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Lifting-Line Theory for Cascade of Blades in Subsonic Shear Flow

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Summary. A theory of lifting-lines for cascade of blades in subsonic shear flow is developed by assuming that the dependence of the local lift force upon the local upstream Mach number in shear flow is same as that in two-dimensional flow. By comparing theoretical results with experimental ones, the physical validity of the present theory is discussed.

The effect of compressibility upon the mean characteristics of a cascade is found to depend upon the root of harmonic mean square of upstream Mach number ($\bar{M}^*_{-\infty}$).

Experimental correlation is satisfactory as far as $\bar{M}^*_{-\infty}$ of the shear flow lies below a certain limit value, and this limit value corresponds to the two-dimensional Mach number beyond which the small disturbance theory for two-dimensional cascade shows poor validity. Significance of $\bar{M}^*_{-\infty}$ as a criterion for the mean characteristics is also confirmed experimentally.

SYMBOLS

a: shear parameter $=(\lambda/M_{-\infty})dM_{-\infty}/dy$
$c_{-\infty}$: speed of sound at far upstream station
C_l: local lift coefficient; Eq. (14)
C_L: lift coefficient; Eq. (63)
$\bar{C}_l$: mean local lift coefficient; Eq. (62)
$\bar{C}_L$: total lift coefficient; Eq. (65)
C_{Di}: total induced drag coefficient; Eq. (74)
c_p: specific heat at constant pressure
C_p: pressure coefficient on blade surface; Eq. (100)
$C_{p,cr}$: critical pressure coefficient
$C_{\Delta p}$: pressure-rise coefficient across cascade; Eq. (66)
$C_n(\alpha)$: function defined by Eq. (44)
$F_n(\alpha)$: coefficient in the series of the orthogonal functions
K: slope of lift coefficient curve in uniform flow; Eq. (37)
$M_{-\infty}$: upstream Mach number at far upstream station
M_1: upstream Mach number at inlet station (experimental value)
$\bar{M}_{-\infty}$: mean upstream Mach number; Eq. (64)
$\bar{M}^*_{-\infty}$: harmonic mean upstream Mach number; Eq. (50)

M_{cr}: critical Mach number
p: perturbed pressure from the arithmetic mean of far up- and downstream pressures
$p_{-\infty}$: pressure at far upstream station
Δp: pressure increase across cascade
Δq_{∞}: variation of axial velocity across cascade
$\overline{\Delta q'_{\infty}}$: mean value of Δq_{∞} in a streamsurface of $M_{-\infty}(y)$=constant; Eq. (79)
$\overline{\overline{\Delta q}}_{\infty}$: mean value of Δq_{∞} in the whole Trefftz plane; Eq. (80)
$S_n(\alpha)$: function defined by Eq. (45)
$s_{-\infty}$: entropy of far upstream flow
t: pitch chord ratio
u, v, w: x-, y- and z-component of disturbance velocity
U: velocity of far upstream flow
x, y, z: dimensionless rectangular coordinates defined in Fig. 1
Δy^*_{∞}: mean spanwise displacement of streamline
Y_n: eigenfunction for the boundary value problem of Eqs. (23) and (24)
α: variable in Fourier integral given by Eq. (30)
$\alpha_{-\infty}$: angle of attack with respect to far upstream flow direction measured from zero-lift angle
α_0: effective angle of attack with respect to vector mean of far up- and downstream velocities measured from zero-lift angle
α_g: geometrical angle of attack with respect to far upstream flow measured from direction of blade chord
β_n: eigenvalues for the boundary value problem given by Eqs. (23) and (24)
γ: stagger angle
$\bar{\theta}$: mean turning angle; Eq. (73)
κ: ratio of specific heats
λ: aspect ratio
$\mu = 1 - \overline{M}^{*2}_{-\infty}$
ρ: disturbance fluid density
$\rho_{-\infty}$: far upstream fluid density

Subscript

$-\infty$: value at far upstream station
∞: value at far downstruam station
i: value for incompressible flow
$*$: value at the uniform Mach number of $M^{(2D)}_{-\infty} = \overline{M}^*_{-\infty}$

Superscript

$(2D)$: value for two-dimensional (or uniform) flow
(0): value for $\alpha = 0$

1. Introduction

The three-dimensional flow in axial-flow turbomachinery has been usually treated by means of so-called strip theory approximation. As pointed out by Smith [*6*], however, in the case of incompressible flow the flow pattern predicted by the strip theory can be realized only when the peripheral fluid velocity distribution is everywhere that of a free vortex, i.e. the blade circulation is uniform along the blade span and the oncoming flow is everywhere irrotational. In general, however, the actual flow in turbomachines contains so-called secondary flow, because more general flow patterns departing from that of free vortex are usually adopted or because the wall boundary layers in the oncoming flow will destroy the irrotational condition, even if the design is intended to produce zero upstream vorticity. Moreover in recent years, axial flow compressors, especially of lift jet engines for V/STOL aircraft, have been designed to have a small boss ratio and operate at high subsonic or transonic Mach numbers, so as to make the thrust-weight ratio as high as possible. Then, not only the secondary flow in the above mentioned category has attained greater magnitude, but also that due to non-uniform entropy and nonuniform resultant Mach number along the span has become a matter of no little importance. Nevertheless, we have as yet very little information as to the characteristics of cascade in subsonic shear flow.

From an aerodynamical point of view, the subject of this paper belongs to the problem of the three-dimensional disturbances produced by obstacles in nonuniform flow, or in other words, in shear flow. This shear flow problem has been attacked theoretically along the two lines of approach, i.e. by means of the secondary flow theory, for example, by Squire and Winter [*2*], Hawthorne [*3*],[*4*] and Smith [*6*], and by means of the small disturbance theory by v. Kármán and Tsien [*1*], Lighthill [7] and Honda [*11*].

All the studies mentioned above, however, limit themselves to incompressible flow. Therefore only the effects of upstream vorticity and nonuniform blade circulation are dealt with. McCune [9] investigated the three-dimensional flow in compressor blade rows in subsonic, supersonic and transonic flow regimes. But the flow pattern to be treated by his method is confined only to a solid-rotational flow, and hence the method can not be applied directly to the shear flow with arbitrary nonuniformity. In his papers, moreover, only the problem of nonlifting blades is dealt with. In the case of zero blade circulation, however, the Mach number nonuniformity seems to play no important role in occurrence of the secondary flow, because the spanwise gradient of the upstream Mach number seems to contribute mainly to steepening the spanwise gradient of blade circulation. Therefore rather interesting and important problem of lifting blades has been left to research. Thus so far as is known, the present work is the first specifically designed to make clear the effect of compressibility upon three-dimensional disturbances due to lifting blades in subsonic shear flow.

Recently experiments were made by the authors [*13*] to investigate the characteristics of cascade of blades in high subsonic or transonic shear flows. In these works it was found that there is no essential discrepancy between the spanwise lift distribution predicted by the theory for incompressible shear flow and that obtained experimentally in high speed shear flow as far as the maximum Mach number is lower than about 1.1. Therefore it seems reasonable to extend the lifting-line theory for incompressible shear flow given by v. Kármán and Tsien [1] to the problem for the cascade of blades in subsonic shear flow as a first step to estimation of effect of compressibility.

From this aspect the lifting-line theory is developed in this paper for cascade of blades in subsonic shear flow. In this theory, the general expressions for three-dimensional small disturbances induced by an isolated lifting-line or cascade of lifting-lines in subsonic shear flow between two parallel walls are deduced in Fourier integral forms on the assumptions of inviscid, ideal gas and isentropic flow. Besides, extending the Prandtl's lifting-line method to subsonic shear flow, we have assumed that the characteristics of any blade element in subsonic shear flow can be associated with those of the cascade in the two-dimensional subsonic flow with an attack angle corrected by downwash and with an upstream Mach number corresponding to the local one in the shear flow.

The numerical calculations are worked out for several cascade conditions and the aerodynamic characteristics are studied theoretically. Some experimental works are also conducted, so as to evaluate the reasonableness of various simplifying assumptions in the theoretical analysis and to estimate the physical validity of analytically obtained results.

2. Basic Equations and Solution

2.1 *Linearized Equation of Compressible Shear Flow*

In the present analysis it is assumed that the fluid is a perfect gas of constant specific heat with no viscosity and zero thermal conductivity and that it flows in a steady three-dimensional pattern, which differs only by a small disturbance from the undisturbed two-dimensional parallel shear flow. Thus the entropy remains constant along each streamline.

Let a rectangular cartesian coordinate system with coordinates x, y, z be chosen so that the x-axis is parallel to the direction of the undisturbed stream, then this undisturbed shear flow can be specified by the velocity vector $(U(y), 0, 0)$. The velocity of the main flow $U(y)$ is a function of y alone in this case, because our interest lies mainly in the shear flow with the velocity varying in the direction of the blade span. If $\rho_{-\infty}$ and $p_{-\infty}$ are the density and the pressure of the undisturbed stream respectively, then $\rho_{-\infty}$ is a function of y alone and $p_{-\infty}$ is constant. Accordingly in the case of a perfect gas with constant adiabatic index κ the local speed of sound in the undisturbed stream is given by

$$c_{-\infty}=(\kappa p_{-\infty}/\rho_{-\infty})^{\frac{1}{2}} \tag{1}$$

as a function of y only, and hence the upstream Mach number which is given by

$$M_{-\infty}(y) = U(y)/c_{-\infty}(y) \tag{2}$$

is also a function of y alone.

Now let (u, v, w) be the disturbance velocity vector and let ρ and p be the disturbance fluid density and pressure respectively. Then, neglecting the higher order terms of u, v, w, p and ρ according to the assumption of small disturbance, we obtain the following linearized equations of continuity, motion and isentropic change:

$$\rho_{-\infty}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+U\frac{\partial \rho}{\partial x}+v\frac{d\rho_{-\infty}}{dy}=0, \tag{3}$$

$$\left.\begin{aligned} &\rho_{-\infty}\left(U\frac{\partial u}{\partial x}+v\frac{dU}{dy}\right)+\frac{\partial p}{\partial x}=0,\\ &\rho_{-\infty}U\frac{\partial v}{\partial x}+\frac{\partial p}{\partial y}=0,\\ &\rho_{-\infty}U\frac{\partial w}{\partial x}+\frac{\partial p}{\partial z}=0, \end{aligned}\right\} \tag{4}$$

$$U\frac{\partial p}{\partial x}-c_{-\infty}^2\left(U\frac{\partial \rho}{\partial x}+v\frac{d\rho_{-\infty}}{dy}\right)=0. \tag{5}$$

By eliminating u, v, w and ρ from Eqs. (3), (4) and (5), we can get an equation for a single dependent variable p as follows:

$$\{1-M_{-\infty}^2(y)\}\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}-\frac{2}{M_{-\infty}}\frac{dM_{-\infty}}{dy}\frac{\partial p}{\partial y}=0. \tag{6}$$

Before solving the above equation, something should be said of the significance of the fourth term on the left-hand side of Eq. (6). Using the relations from thermodynamics for the perfect gas and considering the constancy of $p_{-\infty}$, we ge the following relation:

$$\frac{2}{M_{-\infty}}\frac{dM_{-\infty}}{dy}=\frac{2}{U}\frac{dU}{dy}+\frac{1}{\rho_{-\infty}}\frac{d\rho_{-\infty}}{dy}=\frac{2}{U}\frac{dU}{dy}-\frac{d}{dy}\left(\frac{s_{-\infty}}{c_p}\right), \tag{7}$$

where $s_{-\infty}$ is the entropy of the fluid far upstream and c_p is the specific heat at constant pressure. Eq. (7) shows that the effects of both vorticity $(2/U)\,dU/dy$ and entropy gradient $-d(s_{-\infty}/c_p)/dy$ are always exercised as the coupled form of Eq. (7) and if there exists no entropy gradient, the factor $(2/M_{-\infty})dM_{-\infty}/dy$ is equivalent to the vorticity term. Thus it should be noticed that the effect of compressibility depends entirely upon the factor $1-\{M_{-\infty}(y)\}^2$ in the first term of Eq. (6) and not upon the factor $(2/M_{-\infty})dM_{-\infty}/dy$.

2.2 *Solution for an Isolated Lifting-Line*

Before dealing with the case of cascade illustrated in Fig. 1, we had better to solve the problem for an isolated blade which can be regarded as a special case of the cascade. In substitution for a single blade spanning between two walls parallel to (x, z) plane, a lifting-line with lift distribution of $l(y)$ is oriented along the y-axis from $y=0$ to $y=\lambda$, where all the coordinates x, y and z are made dimensionless by setting the chord length of the blade equal to unity. Therefore λ is itself equal to the aspect ratio of the blade.

Let the flow be subsonic everywhere, namely

$$M_{-\infty}(y)<1 \qquad \text{for} \qquad 0\leqq y\leqq \lambda, \tag{8}$$

then the disturbance pressure p must vanish at infinity, so that

$$p=0 \qquad \text{as } |x| \text{ or } |z|\rightarrow\infty. \tag{9}$$

Next, the condition that the velocity component normal to the walls vanishes at the wall surfaces can be written as

$$\partial p/\partial y=0 \qquad \text{at} \qquad y=0,\ \lambda. \tag{10}$$

The lifting-line corresponds to a filament of pressure dipole of strength $l(y)$ with axis in the z-direction, and hence p must satisfy the condition

$$\lim_{z\rightarrow\pm 0} p = \mp\frac{1}{2}l(y)\delta(x), \tag{11}$$

where $\delta(x)$ is the delta function of Dirac.

Now, in order that the local lift distribution $l(y)$ is related to the geometry of the given blade, we assume that the lift characteristics of the blade element at any given span-station y in the shear flow are similar to those in the two-dimensional flow with the main stream direction inclined by an angle equal to the downwash at the lifting-line and with the Mach number equal to the local one at the corresponding span-station in the shear flow. As is well-known, the lift coefficient $C_L^{(2D)}$ of a thin airfoil in two-dimensional flow can be given for small angle of attack in the form

$$C_L^{(2D)}=K(M_{-\infty}^{(2D)})\alpha_{-\infty}, \tag{12}$$

where K is the slope of the lift coefficient curve given by

$$K=\left(\frac{\partial C_L^{(2D)}}{\partial\alpha_{-\infty}}\right)_{\alpha_{-\infty}=0}, \tag{13}$$

which is a function of the free stream Mach number $M_{-\infty}^{(2D)}$, and $\alpha_{-\infty}$ is the angle of attack measured from the zero-lift angle. Then, on the assumption mentioned above, the local lift coefficient defined by

$$C_l(y)\equiv\frac{l(y)}{\frac{1}{2}\kappa p_{-\infty}M_{-\infty}^2(y)} \tag{14}$$

can be expressed for small angle of attack as follows:

$$C_l(y) = K[M_{-\infty}(y)]\left[\alpha_{-\infty} + \left[\frac{w}{U}\right]_{\substack{x=0\\z=0}}\right]. \tag{14'}$$

Here $\alpha_{-\infty}$ is the angle of attack relative to the undisturbed stream direction measured from the zero-lift angle, and $[w/U]_{x=z=0}$ is the upwash at the lifting-line. As is the case with the ordinary lifting-line theory for incompressible flow, the upwash $[w/U]_{x=z=0}$ is a function of y. It should be noted, however, that in the present case the slope of the lift coefficient K is also a function of y because the upstream Mach number $M_{-\infty}$ varies along the span.

In order to solve Eq. (6) under the boundary conditions given by Eqs. (9), (10) and (11), the method of separation of the variables can be successfully applied, which is adopted also by v. Kármán and Tsien [*1*], Lighthill [*7*] and Honda [*11*]. As is well-known, a bound vortex of strength of Γ in two-dimensional uniform flow is equivalent to a pressure dipole of strength $\rho U\Gamma$ with its axis normal to the main flow. Then the disturbance pressure field due to a pressure dipole at the origin can be given by the following Fourier integral:

$$\begin{aligned} p &= -\frac{\rho U\Gamma}{2\pi}\frac{\partial}{\partial z}\log\sqrt{x^2+z^2} \\ &= -\operatorname{sgn} z\,\frac{\rho U\Gamma}{2\pi}\int_0^\infty \cos\alpha x\cdot e^{-\alpha|z|}d\alpha. \end{aligned} \tag{15}$$

On the analogy of this case, we assume the required solution of Eq. (6) as

$$p = -\operatorname{sgn} z\int_0^\infty \cos\alpha x\; P(y,z;\alpha)d\alpha\,. \tag{16}$$

Then the equation to be satisfied by $P(y, z; \alpha)$ is

$$\frac{\partial^2 P}{\partial z^2} + \frac{\partial^2 P}{\partial y^2} - \frac{2}{M_{-\infty}}\frac{dM_{-\infty}}{dy}\frac{\partial P}{\partial y} - \{1-M_{-\infty}^2\}\alpha^2 P = 0, \tag{17}$$

and the conditions imposed on P are led from the boundary conditions (9) to (11) as follows:

$$\int_0^\infty |P(y,z;\alpha)|d\alpha < \infty, \tag{18}$$

$$P(y,z;\alpha)\to 0 \quad \text{as} \quad z\to\pm\infty, \tag{19}$$

$$P(y,z;\alpha)\to\frac{1}{2\pi}l(y) \quad \text{as} \quad z\to 0\,, \tag{20}$$

$$\partial P/\partial y = 0 \quad \text{at} \quad y=0,\ \lambda. \tag{21}$$

It should be noted that the condition (18) ensures the convergence of the Fourier integral given by Eq. (16) as well as vanishing of p at $x=\pm\infty$ from the Riemann-Lebesgue theorem. Furthermore, the condition (20) corresponds to the equation (11), because the delta function can be expressed in the following Fourier integral form:

$$\delta(x)=\frac{1}{\pi}\int_0^\infty \cos\alpha x\, d\alpha. \tag{22}$$

Applying, once more, the method of separation of the variables to P, we assume the elementary solution of Eq. (17) as

$$P=e^{\beta z}Y(y;\alpha).$$

Then Y must satisfy the following equation

$$\frac{d}{dy}\left(\frac{1}{M_{-\infty}^2}\frac{dY}{dy}\right)+\left(-\frac{1-M_{-\infty}^2}{M_{-\infty}^2}\alpha^2+\frac{1}{M_{-\infty}^2}\beta^2\right)Y=0, \tag{23}$$

together with the boundary condition

$$dY/dy=0 \quad \text{at} \quad y=0,\lambda. \tag{24}$$

To solve the ordinary differential equation (23) under the boundary condition (24) belongs to the well-known Sturm-Liouville type boundary value problem. Therefore the sequences of eigenvalues $\beta=\beta_n(\alpha)$ and eigenfunctions $Y=Y_n(y;\alpha)$ $(n=0, 1, 2, \ldots)$ exist for any given real values of α. Consequently the eigenvalues $\beta_n(\alpha)$ and the eigenfunctions $Y_n(y;\alpha)$ are continuous functions of α. The latters constitute a complete set of orthogonal functions in the range $0\leqq y\leqq\lambda$; namely

$$\int_0^\lambda \frac{Y_nY_m}{M_{-\infty}^2}dy=0 \quad (n\neq m). \tag{25}$$

As shown in Appendix I, the eigenvalues β_n^2 are always positive over $0<\alpha<\infty$, as far as the flow is subsonic everywhere. Therefore β_n are real numbers and can be appointed to be positive. Then if we express P in the following infinite series:

$$P(y,z;\alpha)=\sum_{n=0}^\infty e^{-\beta_n(\alpha)|z|}F_n(\alpha)Y_n(y;\alpha), \tag{26}$$

the condition (19) is satisfied. Furthermore the condition (20) reduces to

$$l(y)=2\pi\sum_{n=0}^\infty F_n(\alpha)Y_n(y;\alpha). \tag{27}$$

Therefore $F_n(\alpha)$ are given by

$$F_n(\alpha)=\frac{1}{2\pi}\frac{1}{\lambda}\int_0^\lambda l(y)\frac{Y_n(y;\alpha)}{M_{-\infty}^2}dy, \tag{28}$$

where Y_n are normalized by

$$\frac{1}{\lambda}\int_0^\lambda \frac{Y_n^2}{M_{-\infty}^2}dy = 1. \tag{29}$$

Finally, as demonstrated in Appendix II, P given by Eq. (26) satisfies the condition (18). Consequently we get the required solution of Eq. (6) in the form

$$p = -\operatorname{sgn} z \int_0^\infty \cos\alpha x \sum_{n=0}^\infty e^{-\beta_n(\alpha)|z|} F_n(\alpha) Y_n(y;\,\alpha) d\alpha\,. \tag{30}$$

