Institute of Space and Aeronautical Science, University of Tokyo
Report No. 416, October 1967

The Effect of Atmospheric Nonuniformity On Sonic Boom Intensities

By

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Summary: A theoretical investigation is presented on the effect of atmospheric nonuniformity on the decay of shock waves generated by a supersonic aircraft in steady level flight with special reference to sonic boom problem.

At the outset, analysis is made for an axisymmetric body in an adiabatic atmosphere, together with two-dimensional body case for the sake of reference. The decay rule of a bow shock due to atmospheric nonuniformity is obtained by the application of Whitham's theory on the behavior of shock waves at large distances from the body in a uniform medium to the case of a stratified atmosphere. Numerical calculations are carried out at flight Mach numbers from 1.3 to 3.0 and flight altitudes up to 10 km.

Extension of the above-mentioned technique makes it possible to obtain the decay of a bow shock in an arbitrary stratified atmosphere and a numerical calculation is made for the case of an axisymmetric body in a standard atmosphere at the same flight Mach numbers as above and flight altitudes up to 20 km. The result shows good agreement with existing flight test-data and other theoretical result by different method.

SYMBOLS

- c: speed of sound
- c_p, c_v: specific heats at constant pressure and volume
- g: acceleration due to gravity
- $h(\xi)$: nondimensional shape function of two-dimensional body
- i: specific enthalpy
- K_r: reflection factor
- l: body length
- m, m': $1/(\gamma-1)$, $1/(n-1)$
- M: Mach number
- n: polytropic exponent
- p: pressure
- Δp: pressure jump across a shock wave
- $\boldsymbol{v}$, q: flow velocity vector and its magnitude
- $r_b(\xi)$: nondimensional radius of the body of revolution
- R: wave decay rate due to atmospheric nonuniformity
- $\mathscr{R}$: gas constant of air

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s: specific entropy
T: absolute temperature
U: free stream velocity
v_x, v_y, v_z: components of nondimensional velocity in Cartesian coordinate system
v_x, v_r, v_θ: components of nondimensional velocity in cylindrical coordinate system
(x, y, z): Cartesian coordinate system
(x, r, θ): cylindrical coordinate system
z^*: distance measured downwards from the upper boundary of a polytropic atmosphere
α: $\beta l/T_a$
β: atmospheric lapse rate
γ: ratio of specific heats
δ: flow deflection angle
μ: Mach angle
ξ, ζ, η: nondimensional coordinates, x/l, z/l and r/l, respectively
η_c: nondimensional distance between aircraft and tropopause
ρ: density
τ: exact characteristic variable
Φ: velocity potential
$U\phi$: disturbance velocity potential
ω: function related to Mach line
$\boldsymbol{\omega}$: vorticity
Ω: gravitation potential

SUBSCRIPTS

∞: value in undisturbed field
a: value at flight altitude
g: value on the ground
x, y, z: differentiation with respect to x, y and z, respectively

A prime denotes disturbance quantity or differentiation in case by case, and a bar on a symbol indicates the value in the vertical plane including flight path.

1. INTRODUCTION

The importance of sonic boom phenomena has recently been taken cognizance of increasingly prior to the introduction of the supersonic transport into commercial air service. This is understood partly from the fact that the number of papers on sonic boom problems published in the past ten years amounts to near one handred involving ones on community reaction and the like. Most of the studies on generation and propagation of sonic boom made so far, however, do not take into account the complete influences of variation of atmospheric condition with altitude, except for acoustic treatment of the refraction of rays due to the gradient of atmospheric temperature [*1*], [*2*].

The far-field behavior of shock waves in a stationary, uniform medium produced by a supersonic projectile in steady level flight was investigated by G. B. Whitham in 1952. The essential part of his theory lies in deriving a higher approximation than the linearized one by the use of so-called Whitham's technique. He applied the method first to the case of body of revolution in steady flight [*3*], and subsequently extended it to the case of thin symmetric wing[*4*]. P. S. Rao is the first one who applied Whitham's method to the sonic boom problem. He dealt with the case of accelerating flight of a body of revolution in a uniform atmosphere [*5*].

Although Whitham's theory itself was not directly derived for the solution of the sonic boom problem, it is quite useful in constructing sonic boom theory because of its well description of the behavior of weak shock waves in the far-field from the origin. Thus, his theory has led up the approach to estimate sonic boom intensities caused by supersonic aircrafts in steady level flight. However, since his theory originally refers to a uniform atmosphere, it can not be applied without modification to the actual atmosphere in which pressure, temperature and density vary with altitude.

The modification mainly used so far is simply to replace the uniform reference pressure in Whitham's theory by the geometric mean of the atmospheric pressures at flight altitude, p_a, and on the ground, p_g. As a result, the following formula generally used to estimate sonic boom intensities on the ground is obtained:

$$\frac{\Delta p}{p_g}=K_r\sqrt{\frac{p_a}{p_g}}\frac{2^{\frac{1}{4}}\gamma}{(\gamma+1)^{\frac{1}{2}}}(M_a^2-1)^{\frac{1}{8}}\left[\int_0^{\lambda_0}F(\lambda)\,d\lambda\right]^{\frac{1}{2}}h^{-\frac{3}{4}} \tag{1.1}$$

where K_r is the reflection factor on the ground, M_a the flight Mach number, γ the specific heats ratio, h the flight altitude, and the integral is a quantity depending only on the aircraft shape. The above formula participates in a so-called volume boom.

Although it has been well confirmed by various flight tests [*6*] that Eq. (1.1) gives us fairly good estimation of sonic boom intensity, it is still felt that the modification mentioned above is in short of full theoretical foundation, and, therefore, the present study aims at developing a theory to account for the effect of atmospheric nonuniformity on shock wave propagation. In 1963, M. P. Friedman and his coworkers constructed a numerical method of treating sonic boom problem in general form by use of ray-shock theory [*7*]. Comparison of the result of the present study with that of Friedman's will be made later.

In the present study, an attempt is made to extend Whitham's theory to the case of nonuniform atmosphere and obtain theoretical correction factor for atmospheric nonuniformity in the sonic boom problem. Attention is restricted only to volume boom, since lift boom may be treated by introducing the concept of an equivalent body of revolution by the use of supersonic area rule [*8*]. In the present analysis the influence of atmospheric winds is not considered.

In Chapter 2 survey is made of the structure and the state of the atmosphere, and in Chapter 3 the relation between atmospheric nonuniformity and creation of vorticity due to disturbances is discussed. As a result, it is verified that a potential flow is possible in the whole field only in case of a small disturbance flow in an adiabatic atmosphere with a special lapse rate as to make the whole flow field isoenergetic. In Chapter 4 the fundamental equation common to various kinds of atmosphere is derived. In the first place, the assumption of an adiabatic atmosphere is made in order to make analysis easy by use of velocity potential, and linearized potential equations including the effect of atmospheric nonuniformity is derived for a thin two-dimensional body as well as a slender axisymmetric body. Next, the linearized equation for an actual atmosphere is derived for a slender body of revolution. In Chapter 5 the solutions are obtained for an adiabatic atmosphere by the application of Whitham's technique and the expressions of sonic boom intensities are given for two-dimensional and axisymmetric bodies. In the latter case, analysis is valid only in the field near the vertical plane including the flight path. In Chapter 6, is presented a general treatment of the problem in an arbitrarily nonuniform atmosphere where velocity potential no more exists. Numerical results is obtained for the case of an axisymmetric body in the standard atmosphere including the stratosphere.

2. Atmospheric States

The phenomena of sonic boom are brought about due to the propagation of shock waves through an atmosphere. Therefore, the study of the phenomena requires the knowledge of atmospheric states. An atmosphere is in stratified layers and consists of four spheres which are called, upwards from the earth, troposphere, stratosphere, ionosphere and exosphere. The service flight altitude of aircrafts is anticipated up to about 20 km even though SST's are considered of. Therefore, it is sufficient in the present study to deal with a troposphere and a stratosphere about 35 km above the sea level. If aircrafts would fly at higher altitudes, sonic booms would become much weaker on the ground, and so do not give rise to any public discussion.

