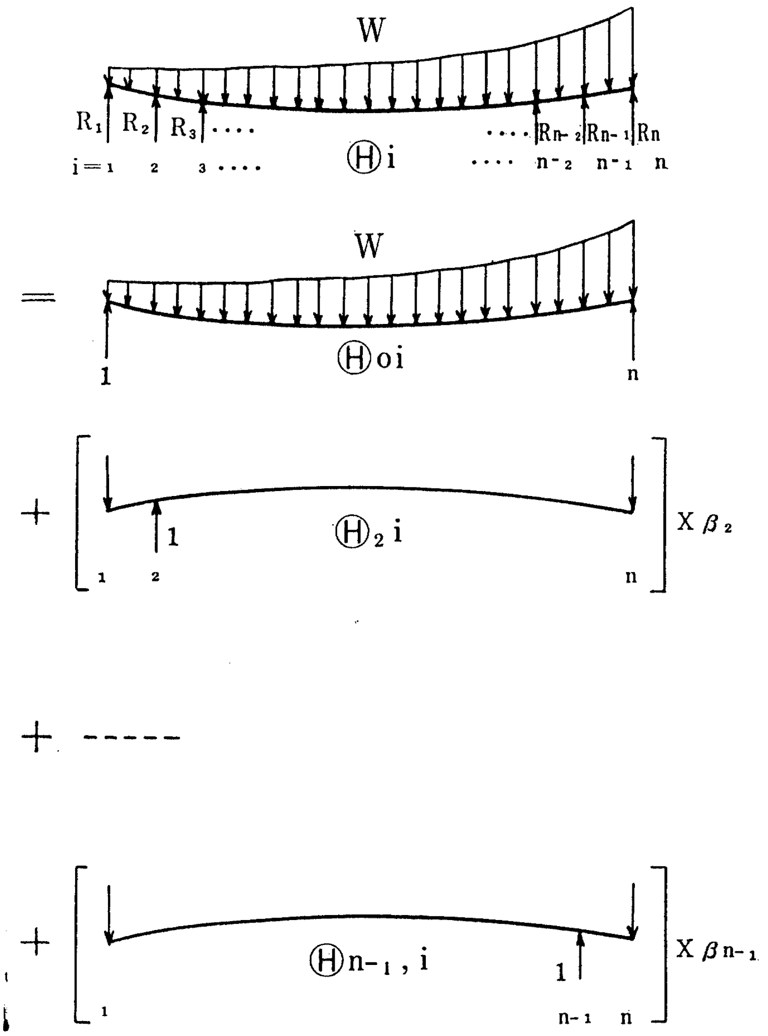

FIG. 1. Determination of the slope of the deflection

the difference in angle between the slope of the deflection at i-point and the reference point, when the weight of the body is supported at 1-point and n-point. Θ_{2i} denotes the difference between the slope at i-point and the reference point, when the body is supported at 1-point and n-point and the unit load is applied at 2-point. $\Theta_{n-1,i}$ is the difference between the slope of the deflection at i-point and the reference point, when the unit load operates on $(n-1)$-point. R_i and β_j in Fig. 1 are given in the following expressions.

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$$R_i=\frac{C_{L\alpha i}\alpha_i}{\sum_{i=1}^{n} C_{L\alpha_i}\alpha_i}W \quad (i=1, 2, \cdots, n) \tag{4}$$

$$\beta_j=\frac{C_{L\alpha_j}\alpha_j}{\sum_{i=1}^{n} C_{L\alpha_i}\alpha_i}W \quad (j=2, 3, \cdots, n-1) \tag{5}$$

The difference angle of i-point can be written in the following form.

$$\Theta_i=\Theta_{0i}+\sum_{j=2}^{n-1}\beta_j\Theta_{ji} \tag{6}$$

The loading factor is $(\sum_{i=1}^{n} C_{L\alpha_i}\alpha_i)\rho v^2A/2W$, when the body is loaded by the aerodynamic load. When the angle of attack of the reference point is denoted by α_0, the angle of attack of i-point is given in the next equation.

$$\alpha_i-\alpha_0=\frac{\sum_{i=1}^{n} C_{L\alpha_i}\alpha_i}{W}\frac{1}{2}\rho v^2A\Theta_i \tag{7}$$

This equation can be rewritten in another form.

$$\alpha_i-\alpha_0=\frac{\sum_{i=1}^{n} C_{L\alpha_i}\alpha_i}{W}\frac{1}{2}\rho v^2A\left(\Theta_{0i}+\frac{W}{\sum_{i=1}^{n} C_{L\alpha_i}\alpha_i}\sum_{j=2}^{n-1} C_{L\alpha_j}\alpha_j\Theta_{ji}\right) \tag{8}$$

The divergence speed is obtained by the following condition, when the distance of i-point from the center of the gravity is written as x_i.

$$\sum_{i=1}^{n} L_i x_i=0 \tag{9}$$

or

$$\sum_{i=1}^{n} C_{L\alpha_i}\alpha_i x_i=0 \tag{10}$$

Eqs. (7) and (10) give the divergence speed.

a) Case of single-stage rocket ($n=2$).