Now we are in a position to determine $l(y)$ or in other words unknown coefficients $F_n(\alpha)$ in the series (27). For this purpose we must derive the expression of the upwash at the lifting-line, which is obtained by integrating the x-component of Eq. (4) in the form

$$\left[\frac{w}{U}\right]_{\substack{x=0\\z=0}} = -\frac{1}{\kappa p_{-\infty} M_{-\infty}^2}\int_{-\infty}^0 \left[\frac{\partial p}{\partial z}\right]_{z=0} dx. \tag{31}$$

As shown in Appendix III, this is represented by

$$\left[\frac{w}{U}\right]_{\substack{x=0\\z=0}} = -\frac{1}{2\kappa p_{-\infty} M_{-\infty}^2}\sum_{n=0}^\infty \beta_n(0) F_n(0) Y_n(y;\,0)\,. \tag{31'}$$

On the other hand from the fact that Eq. (27) holds for any real value of α, it follows that

$$l(y) = 2\pi \sum_{x=0}^\infty F_n(0) Y_n(y;\,0)\,. \tag{27'}$$

Substituting Eqs. (27′) and (31′) into Eq. (14′), we get the equation for determining $F_n(0)$ ($n=0, 1, 2, \cdots$) as follows;

$$\frac{4\pi}{\kappa p_{-\infty} M_{-\infty}^2(y)} \sum_{n=0}^\infty \left[1 + \frac{1}{8} K\{M_{-\infty}(y)\}\beta_n(0)\right] F_n(0) Y_n(y;\,0) = K\{M_{-\infty}(y)\}\alpha_{-\infty}\,. \tag{32}$$

In the case of incompressible shear flow, $K[M_{-\infty}(y)]$ becomes independent of y, because $M_{-\infty}(y)=0$ over the whole span. Then it is possible to obtain $F_n(0)$ in explicit analytical forms by using the orthogonality of $Y_n(y;\alpha)$. In the case of compressible shear flow, however, $K[M_{-\infty}(y)]$ is no longer independent of y. In such cases $F_n(0)$ can not be given in explicit mathematical expressions, and only approximate solution of Eq. (32) is obtainable by some numerical methods. The method adopted in this work for the numerical calculation consists in neglecting all coefficients $F_n(0)$ of order greater than some specific n ($n=10$ say). The remaining $F_n(0)$ ($n=0, 1, 2, \ldots$) are then determined without great difficulty by the condition that Eq. (32) be satisfied at the $n+1$ points (e.g. equally spaced) along the span, and therefore the problem reduces to solving a system of $n+1$ linear equations for $n+1$ unknowns $F_0(0)$, $F_1(0)$, $F_2(0)$, $\cdots$, $F_n(0)$.

2.3 *Solution for Cascade of Lifting-Lines*

Here, as is shown in Fig. 1, it is supposed that a linear cascade of blades is replaced by a cascade of lifting-lines which can be indicated by

$$x = mt\sin\gamma,\quad z = mt\cos\gamma,\quad 0 \leqq y \leqq \lambda,\quad m = 0, \pm 1, \pm 2, \cdots,$$

where t is the pitch-chord ratio and γ is the stagger angle. The x-axis is chosen to be parallel to the direction of the flow far upstream. For convenience let the disturbance pressure p be the perturbation from the arithmetical mean of the pressures far upstream and far downstream. Then the boundary conditions for the cascade are written as

$$p = \pm\frac{1}{2}\Delta p \qquad \text{at} \qquad x = \pm\infty, \tag{33}$$

$$\partial p/\partial y = 0 \qquad \text{at} \qquad y = 0, \lambda, \tag{34}$$

and

$$\lim_{\epsilon\to 0}\{[p]_{z=mt\cos\gamma-\epsilon} - [p]_{z=mt\cos\gamma+\epsilon}\}$$
$$= \delta(x - mt\sin\gamma)l(y), \quad m=0,\ \pm 1,\ \pm 2,\ \pm 3\cdots, \tag{35}$$

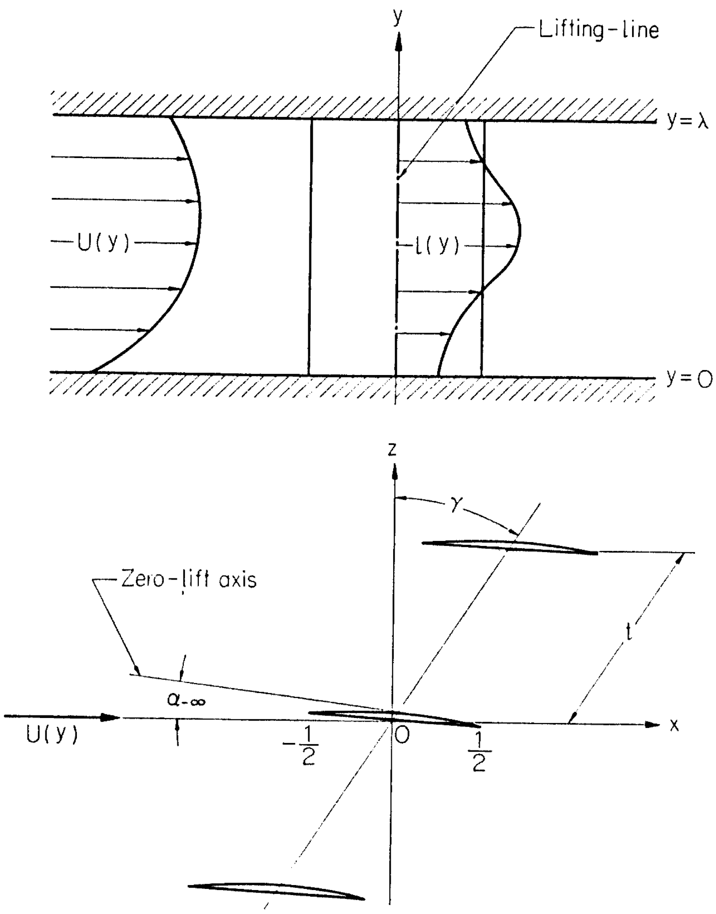

FIG. 1. Coordinate system

where Δp is a finite value corresponding to the increase in pressure across the cascade.

Then, on the same hypothesis as made in the case of an isolated blade, the local lift coefficient in the shear flow is given by

$$C_l(y) \equiv \frac{l(y)}{\frac{1}{2}\kappa p_{-\infty} M_{-\infty}^2} = K\{M_{-\infty}(y), \gamma, t\}\left(\alpha_{-\infty} + \left[\frac{w}{U}\right]_{\substack{x=mt\sin\gamma \\ z=mt\cos\gamma}}\right)$$

$$m=0, \pm 1, \pm 2, \pm 3, \cdots, \tag{36}$$

where $\alpha_{-\infty}$ is the angle between the x-axis and the zero-lift axis. It should be noticed that K in this case is defined by

$$K[M_{-\infty}^{(2D)}, \gamma, t] = [\partial C_L^{(2D)} / \partial \alpha_0^{(2D)}]_{\alpha_0^{(2D)}=0}, \tag{37}$$

where $C_L^{(2D)}$ is the lift coefficient in two-dimensional flow with the Mach number of $M_{-\infty}^{(2D)}$, and $\alpha_0^{(2D)}$ is the effective angle of attack, defined as the angle between the direction of the vector mean of the far upstream and far downstream flow and the zero-lift axis. Since the x-axis is chosen to be parallel to the direction of the far upstream flow, the downwash $[w/U]_{x=z=0}$ in the case of cascade has non-zero value even in uniform flow, in which $\alpha_{-\infty}+[w/U]_{x=z=0}$ reduces to $\alpha_0^{(2D)}$.

The solution for the cascade can be obtained by superposing the solution for an isolated lifting-line given by Eq. (30) in the form

$$p = -\sum_{m=-\infty}^{\infty} \operatorname{sgn}(z-mt\cos\gamma)$$
$$\times \int_0^{\infty} \cos\{\alpha(x-mt\sin\gamma)\} \sum_{n=0}^{\infty} e^{-\beta_n(\alpha)|z-mt\cos\gamma|} F_n(\alpha) Y_n(y;\alpha) d\alpha. \tag{38}$$

It should be noted that the expression (38) obtained merely from superposition of the solution for an isolated lifting-line neglects the effect of displacement of the trailing vortex sheets, that is to say, all the trailing vortex sheets are assumed to remain parallel to the (x, y) plane. Actually, however, the main flow is turned through a cascade and hence the trailing vortex sheets should run at an angle of the turning angle with the (x, y) plane. The present theory, however, is based on the assumption of the small disturbance. Therefore the turning angle is supposed to be so small that the effect of the turning of the vortex sheets upon the induced velocities should be negligible within the extent of approximation of the present theory.

It is evident that the periodicity allows us to confine the range for the expression of the flow field only to

$$-\frac{1}{2} t\cos\gamma \leqq z \lesseqgtr \frac{1}{2} t\cos\gamma, \tag{39}$$

without losing generality. Then after some calculations the following expressions are obtained from Eq. (38) for the range (39):

$$p = p_{\mathrm{I}} + p_{\mathrm{II}} + p_{\mathrm{III}}, \tag{40}$$

where

$$p_{\mathrm{I}} = -\operatorname{sgn} z \int_0^\infty \cos \alpha x \sum_{n=0}^{\infty} e^{-\beta_n(\alpha)|z|} F_n(\alpha) Y_n(y;\, \alpha) d\alpha\,, \tag{41}$$

$$p_{\mathrm{II}} = \int_0^\infty \cos \alpha x \sum_{n=0}^{\infty} C_n(\alpha) \sinh\{\beta_n(\alpha)z\} F_n(\alpha) Y_n(y;\, \alpha) d\alpha\,, \tag{42}$$

$$p_{\mathrm{III}} = \int_0^\infty \sin \alpha x \sum_{n=0}^{\infty} S_n(\alpha) \cosh\{\beta_n(\alpha)z\} F_n(\alpha) Y_n(y;\, \alpha) d\alpha\,, \tag{43}$$

and

$$C_n(\alpha) = \frac{\sinh\{\beta_n(\alpha)t \cos\gamma\}}{\cosh\{\beta_n(\alpha)t\cos\gamma\} - \cos(\alpha t \sin\gamma)} - 1\,, \tag{44}$$

$$S_n(\alpha) = \frac{\sin(\alpha t \sin\gamma)}{\cosh\{\beta_n(\alpha)t\cos\gamma\} - \cos(\alpha t \sin\gamma)}\,. \tag{45}$$

Obviously p_{I} denotes the contribution of the zero-th lifting-line itself, and p_{II} and p_{III} are associated with the additional influences due to all the other lifting-lines. Therefore p_{I} satisfies the following relation:

$$\lim_{z \to \pm 0} p_{\mathrm{I}} = \mp \frac{1}{2}\, l(y)\delta(x), \tag{46}$$

while p_{II} and p_{III} have no singularity in the range (39).

Now we shall proceed to get the expression of the pressure increase Δp across the cascade. From the results for an isolated lifting-line, we have

$$\lim_{x \to \pm\infty} p_{\mathrm{I}} = 0\,. \tag{47}$$

On the other hand it is found from Appendix IV that

$$\beta_n(\alpha) \to \begin{cases} \beta_n(0) > 0 & (n = 1, 2, 3, \cdots), \\ \mu^{\frac{1}{2}}\alpha \to 0 & (n = 0) \end{cases} \quad \text{as} \quad \alpha \to 0\,, \tag{48}$$

where

$$\mu = 1 - \overline{M}{}^{*2}_{-\infty}\,, \tag{49}$$

$$\overline{M}{}^{*}_{-\infty} = \left(\frac{1}{\lambda} \int_0^\lambda \frac{1}{M_{-\infty}^{\,2}} dy \right)^{-1/2}. \tag{50}$$

We shall call $\overline{M}{}^{*}_{-\infty}$ as the harmonic mean Mach number, which will prove of great significance in relation to the mean characteristics of the cascade in compressible shear flow.

By using the relation (48), we have

$$\lim_{\alpha\to 0} C_n(\alpha)\sinh\{\beta_n(\alpha)z\} = \begin{cases} \left[\coth\left\{\frac{1}{2}\beta_n(0)t\cos\gamma\right\} - 1\right]\sinh\{\beta_n(0)z\} & (n=1,2,3,\cdots), \\ \frac{2\mu\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{z}{t} & (n=0) \end{cases} \tag{51}$$

$$\lim_{\alpha\to 0} \alpha S_n(\alpha)\cosh\{\beta_n(\alpha)z\} = \begin{cases} 0 & (n=1,2,3,\cdots) \\ \frac{2\sin\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t} & (n=0). \end{cases} \tag{52}$$

From these limiting behaviours, we have

$$\lim_{x\to\pm\infty} p_{\text{II}} = 0, \tag{53}$$

$$\lim_{x\to\pm\infty} p_{\text{III}} = \pm\frac{\pi}{2}\frac{2\sin\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}F_0(0)Y_0(y;0). \tag{54}$$

Therefore the pressure increase across the cascade Δp can be given by

$$\Delta p = \frac{2\pi\sin\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}F_0(0)Y_0(y;0). \tag{55}$$

Here it should be noted that the zero-th eigenfunction $Y_0(y;0)$ becomes constant when $\alpha=0$ and besides from the normalization (29) we have

$$Y_0(y;0) = \overline{M}^*_{-\infty}. \tag{56}$$

Therefore Δp given by Eq. (55) is independent of y, which is consistent with the flow conditions far up- and downstream.

We are now in a position to get the expression of the upwash distribution on the lifting-lines, which can be obtained by substituting Eq. (40) into Eq. (31). Obviously the contribution of p_{I} to the downwash is given by Eq. (31′), while that of p_{II} becomes zero on the lifting-lines. On the other hand the contribution of p_{III} is given by

$$\begin{aligned} \left[\int_{-\infty}^{0}\frac{\partial p_{\text{II}}}{\partial z}dx\right]_{z=0} &= \lim_{\substack{z\to 0\\ x\to\infty}}\int_0^\infty \frac{\sin\alpha x}{\alpha}\sum_{n=0}^\infty C_n(\alpha)\beta_n(\alpha)\cosh\{\beta_n(\alpha)z\}F_n(\alpha)Y_n(y;\alpha)d\alpha \\ &= \frac{\pi}{2}\left[\frac{2\mu\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}F_0(0)Y_0(y;0)\right. \\ &\quad \left.+\sum_{n=1}^\infty \beta_n(0)\left[\coth\left\{\frac{1}{2}\beta_n(0)t\cos\gamma\right\}-1\right]F_n(0)Y_n(y;0)\right], \end{aligned} \tag{57}$$

where the following relations are taken into considerations:

$$\lim_{\alpha\to 0} C_n(\alpha)\beta_n(\alpha)\cosh\{\beta_n(\alpha)z\} = \begin{cases} \beta_n(0)\left[\coth\left\{\frac{1}{2}\beta_n(0)t\cos\gamma\right\} - 1\right]\cosh\{\beta_n(0)z\} & (n=1,2,3,\cdots) \\ \frac{2\mu\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t} & (n=0). \end{cases} \tag{58}$$

Consequently we obtain the upwash distribution in the form

$$\left[\frac{w}{U}\right]_{\substack{x=0\\z=0}}=-\frac{\pi}{\kappa p_{-\infty}M_{-\infty}^2}\left[\frac{\mu\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}F_0(0)Y_0(y;0)\right.$$
$$\left.+\frac{1}{2}\sum_{n=1}^{\infty}\beta_n(0)\coth\left\{\frac{1}{2}\beta_n(0)t\cos\gamma\right\}F_n(0)Y_n(y;0)\right]. \tag{59}$$

Finally we can get from Eqs. (27′), (36) and (59) the equation for determining $F_n(0)$ for cascade as follows:

$$\left[1+K\{M_{-\infty}(y),\gamma,t\}\frac{\mu\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{4t}\right]F_0(0)Y_0(y;0)$$
$$+\sum_{n=1}^{\infty}\left[1+\frac{1}{8}K\{M_{-\infty}(y),\gamma,t\}\beta_n(0)\coth\left\{\frac{1}{2}\beta_n(0)t\cos\gamma\right\}\right]F_n(0)Y_n(y;0)$$
$$=\frac{1}{4\pi}\kappa p_{-\infty}\{M_{-\infty}(y)\}^2K\{M_{-\infty}(y),\gamma,t\}\alpha_{-\infty}. \tag{60}$$

From Eq. (60) the unknown coefficients $F_n(0)$ can be determined numerically by the same method as mentioned previously in connection with Eq. (32) for an isolated lifting-line. It is clear that Eq. (60) reduces to Eq. (32) when t tends to infinity. Therefore the discussions for cascade to be made hereafter include the case of an isolated lifting-line as a special case.

3. Expressions of Aerodynamic Forces and Flow Characteristics

Once the coefficients $F_n(0)$ are determined from Eq. (60), we can calculate the aerodynamic forces and the characteristics of flow field with the expressions which are to be derived in this section. For shortness $Y_n(y;0)$, $\beta_n(0)$ and $F_n(0)$ are hereafter written as $Y_n^{(0)}$, $\beta_n^{(0)}$ and $F_n^{(0)}$ respectively.

The local lift coefficient defined by Eq. (14) as a ratio of local lift force per unit span to the upstream local dynamic pressure, is given by

$$C_l(y)=\bar{C}_l\left(\frac{\bar{M}^*_{-\infty}}{M_{-\infty}(y)}\right)^2\left(1+\sum_{n=1}^{\infty}\frac{F_n^{(0)}}{F_0^{(0)}}\frac{Y_n^{(0)}(y)}{Y_0^{(0)}}\right). \tag{61}$$

Here $\bar{C}_l$ is the mean local lift coefficient given by

$$\bar{C}_l\equiv\frac{1}{\lambda}\int_0^{\lambda}C_l(y)dy=\frac{4\pi F_0^{(0)}Y_0^{(0)}}{\kappa p_{-\infty}\bar{M}^{*2}_{-\infty}}. \tag{62}$$

On the other hand, if we define the lift coefficient C_L as a ratio of local lift force per unit span to the upstream mean dynamic pressure, then it can be written as

$$C_L\equiv\frac{l(y)}{\frac{1}{2}\kappa p_{-\infty}\bar{M}_{-\infty}^2}=\bar{C}_l\left(\frac{\bar{M}^*_{-\infty}}{\bar{M}_{-\infty}}\right)^2\left(1+\sum_{n=1}^{\infty}\frac{F_n^{(0)}}{F_0^{(0)}}\frac{Y_n^{(0)}(y)}{Y_0^{(0)}}\right), \tag{63}$$

where $\bar{M}_{-\infty}$ is defined by

$$\bar{M}_{-\infty}\equiv\left(\frac{1}{\lambda}\int_0^\lambda M_{-\infty}^2 dy\right)^{1/2}, \tag{64}$$

which is hereafter referred to as the mean upstream Mach number in contrast to the harmonic mean Mach number $\bar{M}^*_{-\infty}$ defined by Eq. (50).