According to the standard atmosphere established by ICAO in 1952, a troposphere is defined as a layer 11 km in thickness above the sea level, in which the atmospheric temperature decrease with altitude at the rate of 0.0065°C/m from 15°C at the sea level, and the temperature in the bottom region of a stratosphere is prescrived to be −56.5°C independent of altitude. In addition of such temperature variations, its pressure and density also change keeping static equilibrium everywhere under the action of gravity. Since atmospheric phenomena change in a very complicated way, it is difficult to describe the state of an actual atmosphere exactly by theoretical method.

For the sake of simplicity of calculations are often treated hypothetical atmospheres composed of perfect gases, namely, homogeneous, isothermal, adiabatic and polytropic atmospheres. An isothermal atmosphere corresponds to the state of stratosphere, and a polytropic atmosphere with a constant lapse rate does to that of a troposphere. The atmosphere with a temperature gradient can not be in thermal equilibrium, but in mechanical equilibrium. Then a question arises whether or not the equilibrium of state can be stable; when disturbances are added to break the equilibrium, it is said to be stable if the displaced air particle tends to return to its original position. Otherwise, the equilibrium is said to be unstable, and this leads to the occurrence of convection which tends to mix the medium in the direction of a uniform state.

For a polytropic atmosphere there is a relation $p=K\rho^n$ between pressure and density, and other kinds of atmospheres are special cases of this atmosphere. For instance, the case of $n=1$ corresponds to an isothermal atmosphere, and the case of $n=\gamma$ does to an adiabatic atmosphere where γ is the ratio of specific heats for air. By simple calculation, the lapse rate β of a polytropic atmosphere is found to be $(n-1)g/(n\mathscr{R})$, and further investigation of equilibrium shows that the mechanical equilibrium is stable only when $\beta<(\gamma-1)g/(\gamma\mathscr{R})$. From this fact it can be said that an adiabatic atmosphere is in neutral equilibrium. For $\gamma=1.40$ and for $\mathscr{R}=287.0\ \mathrm{m^2/sec^2\ {}^\circ C}$, the lapse rate of an adiabatic atmosphere becomes 0.00976°C/m which is larger than the value 0.0065°C/m of troposphere in a standard atmosphere. In the troposphere of a standard atmosphere with a polytropic exponent $n=1.24$, no convection is caused by a disturbance, because the equilibrium is stable for the case $n \leqq 1.40$ from the discussion given above.

The fact that the atmospheric temperature in a polytropic atmosphere decreases with altitude suggests the existance of the upper limit of the atmosphere. Now let z^* be a distance measured vertically downwards from the level where the absolute temperature of the atmosphere vanishes (Fig. 1), T_∞ be the absolute temperature at the point z^* and β be the atmospheric temperature gradient. Then we can write

$$T_\infty=\beta z^*. \tag{2.1}$$

Therefore, the speed of sound at this point c_∞ becomes

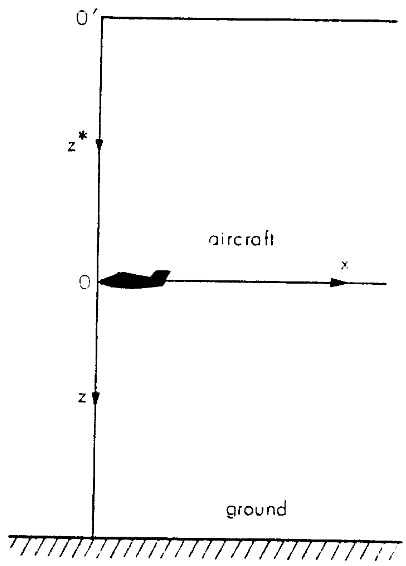

FIG. 1. Positional relation between coordinate systems for analysis of sonic boom and for expression of atmospheric state.

$$c_{\infty}^2=\gamma\mathscr{R}T_{\infty}=\gamma\mathscr{R}\beta z^*. \tag{2.2}$$

Further, if the origin of the coordinate system (x, y, z) is chosen at the nose of a body as shown in Fig. 1, Eq. (2.2) is rewritten by use of the temperature T_a and speed of sound c_a at the flight altitude as

$$c_{\infty}^2=c_a^2\left(1+\frac{\beta z}{T_a}\right) \tag{2.3}$$

where β is equal to the aforementioned lapse rate $(n-1)g/(n\mathscr{R})$.

3. Relation Between Vorticity and Atmospheric Nonuniformity

In case of a uniform atmosphere, entropy change through a shock wave is of order of the cube of the velocity or pressure change. When the shock wave is weak, entropy change may be safely neglected in the approximation where smaller order than the cube of the changes of these quantities is allowed to be neglected. Hence the flow field can be considered as an isentropic one and can be treated as a potential flow without vorticity. However, in considering more general cases, such as an actual atmosphere, the treatment of potential flow is not always permitted even in the same approximation as mentioned above. Next, discussions are made on this point.

The atmosphere considered is assumed to be composed of a perfect gas with constant specific heats and without viscosity and heat conductivity. Such a phenomenon as a sonic boom takes place in a so extensive region that the variation of density due to gravity force has to be taken into account. Therefore, the gravity force term must be added to the fundamental equations in the ordinary aerodynamic. The required equations are given as follows:

$$\text{continuity:} \quad \frac{\partial \rho}{\partial t}+\operatorname{div}\rho\boldsymbol{v}=0, \tag{3.1}$$

$$\text{momentum:} \quad \frac{D\boldsymbol{v}}{Dt}=-\operatorname{grad}\Omega-\frac{1}{\rho}\operatorname{grad}p, \tag{3.2}$$

$$\text{entropy:} \quad \frac{Ds}{Dt}=0, \tag{3.3}$$

$$\text{state:} \quad p=k\rho^{\gamma}\exp(s/c_v) \tag{3.4}$$

where Ω is gravitation potential, and s is specific entropy. Assuming the atmosphere to be adiabatic, let the coordinate system shown in Fig. 1 move in the holizontal direction of the negative x-axis at a velocity U and we observe the phenomenon from this system, where y-axis is taken perpendicularly to the x-z plane and the origin at the nose of the body.

Now, using Eq. (3.2) and the thermal relation, we get

$$\boldsymbol{v}\times\boldsymbol{\omega}-\operatorname{grad}\left(\frac{1}{2}q^2+\Omega+i\right)+T\operatorname{grad}s=0 \tag{3.5}$$

for the steady state. This equation is called Crocco's theorem which gives the relation between entropy and vorticity as $\boldsymbol{\omega} = \mathrm{rot}\, \boldsymbol{v}$. Provided that the strength of a shock wave is so weak that the entropy jump across it is safely neglected, the entropy becomes constant everywhere in the whole flow field by Eq. (3.3) because the entropy ahead of the shock wave is uniform in the adiabatic atmosphere and the last term in the left hand side of Eq. (3.5) vanishes. Making scalar product of Eq. (3.5) and $\boldsymbol{v}$, we get

$$\boldsymbol{v} \cdot \mathrm{grad}\left(\frac{1}{2} q^2 + \Omega + i\right) = 0. \tag{3.6}$$

Hence, along a streamline,

$$\frac{1}{2} q^2 + \Omega + i = \text{const.} \tag{3.7}$$

Here, the gravitation potential Ω is expressed as

$$\Omega = -gz^* \tag{3.8}$$

and the enthalpy i is

$$i = c_p T + \text{const} \tag{3.9}$$

where c_p is the specific heat at constant pressure. From Eq. (2.1)

$$T_\infty = \frac{(\gamma - 1)g}{\gamma \mathscr{R}} z^* \tag{3.10}$$

for an adiabatic atmosphere, and therefore in the upper stream with constant velocity U, the left hand side of Eq. (3.7) becomes, irrespective of the altitude,

$$\frac{1}{2} U^2 - gz^* + c_p T_\infty \equiv \frac{1}{2} U^2. \tag{3.11}$$

Finally we conclude that Eq. (3.7) is valid in the whole flow field and the constant in Eq. (3.7) is equal to $\frac{1}{2}U^2$. Rewriting Eq. (3.7) by use of the local speed of sound c,

$$\frac{1}{2} q^2 - gz^* + \frac{c^2}{\gamma - 1} = \frac{1}{2} U^2 \tag{3.12}$$

which holds in the whole flow field. In this case Eq. (3.5) becomes simply

$$\boldsymbol{v} \times \boldsymbol{\omega} = 0. \tag{3.13}$$

This formula implies either the flow is irrotational or the velocity and vorticity vectors are parallel to each other. Only in two-dimensional flow, where the vorticity vector is perpendicular to the velocity vector, Eq. (3.13) implies the flow is irrotational.