In this case Θ_i is

$$\Theta_i=\Theta_{0i} \tag{11}$$

Eqs. (8) can be written in the following form.

$$\alpha_1-\alpha_0=\frac{\sum_{i=1}^{2} C_{L\alpha_i}\alpha_i}{W}\frac{1}{2}\rho v^2A\Theta_{01} \tag{12.1}$$

$$\alpha_2-\alpha_0=\frac{\sum_{i=1}^{2} C_{L\alpha_i}\alpha_i}{W}\frac{1}{2}\rho v^2A\Theta_{02} \tag{12.2}$$

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Eq. (12) may be written as

$$\left(\frac{C_{L\alpha 1}}{W}qA\Theta_{01}-1\right)\alpha_1+\frac{C_{L\alpha 2}}{W}qA\Theta_{01}\alpha_2=-\alpha_0 \tag{13.1}$$

$$\frac{C_{L\alpha 1}}{W}qA\Theta_{02}\alpha_1+\left(\frac{C_{L\alpha 2}}{W}qA\Theta_{02}-1\right)\alpha_2=-\alpha_0 \tag{13.2}$$

Eq. (10) becomes

$$C_{L\alpha_1}\alpha_1 x_1+C_{L\alpha_2}\alpha_2 x_2=0 \tag{14}$$

From Eqs. (13.1) and (13.2) α_1 and α_2 are obtained and Eq. (14) can be solved in terms of q. The divergence speed is given by the following equation

$$q=\frac{bC_{L\alpha_2}-aC_{L\alpha_1}}{C_{L\alpha_1}C_{L\alpha_2}}\,\frac{W}{A(a+b)}\,\frac{1}{\Theta} \tag{15}$$

where $a=x_1$, $b=-x_2$, $\Theta=\Theta_{01}-\Theta_{02}$

b) Case of two-stage rocket ($n=3$).

From Eq. (6)

$$\Theta_i=\Theta_{0i}+\beta_2\Theta_{2i} \tag{16}$$

$$\beta_2=\frac{C_{L\alpha_2}\alpha_2}{\sum_{i=1}^{3}C_{L\alpha_i}\alpha_i}W \tag{17}$$

Eq. (8) can be written in the following form.

$$\alpha_1-\alpha_0=\frac{\sum_{i=1}^{3}C_{L\alpha_i}\alpha_i}{W}\,\frac{1}{2}\rho v^2A \times\left(\Theta_{01}+\frac{W}{\sum_{i=1}^{3}C_{L\alpha_i}\alpha_i}C_{L\alpha_2}\alpha_2\Theta_{21}\right) \tag{18.1}$$

$$\alpha_2-\alpha_0=\frac{\sum_{i=1}^{3}C_{L\alpha_i}\alpha_i}{W}\,\frac{1}{2}\rho v^2A \times\left(\Theta_{02}+\frac{W}{\sum_{i=1}^{3}C_{L\alpha_i}\alpha_i}C_{L\alpha_2}\alpha_2\Theta_{22}\right) \tag{18.2}$$

$$\alpha_3-\alpha_0=\frac{\sum_{i=1}^{3}C_{L\alpha_i}\alpha_i}{W}\,\frac{1}{2}\rho v^2A \times\left(\Theta_{03}+\frac{W}{\sum_{i=1}^{3}C_{L\alpha_i}\alpha_i}C_{L\alpha_2}\alpha_2\Theta_{23}\right) \tag{18.3}$$

The values of α_1, α_2 and α_3 can be obtained from Eqs. (18.1), (18.2) and (18.3). The equation to determine the divergence speed is derived from Eq. (10). This equation is