Integration of Eq. (63) with respect to y yields the total lift coefficient as follows:

$$\bar{C}_L\equiv\frac{1}{\lambda}\int_0^\lambda C_L dy=\bar{C}_l\left(\frac{\bar{M}^*_{-\infty}}{\bar{M}_{-\infty}}\right)^2\left(1+\sum_{n=1}^{\infty}\frac{F_n^{(0)}}{F_0^{(0)}}\frac{1}{\lambda}\int_0^\lambda\frac{Y_n^{(0)}}{Y_0^{(0)}}dy\right). \tag{65}$$

One of the most important aerodynamic characteristics for a cascade is the increase in pressure across the cascade, which has been already given by Eq. (55). Then the static pressure-rise coefficient $C_{\Delta p}$ can be expressed as

$$C_{\Delta p}\equiv\frac{\Delta p}{\frac{1}{2}\kappa p_{-\infty}\bar{M}_{-\infty}^2}=\bar{C}_l\left(\frac{\bar{M}^*_{-\infty}}{\bar{M}_{-\infty}}\right)^2\frac{\sin\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}. \tag{66}$$

The effective angle of attack defined by

$$\alpha_0\equiv\alpha_{-\infty}+\left[\frac{w}{U}\right]_{\substack{x=0\\z=0}}, \tag{67}$$

means the angle of attack with respect to the flow direction at the lifting-line, which is a function of y due to nonuniform upwash distribution. The upwash given by Eq. (59) can be rewritten as

$$\left[\frac{w}{U}\right]_{\substack{x=0\\z=0}}=-\bar{C}_l\left(\frac{\bar{M}^*_{-\infty}}{M_{-\infty}(y)}\right)^2\frac{1}{4}\left[\frac{\mu\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}\right.$$
$$\left.+\frac{1}{2}\sum_{n=1}^{\infty}\beta_n^{(0)}\coth\left\{\frac{1}{2}\beta_n^{(0)}t\cos\gamma\right\}\frac{F_n^{(0)}}{F_0^{(0)}}\frac{Y_n^{(0)}}{Y_0^{(0)}}\right]. \tag{59'}$$

The mean effective angle of attack is given by

$$\bar{\alpha}_0\equiv\frac{1}{\lambda}\int_0^\lambda\alpha_0\,dy=\alpha_{-\infty}-\bar{C}_l\frac{\mu\cos\gamma}{u\cos^2\gamma+\sin^2\gamma}\frac{1}{4t}. \tag{68}$$

Next we proceed to obtain the expressions of the flow characteristics in the Trefftz plane. The z-component of the induced velocity is obtained by integrating the z-component of the equation of motion (4) as follows:

$$\frac{w}{U}=-\frac{1}{\kappa p_{-\infty}M_{-\infty}^2}\int_{-\infty}^{x}\frac{\partial p}{\partial z}dx$$
$$=\frac{1}{2}\frac{w_\infty}{U}-\frac{1}{\kappa p_{-\infty}M_{-\infty}^2}\int_0^\infty\left[\frac{\sin\alpha x}{\alpha}\sum_{n=0}^{\infty}\left\{e^{-\beta_n(\alpha)|z|}+C_n(\alpha)\cosh(\beta_n(\alpha)z)\right\}\right.$$
$$\left.-\cos\alpha x\sum_{n=0}^{\infty}\frac{S_n(\alpha)}{\alpha}\sinh(\beta_n(\alpha)z)\right]\beta_n(\alpha)F_n(\alpha)Y_n(y;\alpha)d\alpha, \tag{69}$$

where w/U at far downstream-station is given by

$$\frac{w_\infty}{U} = -\bar{C}_l\left(\frac{\bar{M}^*_{-\infty}}{M_{-\infty}(y)}\right)^2 \frac{1}{2}\left[\frac{u\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t} + \frac{1}{2}\sum_{n=1}^{\infty}\beta_n^{(0)}\left[e^{-\beta_n^{(0)}|z|} + \left\{\coth\left(\frac{1}{2}\beta_n^{(0)}t\cos\gamma\right)-1\right\}\cosh(\beta_n^{(0)}z)\right]\frac{F_n^{(0)}}{F_0^{(0)}}\frac{Y_n^{(0)}(y)}{Y_0^{(0)}}\right],$$

$$\left(|z|\leqq\frac{1}{2}t\cos\gamma\right). \tag{70}$$

Similarly we can get the y-component of the induced velocity in the form

$$\begin{aligned}\frac{v}{U} &= -\frac{1}{\kappa p_{-\infty}M_{-\infty}^2}\int_{-\infty}^{x}\frac{\partial p}{\partial y}dx \\ &= \frac{1}{2}\frac{v_\infty}{U} \\ &+ \frac{1}{\kappa p_{-\infty}M_{-\infty}^2}\int_0^\infty\sum_{n=0}^{\infty}\left[\frac{\sin\alpha x}{\alpha}\left\{\operatorname{sgn} z\cdot e^{-\beta_n(\alpha)|z|}-C_n(\alpha)\sinh(\beta_n(\alpha)z)\right\}\right. \\ &\left.+\cos\alpha x\cdot\frac{S_n(\alpha)}{\alpha}\cosh(\beta_n(\alpha)z)\right]F_n(\alpha)\frac{dY_n(y;\alpha)}{dy}d\alpha,\end{aligned} \tag{71}$$

and

$$\frac{v_\infty}{U} = \operatorname{sgn} z\,\bar{C}_l\left(\frac{\bar{M}^*_{-\infty}}{M_{-\infty}(y)}\right)^2\frac{1}{4}\sum_{n=1}^{\infty}\frac{\sinh\left\{\beta_n^{(0)}\left(\frac{1}{2}t\cos\gamma-|z|\right)\right\}}{\sinh\left\{\frac{1}{2}\beta_n^{(0)}t\cos\gamma\right\}}$$

$$\times\frac{F_n^{(0)}}{F_0^{(0)}}\frac{d}{dy}\left(\frac{Y_n^{(0)}}{Y_0^{(0)}}\right),\qquad\left(|z|\leqq\frac{1}{2}t\cos\gamma\right). \tag{72}$$

It can easily be found that as is the case with incompressible shear flow the induced velocities in the cascade plane are just one-half of those in the Trefftz plane.

The turning angle can be expressed by $-w_\infty/U$ in the coordinate system adopted here. In the shear flow, however, $-w_\infty/U$ varies from one streamline to another because of the permanently remaining secondary flow. If we define the mean turning and $\bar{\theta}$ as

$$\bar{\theta}\equiv\frac{1}{\lambda t\cos\gamma}\int_0^\lambda dy\int_{-(1/2)t\cos\gamma}^{(1/2)t\cos\gamma}\left(-\frac{w_\infty}{U}\right)dz, \tag{73}$$

then we obtain by using Eq. (70)

$$\bar{\theta}=\bar{C}_l\frac{\mu\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{2t}. \tag{73'}$$

Now let us adopt as a reference direction the direction of the spanwisely averaged velocity vector of the stream at the lifting-line and define the induced drag as an aerodynamic force component parallel to that direction, then the total induced drag coefficient can be written as

$$C_{Di}\equiv\int_0^{\lambda}\{\bar{\alpha}_0-\alpha_0(y)\}\,l(y)dy\Big/\left(\frac{1}{2}\kappa p_{-\infty}\int_0^{\lambda}M_{-\infty}^2dy\right), \tag{74}$$

and using Eqs. (67), (59′), (68) and (27′), we finally obtain

$$C_{Di}=\frac{1}{8}\bar{C}_l^2\left(\frac{\bar{M}^*_{-\infty}}{\bar{M}_{-\infty}}\right)^2\left[\sum_{n=1}^{\infty}\beta_n^{(0)}\coth\left(\frac{1}{2}\beta_n^{(0)}t\cos\gamma\right)\left(\frac{F_n^{(0)}}{F_0^{(0)}}\right)^2\right.$$
$$\left.-\frac{2\mu\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}\sum_{n=1}^{\infty}\frac{F_n^{(0)}}{F_0^{(0)}}\frac{1}{\lambda}\int_0^{\lambda}\frac{Y_n^{(0)}}{Y_0^{(0)}}dy\right], \tag{74′}$$

We are now to consider the variation of the axial velocity Δq_∞ which is given by

$$\Delta q_\infty=u_\infty\cos\gamma-w_\infty\sin\gamma. \tag{75}$$

The x-component of the induced velocity u_∞ in the Trefftz plane can be obtained by integrating the x-component of the equation of motion (4) in the form

$$\frac{u_\infty}{U}=-\frac{1}{\kappa p_{-\infty}M_{-\infty}^2}\Delta p-\frac{1}{U}\frac{dU}{dy}\int_{-\infty}^{\infty}\frac{v}{U}dx. \tag{76}$$

The integral in the second term on the right-hand side of Eq. (76) denotes the spanwise displacement of the streamline in the Trefftz plane. This integral, however, doesn't converge due to the presence of the secondary flow. Let us consider, however, the finite part of the integral and indicate it by Δy^*_∞, then it means the mean spanwise displacement of the streamline in the Trefftz plane, and only the part of $p_{\rm III}$ in Eq. (40) is associated with it. Hence,

$$\Delta y^*_\infty=-\frac{1}{\kappa p_{-\infty}M_{-\infty}^2}\lim_{x'\to\infty}\int_{-z\tan\gamma-x'}^{z\tan\gamma+x'}dx\int_{-\infty}^{x}\frac{\partial p_{\rm III}}{\partial y}dx. \tag{77}$$

Substitution of Eq. (43) into Eq. (77) yields

$$\Delta y^*_\infty=\bar{C}_l\left(\frac{\bar{M}^*_{-\infty}}{M_{-\infty}(y)}\right)^2\frac{1}{4}\left[\frac{2\sin\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}\bar{M}^{*2}_{-\infty}\frac{d}{dy}\frac{Y_0^{(1)}}{Y_0^{(0)3}}\right.$$
$$\left.+\tan\gamma\sum_{n=1}^{\infty}\varphi_n(z)\frac{F_n^{(0)}}{F_0^{(0)}}\frac{d}{dy}\frac{Y_n^{(0)}}{Y_0^{(0)}}\right], \tag{77′}$$

where

$$\varphi_n(z)=\frac{|z|\cosh\{\beta_n^{(0)}(t\cos\gamma-|z|)\}+(t\cos\gamma-|z|)\cosh\{\beta_n^{(0)}z\}}{\cosh(\beta_n^{(0)}t\cos\gamma)-1},$$

$$\frac{Y_0^{(1)}}{Y_0^{(0)3}}=\sum_{n=1}^{\infty}\frac{1}{\beta_n^{(0)2}}\frac{Y_n^{(0)}(y)}{Y_0^{(0)}}\frac{1}{\lambda}\int_0^{\lambda}\frac{Y_n^{(0)}}{Y_0^{(0)}}dy,\quad\left(|z|\leqq\frac{1}{2}t\cos\gamma\right).$$

Therefore from Eqs. (75), (76) and (77′) we obtain Δq_∞ in the form

$$\frac{\Delta q_\infty}{U}=-\left(\frac{1}{\kappa p_{-\infty}M_{-\infty}^2}\Delta p+\frac{1}{U}\frac{dU}{dy}\Delta y_\infty^*\right)\cos\gamma-\frac{w_\infty}{U}\sin\gamma$$

$$=-\bar{C}_l\frac{1}{4}\left(\frac{\bar{M}_{-\infty}^*}{M_{-\infty}(y)}\right)^2\sin\gamma\left[\frac{2\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}\bar{M}_{-\infty}^{*2}\left(1+\frac{1}{U}\frac{dU}{dy}\frac{d}{dy}\frac{Y_0^{(1)}(y)}{Y_0^{(0)3}}\right)\right.$$

$$-\sum_{n=1}^{\infty}\frac{F_n^{(0)}}{F_0^{(0)}}\left\{\frac{\cosh\left\{\beta_n^{(0)}\left(\frac{1}{2}t\cos\gamma-|z|\right)\right\}}{\sinh\left(\frac{1}{2}\beta_n^{(0)}t\cos\gamma\right)}\beta_n^{(0)}\frac{Y_n^{(0)}(y)}{Y_0^{(0)}}\right.$$

$$\left.\left.-\varphi_n(z)\frac{1}{U}\frac{dU}{dy}\frac{d}{dy}\frac{Y_n^{(0)}(y)}{Y_0^{(0)}}\right\}\right]. \tag{78}$$

The first term in the square bracket [ ] on the right-hand side of Eq. (78) vanishes in the case of incompressible flow where $\bar{M}_{-\infty}^{*2}$ becomes zero.

Let $\overline{\Delta q'_\infty}$ be the mean variation of the axial velocity in the streamsurfaces of constant upstream Mach number, then it is given by

$$\frac{\overline{\Delta q'_\infty}}{U}\equiv\frac{1}{t\cos\gamma}\int_{-(t/2)\cos\gamma}^{(t/2)\cos\gamma}\frac{\Delta q_\infty}{U}dz$$

$$=-\bar{C}_l\frac{1}{2}\left(\frac{\bar{M}_{-\infty}^*}{M_{-\infty}(y)}\right)^2\sin\gamma\left[\frac{\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}\bar{M}_{-\infty}^{*2}\left(1+\frac{1}{U}\frac{dU}{dy}\frac{d}{dy}\frac{Y_0^{(1)}(y)}{Y_0^{(0)3}}\right)\right.$$

$$\left.-\frac{1}{t\cos\gamma}\sum_{n=1}^{\infty}\frac{F_n^{(0)}}{F_0^{(0)}}\left\{\frac{Y_n^{(0)}}{Y_0^{(0)}}-\frac{1}{\beta_n^{(0)2}}\frac{1}{U}\frac{dU}{dy}\frac{d}{dy}\frac{Y_n^{(0)}(y)}{Y_0^{(0)}}\right\}\right]. \tag{79}$$

When the upstream entropy is uniform, we have from Eq. (7)

$$\frac{1}{U}\frac{dU}{dy}=\frac{1}{M_{-\infty}}\frac{dM_{-\infty}}{dy}$$

and as shown by Honda[*11*] the following relation holds:

$$\int_0^\lambda\left(\frac{Y_n^{(0)}}{M_{-\infty}}-\frac{1}{\beta_n^{(0)2}}\frac{1}{M_{-\infty}^2}\frac{dM_{-\infty}}{dy}\frac{dY_n^{(0)}}{dy}\right)dy=0.$$

Then by integrating Eq. (79) and using the above relation we obtain the total variation of the axial velocity $\overline{\overline{\Delta q}}_\infty$ in the form

$$\overline{\overline{\Delta q}}_\infty\equiv\frac{1}{\lambda}\int_0^\lambda\overline{\Delta q'_\infty}dy$$

$$=-\bar{C}_l\frac{1}{2}\bar{M}_{-\infty}^*\left(\frac{\kappa p_{-\infty}}{\rho_{-\infty}}\right)^{\frac{1}{2}}\frac{\sin\gamma\cdot\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}\bar{M}_{-\infty}^{*2}E, \tag{80}$$

where

$$E=\frac{1}{\lambda}\int_0^\lambda\frac{\bar{M}_{-\infty}^*}{M_{-\infty}}dy+\sum_{n=1}^{\infty}\frac{1}{\lambda}\int_0^\lambda\frac{Y_n^{(0)}}{Y_0^{(0)}}dy\frac{1}{\lambda}\int_0^\lambda\frac{\bar{M}_{-\infty}^*}{M_{-\infty}}\frac{Y_n^{(0)}}{Y_0^{(0)}}dy. \tag{81}$$

It is evident that $\overline{\overline{\Delta q}}_\infty$ becomes zero in the case of incompressible flow because $\bar{M}_{-\infty}^{*2}=0$.

Now on the analogy of Δy^*_∞, if we define Δz^*_∞ as

$$\Delta z^*_\infty = \text{f.p.} \int_{-\infty}^{\infty} \frac{w}{U} dx, \tag{82}$$

where f.p. denotes the finite part of the integral, then Δz^*_∞ gives the mean vertical displacement of a streamline relative to the trailing vortex sheets. As is the case with Δy^*_∞, the finite part of the integral (82) can be obtained from the part of p_{III}. Therefore,

$$\begin{aligned}\Delta z^*_\infty &= -\frac{1}{\kappa p_{-\infty} M_{-\infty}^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} \frac{\partial p_{\text{III}}}{\partial z} dx \\ &= -\operatorname{sgn} z\, \overline{C}_l \left(\frac{\overline{M}^*_{-\infty}}{M_{-\infty}(y)} \right)^2 \frac{1}{4} \tan \gamma \sum_{n=1}^{\infty} \phi_n(z) \beta_n^{(0)} \frac{F_n^{(0)}}{F_0^{(0)}} \frac{Y_n^{(0)}}{Y_0^{(0)}}, \end{aligned} \tag{82'}$$

where

$$\phi_n(z) = \frac{|z| \sinh \{\beta_n^{(0)}(t \cos \gamma - |z|)\} - (t \cos \gamma - |z|) \sinh (\beta_n^{(0)} |z|)}{\cosh(\beta_n^{(0)} t \cos \gamma) - 1}.$$

4. Discussion

4.1 *Factors Influenced by Mach Number*

At first, a remark should be made upon the reason why the flow features in the Trefftz plane or in the cascade plane as well as the aerodynamic forces acting on the lifting-lines can be all expressed with the values of $Y_n(y;\alpha)$, $F_n(\alpha)$ or $\beta_n(\alpha)$ for $\alpha=0$ only, namely with $Y_n^{(0)}$, $F_n^{(0)}$ or $\beta_n^{(0)}$. Eq. (23) indicates that $Y_n(y;\alpha)$ and $\beta_n(\alpha)$ are independent of the factor $1-M_{-\infty}^2$ when $\alpha=0$. On the other hand we can see from Eq. (16) that $\alpha=0$ means that the wave length of the disturbance is infinitive. If we assume that the flow field becomes uniform in the x-direction downstream at infinity, the first term on the right-hand side of Eq. (6) vanishes as $x \to \infty$ because $\partial^2 p/\partial x^2$ becomes zero. Therefore the flow field in the Trefftz plane is independent of the factor $1-M_{-\infty}^2$ in Eq. (6). Owing to the fact that as pointed out previously the induced velocities in the cascade plane are just one-half of those in the Trefftz plane, the flow field in the cascade plane is also independent of the factor $1-M_{-\infty}^2$ in Eq. (6). This is the reason why the expressions for the flow field in the Trefftz plane and in the cascade plane and the aerodynamic forces can be given with the values for $\alpha=0$.

This in no ways means, however, that there exists no effect of compressibility on the induced velocities or the aerodynamic forces, since the coefficients $F_n(0)$ are subject to the compressibility effect through the condition (14′) or (36).

We have hitherto obtained several expressions for the aerodynamic characteristics of cascade in subsonic shear flow, which are all arranged in such forms as are in proportion to the mean local lift coefficient $\overline{C}_l$. The remaining terms multiplied by $\overline{C}_l$ include the several factors, such as $\overline{M}^*_{-\infty}/M_{-\infty}$, $\overline{M}^*_{-\infty}/\overline{M}_{-\infty}$, $Y_n^{(0)}/Y_0^{(0)}$,

$F_n^{(0)}/F_0^{(0)}$, $\beta_n^{(0)}$ and $\overline{M}_{-\infty}^{*2}$. Of these factors, $\overline{M}_{-\infty}^{*}/M_{-\infty}$, $\overline{M}_{-\infty}^{*}/\overline{M}_{-\infty}$, $Y_n^{(0)}/Y_0^{(0)}$ are determined only by the shear parameter a defined by

$$a \equiv \frac{\lambda}{M_{-\infty}} \frac{dM_{-\infty}}{dy}.$$

On the other hand the eigenvalues $\beta_n^{(0)}$ depend upon a and the aspect ratio λ. The factors influenced by the compressibility are, therefore, confind to $\overline{C}_l$, $F_n^{(0)}/F_0^{(0)}$ and $\overline{M}_{-\infty}^{*2}$. It should be noted that the mean characteristics are determined mainly by the factors $\overline{C}_l$ and the harmonic mean Mach number $\overline{M}_{-\infty}^{*}$.