Thus, under the assumption of an adiabatic atmosphere where the vorticity is zero in the undisturbed field, Eq. (3.13) indicates that the flow is everywhere irrotational, and we can make use of the velocity potential analysis in approximation of neglecting the entropy change across the shock wave. The entropy for other than adiabatic atmospheres varies with altitude and the field is not homo-energetic. Therefore, when disturbances are added to the flow, the vorticities will be generated even if the state of the undisturbed flow is stable, and the treatment as irrotational flow fails.

These conclusions are also deduced from the application of Kelvin's theorem for a compressible fluid.

4. Fundamental Equations

When the components of the velocity vector $\boldsymbol{v}$ along each axis in the coordinates (x, y, z) shown in Fig. 1 are denoted by (v_x, v_y, v_z), Eqs. (3.1) to (3.4) become in a steady flow

continuity: $$\frac{\partial \rho v_x}{\partial x}+\frac{\partial \rho v_y}{\partial y}+\frac{\partial \rho v_z}{\partial z}=0, \tag{4.1}$$

momentum: $$v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{4.2a}$$

$$v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{4.2b}$$

$$v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}=g-\frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{4.2c}$$

entropy: $$v_x\frac{\partial}{\partial x}(p\rho^{-\gamma})+v_y\frac{\partial}{\partial y}(p\rho^{-\gamma})+v_z\frac{\partial}{\partial z}(p\rho^{-\gamma})=0. \tag{4.3}$$

In the undisturbed stream we have

$$\rho_\infty g-\frac{\partial p_\infty}{\partial z}=0,\ \frac{\partial p_\infty}{\partial y}=0,\ \frac{\partial p_\infty}{\partial x}=0, \tag{4.4}$$

$$p_\infty=\mathscr{R}\rho_\infty T_\infty. \tag{4.5}$$

Next, as shown in Fig. 2, defining a cylindrical coordinate system (x, r, θ) with an origin at the nose of a body, the above equations are converted into the following form by using the relations

$$y=r\sin\theta,\quad z=r\cos\theta,$$

$$r=\sqrt{y^2+z^2},\quad \theta=\tan^{-1}(y/z).$$

Continuity: $$\frac{\partial(\rho v_x)}{\partial x}+\frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta}=0, \tag{4.6}$$

momentum: $$v_x\frac{\partial v_x}{\partial x}+v_r\frac{\partial v_x}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_x}{\partial \theta}=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{4.7a}$$

$$v_x\frac{\partial v_r}{\partial x}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}=g\cos\theta-\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{4.7b}$$

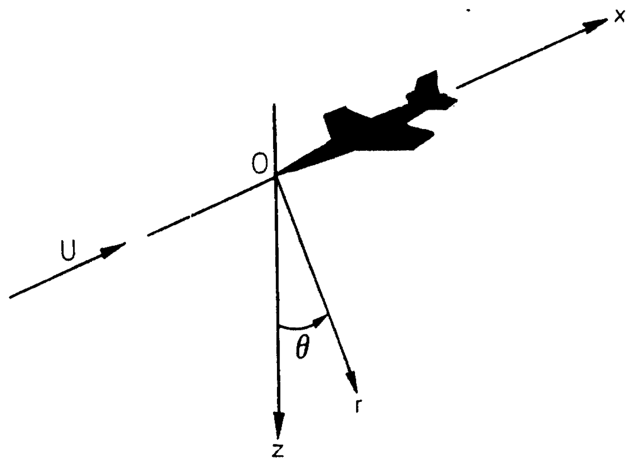

FIG. 2. Coordinate system for an axisymmetric body.

$$v_x \frac{\partial v_\theta}{\partial x} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = -g \sin\theta - \frac{1}{\rho} \frac{\partial p}{r \partial \theta}, \qquad (4.7c)$$

entropy: $$v_x \frac{\partial}{\partial x}(p\rho^{-\gamma}) + v_r \frac{\partial}{\partial r}(p\rho^{-\gamma}) + \frac{v_\theta}{r} \frac{\partial}{\partial \theta}(p\rho^{-\gamma}) = 0, \qquad (4.8)$$

$$\rho_\infty g \sin\theta + \frac{1}{r} \frac{\partial p_\infty}{\partial \theta} = 0, \quad -\rho_\infty g \cos\theta + \frac{\partial p_\infty}{\partial r} = 0, \qquad (4.9)$$

$$p_\infty = \mathscr{R} \rho_\infty T_\infty. \qquad (4.5)$$

Next, these equations will be linearized for the cases of an adiabatic atmosphere and the standard atmosphere.

I. The Case of an Adiabatic Atmosphere

Since in this case the flow field is homentropic according to the assumption made in Chapter 3, the energy equation is always satisfied and velocity potential exists in a small disturbance flow.

(i) Two-Dimensional Body Case

Prior to more general case, two-dimensional problem, though not realistic, is considered here. In this case the flow does not change in the direction of y-axis. Using the relations

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{1}{\rho} \left(\frac{\partial p}{\partial \rho} \right)_s \frac{\partial \rho}{\partial x} = \frac{c^2}{\rho} \frac{\partial \rho}{\partial x}, \qquad (4.10a)$$

$$\frac{1}{\rho} \frac{\partial p}{\partial z} = \frac{c^2}{\rho} \frac{\partial \rho}{\partial z}, \qquad (4.10b)$$

the following gas-dynamic equation is obtained by eliminating p and ρ from the equation of motion (4.2) and the continuity equation (4.1).

$$(c^2 - v_x^2) \frac{\partial v_x}{\partial x} + (c^2 - v_z^2) \frac{\partial v_z}{\partial z} - v_x v_z \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) + g v_z = 0. \qquad (4.11)$$

Let the velocity potential be denoted by Φ, and the velocity components are expressed as

$$v_x=\frac{\partial\Phi}{\partial x},\qquad v_z=\frac{\partial\Phi}{\partial z}.$$

Then Eq. (4.11) becomes

$$(c^2-\Phi_x^2)\Phi_{xx}+(c^2-\Phi_z^2)\Phi_{xx}-2\Phi_x\Phi_z\Phi_{zz}+g\Phi_z=0 \tag{4.12}$$

where the subscripts indicate differentiation with respect to each variable. Further by using a disturbance potential, $U\phi$, which satisfies

$$\Phi=U(x+\phi), \tag{4.13}$$

Eq. (4.12) is written in the following from:

$$\frac{c^2}{U^2}(\phi_{xx}+\phi_{zz})-(1+\phi_x)^2\phi_{xx}-\phi_z^2\phi_{zz}-2\phi_z(1+\phi_x)\phi_{zx}+\frac{g}{U^2}\phi_z=0. \tag{4.14}$$