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$$\left\{C_{L\alpha_1}C_{L\alpha_2}C_{L\alpha_3}\frac{1}{W}\begin{vmatrix}1, \Theta_{21}, \Theta_{01}\\ 1, \Theta_{22}, \Theta_{02}\\ 1, \Theta_{23}, \Theta_{03}\end{vmatrix}(x_1-x_3)\right\}(qA)^2$$
$$-\left\{C_{L\alpha_1}C_{L\alpha_2}\frac{1}{W}\begin{vmatrix}1, \Theta_{01}\\ 1, \Theta_{02}\end{vmatrix}(x_1-x_2)+C_{L\alpha_1}C_{L\alpha_3}\frac{1}{W}\begin{vmatrix}1, \Theta_{01}\\ 1, \Theta_{03}\end{vmatrix}(x_1-x_3)\right.$$
$$+C_{L\alpha_2}C_{L\alpha_3}\frac{1}{W}\begin{vmatrix}1, \Theta_{02}\\ 1, \Theta_{03}\end{vmatrix}(x_2-x_3)+C_{L\alpha_1}C_{L\alpha_2}\begin{vmatrix}1, \Theta_{21}\\ 1, \Theta_{22}\end{vmatrix}x_1$$
$$\left.+C_{L\alpha_2}C_{L\alpha_3}\begin{vmatrix}\Theta_{22}, 1\\ \Theta_{23}, 1\end{vmatrix}x_3\right\}(qA)+C_{L\alpha_1}x_1+C_{L\alpha_2}x_2+C_{L\alpha_3}x_3=0 \tag{19}$$

Eq. (19) is the second degree equation for q.

c) Case of multi-stage rocket.

When the rocket is N-stage, the number of points on which aerodynamic loads operate is $(N+1)$ and the equation for the divergence speed is the N-th degree. The value of $\alpha_1, \alpha_2, \cdots, \alpha_n$ can be obtained from Eq. (8). Eq. (8) is the system of linear equations for $\alpha_1, \alpha_2, \cdots, \alpha_n$. Using Cramer's formula, α_i can be expressed in the following equation

$$\alpha_i=D_i/D \tag{20}$$

where D is the determinant of the coefficients of Eq. (8). The determinant D can be written down in the following form.

$$D=\begin{pmatrix}
\frac{C_{L\alpha_1}}{W}qA\Theta_{01}-1, \frac{C_{L\alpha_2}}{W}qA\Theta_{01}+C_{L\alpha_2}qA\Theta_{21}, \cdots, \frac{C_{L\alpha_i}}{W}qA\Theta_{01}+C_{L\alpha_i}qA\Theta_{i1}, \cdots, \frac{C_{L\alpha_n}}{W}qA\Theta_{01}\\
\frac{C_{L\alpha_1}}{W}qA\Theta_{02}, \frac{C_{L\alpha_2}}{W}qA\Theta_{02}+C_{L\alpha_2}qA\Theta_{22}-1, \cdots, \frac{C_{L\alpha_i}}{W}qA\Theta_{02}+C_{L\alpha_i}qA\Theta_{i2}, \cdots, \frac{C_{L\alpha_n}}{W}qA\Theta_{02}\\
\frac{C_{L\alpha_1}}{W}qA\Theta_{03}, \frac{C_{L\alpha_2}}{W}qA\Theta_{03}+C_{L\alpha_2}qA\Theta_{23}, \cdots, \frac{C_{L\alpha_i}}{W}qA\Theta_{03}+C_{L\alpha_i}qA\Theta_{i3}, \cdots, \frac{C_{L\alpha_n}}{W}qA\Theta_{03}\\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots\\
\frac{C_{L\alpha_1}}{W}qA\Theta_{0i}, \frac{C_{L\alpha_2}}{W}qA\Theta_{0i}+C_{L\alpha_2}qA\Theta_{2i}, \cdots, \frac{C_{L\alpha_i}}{W}qA\Theta_{0i}+C_{L\alpha_i}qA\Theta_{ii}-1, \cdots, \frac{C_{L\alpha_n}}{W}qA\Theta_{0i}\\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots\\
\frac{C_{L\alpha_1}}{W}qA\Theta_{0n}, \frac{C_{L\alpha_2}}{W}qA\Theta_{0n}+C_{L\alpha_2}qA\Theta_{2n}, \cdots, \frac{C_{L\alpha_i}}{W}qA\Theta_{0n}+C_{L\alpha_i}qA\Theta_{in}, \cdots, \frac{C_{L\alpha_n}}{W}qA\Theta_{0n}-1
\end{pmatrix} \tag{21}$$

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The divergence speed can be obtained, when the value of α_i given by Eq. (20) are substituted into Eq. (10). For the N-stage rocket Eq. (10) is of the N-th degree.

3. Numerical Example

For example, let's derive the divergence speed of LS-A rocket which was fired by National Aerospace Laboratory. The aerodynamic characteristics and the bending stiffness of this rocket is necessary to get the divergence speed. These necessary values are calculated from the model wind-tunnel test results and the bending test results of the rocket which was carried out by National Aerospace Laboratory.

a) Details of the rocket.