4.2 *Similarity between the Characteristics in Uniform Flow and the Mean Characteristics in Shear Flow*

In uniform flow with $M_{-\infty}=M_{-\infty}^{(2D)}$, the lift coefficient $C_L^{(2D)}$ can be written as

$$C_L^{(2D)}=K(M_{-\infty}^{(2D)}, \gamma, t)\alpha_{-\infty}\left\{1+K(M_{-\infty}^{(2D)}, \gamma, t)\frac{\mu\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{4t}\right\}^{-1},$$

where μ reduces to $1-M_{-\infty}^{(2D)2}$. Then we have the expressions of the characteristics in uniform flow as follows:

Effective angle of attack

$$\alpha_0^{(2D)}=\alpha_{-\infty}-C_L^{(2D)}\frac{\mu\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{4t}, \tag{83}$$

Turning angle

$$\theta^{(2D)}=C_L^{(2D)}\frac{\mu\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{2t}, \tag{84}$$

Static pressure-rise coefficient

$$C_{\Delta p}^{(2D)}=C_L^{(2D)}\frac{\sin\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}, \tag{85}$$

Variation of axial velocity

$$\frac{\Delta q_\infty^{(2D)}}{U}=-C_L^{(2D)}\frac{1}{2t}\frac{\sin\gamma\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}M_{-\infty}^{(2D)2}, \tag{86}$$

where the superscript $(2D)$ denotes the value in two-dimensional flow.

From comparison of the above equations with the mean characteristics in shear flow given by Eqs. (68), (73′), (66) and (80) the following similarity relations are drawn out:

$$\frac{\overline{\alpha}_0-\alpha_{-\infty}}{\alpha_{0,*}^{(2D)}-\alpha_{-\infty}}\Bigg/\frac{\overline{C}_l}{C_{L,*}^{(2D)}}=1, \tag{87}$$

$$\frac{\overline{\theta}}{\theta_*^{(2D)}}\Bigg/\frac{\overline{C}_l}{C_{L,*}^{(2D)}}=1, \tag{88}$$

$$\frac{C_{\Delta p}}{C_{\Delta p,*}^{(2D)}} \Big/ \frac{\overline{C}_l}{C_{L,*}^{(2D)}} = \left(\frac{\overline{M}^*_{-\infty}}{\overline{M}_{-\infty}}\right)^2 \tag{89}$$

$$\frac{\overline{\overline{\Delta q}}_\infty}{\Delta q_{\infty,*}^{(2D)}} \Big/ \frac{\overline{C}_l}{C_{L,*}^{(2D)}} = \frac{\overline{M}^*_{-\infty}\cdot c_{-\infty}}{U^{(2D)}} E, \tag{90}$$

where the subscript $*$ denotes the value at $M_{-\infty}^{(2D)} = \overline{M}^*_{-\infty}$.

4.3 *Secondary Vorticity*

In this section we shall obtain the expressions of the component of vorticity in the direction of flow, and show how the results based on the secondary flow theory for incompressible shear flow should be modified for the case of subsonic shear flow.

We can determine the vorticity component contained in the trailing vortex sheets from the velocity jump across the sheets. The velocity jump Δv_s defined by

$$\Delta v_s = (v)_{z=+0} - (v)_{z=-0}, \tag{91}$$

can be obtained from Eqs. (71) and (61) in the form

$$\Delta v_s = \begin{cases} 0 & (x<0) \\ \dfrac{U}{2M_{-\infty}^2}\dfrac{d}{dy}(M_{-\infty}^2 C_l) = UC_l \dfrac{1}{M_{-\infty}}\dfrac{dM_{-\infty}}{dy} + \dfrac{U}{2}\dfrac{dC_l}{dy} & (x<0). \end{cases} \tag{92}$$

By using the relation (7), Δv_s for $x>0$ can be rewritten as

$$\Delta v_s = \frac{d}{dy}\left(\frac{UC_l}{2}\right) + \left(\frac{UC_l}{2}\right)\frac{1}{U}\frac{dU}{dy} + \frac{UC_l}{2}\frac{1}{\rho_{-\infty}}\frac{d\rho_{-\infty}}{dy} \quad (x>0). \tag{93}$$

The first term on the right-hand side of Eq. (93) indicates the vorticity component due to change in circulation about the blade, the second term is that due to the stretching of the vortex filaments carried with the flow between the upper and lower stagnation streamlines in the wake of each blade, and the third term corresponds to the vorticity component caused by the gradient of fluid density or in other words, produced by the spanwise gradient of entropy. It is worth noticing that within the approximation of the present theory Eq. (93) is completely similar to the expression deduced by Smith [6] along the line of approach of the secondary flow theory.

Let us calculate, now, the secondary circulation in a segment of unit span far downstream. Within the approximation of our linearized theory this is given by

$$\delta\Gamma = \int_{-1/2\,t\cos\gamma}^{1/2\,t\cos\gamma}\left(\frac{\partial w_\infty}{\partial y} - \frac{\partial v_\infty}{\partial z} - \frac{w_\infty}{U}\frac{dU}{dy}\right)dz, \tag{94}$$

which, by using Eqs. (70), (72) and (61), can be written as

$$\delta\Gamma = -\frac{U}{2M_{-\infty}^2}\frac{d}{dy}(M_{-\infty}^2 C_l) + \frac{UC_l}{2}\frac{2}{M_{-\infty}}\frac{dM_{-\infty}}{dy}\left[\left(\frac{\mu\cos^2\gamma}{\mu\cos^2\gamma+\sin^2\gamma}-1\right)\frac{\overline{C}_l}{C_l}\left(\frac{\overline{M}^*_{-\infty}}{M_{-\infty}}\right)^2+1\right]. \tag{94'}$$

Obviously the first term on the right-hand side of Eq. (94′) corresponds to the vorticity component in the trailing vortex sheet and hence the second term denotes the contribution of the vorticity component distributed between the wake surfaces.

If we consider the case of incompressible flow with no density gradient and further if the shear is so weak that the approximation of

$$\frac{\overline{C}_l}{C_l}\left(\frac{\overline{M}^*_{-\infty}}{M_{-\infty}}\right)^2 \sim 1$$

is permissible, then the second term on the right hand side of Eq. (94′) degenerates to

$$C_l \frac{dU}{dy}\cos^2\gamma$$

which coincide to the expression deduced by Hawthorne [4] as the component due to the curving of the flow in a passage between the blades. It should be noted further that with the additional approximation of $dC_l/dy \sim 0$, $\delta\Gamma$ vanishes when the stagger angle γ is zero. This result corresponds to that obtained by Preston [5], whose investigation is confined to the case of zero stagger angle.

4.4 *Numerical Examples*

As the present theory is based on the lifting-line, the two-dimensional characteristics $K(M^{(2D)}_{-\infty}, \gamma, t)$ must be known. Practically $K(M^{(2D)}_{-\infty}, \gamma, t)$ can be determined by the two-dimensional theory or experiment. In this work we shall use the well-known linearized two-dimensional subsonic cascade theory based on the Prandtl-Glauert rule as shown for example by Schlichting [8]. Moreover in order to simplify the numerical work we apply to this theory the so-called Weissinger's method, in which each blade in cascade is replaced by a bound vortex at a quarter-chord point and the strength of the bound vortex is determined by making the flow direction at three-quarters-chord point satisfy the boundary condition. Then as shown in Appendix V we have

$$K = 2\pi\mu^{-1/2}\left(1+\frac{1}{2}I\right)^{-1}, \tag{95}$$

where

$$I = \sum_{n=1}^{\infty} \frac{n^2t^2(\mu\cos^2\gamma+\sin^2\gamma)+\dfrac{1}{4}-2n^2t^2\cos^2\gamma}{\left\{n^2t^2(\mu\cos^2\gamma+\sin^2\gamma)+\dfrac{1}{4}\right\}^2-n^2t^2\sin^2\gamma}.$$

Of course the expression (95) can not be applied to such a high subsonic flow as is too close to a sonic flow. The limit of the Mach number beyond which the reliability of Eq. (95) becomes too poor seems nearly equal to the critical Mach number in the case of an isolated airfoil ($t=\infty$), but much lower in the case of a cascade with a higher stagger angle. However we have no reliable informations as to the limit of the Mach number for cascade as well as the value of K above the limit. For the time being we shall assume that the limit is 0.8 and above this limit K is given by

$$K=\frac{1}{2}[K]_{M_{-\infty}^{(2D)}=0}+\frac{1-M_{-\infty}^{(2D)}}{0.2}\left\{[K]_{M_{-\infty}^{(2D)}=0.8}-\frac{1}{2}[K]_{M_{-\infty}^{(2D)}=0}\right\}\quad(0.8\leqq M_{-\infty}^{(2D)}\leqq 1),\tag{96}$$

which means that in this range K decreases linearly. Although there remains a considerable doubt in regard to whether the relation (96) is practical or not, we shall be able to obtain some information about the flow features where the decrease in the lift force occurs at some portion of the span due to shock waves.

For the numerical examples we adopt a profile of the upstream Mach number with a constant shear parameter a such that

$$M_{-\infty}(y)=M_{-\infty,0}e^{ay/\lambda},\tag{97}$$

since the calculation for various values of $M_{-\infty,0}$ with a kept constant will give

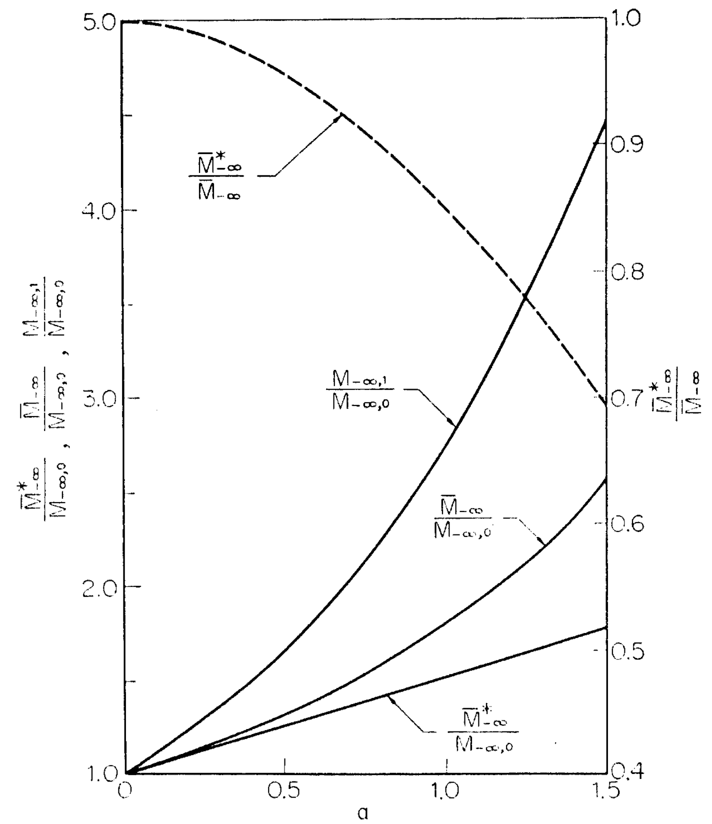

FIG. 2. Characteristics of upstream Mach number with profile of Eq. (97); $M_{-\infty,1}=M_{-\infty}(\lambda)$

explicit information about the effect of compressibility. For this shear flow we have the following expressions:

$$\overline{M}_{-\infty}=M_{-\infty,0}\{(e^{2a}-1)/2a\}^{1/2}, \qquad \overline{M}^*_{-\infty}=M_{-\infty,0}\{2ae^{2a}/(e^{2a}-1)\}^{1/2},$$

$$\left.\begin{aligned} &Y_0^{(0)}=\overline{M}^*_{-\infty}, \\ &Y_n^{(0)}=M_{-\infty,0}\sqrt{2}\,a(a^2+n^2\pi^2)^{-1/2}e^{ay/\lambda}\left\{\sin\left(n\pi\frac{y}{\lambda}\right)\right. \\ &\qquad\left.-\frac{n\pi}{a}\cos\left(n\pi\frac{y}{\lambda}\right)\right\} \quad (n=1,2,3,\cdots), \end{aligned}\right\} \tag{98}$$

$$\beta_0^{(0)}=0, \qquad \beta_n^{(0)}=(a^2+n^2\pi^2)^{1/2}/\lambda \qquad (n=1,2,3,\cdots). \tag{99}$$

Fig. 2 shows various flow properties for the Mach number profile based on Eq. (97).

In the computational work the terms for $n\geqq 11$ are neglected and the calculations were carried out by using the OKITAC 5090 at the Computing Center in University of Tokyo.

According to the discussion in 4.1 and 4.2 we first consider how $\overline{C}_l$ and $F_n^{(0)}/F_0^{(0)}$ are influenced by the compressibility. In Figs. 3 and 4 are shown the dependence of $\overline{C}_l/C_{L,i}^{(2D)}$ upon $\overline{M}_{-\infty}$ and $\overline{M}^*_{-\infty}$ respectively, where the subscript i denotes the value for incompressible flow. It is found that the relation between $\overline{C}_l/C_{L,i}^{(2D)}$ and $\overline{M}_{-\infty}$ or $\overline{M}^*_{-\infty}$ even for such a great shear parameter as $a=1.2$ is close to that for the two-dimensional flow of $a=0$, and moreover $\overline{C}_l$ is close to $C_L^{(2D)}$ at $M_{-\infty}^{(2D)}=\overline{M}^*_{-\infty}$ rather than to $C_L^{(2D)}$ at $M_{-\infty}^{(2D)}=\overline{M}_{-\infty}$ because $\overline{M}^*_{-\infty}\leqq\overline{M}_{-\infty}$ holds according to the so-called Schwarz's theorem. It seems possible, therefore, with a high degree of approximation to evaluate $\overline{C}_l$ in a compressible shear flow by

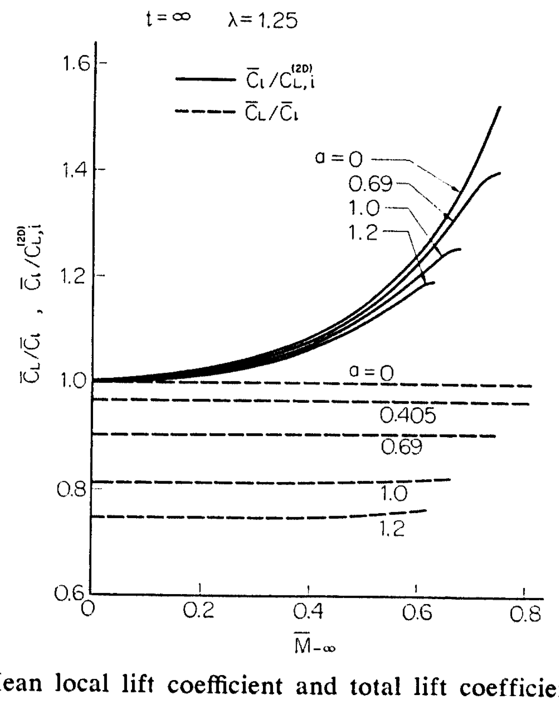

FIG. 3. Mean local lift coefficient and total lift coefficient against mean upstream Mach number; $t=\infty$, $\lambda=1.25$

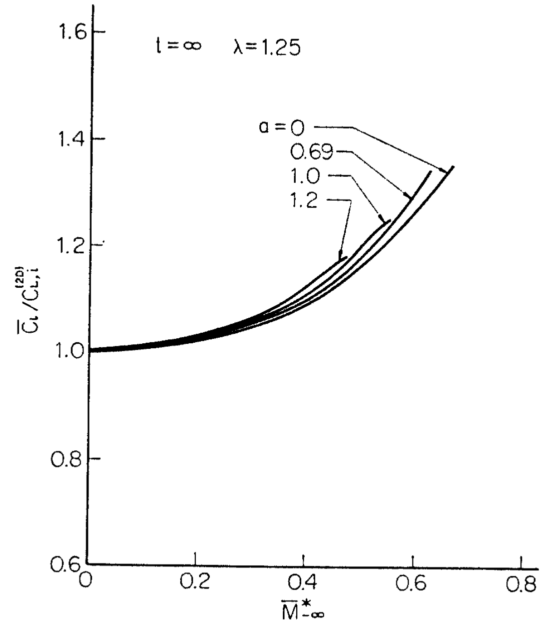

FIG. 4. Mean local lift coefficient against harmonic mean upstream Mach number; $t=\infty$, $\lambda=1.25$

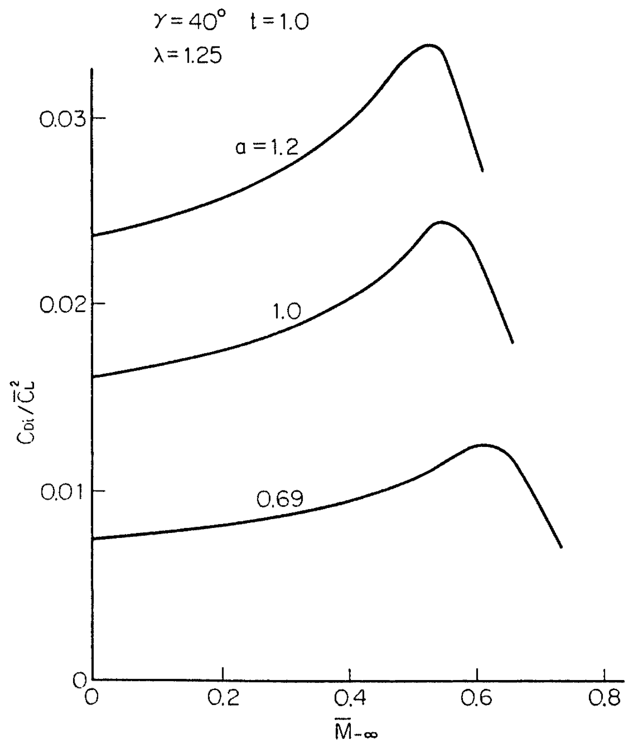

FIG. 5. Total induced drag coefficient; $\gamma=40°$, $t=1.0$, $\lambda=1.25$

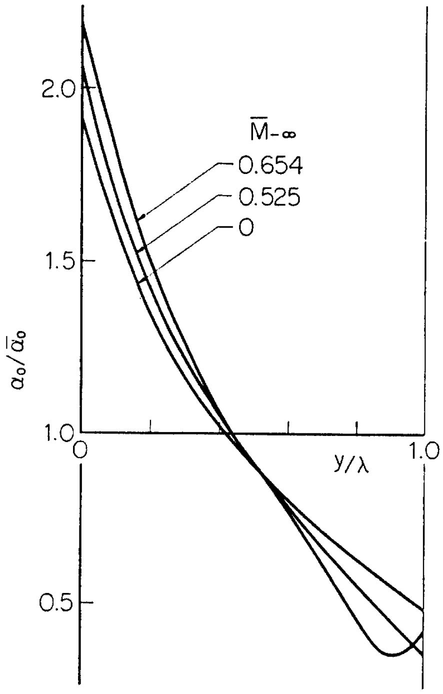

FIG. 6. Spanwise distribution of effective angle of attack; $a=1.0$, $t=\infty$, $\lambda=1.25$

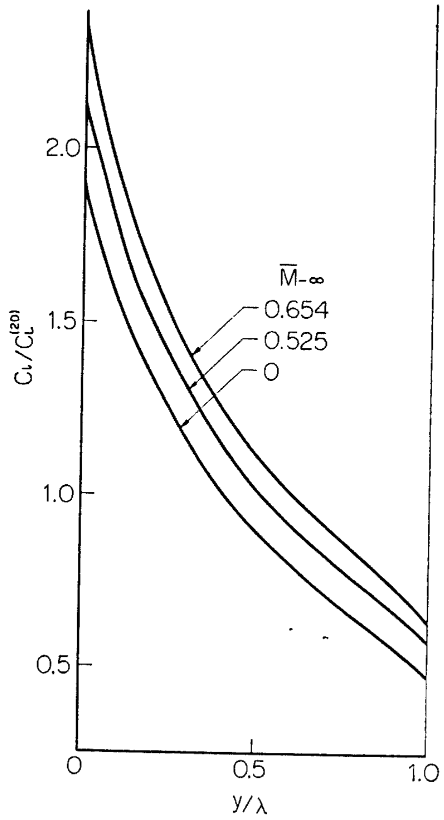

FIG. 7. Spanwise distribution of local lift coefficient; $a=1.0$, $t=\infty$, $\lambda=1.25$

$$\bar{C}_l \sim [C_L^{(2D)}]_{M_{-\infty}^{(2D)}=\bar{M}^*_{-\infty}}.$$

The reason why the curves of $\bar{C}_l/C_{L,i}^{(2D)}$ in the figures show gradual decline above certain values of $\bar{M}_{-\infty}$ or $\bar{M}^*_{-\infty}$ can be ascribed to the stall at the span-station for $M_{-\infty}>0.8$ according to the assumption of Eq. (96).