From Eq. (3.12), which is Bernoulli's equation, the local speed of sound is given as

$$c^2=c_\infty^2-\frac{\gamma-1}{2}(q^2-U^2)=c_\infty^2-\frac{\gamma-1}{2}U^2(2\phi_x+\phi_x^2+\phi_z^2) \tag{4.15}$$

where the relation $q^2=\Phi_x^2+\Phi_z^2$ is used in the last term. The speed of snund at z in the undisturbed field is given by Eq. (2.3) as

$$c_\infty^2=c_a^2\left(1+\frac{\beta z}{T_a}\right). \tag{2.3}$$

If higher than the second order terms of the perturbation velocity are neglected after substituting Eq. (4.15) into Eq. (4.14), Eq. (4.14) reduces to the following linearized equation:

$$\left(\frac{U^2}{c_\infty^2}-1\right)\phi_{xx}-\phi_{zz}-\frac{g}{c_\infty^2}\phi_z=0. \tag{4.16}$$

This equation can be written in a convenient nondimensional form, by introducing following nondimensional quantities:

$$\xi=\frac{x}{l},\quad \zeta=\frac{z}{l},\quad m=\frac{1}{\gamma-1},\quad \alpha=\frac{\beta l}{T_a},\quad M_\infty^2=\frac{U^2}{c_\infty^2}=M_a^2(1+\alpha\zeta)^{-1}$$
$$M_a=\frac{U^2}{c_a^2} \tag{4.17}$$

where l is the body length. With these notations, Eq. (4.16) becomes

$$(M_\infty^2-1)\phi_{\xi\xi}-\phi_{\zeta\zeta}-m\alpha\frac{M_\infty^2}{M_a^2}\phi_\zeta=0. \tag{4.18}$$

(ii) Axisymmetric Body Case

In this case, the velocity components in Eqs. (4.6) and (4.7) are written by use of a potential function as follows:

$$v_x=\frac{\partial \Phi}{\partial x}, \quad v_r=\frac{\partial \Phi}{\partial r}, \quad v_\theta=\frac{\partial \Phi}{r\partial \theta}.$$

If the potential function is given as $\Phi=U(x+\phi)$, Eqs. (4.6) and (4.7) are linearized as in the case of two-dimensional body and result in

$$\left(\frac{U^2}{c_\infty^2}-1\right)\phi_{xx}-\phi_{rr}-\frac{1}{r}\phi_r-\frac{1}{r^2}\phi_{\theta\theta}-\frac{g}{c_\infty^2}\left(\phi_r\cos\theta-\frac{1}{r}\phi_\theta\sin\theta\right)=0. \quad (4.19)$$

Nondimensionalization by the relation

$$\xi=\frac{x}{l}, \quad \eta=\frac{r}{l}, \quad m=\frac{1}{\gamma-1}, \quad \alpha=\frac{\beta l}{T_a}, \quad M_a^2=\frac{U^2}{c_a^2}, \quad M_\infty^2=M_a^2(1+\alpha\eta\cos\theta)^{-1}$$

leads Eq. (4.19) to the following:

$$(M_\infty^2-1)\phi_{\xi\xi}-\phi_{\eta\eta}-\frac{1}{\eta}\phi_\eta-\frac{1}{\eta^2}\phi_{\theta\theta}-m\alpha\frac{M_\infty^2}{M_a^2}\left(\phi_\eta\cos\theta-\frac{1}{\eta}\phi_\theta\sin\theta\right)=0. \quad (4.20)$$

II. The Fundamental Equations in a Standard Atmosphere

In this case the velocity potential no more exists and the analysis should be made over the two regions, a troposphere and a stratosphere. The two-dimensional body case is so unrealistic that it is not treated here, and axisymmetric body case alone is considered. The equations are linearized under the assumption of small disturbance in the same manner as before.

Considering the following expression for velocity components, pressure and density

$$v_x=U(1+v'_x), \quad v_r=Uv'_r, \quad v_\theta=Uv'_\theta,$$
$$p=p_\infty+p', \quad \rho=\rho_\infty+\rho',$$

and nondimensionalization

$$x=l\xi, \quad r=l\eta,$$

Eqs. (4.6) to (4.8) are linearized in the forms

$$\rho_\infty\frac{\partial v'_x}{\partial\xi}+\frac{\partial\rho'}{\partial\xi}+\rho_\infty\frac{\partial v'_r}{\partial\eta}+v'_r\frac{\partial\rho_\infty}{\partial\eta}+\frac{1}{\eta}\left(\rho_\infty v'_r+\rho_\infty\frac{\partial v'_\theta}{\partial\theta}+v'_\theta\frac{\partial p_\infty}{\partial\theta}\right)=0, \quad (4.21)$$

$$\rho_\infty U^2\frac{\partial v'_x}{\partial\xi}=-\frac{\partial p'}{\partial\xi}, \quad (4.22a)$$

$$\rho_\infty U^2\frac{\partial v'_r}{\partial\xi}=\rho' gl\cos\theta-\frac{\partial p'}{\partial\eta}, \quad (4.22b)$$

$$\rho_\infty U^2\frac{\partial v'_\theta}{\partial\xi}=-\rho' gl\sin\theta-\frac{1}{\eta}\frac{\partial p'}{\partial\theta}, \quad (4.22c)$$

$$\frac{\partial p'}{\partial\xi}+v'_r\frac{\partial p_\infty}{\partial\eta}+\frac{v'_\theta}{\eta}\frac{\partial p_\infty}{\partial\theta}+\gamma p_\infty\left(\frac{\partial v'_x}{\partial\xi}+\frac{\partial v'_r}{\partial\eta}+\frac{v'_r}{\eta}+\frac{1}{\eta}\frac{\partial v'_\theta}{\partial\theta}\right)=0 \quad (4.23)$$

where Eq. (4.9) is used.

5. Intensities of Sonic Booms in an Adiabatic Atmosphere

I. Two-Dimensional Body Case

(i) The Solution of the Linearized Equation

In this case the fundamental equation is Eq. (4.18).

$$(M_\infty^2-1)\phi_{\xi\xi}-\phi_{\zeta\zeta}-m\alpha\frac{M_\infty^2}{M_a^2}\phi_\zeta=0, \qquad M_\infty^2=M_a^2(1+\alpha\zeta)^{-1}. \tag{4.18}$$

Now, the solution of this equation is assumed to be in the form

$$\phi=\chi\cdot f(\xi-\omega)+\chi_1\cdot f_1(\xi-\omega)+\chi_2\cdot f_2(\xi-\omega)+\cdots \tag{5.1}$$

as Rao did in his work dealing with the acceleration effect on sonic booms[5]. Here χ, χ_1, $\chi_2 \cdots$ and ω are functions of ζ only, and the following relations are assumed to hold:

$$f_1(t)=\int_0^t f(\lambda)d\lambda, \qquad f_2(t)=\int_0^t f_1(\lambda)d\lambda, \qquad f_3(t)=\int_0^t f_2(\lambda)d\lambda, \cdots. \tag{5.2}$$

In the above relations f is an arbitrary function which depends on body configuration. Substituting Eq. (5.1) into Eq. (4.18) and putting the coefficients of f', f, $f_1 \cdots$ equal to zero because of the reason that the function f is arbitrary, the following equations with respect to ω, χ, χ_1, $\cdots$ are obtained.

$$\omega_\zeta=\pm(M_\infty^2-1)^{\frac{1}{2}}, \tag{5.3}$$

$$\chi_\zeta+P\chi=0, \tag{5.4a}$$

$$\chi_{1\zeta}+P\chi_1=Q_1, \tag{5.4b}$$

$$\chi_{2\zeta}+P\chi_2=Q_2, \tag{5.4c}$$

$$\cdots\cdots\cdots\cdots$$

where

$$P=\frac{1}{2}\left(\frac{\omega_{\zeta\zeta}}{\omega_\zeta}+A\right), \qquad A=m\alpha\frac{M_\infty^2}{M_a^2}, \tag{5.5}$$

$$Q_1=\frac{\chi_{\zeta\zeta}+A\chi_\zeta}{2\omega_\zeta}, \tag{5.6a}$$

$$Q_2=\frac{\chi_{1\zeta\zeta}+A\chi_{1\zeta}}{2\omega_\zeta}, \tag{5.6b}$$

$$\cdots\cdots\cdots\cdots.$$

From Eq. (5.3) $\xi-\omega=$const turns out to describe a Mach line. The integration of Eqs. (5.3) and (5.4) is easily carried out.