The dimensional details of the rocket are shown in Fig.2. The weight of the rocket is 760 kg, when the fuel is full. The center of the gravity locates at 56% of the total length from the most forward point of the body.

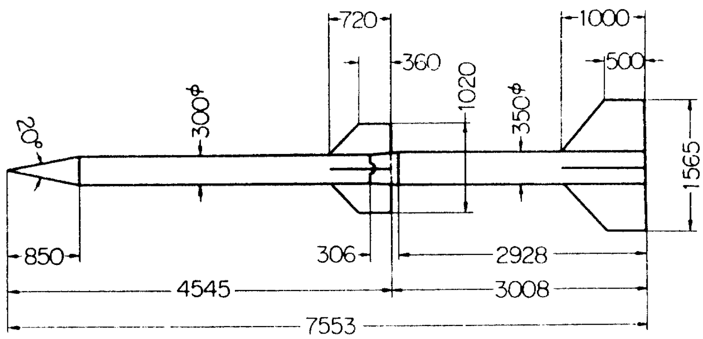

Fig. 2. Dimensional details of LS-A rocket

b) Aerodynamic characteristics.

The wind-tunnel tests were carried out using 1/5 and 1/7 scale models. The aerodynamic characteristics obtained are transformed into the convenient form which is necessary to calculate the divergence speed. That is to consider the lift operate on the nose and the fin of the main rocket and the fin of the booster, and to distribute the total lift to these points.

Table 1. Aerodynamic characteristics of LS-A rocket

	center of pressure (percent of the total lenght)	lift-curve slope (1/rad.)
total	73.5	33.7
component at the nose of the main rocket	2	2
component at the fin of the main rocket	57	13.5
component at the fin of the booster	95	18.2

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The aerodynamic characteristics are variable with Mach number. In this calculation we use the characteristics at $M=2$. This value is selected only because the necessary experimental data can be obtainable at $M=2$. The lift-curve slope and the position of the center of the pressure are shown in Table 1.

c) Bending stiffness.

The values of Θ_{01}, Θ_{02}, Θ_{03}, Θ_{21}, Θ_{22} and Θ_{23} are also necessary. The experiment to measure these values directly is most favorable. But we have only the bending stiffness of the body which was obtained by the bending test. So the necessary slopes of the deflection are calculated using the bending stiffness of the main rocket and the booster. The calculated values of Θ_{01}, Θ_{02} etc. are shown in Table 2.

TABLE 2. Slope of the deflection of the body (in radian)

Θ_{01}	9.88×10^{-3}	Θ_{21}	-1.28×10^{-5}
Θ_{02}	2.26×10^{-3}	Θ_{22}	-3.04×10^{-6}
Θ_{03}	0	Θ_{23}	0

(Reference is taken at the center of the pressure of the fin of the booster.)

d) Divergence speed.

The divergence speed is now calculated from Eq. (19). Substituting the above data, Eq. (19) becomes

$$5.45\times10^{-6}(qA)^2+3.28(qA)-4.66\times10^4=0 \tag{22}$$

The positive and the negative solution are obtained from this equation, but only the positive one has the meaning. This solution is

$$q=1.47\times10^{-1} \quad (\mathrm{kg/m^2}) \tag{23}$$

When ρ equals $1/8\,(\mathrm{kg\cdot s^2/m^4})$, the divergence speed is

$$v_D=1.53\times10^3 \quad (\mathrm{m/s}) \tag{24}$$

This speed is $M\fallingdotseq4.5$. So that it is not correct to use the aerodynamic characteristics at $M=2.0$. To get the more accurate result, this calculation should be repeated until Mach number of the assumed and the calculated value are the same.

4. CONCLUSIONS

The equations for calculating the divergence speed of rocket are obtained in this paper. The analysis is carried out in such a way that the total lift is assumed to be separated into components on several points of the body and the inertia force of the body is equated with these separated lifts. The necessary characteristic values to calculate the divergence speed for an actual rocket can be obtained by the wind-tunnel test of the model and the simple bending test of the body. So this method is

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very useful and simple to calulate the divergence speed.

Eq. (10) is of the N-th degree and theoretically N different divergence speeds are obtained, when the rocket is N-stage. Even if the negative value of q is meaningless, it is provable to get more than one divergence speed. In all the positive values of q the smallest one would be experienced in the actual flight. The other values of q could be considered to correspond to the higher modes of the deflection. These divergence speeds would give the boundary between the stable and the unstable region. The region below the smallest value of q is stable and the region between this value and the secondly smallest value of q would be unstable. If this unstable region is narrow, this smallest value of q would not have the actual meaning and the lager value of q would give the divergence speed in an actual flight. Nevertheless, it is on the safe side to take the smallest value of q.

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