The effect of compressibility upon $F_n^{(0)}/F_0^{(0)}$ can be known from the dependence of $\bar{C}_L/\bar{C}_l$ upon $\bar{M}_{-\infty}$, since the effect of compressibility upon $\bar{C}_L/\bar{C}_l$ appears only through the factors $F_n^{(0)}/F_0^{(0)}$ as known from Eq. (65). As shown with dashed lines in Fig. 3, $\bar{C}_L/\bar{C}_l$ is found to depend very weakly upon $\bar{M}_{-\infty}$.

Fig. 5 shows the dependence of the total induced drag upon the mean Mach number. We can consider $C_{Di}/\bar{C}_L^2$ as a cross product of $C_{Di}/\bar{C}_l^2$ and $(\bar{C}_l/\bar{C}_L)^2$, and further as mentioned above $(\bar{C}_l/\bar{C}_L)^2$ shows only weak dependence upon $\bar{M}_{-\infty}$. Therefore the phenomena of increase of $C_{Di}/\bar{C}_l^2$ with $\bar{M}_{-\infty}$ for lower Mach number range seems due to the factor $\mu=1-\bar{M}^{*2}_{-\infty}$ contained in the second term in the square bracket [ ] on the right-hand side of Eq. (74'). As seen from Fig. 5, $C_{Di}/\bar{C}_L^2$ decreases with $\bar{M}_{-\infty}$ in higher Mach number range. This is ascribed to the fact that the lift and the downwash decrease at the span-station for $M_{-\infty}>0.8$.

Hitherto we have made clear the effect of compressibility upon $\bar{C}_l/C_{D,i}^{(2D)}$ and $F_n^{(0)}/F_0^{(0)}$. Then we can know the mean characteristics in shear flow from both the

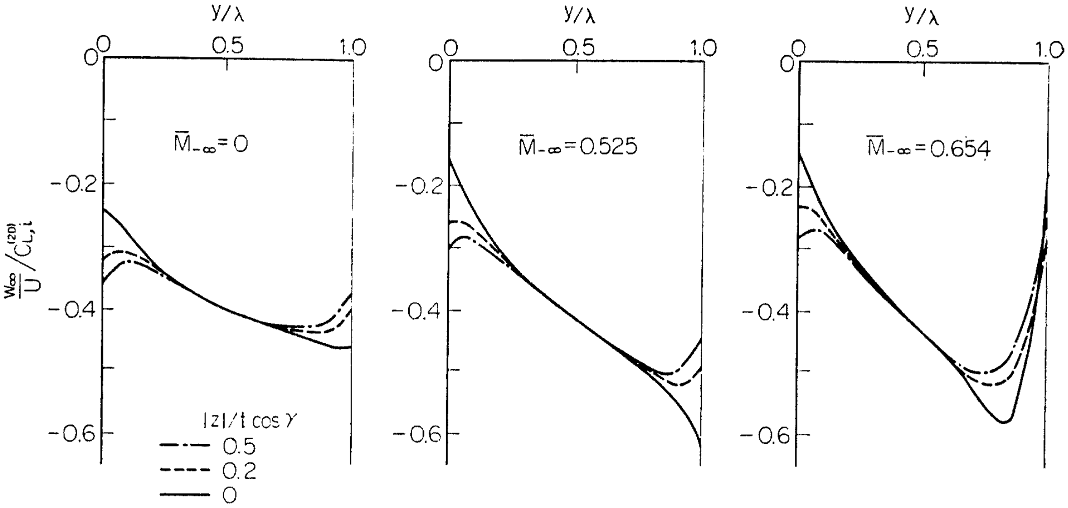

FIG. 8 (a). The z-component of induced velocity in the Trefftz plane; $a=1.0$, $\gamma=40°$, $t=1.0$, $\lambda=2.5$

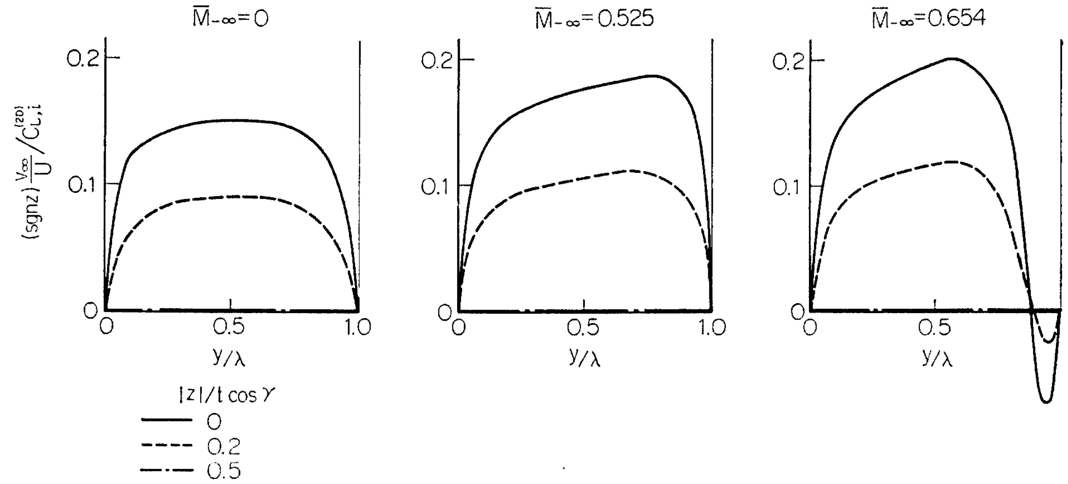

FIG. 8 (b). The y-component of induced velocity in the Trefftz plane; $a=1.0$, $\gamma=40°$, $t=1.0$, $\lambda=2.5$

similarity relations (87) to (90) and the two-dimensional characteristics (83) to (86). The fact that the relation between $\bar{C}_l$ and $\bar{M}^*_{-\infty}$ is close to that between $C_L^{(2D)}$ and $M_{-\infty}^{(2D)}$ allows us approximately to predict $\bar{\alpha}_0$, $\bar{\theta}$, $C_{\Delta p}$ and $\Delta q_\infty/U$ only from the informations about the two-dimensional cascade.

As seen from the expressions derived in the chapter 3 the effect of compressibility upon the patterns of the spanwise variation of the aerodynamic characteristics appears through the factor $F_n^{(0)}/F_0^{(0)}$, while that of the mean values appears through the factors $\bar{C}_l$ and $\bar{M}^{*2}_{-\infty}$. From the fact that the factor $F_n^{(0)}/F_0^{(0)}$ depends only weakly upon $\bar{M}_{-\infty}$, it is suggested that the patterns of the spanwise distribution of the aerodynamic characteristics show no essential change with the mean Mach number as far as the shear parameter a remains unchanged.

In fact this is confirmed by the examples in Figs. 6 and 7. It is mainly due to

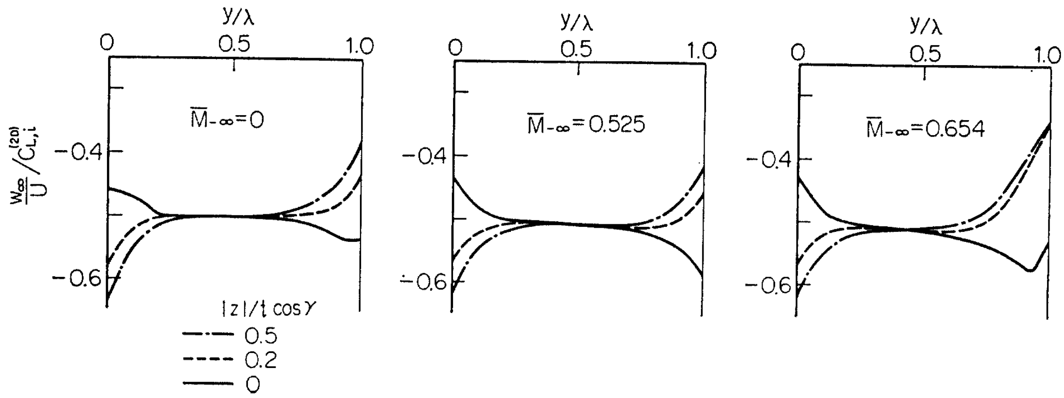

FIG. 9 (a). The z-component of induced velocity in the Trefftz plane; $a=1.0$, $\gamma=0°$, $t=1.0$, $\lambda=2.5$

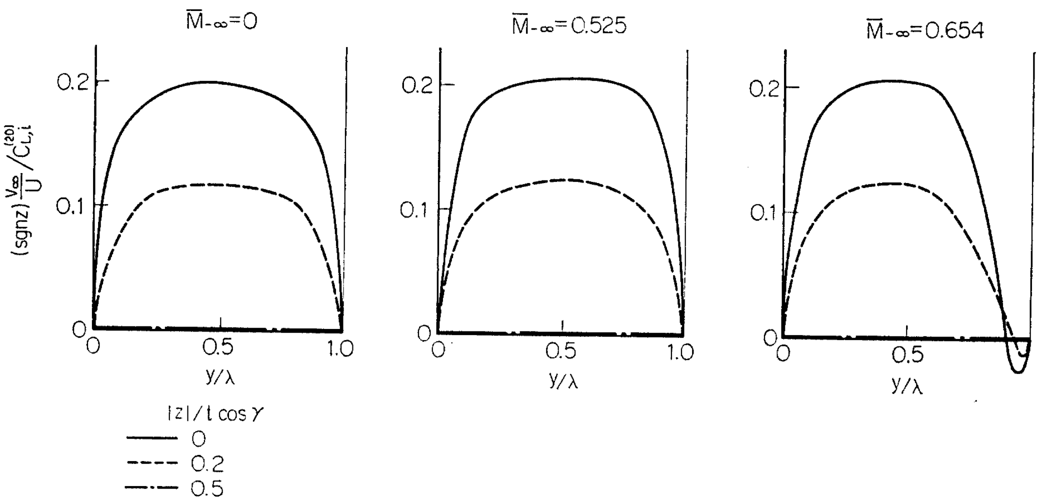

FIG. 9 (b). The y-component of induced velocity in the Trefftz plane; $a=1.0$, $\gamma=0°$, $t=1.0$, $\lambda=2.5$

the compressibility effect upon the factor $\bar{C}_l$ that the effective angle of attack (Fig. 6) becomes much more non-uniform as $\bar{M}_{-\infty}$ increases. In the case of $\bar{M}_{-\infty}=0.654$, however, where some portion of the span is attacked by the upstream Mach number of $M_{-\infty}>0.8$, the local lift of that part of the span decreases according to the assumption of Eq. (96), and hence the strength of downwash decreases locally as seen in Fig. 6.

Figs. 8 and 9 show the distribution of the induced velocities in the Trefftz plane. It will be seen that the total secondary circulation is small when the stagger angle $\gamma=0°$, because the vorticity component contained in the trailing vortex sheets and that distributed between the vortex sheets are in the opposite direction to each other. This result may afford a numerical evidence for the theoretical analysis in the previous section.

4.5 *Comparison with Experimental Results*

In order to evaluate the physical validity of the results obtained analytically from the present theory, we have conducted the experiment on cascade of blades in subsonic shear flow in addition to the previous experiment [*13*] for transonic shear flow. The technic and the apparatus employed are essentially the same as those in reference [*13*], and only a brief outline need be given here.

The shear flows with peaked profile of the Mach number were formed artificially in a high speed linear cascade wind tunnel by using parallel plates (Fig. 10) installed upstream the cascade. The tested cascade is composed of eight steel blades with a profile of double circular arc, the chord length of 30 mm, the span length of 75 mm, the maximum thickness ratio of 5.42 per cent and the mean camber angle of 16°46′ (Fig. 11). The static pressures on the blade surface at various span-stations were measured by traversing spanwisely the central two blades having the seven pressure holes distributed along the chord.

First, it must be confirmed to what extent the two-dimensional characteristics given by Eq. (95) are satisfied by the tested cascade in uniform flow. In Figs. 12(a)

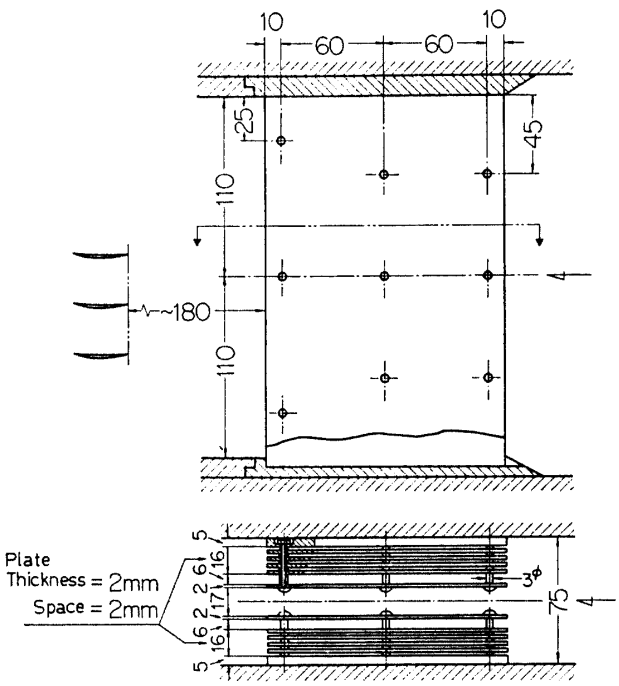

FIG. 10. Velocity profile generator

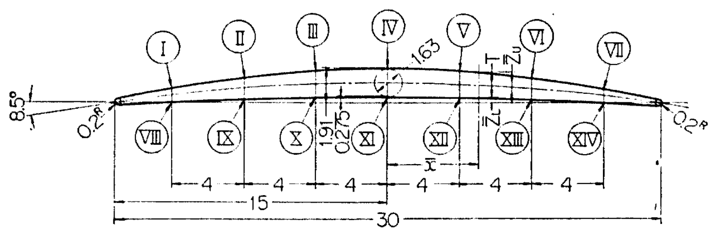

FIG. 11. Profile of tested blade. Roman numerals indicate positions of static pressure holes

and 12(b) the lift coefficient in uniform flow $C_L^{(2D)}$ from the experiment (shown by open circles) is compared with that based on Eq. (95) (shown by solid lines). Figs. 13 and 14 show the chordwise distribution of the blade surface pressure coefficient $C_p^{(2D)}$ in uniform flow obtained experimentally at various Mach numbers $M_{-\infty}^{(2D)}$. As will be seen from Fig. 13 the critical Mach number $M_{cr}^{(2D)}$ for the isolated blade under the given condition is about 0.75, then it may be evident from Fig. 12(a) that the linearized two-dimensional theory can estimate fairly well the lift force of the isolated blade in uniform flow as far as the Mach number is below the critical Mach number. As seen from Fig. 14 the critical Mach number $M_{cr}^{(2D)}$ for this cascade ($r=40^\circ$, $t=1.4$ and $\alpha_g=2^\circ$) seems also close to 0.8. Fig. 12(b) shows, however, Eq. (95) can be applied to this cascade only for lower Mach numbers. This results seems in agreement with the view of Grewe [*10*] who pointed out that the extent of the Mach number in which the linearized cascade theory can be applied to compressor cascade becomes more narrow for the higher stagger angle. At the same time it is found that $C_p^{(2D)}$ distribution shows essential change in its pattern beyond a certain Mach number level which is about 0.75 for the isolated blade in Fig. 13 and about 0.4 for the cascade in Fig. 14. This phenomenon

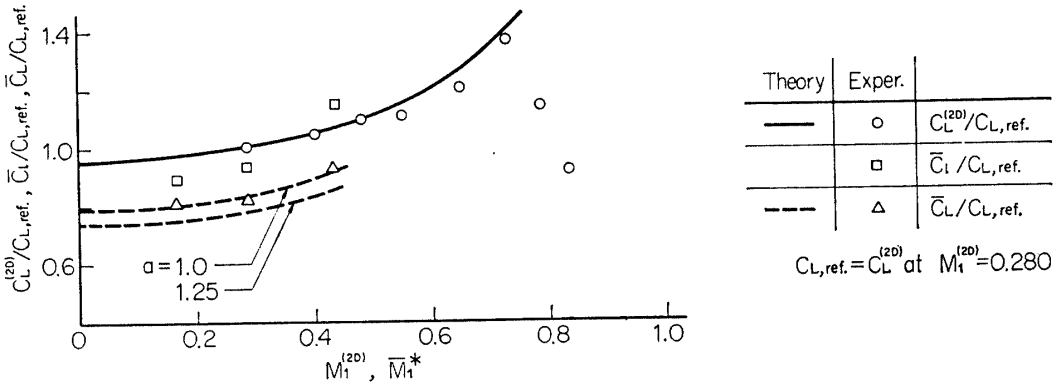

FIG. 12 (a). Lift coefficient in uniform flow, and mean local lift coefficient and total lift coefficient in the shear flows shown in FIG. 17 (a); $t=\infty$, $\lambda=1.25$

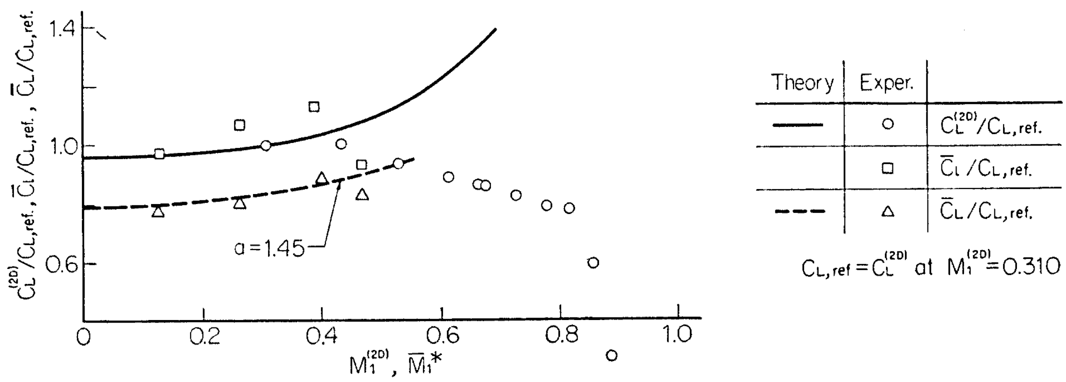

FIG. 12 (b). Lift coefficient in uniform flow, and mean local lift coefficient and total lift coefficient in the shear flows shown in FIG. 18 (a); $\gamma=40^\circ$, $t=1.4$, $\lambda=1.25$