$$\omega=\pm\int(M_\infty^2-1)^{\frac{1}{2}}d\zeta+C, \tag{5.7}$$

$$\chi=Ke^{-\int Pd\zeta}, \tag{5.8a}$$

$$\chi_1=e^{-\int Pd\zeta}\left[\int Q_1e^{\int Pd\zeta}d\zeta+K_1\right], \tag{5.8b}$$

$$\chi_2 = e^{-\int P d\zeta}\left[\int Q_2 e^{\int P d\zeta} d\zeta + K_2\right], \tag{5.8c}$$

$$\cdots\cdots\cdots\cdots .$$

In Eq. (5.7) the plus sign is taken because only the Mach lines pointing downstream from the body are to be considered from the physical view point. Using the relation $M_\infty^2 = M_a^2(1+d\zeta)^{-1}$ the integration of Eq. (5.7) is performed to give the expression of ω in the form of a power series of $\alpha\zeta$ as follows:

$$\omega = \zeta\Bigg[(M_a^2-1)^{\frac{1}{2}} - \frac{M_a^2}{4(M_a^2-1)^{\frac{1}{2}}}\alpha\zeta + \frac{1}{6}\left\{\frac{M_a^4}{4(M_a^2-1)^{\frac{3}{2}}} + \frac{M_a^2}{(M_a^2-1)^{\frac{1}{2}}}\right\} \times(\alpha\zeta)^2 + \cdots\cdots\Bigg] + C. \tag{5.9}$$

The integration constant C in Eq. (5.7) becomes zero by the condition that Eq. (5.9) should be reduced to the case of a uniform atmosphere by putting $\alpha=\beta=0$. From Eqs. (5.3) and (5.5), the integration of P in Eqs. (5.8) is given by

$$\int P d\zeta = \ln\left\{\left(\frac{M_\infty^2-1}{M_a^2-1}\right)^{\frac{1}{4}}\left(\frac{M_a}{M_\infty}\right)^m\right\}. \tag{5.10}$$

Hence, from Eq. (5.8*a*),

$$\chi = K\left(\frac{M_a^2-1}{M_\infty^2-1}\right)^{\frac{1}{4}}\left(\frac{M_\infty}{M_a}\right)^m. \tag{5.11a}$$

Thus, $\chi_1, \chi_2, \cdots\cdots$ are successively determined from Eqs. (5.8*b*), (5.8*c*), $\cdots$ to result in

$$\chi_1 = \left(\frac{M_a^2-1}{M_\infty^2-1}\right)^{\frac{1}{4}}\left(\frac{M_\infty}{M_a}\right)^m\Bigg[K_1 + K\frac{\alpha}{4M_a^2}(M_\infty^2-1)^{\frac{1}{2}}\left\{\frac{3}{4}+2m\left(\frac{m}{2}-1\right) + \frac{1}{2}\frac{1}{(M_\infty^2-1)} + \frac{5}{12}\frac{1}{(M_\infty^2-1)^2}\right\}\Bigg], \tag{5.11b}$$

$$\begin{aligned}\chi_2 = \left(\frac{M_a^2-1}{M_\infty^2-1}\right)^{\frac{1}{4}}\left(\frac{M_\infty}{M_a}\right)^m\Bigg[&K_2 + K_1\frac{\alpha}{4M_a^2}(M_\infty^2-1)^{\frac{1}{2}}\left\{\frac{3}{4}+2m\left(\frac{m}{2}-1\right) + \frac{1}{2}\frac{1}{(M_\infty^2-1)} + \frac{5}{12}\frac{1}{(M_\infty^2-1)^2}\right\} \\ &+ K\frac{\alpha^2}{16M_a^4}\left\{m\left(\frac{m}{2}-1\right)\left(\frac{3}{4}+2m\cdot\overline{\frac{m}{2}-1}\right)M_\infty^2\right. \\ &-\frac{5}{8}\left(\frac{3}{4}+2m\cdot\overline{\frac{m}{2}-1}\right)(M_\infty^2-1) + \left(\frac{31}{16}-\frac{7m}{6}\cdot\overline{\frac{m}{2}-1}\right)\frac{1}{(M_\infty^2-1)} \\ &\left.+\frac{77}{24}\frac{1}{(M_\infty^2-1)^2} + \frac{385}{288}\frac{1}{(M_\infty^2-1)^3}\right\}\Bigg].\end{aligned} \tag{5.11c}$$

The integration constants $K, K_1, K_2 \cdots$ in Eqs. (5.11) are determined so that the solutions just obtained may be reduced to the solution for a uniform atmosphere by putting α equal to zero.

In case of α being zero, the present solution becomes

$$\phi=Kf(\xi-\sqrt{M_a^2-1}\,\zeta)+K_1f_1(\xi-\sqrt{M_a^2-1}\,\zeta)+K_2f_2(\xi-\sqrt{M_a^2-1}\,\zeta)+\cdots \quad (5.12)$$

while in case of a uniform atmosphere the solution obviously takes the familiar form:

$$\phi=f(\xi-\sqrt{M_a^2-1}\,\zeta). \quad (5.13)$$

Since Eqs. (5.12) and (5.13) are to be coincident, $K, K_1, K_2, \cdots$ are determined as follows

$$K=1, \quad K_1=K_2=\cdots\cdots=0. \quad (5.14)$$

In the solution thus obtained f is related to the coordinate $h(\xi)$ of the body surface by the following equation

$$f(\xi)=-\frac{l}{\sqrt{M_a^2-1}}h(\xi). \quad (5.15)$$

(ii) Pressure Jump across the Shock Wave in the Far-Field

In the following analysis, Whitham's higher approximation technique is applied to the linear solution just obtained. This theory is based on the cardinal hypothesis as follows: *the linearized theory gives a correct first approximation to the flow everywhere provided that the value which it predicts for any physical quantity, at a given distance* ζ *from the axis on the approximate Mach line* $\xi-\omega=$const, *pointing downstream from a given point on the body surface, is interpreted as the value, at that distance from the axis, on the exact Mach line which points down-stream from the said point.* According to this hypothesis, the higher approximate solution of the problem can be obtained by replacing $(\xi-\omega)$ in Eq. (5.1) by $\tau(\xi,\zeta)$, where $\tau(\xi,\zeta)=$ const denoted the exact Mach line. The practical value of the hypothesis exists in the fact that if a sufficiently good approximation to the exact Mach lines, e.g. a second approximation, is used, instead of the exact one, the approximation is still effective in improving the solution. If the linear solution is known, the second approximation to the Mach lines may be deduced.