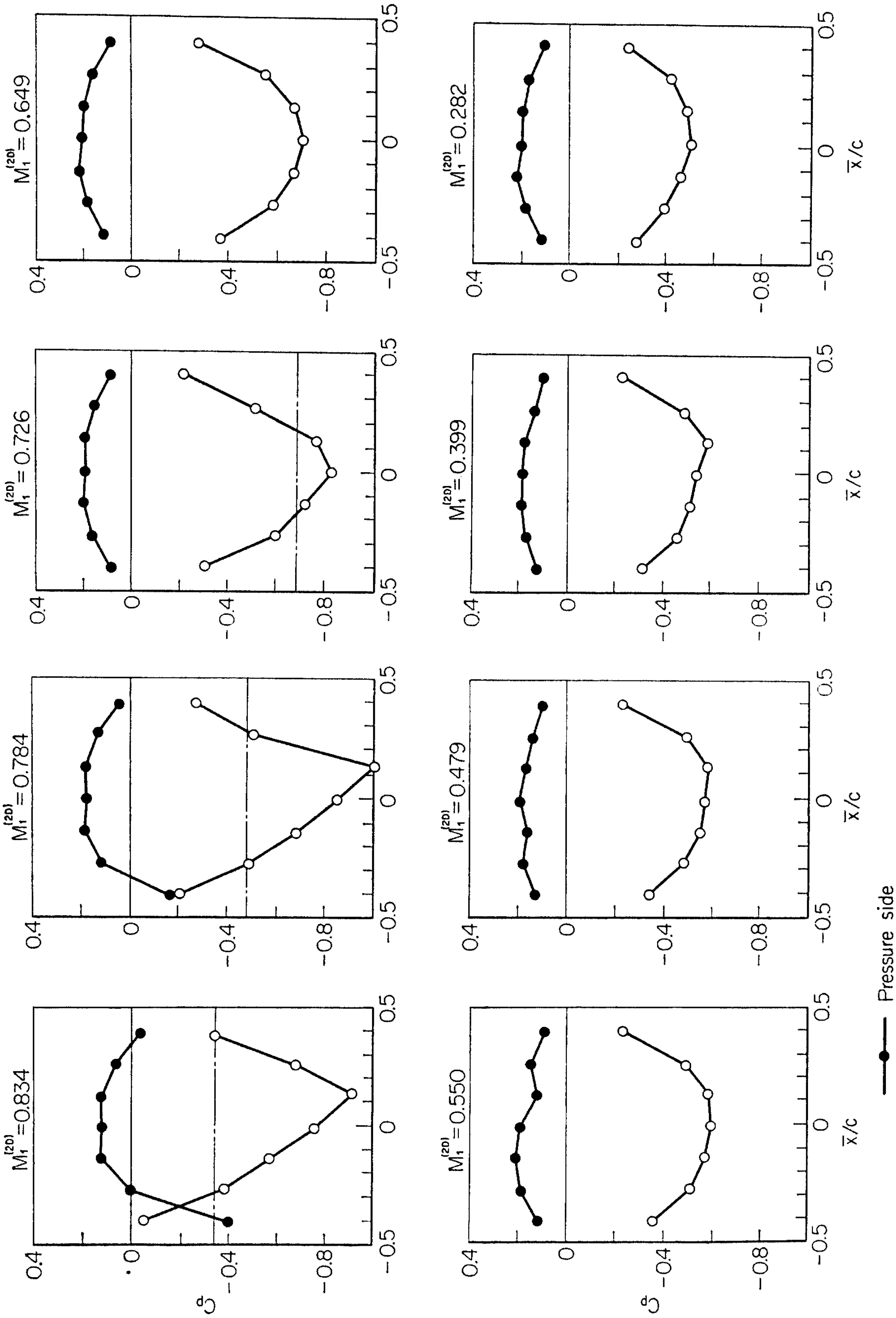

FIG. 13. Blade surface pressure coefficient in uniform flow; $t=\infty$, $\alpha_g=2°$

of change in the pattern of the $C_p^{(2D)}$ distribution may be due to the occurrence of separation of the boundary layer on the blade surface, which in the case of the compressor cascade seems to begin at a lower Mach number even if no shock wave yet exists.

Figs. 15(a), 15(b) and 16(a), 16(b) show the chordwise distributions of the blade surface pressure coefficient C_p at various span-stations in each of the shear flows whose flow patterns are illustrated in Figs. 17(a) and 18(a) respectively. Here C_p is defined by

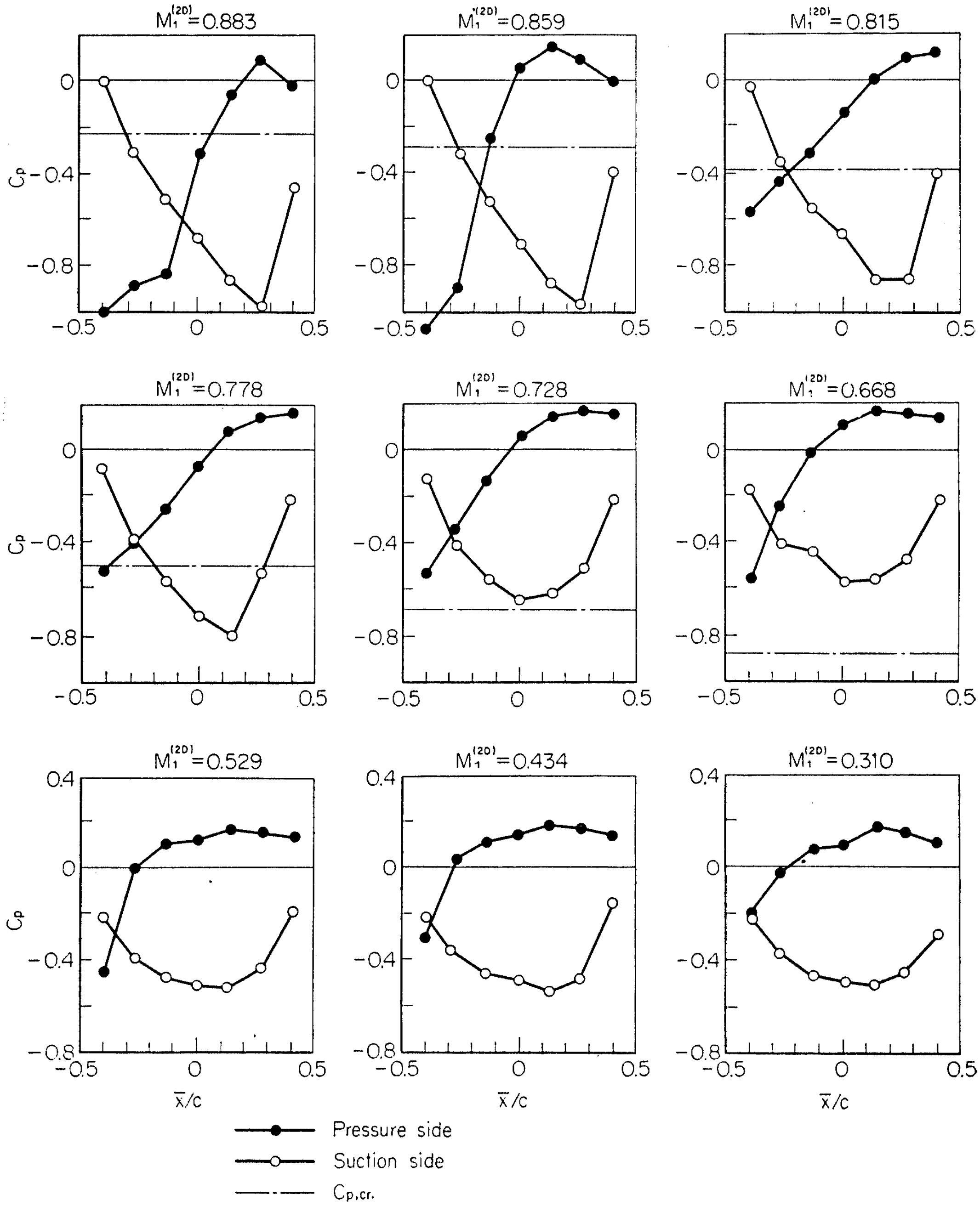

FIG. 14. Blade surface pressure coefficient in uniform flow; $\gamma=40°$, $t=1.4$, $\alpha_g=4°$

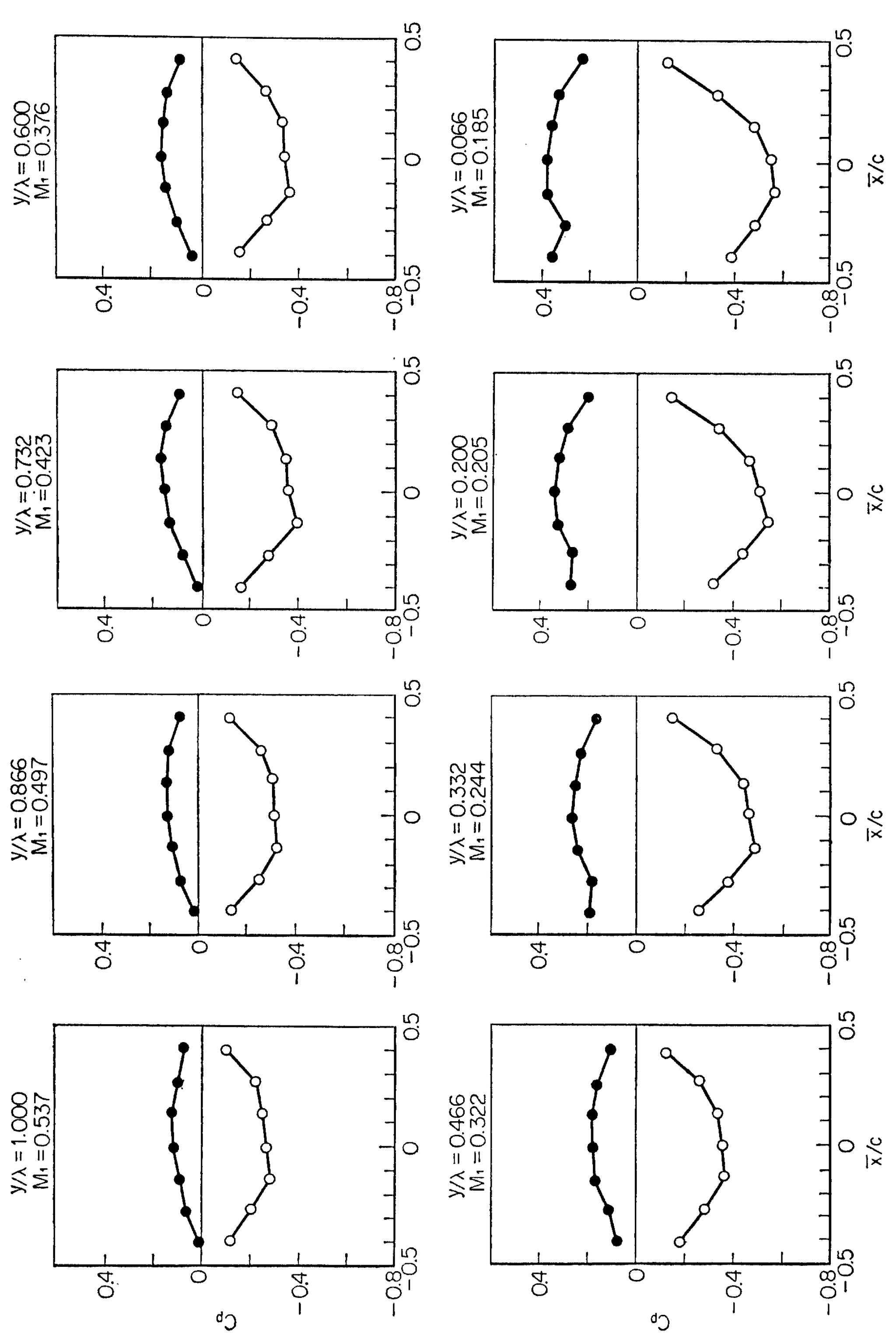

FIG. 15 (a). Blade surface pressure coefficient at various spanstations in the shear flow II shown in FIG. 17 (a); $\lambda=1.25$, $t=\infty$, $\alpha_g=2°$, $\overline{M}_1^*=0.278$

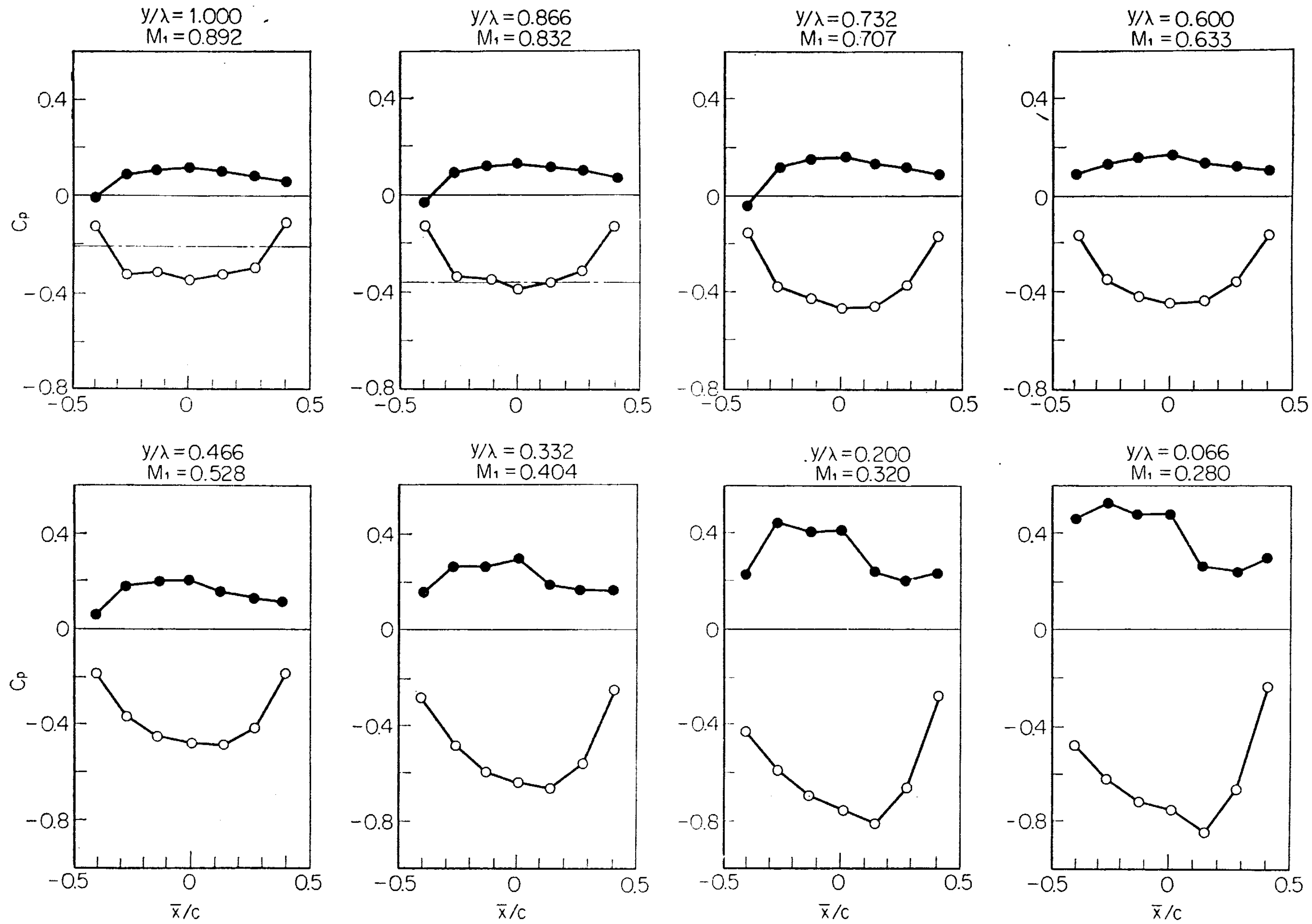

FIG. 15 (b). Blade surface pressure coefficient at various spanstations in the shear flow III shown in FIG. 17 (a).; $\lambda=1.25$, $t=\infty$, $\alpha_g=2$, $M_1^*=0.439$

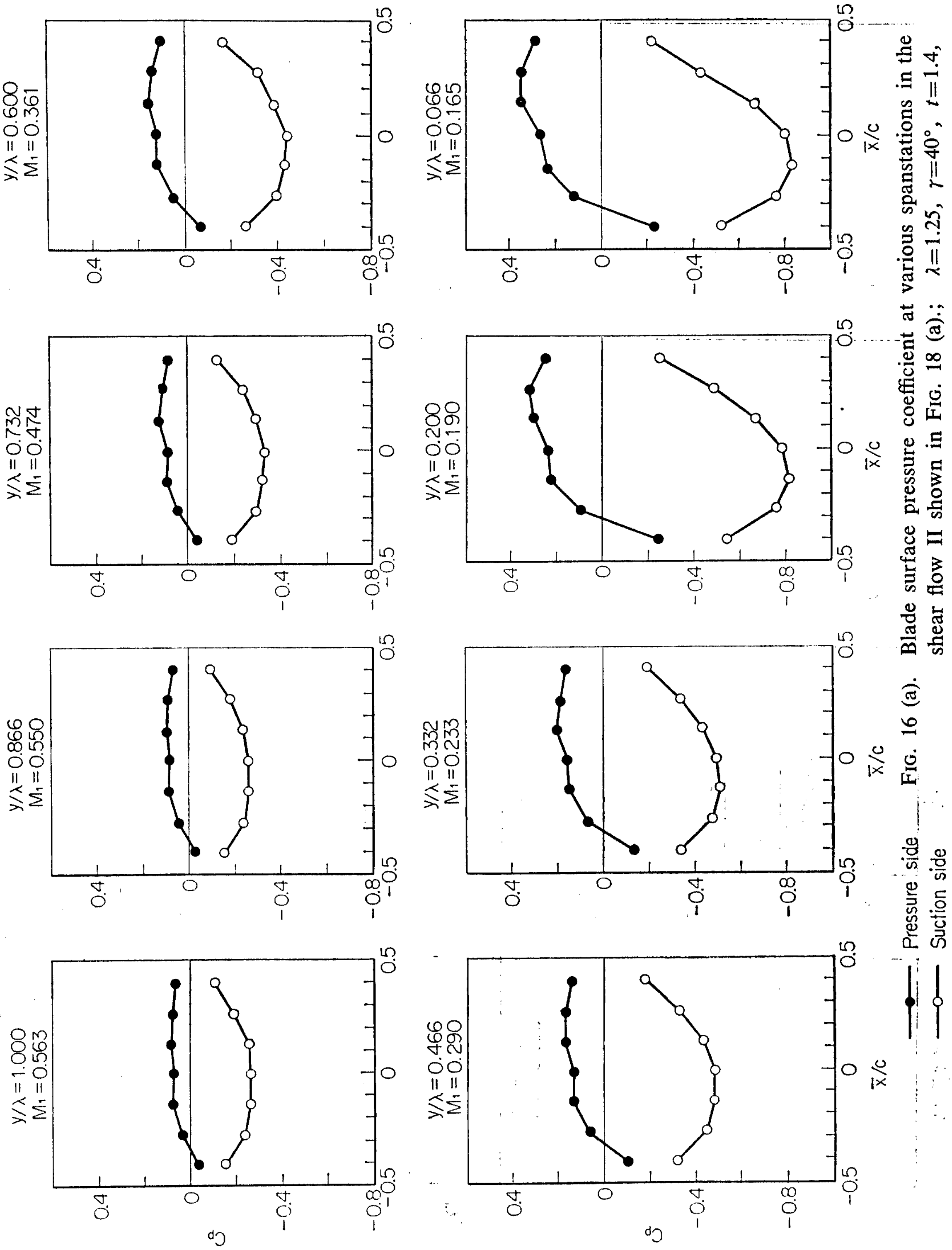

FIG. 16 (a). Blade surface pressure coefficient at various spanstations in the shear flow II shown in FIG. 18 (a).; $\lambda=1.25$, $\gamma=40°$, $t=1.4$, $\alpha_g=4°$, $\overline{M}_1^*=0.265$

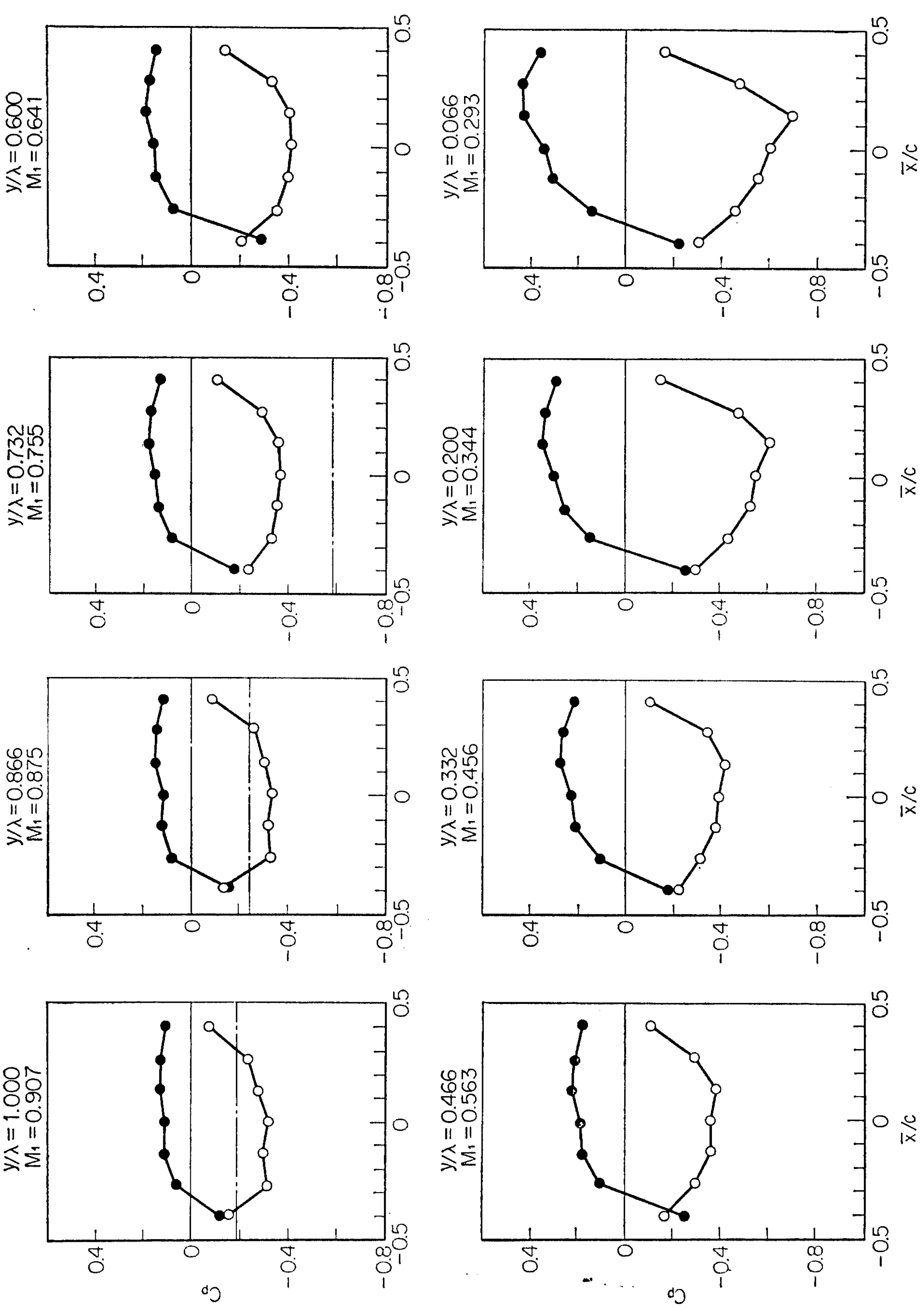

FIG. 16 (b). Blade surface pressure coefficient at various spanstations in the shear flow IV shown in FIG. 18 (a).; $\lambda=1.25$, $\gamma=40°$, $t=1.4$, $\alpha_g=4°$, $\overline{M}_1^*=0.469$

$$C_p \equiv p \Big/ \left[\frac{1}{2}\,\kappa p_{-\infty}\{M_{-\infty}(y)\}^2\right]. \tag{100}$$

The pattern of the chordwise C_p distribution in shear flow shows in contrast to that in uniform flow, no essential change with the local upstream Mach number $M_{-\infty}(y)$, even though $M_{-\infty}(y)$, varies along the span, for example, from 0.293 to 0.907 (Fig. 16(b)). Moreover the pattern of the chordwise C_p distribution in a shear flow is found to correspond to that in the uniform flow at $M_{-\infty}^{(2D)} = \overline{M}^*_{-\infty}$. Thus the importance of the harmonic mean Mach number $\overline{M}^*_{-\infty}$ for determining the mean characteristics in shear flow is confirmed also by the experimental results.

The results mentioned above suggest us that the present lifting-line theory can give fairly good estimation of the lift distribution in subsonic shear flow as far as the harmonic mean Mach number lies below a certain limit value, even if $M_{-\infty}(y)$ at some portion of the span is locally greater than that limit. Moreover this limit value seems to correspond to the limit of the Mach number in uniform flow beyond which the linearized small disturbance theory becomes inapplicable.

The evidence to support this conception will be given by Figs. 17 and 18. In those figures the spanwise distributions of the local lift coefficient C_l and the lift coefficient C_L obtained from the experiment are compared with those based on the present theory. The flow properties of the shear flows shown in Figs. 17(a) and 18(a) are illustrated in Tables 1 and 2 respectively. The C_L distribution for the three shear flows I, II and III in Fig. 17, all of which have $\overline{M}^*_{-\infty}$ lower than the critical Mach number $M_{cr}^{(2D)}$ ($\sim$0.75) for this blade in uniform flow, show good agreement with those calculated on the lifting-line theory for the three shear flows

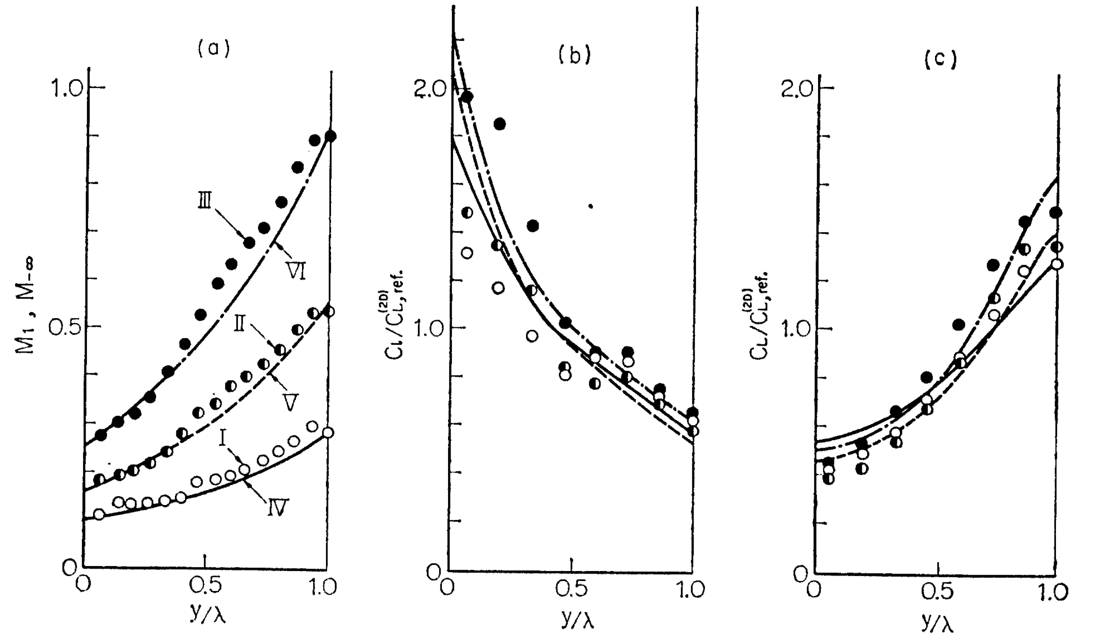

FIG. 17. Spanwise distributions of (a) inlet Mach number, (b) local lift coefficient and (c) lift coefficient; $\lambda=1.25$, $t=\infty$, $\alpha_g=2°$, $C_{L,\mathrm{ref}}^{(2D)}$ denotes $C_L^{(2D)}$ at $M_1^{(2D)}=0.280$, c.f. Table 1

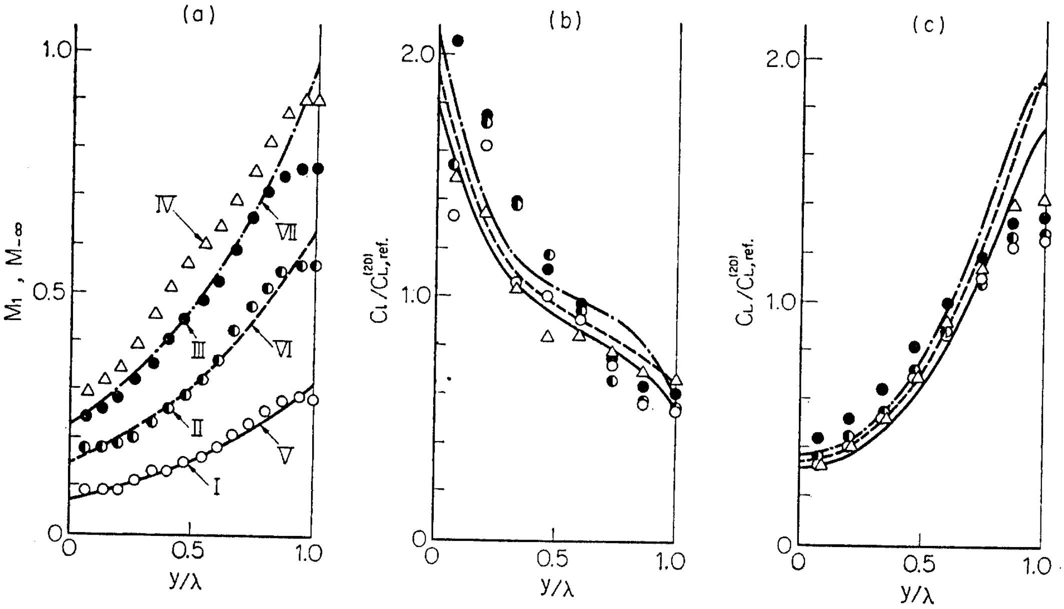

FIG. 18. Spanwise distributions of (a) inlet Mach number, (b) local lift coefficient and (c) lift coefficient; $\lambda=1.25$, $t=1.4$, $\alpha_g=4°$, $C_{L,\mathrm{ref}}^{(2D)}$ denotes $C_L^{(2D)}$ at $M_1^{(2D)}=0.310$, c.f. Table 2

TABLE 1. Characteristics of upstream Much number shown in Fig. 17(a)

			$\bar{M}_1$	$\bar{M}_1^*$	$\bar{M}_1^*/\bar{M}_1$	M
Experiment	I	○	.199	.162	.815	
	II	◐	.356	.278	.781	
	III	●	.593	.439	.740	
Theory	IV	———	.179	.152	.850	0.1 exp (1.0 y/λ)
	V	- - - -	.338	.264	.781	0.16 exp (1.25 y/λ)
	VI	— - —	.550	.429	.781	0.26 exp (1.25 y/λ)

TABLE 2. Characteristics of upstream Mach number shown in Fig. 18(a)

			$\bar{M}_1$	$\bar{M}_1^*$	$\bar{M}_1^*/\bar{M}_1$	M_1
Experiment	I	○	.200	.135	.712	
	II	◐	.370	.265	.717	
	III	●	.521	.382	.733	
	IV	△	.621	.469	.755	
Theory	V	———	.182	.131	.721	0.75 exp (1.45 y/λ)
	VI	- - - -	.365	.263	.721	0.15 exp (1.45 y/λ)
	VII	— - —	.559	.402	.721	0.23 exp (1.45 y/λ)

IV, V and VI respectively. In Fig. 18 also fairly good correspondence between the theory and the experiment can be found with an exception of the result for the shear flow IV. The shear flow IV, however has the harmonic mean Mach number of $\bar{M}^*_{-\infty}=0.469$, which seems higher than the limit of the Mach number for this cascade in uniform flow beyond which the assumption of small disturbance is not permissible.

Finally we need discuss the mean characteristics. In Figs. 12(a) and 12(b) the variations of $\bar{C}_l$ and $\bar{C}_L$ with $\bar{M}^*_{-\infty}$ are also shown. They show good agreement with the theoretical curves as far as $\bar{M}^*_{-\infty}$ is lower than the limit mentioned above. Attention should be drawn to the fact that the decrease of $\bar{C}_l$ and $\bar{C}_L$ for the harmonic mean Mach number higher than the limit value corresponds to that of $C_L^{(2D)}$ obtained experimentally in uniform flow.

5. Conclusion

The theory of lifting-lines for cascade of blades in subsonic shear flow is developed and compared with some experimental results. The general conclusions to be drawn from this work are as follows:

(1) The disturbance pressure due to the lifting-lines in subsonic shear flow between two parallel walls can be expressed in a Fourier integral form and the aerodynamic force coefficients and the flow field in the Trefftz plane can be expressed in the forms of infinite serieses of orthogonal eigenfunctions.

(2) The effect of compressibility upon the mean characteristics of cascade in shear flow can be ascribed to the two factors, i.e., the mean local lift coefficient $\bar{C}_l$ and the harmonic mean Mach number $\bar{M}^*_{-\infty}$.

(3) The pattern of the spanwise distribution of the aerodynamic characteristics of cascade depends only weakly upon the Mach number level but largely upon the shear parameter a, i.e., the upstream vorticity and the entropy gradient.

(4) The mean characteristics in shear flow can be related to the characteristics in uniform flow by simple similarity relations in which the harmonic mean Mach number $\bar{M}^*_{-\infty}$ is used as a parameter.

(5) The dependence of $\bar{C}_l$ upon $\bar{M}^*_{-\infty}$ is very close to that of $C_L^{(2D)}$ upon $M_{-\infty}^{(2D)}$.

(6) The present lifting-line theory can evaluate fairly well the lift distribution of the cascade in subsonic shear flow as far as $\bar{M}^*_{-\infty}$ is below a certain limit, and this limit seems to correspond to the limit of the Mach number for uniform flow beyond which the assumption of small disturbance becomes unsatisfactory.

(7) The chordwise distribution of the static pressure on the blade surface in a shear flow shows no essential change in its pattern along the whole span, and this pattern corresponds to that in the uniform flow at $M_{-\infty}^{(2D)}=\bar{M}^*_{-\infty}$.

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REFERENCES

[1] Th. v. Kármán & H. S. Tsien: Quart. Appl. Math., Vol. 3, No. 1 (1945) P. 1.
[2] H. B. Squire & J. K. G. Winter: J. Aeron. Sci., Vol. 18, No. 4 (1951) P. 271.
[3] W. R. Hawthorne: J. Aeron. Sci., Vol. 21, No. 9 (1954) P. 271.
[4] W. R. Hawthorne: Quart. Journ. Mech. and Appl. Math., Vol. 8, Pt. 3 (1955) P. 266.
[5] J. H. Preston: The Aeron. Quart., Vol. 5 (1954) P. 218.
[6] L. H. Smith: Trans. ASME, Vol. 77 (1955) P. 1065.
[7] M. J. Lighthill: J. Fluid Mech., Vol. 3, No. 2 (1957) P. 113.
[8] H. Schlichting & E. G. Feindt, Forsch. Ing.-Wesen., Bd. 24, Nr. 27 (1958) S. 1335.
[9] J. E. McCune: J. Aero/Space Sci., Vol. 25 (1958) P. 544; P. 616.
[10] K. H. Grewe: Forsch. Ing.-Wesen., Bd. 25, Nr. 27 (1959) S. 1.
[11] M. Honda: Proc. Roy. Soc. Lond., Ser. A, No. 264 (1960) P. 372; No. 265 (1961) P. 46.
[12] R. Courant & D. Hilbert: Method of Mathematical Physics (1964) Vol. 2, P. 111, Tokyo-Tosho Co. (Tokyo).
[13] M. Namba & T. Asanuma: Trans. Japan Soc. Mech. Engrs., Vol. 31, No. 228 (1965) P. 1236; Bulletin of Inst. Space and Aeron. Sci. Univ. Tokyo, Vol. 1, No. 3(B) (1965) P. 164.

APPENDIX I

General Properties of Eigenvalues

By the transformation of independent variable

$$\eta=\int_0^y M_{-\infty}^2 dy \Big/ \int_0^\lambda M_{-\infty}^2 dy, \tag{A-1}$$

Eqs. (23) and (24) reduce to

$$d^2Y/d\eta^2-\{\alpha^2 q(\eta)-\beta^2 r(\eta)\}Y=0, \tag{A-2}$$

$$dY/d\eta=0 \qquad \text{at} \qquad \eta=0,1, \tag{A-3}$$

respectively, where

$$q=(1-M_{-\infty}^2)\left(M_{-\infty}^{-2}\int_0^\lambda M_{-\infty}^2 dy\right)^2, \tag{A-4}$$

$$r=\left(M_{-\infty}^{-2}\int_0^\lambda M_{-\infty}^2 dy\right)^2. \tag{A-5}$$

If the upstream Mach number $M_{-\infty}$ is always positive and a continuous bounded function of η in the domain of $0\leqq\eta\leqq1$, the functions $q(\eta)$ and $r(\eta)$ are also continuous and bounded. Therefore both functions $q(\eta)$ and $r(\eta)$ have minimum and maximum values in the domain of $0\leqq\eta\leqq1$, which are to be denoted by q_m, r_m

and q_M, r_M respectively. Then we have the following inequalities:

$$q_m \leqq q(\eta) \leqq q_M, \qquad r_m \leqq r(\eta) \leqq r_M. \tag{A-6}$$

If $q(\eta)=q_m$ and $r(\eta)=r_M$ everywhere, then we have as the solution of the Sturm-Liouville problem (A-2) and (A-3), a set of eigenvalues β'_n represented by

$$\beta'^2_n = \frac{q_m}{r_M}\alpha^2 + \frac{1}{r_M}n^2\pi^2 \qquad (n=0, 1, 2, \cdots). \tag{A-7}$$

Likewise if $q(\eta)=q_M$ and $r(\eta)=r_m$ everywhere, then the eigenvalues are given by

$$\beta''^2_n = \frac{q_M}{r_m}\alpha^2 + \frac{1}{r_m}n^2\pi^2 \qquad (n=0, 1, 2, \cdots). \tag{A-8}$$

Next, from the variational principle the eigenvalues β^2_n of the boundary value problem (A-2) and (A-3) are equal to the stationary values of $D[\varphi]/H[\varphi]$, where $D[\varphi]$ and $H[\varphi]$ are functionals defined by

$$D[\varphi] \equiv \int_0^1 (\varphi'^2 + \alpha^2 q(\eta)\varphi^2)d\eta, \tag{A-9}$$

$$H[\varphi] \equiv \int_0^1 \varphi^2 r(\eta)d\eta. \tag{A-10}$$

Now according to Courant and Hilbert[*12*], if the function $q(\eta)$ varies in one direction at each point in the whole domain, then the functional $D[\varphi]$ varies in the same direction. On the other hand, if the function $r(\eta)$ varies in one direction everywhere in the domain, then $H[\varphi]$ varies in the same direction and accordingly $D[\varphi]/H[\varphi]$ varies in the opposite direction.