Next, we determined this second approximation to the Mach lines. The direction of a Mach line is given as

$$\frac{d\xi}{d\zeta}=\cot(\mu+\delta), \quad (5.16)$$

where μ is the local Mach angle and δ is the local flow angle. Using the local speed of sound c determined from Bernoulli's equation together with the magnitude of the velocity q,

$$\mu=\sin^{-1}\frac{c}{q}=\mu_\infty-(M_\infty^2-1)^{-\frac{1}{2}}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)v'_x+O(v'^2_x+v'^2_z), \quad (5.17)$$

$$\delta=\tan^{-1}[v'_z(1+v'_x)^{-1}]=v'_z+O(v'^2_x+v'^2_z). \quad (5.18)$$

Hence, Eq. (5.16) may be reduced to the form

$$\frac{d\xi}{d\zeta}=(M_\infty^2-1)^{\frac{1}{2}}+\frac{(\gamma+1)M_\infty^4}{2(M_\infty^2-1)^{\frac{1}{2}}}v_x'-M_\infty^2[v_z'+(M_\infty^2-1)^{\frac{1}{2}}v_x']+O(v_x'^2+v_z'^2). \tag{5.19}$$

The values for v_x' and v_z' are given in the following forms by differentiating Eq. (5.1) with respect to ξ and ζ respectively.

$$v_x'l=\phi_\xi=\chi f'(\xi-\omega)+\chi_1 f(\xi-\omega)+\chi_2 f_1(\xi-\omega)+\cdots, \tag{5.20a}$$

$$v_z'l=\phi_\zeta=-\omega_\zeta\chi f'(\xi-\omega)+(\chi_\zeta-\omega_\zeta\chi_1)f(\xi-\omega)+(\chi_{1\zeta}-\omega_\zeta\chi_2)f_1(\xi-\omega)\cdots. \tag{5.20b}$$

In the right side of Eqs. (5.20), those terms other than the first are small by the factors α or high powers of α compared with the first. Hence, retaining only the first term in Eqs. (5.20), we get

$$v_x'l=\phi_\xi=\left(\frac{M_a^2-1}{M_\infty^2-1}\right)^{\frac{1}{4}}\left(\frac{M_\infty}{M_a}\right)^m f'(\xi-\omega), \tag{5.21a}$$

$$v_z'l=\phi_\zeta=-(M_\infty^2-1)^{\frac{1}{2}}\phi_\xi. \tag{5.21b}$$

The substitution of Eqs. (5.21) in Eq. (5.19) gives, on replacing $(\xi-\omega)$ with τ and neglecting smaller terms than $O(v_x'^2+v_z'^2)$, the equation of direction of the higher approximate Mach line. This equation holds all along any downstream-pointing Mach line along which τ is constant, starting from the initial value given by $\xi-(M_a^2-1)\zeta$ at the body. The integration is performed at $\tau=$const resulting in

$$\xi=\int_0^\zeta(M_\infty^2-1)^{\frac{1}{2}}d\zeta+kf'(\tau)\cdot I(\zeta)+\tau, \tag{5.22}$$

where

$$k=\frac{(\gamma+1)}{2l}\cdot\frac{(M_a^2-1)^{\frac{1}{4}}}{M_a^m},\quad I(\zeta)=\int_0^\zeta\frac{M_\infty^{m+4}}{(M_\infty-1)^{\frac{3}{4}}}d\zeta.$$

Eq. (5.22) is the second approximation to the equation of the Mach line which emanates from a point on the axis at a distance τ from the nose.

The next step is to determine the shape of the front shock. As is easily obtained from the shock conditions, a weak shock wave has the following geometrical property: its direction bisects, to the first order in the strength, the Mach directions ahead of and behind it. The Mach direction angle ahead of the shock wave where the flow is undisturbed is obviously equal to μ_∞. The gradient of the Mach line behind the shock is calculated from Eq. (5.22) as

$$\frac{d\zeta}{d\xi}=\frac{1}{(M_\infty^2-1)^{\frac{1}{2}}+kf'(\tau)\cdot I'(\zeta)}\equiv\tan(\mu_\infty+\varepsilon_1), \tag{5.23}$$

where $\mu_\infty+\varepsilon_1$ is the Mach direction angle behind the shock. Hence,

$$\varepsilon_1=-kf'(\tau)\cdot I(\zeta)\sin^2\mu_\infty. \tag{5.24}$$

Assuming the shape of the shock in the form

$$\xi=\int_0^\zeta(M_\infty^2-1)^{\frac{1}{2}}d\zeta-G(\zeta), \tag{5.25}$$

the direction of shock, $(\mu+\varepsilon_2)$, is given as

$$\tan(\mu_\infty+\varepsilon_2)=\frac{d\zeta}{d\xi}=\frac{1}{(M_\infty^2-1)^{\frac{1}{2}}-G'(\zeta)}\,. \tag{5.26}$$

Hence,

$$\varepsilon_2=G'(\zeta)\sin^2\mu_\infty\,. \tag{5.27}$$

According to the angle property of shock wave stated above, ε_1 is equal to $2\varepsilon_2$, and therefore, from Eqs. (5.24) and (5.27),

$$G'(\zeta)=-\frac{1}{2}kf'(\tau)\cdot I'(\zeta)\,. \tag{5.28}$$

The point (ξ, ζ) under consideration is a cross point of the shock wave and the Mach line behind it; hence eliminating $\xi-\int_0^\zeta (M_\infty^2-1)^{\frac{1}{2}}d\zeta$ from Eqs. (5.22) and (5.25),

$$G(\zeta)+kf'(\tau)I(\zeta)+\tau=0\,. \tag{5.29}$$

Since ζ also can be considered as a function of τ, the equation for the function $\zeta(\tau)$ is obtained by differentiating Eq. (5.29) with respect to τ and eliminating $G'(\zeta)$ from Eq. (5.28). The result is

$$\frac{1}{2}kf'(\tau)I'(\zeta)\frac{d\zeta}{d\tau}+kf''(\tau)I(\zeta)+1=0\,, \tag{5.30}$$

or

$$\frac{d}{d\tau}\left[\frac{1}{2}kf'^2(\tau)I(\zeta)\right]=-f'(\tau). \tag{5.31}$$

The shock wave must start at the leading edge of the body. Hence, Eq. (5.31) is integrated, under the intial condition $\tau=0$ when $\zeta=0$, resulting in

$$I(\zeta)=-\frac{2}{k}\frac{\int_0^\tau f'(\nu)d\nu}{f'^2(\tau)} \tag{5.32}$$

This formula relates the position on the shock wave with that on the body axis.

From Eq. (5.32) it is possible to find the value of τ in the far-field where ζ is large. A little investigation of Eq. (5.32) may show that, because $I(\zeta)$ is a monotonic increasing function, ζ becomes very large when τ takes the value near τ_0 which is the first zero of $f'(\tau)$ apart from $\tau=0$ itself. The asymptotic expression of Eq. (5.31) thus obtained is

$$[f'(\tau)]_{\tau\approx\tau_0}=-\left[\frac{-2\int_0^{\tau_0}f'(\nu)d\nu}{kI(\zeta)}\right]^{\frac{1}{2}} \tag{5.33}$$

For a closed body, τ_0 exists since $\int_0^\infty f'(\nu)d\nu=0$ and hence there must be at least one zero. Detailed discussions on this point is beyond the scope of this paper and should be refered to Whitham's original paper [3]. Thus from Eq. (5.29) using the above result,

$$G(\zeta)=[-2k\textstyle\int_0^{\tau_0}f'(\nu)d\nu]^{\frac{1}{2}}[I(\zeta)]^{\frac{1}{2}}-\tau_0. \tag{5.34}$$

The pressure jump across the shock wave is easily found using the shock condition together with the shock inclination as

$$\frac{\Delta p}{p_\infty}=\frac{p-p_\infty}{p_\infty}=\frac{4\gamma}{\gamma+1}\frac{(M_\infty^2-1)^{\frac{1}{2}}}{M_\infty^2}G'(\zeta). \tag{5.35}$$

Combining Eqs. (5.28) and (5.33) with Eq. (5.35),

$$\frac{\Delta p}{p_\infty}=\frac{2\gamma}{(\gamma+1)^{\frac{1}{2}}}(M_a^2-1)^{\frac{1}{4}}\left(\frac{M_a^2-1}{M_\infty^2-1}\right)^{\frac{1}{4}}\left(\frac{M_\infty}{M_a}\right)^{m+2}\left[\int_0^\zeta J(\zeta)d\zeta\right]^{-\frac{1}{2}}$$
$$\times\left[-\frac{1}{l}\int_0^{\tau_0}f'(\nu)d\nu\right]^{\frac{1}{2}} \tag{5.36}$$

Eq. (5.36) expresses the pressure jump across the front shock at large distance from the body flying at supersonic speeds in an adiabatic atmosphere. On the other hand, the pressure jump in the uniform atmosphere can be easily obtained by putting α equal to zero and its final form is

$$\left(\frac{\Delta p}{p_\infty}\right)_{uni}=\frac{2\gamma}{(\gamma+1)^{\frac{1}{2}}}(M_a^2-1)^{\frac{1}{4}}\zeta^{-\frac{1}{2}}\left[-\frac{1}{l}\int_0^{\tau_0}f'(\nu)d\nu\right]^{\frac{1}{2}}. \tag{5.37}$$

Let the ratio of the pressure jump in the adiabatic and the uniform atmospheres be denoted by R. Then

$$R=\left(\frac{M_a^2-1}{M_\infty^2-1}\right)^{\frac{1}{4}}\left(\frac{M_\infty}{M_a}\right)^{m+2}\left[\frac{\zeta}{\int_0^\zeta J(\zeta)d\zeta}\right]^{\frac{1}{2}} \tag{5.38}$$

where

$$J(\zeta)=1+A_1\alpha\zeta+A_2(\alpha\zeta)^2+A_2(\alpha\zeta)^3+\cdots\cdots\cdots, \tag{5.39}$$

$$A_1=\frac{3}{4}M_a^2(M_a^2-1)^{-1}-\left(2+\frac{m}{2}\right),$$

$$A_2=\frac{1}{2}\left\{\frac{21}{16}M_a^4(M_a^2-1)^{-2}-\frac{3}{2}M_a^2(M_a^2-1)^{-1}\right\}-\frac{3}{4}\left(2+\frac{m}{2}\right)M_a^2(M_a^2-1)^{-1}$$
$$+\frac{1}{2}\left(2+\frac{m}{2}\right)\left(3+\frac{m}{2}\right),$$

$$A_3=\frac{1}{6}\left\{\frac{231}{64}M_a^6(M_a^2-1)^{-3}-\frac{63}{8}M_a^4(M_a^2-1)^{-2}+\frac{9}{2}M_a^2(M_a^2-1)^{-1}\right\}$$
$$-\frac{1}{2}\left(2+\frac{m}{2}\right)\left\{\frac{21}{16}M_a^4(M_a^2-1)^{-2}-\frac{3}{2}M_a^2(M_a^2-1)^{-1}\right\}$$
$$+\frac{3}{8}\left(2+\frac{m}{2}\right)\left(3+\frac{m}{2}\right)M_a^2(M_a^2-1)^{-1}$$
$$-\frac{1}{6}\left(2+\frac{m}{2}\right)\left(3+\frac{m}{2}\right)\left(4+\frac{m}{2}\right).$$

Rewriting Eq. (5.38), the following is obtained.

$$R=\left(\frac{M_a^2-1}{M_\infty^2-1}\right)^{\frac{1}{4}}\left(\frac{M_\infty}{M_a}\right)^{m+2}\left\{1+\frac{A_1}{2}\alpha\zeta+\frac{A_2}{3}(\alpha\zeta)^2+\frac{A_3}{4}(\alpha\zeta)^3+\cdots\cdot\right\}^{-\frac{1}{2}}. \tag{5.40}$$

R thus obtained is the theoretical modification factor which is to replace the aforementioned classical one, $\sqrt{p_a/p_g}$.

II. Axisymmetric Body Case

(i) The Solution of the Linearized Equation

In this case Eq. (4.20) is used and the solution is obtained by the same way as in the two-dimensional case. Thus, the solution is assumed in the form

$$\phi=\chi\cdot f(\xi-\omega)+\chi_1\cdot f_1(\xi-\omega)+\chi_2\cdot f_2(\xi-\omega)+\cdots\cdots \tag{5.41}$$

where

$$f_1(t)=\int_0^t f(\lambda)d\lambda,\quad f_2(t)=\int_0^t f_1(\lambda)d\lambda,\cdots\cdots. \tag{5.42}$$

Here, χ, χ_1, $\chi_2\cdots$ and ω are functions of η and θ. Substituting Eq. (5.41) into Eq. (4.20), and putting the coefficients of f'', f', f, $f_1\cdots$ equal to zero, we get the following equations for ω, χ, χ_1, $\cdots$.

$$(M_\infty^2-1)-\omega_\eta^2-\frac{1}{\eta^2}\omega_\theta^2=0, \tag{5.43a}$$

$$2\omega_\eta\chi_\eta+\omega_{\eta\eta}\chi+\frac{1}{\eta}\omega_\eta\chi+\frac{2}{\eta^2}\omega_\theta\chi_\theta+\frac{1}{\eta^2}\omega_{\theta\theta}\chi+m\alpha\frac{M_\infty^2}{M_a^2}\omega_\eta\chi\cos\theta$$
$$-m\alpha\frac{M_\infty^2}{M_a^2}\frac{1}{\eta}\omega_\theta\chi\sin\theta=0, \tag{5.43b}$$

$$2\omega_\eta\chi_{1\eta}+\omega_{\eta\eta}\chi_1+\frac{1}{\eta}\omega_\eta\chi_1+\frac{2}{\eta^2}\omega_\theta\chi_{1\theta}+\frac{1}{\eta^2}\omega_{\theta\theta}\chi_1+m\alpha\frac{M_\infty^2}{M_a^2}\omega_\eta\chi_1\cos\theta$$
$$-m\alpha\frac{M_\infty^2}{M_a^2}\frac{1}{\eta}\omega_\theta\sin\theta=\chi_{\eta\eta}+\frac{1}{\eta}\chi_\eta+\frac{1}{\eta^2}\chi_{\theta\theta}+m\alpha\frac{M_\infty^2}{M_a^2}\chi_\eta\cos\theta$$
$$-m\alpha\frac{M_\infty^2}{M_a^2}\frac{1}{\eta}\chi_\theta\sin\theta, \tag{5.43c}$$

$$2\omega_\eta\chi_{2\eta}+\omega_{\eta\eta}\chi_2+\frac{1}{\eta}\omega_\eta\chi_2+\frac{2}{\eta^2}\omega_\theta\chi_{2\theta}+\frac{1}{\eta^2}\omega_{\theta\theta}\chi_2+m\alpha\frac{M_\infty^2}{M_a^2}\omega_\eta\chi_2\cos\theta$$
$$-m\alpha\frac{M_\infty^2}{M_a^2}\frac{1}{\eta}\omega_\theta\chi_2\sin\theta=\chi_{1\eta\eta}+\frac{1}{\eta}\chi_{1\eta}+\frac{1}{\eta^2}\chi_{1\theta\theta}+m\alpha\frac{M_\infty^2}{M_a^2}\chi_{1\eta}\cos\theta$$
$$-m\alpha\frac{M_\infty^2}{M_a^2}\frac{1}{\eta}\chi_{1\theta}\sin\theta. \tag{5.43d}$$