Therefore taking the relations (A-6) into account, we get the following inequality with respect to the eigenvalues:

$$\beta'^2_n \leqq \beta^2_n \leqq \beta''^2_n \tag{A-11}$$

or

$$\alpha_m\alpha^2 + b_m n^2 \leqq \beta^2_n \leqq \alpha_M\alpha^2 + b_M n^2, \tag{A-12}$$

where

$$\left.\begin{array}{ll} a_m \equiv q_m/r_M, & a_M \equiv q_M/r_m \\ b_m \equiv \pi^2/r_M, & b_M \equiv \pi^2/r_m \end{array}\right\}. \tag{A-13}$$

From the relation (A-12), the following deductions are drawn out:

The eigenvalues β^2_n can be expressed as

$$\beta^2_n = g(\alpha, n)\alpha^2 + h(\alpha, n)n^2, \tag{A-14}$$

where $g(\alpha, n)$ and $h(\alpha, n)$ are bounded functions of α and n.

Furthermore if the flow is subsonic everywhere, namely if $1 > M_{-\infty} > 0$ everywhere in the range $0 \leqq \eta \leqq 1$, then from Eq. (A-4) we have

$$q_m > 0, \tag{A-15}$$

and therefore from (A-12)

$$\beta_n^2 > 0. \tag{A-16}$$

This means that the eigenvalues β_n are always real, if the flow field is subsonic everywhere.

APPENDIX II

Proof of Inequality (18)

First we assume that the lift distribution function $l(y)$ is bounded over the range $0 \leqq y \leqq \lambda$ and satisfies

$$dl(y)/dy = 0 \qquad \text{at} \qquad y = 0, \lambda. \tag{A-17}$$

Then the infinite series on the right-hand side of Eq. (27) is absolutely and uniformly convergent with respect to y for any real value of α, and hence it is possible to find such an infinite series $\sum_{n=0}^{\infty} A_n(y)$ that is convergent uniformly with respect to y and satisfies

$$|F_n(\alpha)Y_n(y;\alpha)| \leqq A_n(y). \tag{A-18}$$

Consequently, using the relation (A-12), we get the following relation:

$$\begin{aligned}\int_0^\infty |P|\,d\alpha &= \int_0^\infty \Big|\sum_{n=0}^\infty e^{-\beta_n(\alpha)|z|}F_n(\alpha)Y_n(y;\alpha)\Big|\,d\alpha \\ &\leqq \int_0^\infty \sum_{n=0}^\infty e^{-\beta_n(\alpha)|z|}\,|F_n(\alpha)Y_n(y;\alpha)|\,d\alpha \\ &\leqq \int_0^\infty \sum_{n=0}^\infty e^{-\beta_n(\alpha)|z|}A_n(y)\,d\alpha \\ &\leqq \int_0^\infty \sum_{n=0}^\infty e^{-|z|\sqrt{a_m\alpha^2+b_m\cdot n^2}}\,A_n(y)\,d\alpha\,. \end{aligned} \tag{A-19}$$

On the other hand, we have

$$\sum_{n=0}^\infty A_n(y)\int_0^\infty e^{-|z|\sqrt{a_m\alpha^2+b_m n^2}}\,d\alpha = \sum_{n=0}^\infty A_n(y)\sqrt{\frac{b_m z^2 n^2}{a_m}}\,K_1(\sqrt{b_m}\,|z|\,n), \tag{A-20}$$

where K_1 denotes the modified Bessel function of the second kind and of order one. The convergence of the infinite series on the right-hand side of Eq. (A-20) is self-evident from the asymptotic property of K_1 for large argument. Therefore from the Hardy's theorem the convergence of the improper integral on the right-hand side of the inequality (A-19) is proved.

Appendix III

Deduction of Eq. (31′)

Using the expression of p given by Eq. (30), we get the z-component of the induced velocity as follows:

$$\frac{w}{U}=-\frac{1}{\kappa p_{-\infty}M_{-\infty}^2}\int_{-\infty}^{x}dx\int_0^\infty \cos\alpha x\sum_{n=0}^{\infty}\beta_n(\alpha)e^{-\beta_n(\alpha)|z|}F_n(\alpha)Y_n(y;\alpha)d\alpha$$
$$=-\frac{1}{\kappa p_{-\infty}M_{-\infty}^2}\left[\int_0^\infty \frac{\sin\alpha x}{\alpha}\sum_{n=0}^{\infty}\beta_n(\alpha)e^{-\beta_n(\alpha)|z|}F_n(\alpha)Y_n(y;\alpha)d\alpha\right.$$
$$\left.-\lim_{x\to\infty}\int_0^\infty \frac{\sin\alpha x}{\alpha}\sum_{n=0}^{\infty}\beta_n(\alpha)e^{-\beta_n(\alpha)|z|}F_n(\beta)Y_n)y;\alpha)d\alpha\right], \tag{A-21}$$

where the assumption of $w/U=0$ at $x=-\infty$ is used. Then from the Fourier's single integral theorem it follows that

$$\lim_{x\to\infty}\int_0^\infty \frac{\sin\alpha x}{\alpha}\sum_{n=0}^{\infty}\beta_n(\alpha)e^{-\beta_n(\alpha)|z|}F_n(\alpha)Y_n(y;\alpha)d\alpha$$
$$=\frac{\pi}{2}\sum_{n=1}^{\infty}\beta_n(0)e^{-\beta_n(0)|z|}F_n(0)Y_n(y;0), \tag{A-22}$$

where $\beta_n(0)=0$ is taken into account. On the other hand, it is suggested from Appendix I that $\beta_n(\alpha)/\alpha$ tends to some finite value , that is to say,

$$\beta_n(\alpha)/\alpha\to g_n+O(\alpha^{-2}) \qquad \text{as} \quad \alpha\to\infty. \tag{A-23}$$

Then Eq. (A-21) can be rewritten as

$$\frac{w}{U}=-\frac{1}{\kappa p_{-\infty}M_{-\infty}^2}\left(w_1'+w_2'+\frac{\pi}{2}\sum_{n=0}^{\infty}\beta_n(0)e^{-\beta_n(0)|z|}F_n(0)Y_n(y;0)\right), \tag{A-24}$$

where

$$\left.\begin{aligned} w_1'=\int_0^\infty \sin\alpha x\sum_{n=0}^{\infty}\Big[&\frac{\beta_n(\alpha)}{\alpha}e^{-\beta_n(\alpha)|z|}F_n(\alpha)Y_n(y;\alpha)\\ &-g_ne^{-g_n|z|\alpha}F_n(\infty)Y_n(y;\infty)\Big]d\alpha,\end{aligned}\right\} \tag{A-25}$$

$$\left.\begin{aligned} w_2'&=\int_0^\infty \sin\alpha x\sum_{n=0}^{\infty}g_n\cdot e^{-g_n|z|\alpha}F_n(\infty)Y_n(y;\infty)d\alpha\\ &=\sum_{n=0}^{\infty}g_nF_n(\infty)Y_n(y;\infty)\frac{x}{x^2+g_n^2z^2}.\end{aligned}\right\} \tag{A-26}$$

It is seen that w_1' tends to zero uniformly as $x\to0$ for any real value of z, while w_2' has a singularity on the y-axis. In order to get the induced velocity at the lifting-line, the portion of w_2' must be excluded, because w_2' denotes the effect of the lifting-line itself and hence w_2' tends to infinity at $z=0$ as $x\to0$. Therefore we get the expression of the z-component of the induced velocity on the lifting-line as follows:

$$\left[\frac{w}{U}\right]_{\substack{x=0\\z=0}}=-\frac{\pi}{2\kappa p_{-\infty}M_{-\infty}^2}\sum_{n=0}^{\infty}\beta_n(0)F_n(0)Y_n(y;0). \tag{A-27}$$

Appendix IV

Eigenvalues and Eigenfunctions for Small α

Eqs. (23) and (24) can be reproduced as

$$L(Y_n)+\frac{\beta_n^2}{M_{-\infty}^2}Y_n-\frac{1-M_{-\infty}^2}{M_{-\infty}^2}\alpha^2Y_n=0, \tag{A-28}$$

$$\frac{dY_n}{dy}=0 \quad \text{at} \quad y=0,\lambda, \tag{A-29}$$

where

$$L(Y_n)\equiv\frac{d}{dy}\left(\frac{1}{M_{-\infty}^2}\frac{dY_n}{dy}\right). \tag{A-30}$$

We assume that the eigenfunctions Y_n and the eigenvalues β_n^2 can be expanded in the power serieses of α^2 near $\alpha=0$ as follows:

$$Y_n=Y_n^{(0)}+\alpha^2Y_n^{(1)}+\alpha^4Y_n^{(2)}+\cdots, \tag{A-31}$$

$$\beta_n^2=\beta_n^{(0)2}+\alpha^2\beta_n^{(1)2}+\alpha^4\beta_n^{(2)2}+\cdots. \tag{A-32}$$

Substituting Eqs. (A-31) and (A-32) into Eq. (A-28) and equating coefficients of like powers of α^2 to zero, we have a set of the differential equations

$$L(Y_n^{(0)})+\frac{\beta_n^{(0)2}}{M_{-\infty}^2}Y_n^{(0)}=0, \tag{A-33-1}$$

$$L(Y_n^{(1)})+\frac{\beta_n^{(0)2}}{M_{-\infty}^2}Y_n^{(1)}=\frac{1-M_{-\infty}^2}{M_{-\infty}^2}Y_n^{(0)}-\frac{\beta_n^{(1)2}}{M_{-\infty}^2}Y_n^{(0)}, \tag{A-33-2}$$

$$L(Y_n^{(2)})+\frac{\beta_n^{(0)2}}{M_{-\infty}^2}Y_n^{(2)}=\frac{1-M_{-\infty}^2}{M_{-\infty}^2}Y_n^{(1)}-\frac{1}{M_{-\infty}^2}(\beta_n^{(1)2}Y_n^{(1)}+\beta_n^{(2)2}Y_n^{(0)}), \tag{A-33-3}$$

....

The boundary condition (A-29) becomes

$$\frac{dY_n^{(m)}}{dy}=0 \quad \text{at} \quad y=0,\lambda \quad (m=0,1,2,\cdots). \tag{A-34}$$

For convenience, the condition of normalization is to be given by

$$\frac{1}{\lambda}\int_0^{\lambda}\frac{Y_n^{(0)2}}{M_{-\infty}^2}dy=1. \tag{A-35}$$

If the zero-th approximation $\beta_n^{(0)}$ and $Y_n^{(0)}$ are exactly determined from Eqs. (A-33-1), (A-34) and (A-35), then the higher order approximation is obtained as follows:

Multiply Eq. (A-33-1) from the left by $Y_m^{(1)}$. Next, replace n in Eq. (A-33-2) by m and multiply it from the left by $Y_n^{(0)}$. Substract the latter from the former and integrate from 0 to λ, then

$$\int_0^\lambda \{Y_m^{(1)}L(Y_n^{(0)})-Y_n^{(0)}L(Y_m^{(1)})\}\,dy+(\beta_n^{(0)2}-\beta_m^{(0)2})\int_0^\lambda \frac{Y_n^{(0)}Y_m^{(1)}}{M_{-\infty}^2}\,dy$$
$$=-\int_0^\lambda \frac{1-M_{-\infty}^2}{M_{-\infty}^2}\,Y_m^{(0)}Y_n^{(0)}dy+\beta_n^{(1)2}\int_0^\lambda \frac{Y_n^{(0)}Y_m^{(0)}}{M_{-\infty}^2}\,dy. \tag{A-36}$$

The first term on the left-hand side of Eq. (A-36) may be evaluated by application of Green's theorem and the boundary condition (A-34), so that

$$\int_0^\lambda \{Y_m^{(1)}L(Y_n^{(0)})-Y_n^{(0)}L(Y_m^{(1)})\}\,dy=\left[\frac{1}{M_{-\infty}^2}\,Y_m^{(1)}\frac{dY_n^{(0)}}{dy}-Y_n^{(0)}\frac{dY_m^{(1)}}{dy}\right]_0^\lambda=0. \tag{A-37}$$

From the orthogonality of $Y_n^{(0)}$ and the normalization given by Eq. (A-35), we have

$$\frac{1}{\lambda}\int_0^\lambda \frac{Y_n^{(0)}Y_m^{(0)}}{M_{-\infty}^2}\,dy=\delta_{m,n}. \tag{A-38}$$

Inserting these results into Eq. (A-36) yields

$$(\beta_n^{(0)2}-\beta_m^{(0)2})\frac{1}{\lambda}\int_0^\lambda \frac{Y_n^{(0)}Y_m^{(1)}}{M_{-\infty}^2}\,dy=-\frac{1}{\lambda}\int_0^\lambda \frac{1-M_{-\infty}^2}{M_{-\infty}^2}\,Y_m^{(0)}Y_n^{(0)}dy+\beta_n^{(1)2}\delta_{m,n}. \tag{A-39}$$

Therefore we get

$$\beta_n^{(1)2}=\frac{1}{\lambda}\int_0^\lambda \frac{1-M_{-\infty}^2}{M_{-\infty}^2}\,\{Y_n^{(0)}\}^2dy, \tag{A-40}$$

and

$$\frac{1}{\lambda}\int_0^\lambda \frac{Y_m^{(1)}Y_n^{(0)}}{M_{-\infty}^2}\,dy=-\frac{1}{\beta_n^{(0)2}-\beta_m^{(0)2}}\,\frac{1}{\lambda}\int_0^\lambda \frac{1-M_{-\infty}^2}{M_{-\infty}^2}\,Y_m^{(0)}Y_n^{(0)}dy. \tag{A-41}$$

Accordingly, if $Y_n^{(1)}$ is expanded in the series of the orthogonal eigenfunctions $Y_n^{(0)}$, the coefficient for $Y_n^{(0)}$ is to be given by Eq. (A-41), so that

$$Y_n^{(1)}=\sum_{m\neq n} Y_m^{(0)}\frac{1}{\beta_n^{(0)2}-\beta_m^{(0)2}}\,\frac{1}{\lambda}\int_0^\lambda \frac{1-M_{-\infty}^2}{M_{-\infty}^2}\,Y_m^{(0)}Y_n^{(0)}dy. \tag{A-42}$$

Thus the first approximation has been obtained. The higher order approximations can be obtained in the same way.

Now, we can get the first term of the zero-th approximation from Eqs. (A-33-1), (A-34) and (A-35) as

$$Y_0^{(0)}=\left(\frac{1}{\lambda}\int_0^\lambda \frac{1}{M_{-\infty}^2}\,dy\right)^{-1/2}, \tag{A-43}$$

$$\beta_0^{(0)} = 0. \tag{A-44}$$

Therefore substitution of Eq. (A-43) into Eq. (A-40) gives

$$\beta_0^{(1)} = 1 - \left(\frac{1}{\lambda}\int_0^{\lambda}\frac{1}{M_{-\infty}^2}dy\right)^{-1}. \tag{A-45}$$

From these results it follows that, according as α approaches zero,

$$\beta_0^2 \sim \alpha^2\mu + O(\alpha^4), \tag{A-46}$$

where

$$\mu = 1 - \left(\frac{1}{\lambda}\int_0^{\lambda}\frac{1}{M_{-\infty}^2}dy\right)^{-1}, \tag{A-47}$$

or by using the definition of Eq. (50)

$$\mu = 1 - \overline{M}{}^{*2}_{-\infty}. \tag{A-48}$$

APPENDIX V
The Function $K(M_{-\infty}^{(2D)}, \gamma, t)$

If the flow is two-dimentional with the uniform Mach number $M_{-\infty}^{(2D)}$, then the upwash at the point $x = x_0$, $z = 0$ induced by the cascade of lifting-lines like that in Fig. 1 can be obtained by applying Eq. (69) to the two-dimensional flow in the form

$$\left[\frac{w}{U}\right]_{\substack{z=0\\x=x_0}} = -\frac{\pi}{\kappa p_{-\infty} M_{-\infty}^{(2D)2}} \frac{\mu\cos\gamma}{\mu\cos^2\gamma + \sin^2\gamma}\frac{1}{t} + \mu^{-\frac{1}{2}}\left(\frac{1}{x_0} + I\right)F_0 M_{-\infty}^{(2D)}, \tag{A-49}$$

where

$$\mu = 1 - M_{-\infty}^{(2D)2}, \tag{A-50}$$

and

$$\begin{aligned} I &\equiv x_0\sum_{n=1}^{\infty}\int_0^{\infty}\sin(\alpha x_0)\cos(nt\sin\gamma\cdot\alpha)e^{-\sqrt{\mu}\,\alpha n t\cos\gamma}d\alpha \\ &\equiv \sum_{n=1}^{\infty}\frac{n^2t^2(\mu\cos^2\gamma + \sin^2\gamma) + x_0^2 - 2n^2t^2\sin^2\gamma}{\{n^2t^2(\mu\cos^2\gamma + \sin^2\gamma) + x_0^2\}^2 - 4n^2t^2\sin^2\gamma}. \end{aligned} \tag{A-51}$$

The boundary condition which ensures the agreement of the direction of the streamline with the tangential direction of the airfoil at the point $x = x_0$, $z = 0$ can be expressed as

$$\left[\frac{w}{U}\right]_{\substack{z=0\\x=x_0}} = -\alpha_{-\infty}, \tag{A-52}$$

where $\alpha_{-\infty}$ is the angle between the x-axis or far upstream flow direction and the tangential direction of the thin airfoils at the chord point of $x=x_0$, $z=0$. Then from Eqs. (A-49) and (A-52) the coefficient F_0 can be determined as

$$F_0=\kappa p_{-\infty}M_{-\infty}^{(2D)}\cdot\alpha_{-\infty}\left\{\frac{\pi\mu\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}+\mu^{\frac{1}{2}}\left(\frac{1}{x_0}+I\right)\right\}^{-1}. \tag{A-53}$$

Since the lift force in the two-dimensional flow can be given by $2\pi F_0M_{-\infty}^{(2D)}$, we have the expression of the lift coefficient $C_L^{(2D)}$ in the form

$$C_L^{(2D)}=\frac{2\pi F_0M_{-\infty}^{(2D)}}{\frac{1}{2}\kappa p_{-\infty}M_{-\infty}^{(2D)2}}$$

$$=4\pi\alpha_{-\infty}\left\{\frac{\pi\mu\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}+\mu^{\frac{1}{2}}\left(\frac{1}{x_0}\right)\right\}^{-1}. \tag{A-54}$$

On the other hand, the tangent of the streamline at the lifting-lines can be obtained by applying Eq. (59) to the two-dimensional flow in the form

$$\left[\frac{w}{U}\right]_{\substack{x=0\\z=0}}=-\frac{1}{\kappa p_{-\infty}M_{-\infty}^{(2D)}}\frac{\pi u\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}F_0. \tag{A-55}$$

If the zero-lift angle is equal to $-\alpha_{-\infty}$, then the effective angle of attack $\alpha_0^{(2D)}$ with respect to the zero-lift angle is given by

$$\alpha_0^{(2D)}=\alpha_{-\infty}+\left[\frac{w}{U}\right]_{\substack{x=0\\z=0}}$$

$$=\alpha_{-\infty}\mu^{\frac{1}{2}}\left(\frac{1}{x_0}+I\right)\left\{\frac{\pi u\cos\gamma}{\mu\cos^2\gamma+\sin^2\gamma}\frac{1}{t}\mu^{\frac{1}{2}}\left(\frac{1}{x_0}+I\right)\right\}^{-1}. \tag{A-56}$$

From Eqs. (A-54) and (A-56), we have

$$C_L^{(2D)}=\frac{2\pi}{\mu^{\frac{1}{2}}}\left(\frac{1}{2x_0}+\frac{1}{2}I\right)^{-1}\alpha_0^{(2D)}. \tag{A-57}$$

Therefore the function K can be represented by

$$K(M_{-\infty}^{(2D)},\gamma,t)\equiv C_L^{(2D)}/\alpha_0^{(2D)}=\frac{2\pi}{\mu^{\frac{1}{2}}}\left(\frac{1}{2x_0}+\frac{1}{2}I\right)^{-1}. \tag{A-58}$$

If we put $x_0=1/2$, then Eq. (A-58) gives the expression of the slope of the lift coefficient for thin blades in the cascade in the two-dimensional subsonic flow based on the so-called Weissingers's method.