It is not easy to solve Eqs. (5.43) in exact forms succesively, because ω, χ, χ_1, $\cdot\cdot$ are two-variable functions instead of one as in two-dimensional body case. In the practical treatment of the sonic boom problem, of most importance is the pressure jump across the shock in the region directly below the flight path where θ is nearly zero. In this point of view and also for the sake of simplicity, we expand ω, χ, $\chi_1, \cdots$ in power series of θ^2 considering symmetricity of the flow field as

$$\omega=\omega^{(0)}+\theta^2\omega^{(1)}+\theta^4\omega^{(2)}\cdots, \tag{5.44}$$

$$\chi=\chi^{(0)}+\theta^2\chi^{(1)}+\theta^4\chi^{(2)}\cdots, \tag{5.45}$$

$$\chi_1=\chi_1^{(0)}+\theta^2\chi_1^{(1)}+\theta^4\chi_1^{(2)}+\cdots, \tag{5.46}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

where $\omega^{(0)}, \omega^{(1)}, \cdots, \chi^{(0)}, \chi^{(1)}, \cdots, \chi_1^{(0)}, \chi_1^{(1)}, \cdots$ are functions of η only. Substituting these into Eqs. (5.43) with expansion of $\sin\theta$ and $\cos\theta$, and putting the coeficients of θ^n terms equal to zero, the following equations are derived:

$$(\bar{M}_\infty^2-1)-\omega_\eta^{(0)2}=0, \tag{5.47a}$$

$$\omega_\eta^{(1)}=-\frac{2}{\eta^2(\bar{M}_\infty^2-1)^{\frac{1}{2}}}\,\omega^{(1)2}+\frac{\alpha\eta\bar{M}_\infty^4}{4M_a^2(\bar{M}_\infty^2-1)^{\frac{1}{2}}}, \tag{5.47b}$$

$$\begin{aligned}\omega_\eta^{(2)}+\frac{8}{\eta^2}\frac{\omega^{(1)}}{(\bar{M}_\infty^2-1)^{\frac{1}{2}}}\omega^{(2)}=&\frac{\alpha\eta}{48}\frac{1}{(\bar{M}_\infty^2-1)^{\frac{1}{2}}}\frac{\bar{M}_\infty^4}{M_a^2}\left(6\alpha\eta\frac{\bar{M}_\infty^2}{M_a^2}-1\right)\\&-\frac{1}{2}\frac{\omega_\eta^{(1)2}}{(\bar{M}_\infty^2-1)^{\frac{1}{2}}},\end{aligned} \tag{5.47c}$$

$$\chi_\eta^{(0)}+\frac{1}{2}\left(\frac{\omega_{\eta\eta}^{(0)}}{\omega_\eta^{(0)}}+\frac{1}{\eta}+\frac{2}{\eta^2}\frac{\omega^{(1)}}{\omega_\eta^{(0)}}+m\alpha\frac{\bar{M}_\infty^2}{M_a^2}\right)\chi^{(0)}=0, \tag{5.47d}$$

$$\begin{aligned}\chi_\eta^{(1)}+\frac{1}{2}\left(\frac{\omega_{\eta\eta}^{(0)}}{\omega_\eta^{(0)}}+\frac{1}{\eta}+\frac{10}{\eta^2}\frac{\omega^{(1)}}{\omega_\eta^{(0)}}+m\alpha\frac{\bar{M}_\infty^2}{M_a^2}\right)\chi^{(1)}=&\left(m\alpha\frac{\bar{M}_\infty^2}{M_a^2}\frac{1}{\eta}\frac{\omega^{(1)}}{\omega_\eta^{(0)}}\right.\\&-\frac{1}{2}\frac{\omega_{\eta\eta}^{(1)}}{\omega_\eta^{(0)}}-\frac{1}{2\eta}\frac{\omega_\eta^{(1)}}{\omega_\eta^{(0)}}-\frac{6}{\eta^2}\frac{\omega^{(2)}}{\omega_\eta^{(0)}}-\frac{m\alpha}{2}\frac{\bar{M}_\infty^2}{M_a^2}\frac{\omega_\eta^{(1)}}{\omega_\eta^{(0)}}\\&\left.+\frac{m\alpha}{4}\frac{\bar{M}_\infty^2}{M_a^2}-\frac{\alpha\eta}{4}m\alpha\frac{\bar{M}_\infty^4}{M_a^4}\right)\chi^{(0)}-\frac{\omega_\eta^{(1)}}{\omega_\eta^{(0)}}\chi_\eta^{(0)},\end{aligned} \tag{5.47e}$$

$$\begin{aligned}\chi_{1\eta}^{(0)}+\frac{1}{2}\left(\frac{\omega_{\eta\eta}^{(0)}}{\omega_\eta^{(0)}}+\frac{1}{\eta}+\frac{2}{\eta^2}\frac{\omega^{(1)}}{\omega_\eta^{(0)}}+m\alpha\frac{\bar{M}_\infty^2}{M_a^2}\right)\chi_1^{(0)}=&\frac{1}{2}\frac{\chi_{\eta\eta}^{(0)}}{\omega_\eta^{(0)}}\\&+\frac{1}{2\eta}\frac{\chi_\eta^{(0)}}{\omega_\eta^{(0)}}+\frac{1}{\eta^2}\frac{\chi^{(1)}}{\omega_\eta^{(0)}}+\frac{m\alpha}{2}\frac{\bar{M}_\infty^2}{M_a^2}\frac{\chi_\eta^{(0)}}{\omega_\eta^{(0)}}\end{aligned} \tag{5.47f}$$

where

$$\bar{M}_\infty^2=M_a^2(1+\alpha\eta)^{-1}. \tag{5.48}$$

On integrating Eq. (5.47a), $\omega^{(0)}$ is determined as

$$\omega^{(0)}=\pm\int(\bar{M}_\infty^2-1)^{\frac{1}{2}}d\eta+C. \tag{5.49}$$

Here, the plus sign is taken and the integration constants are equal to zero because of the same reason as in two-dimensional body case. Eq. (5.47b) can be integrated to yield

$$\omega^{(1)} = -\frac{\eta}{2}(\bar{M}_{\infty}^2-1)^{\frac{1}{2}}+\eta^2\left\{C_1+\int\frac{2}{(\bar{M}_{\infty}^2-1)^{\frac{1}{2}}}d\eta\right\}^{-1}. \tag{5.50}$$

In case of a uniform atmosphere, $\omega^{(1)}$ should become zero, and hence, $C_1=0$ in Eq. (5.50). Combining Eq. (5.47d) with Eqs. (5.49) and (5.50), we get the following equation for $\chi^{(0)}$:

$$\chi_{\eta}^{(0)}+P\chi^{(0)}=0 \tag{5.51}$$

where

$$P=\frac{1}{2}\left\{\frac{\omega_{\eta\eta}^{(0)}}{\omega_{\eta}^{(0)}}+\frac{1}{(\bar{M}_{\infty}^2-1)^{\frac{1}{2}}\int_0^{\eta}(\bar{M}_{\infty}^2-1)^{\frac{1}{2}}d\eta}+m\alpha\frac{\bar{M}_{\infty}^2}{M_a^2}\right\}.$$

Eq. (5.51) has the solution

$$\chi^{(0)}=K^{(0)}e^{-\int Pd\eta}=K^{(0)}\left(\frac{M_a^2-1}{\bar{M}_{\infty}^2-1}\right)^{\frac{1}{4}}\left(\frac{\bar{M}_{\infty}}{M_a}\right)^{m}\left\{\int_0^{\eta}\left(\frac{M_a^2-1}{\bar{M}_{\infty}^2-1}\right)^{\frac{1}{2}}d\eta\right\}^{-\frac{1}{2}} \tag{5.52}$$

where $K^{(0)}$ is a constant. Similarly, the other functions $\omega^{(2)},\cdots,\ \chi^{(1)},\cdots,\ \chi_1^{(0)},\cdots$ in Eqs. (5.47) can be obtained successively. The integration constants $K^{(0)}$ and others are all determined in the same way as was done in the two-dimensional body case: that is, first, disturbance velocity components in the adiabatic atmosphere are calculated, and then α is reduced to zero in them. The results must be coincide with the disturbance velocity components in case of a uniform atmosphere already obtained in the ordinary supersonic theory.

When α is put equal to zero in the obtained disturbance velocity components in x-direction v_x' and in r-direction v_r', they become

$$\begin{aligned}v_x'l=\phi_{\xi}=&K^{(0)}\eta^{-\frac{1}{2}}f'(\xi-\sqrt{M_a^2-1}\,\eta)-\left[\frac{1}{8}K^{(0)}(M_a^2-1)^{-\frac{1}{2}}\eta^{-\frac{3}{2}}\right.\\&\left.-K_1^{(0)}\eta^{-\frac{1}{2}}\right]f(\xi-\sqrt{M_a^2-}\,\eta)+\left[\frac{9}{128}K^{(0)}(M_a^2-1)^{-1}\eta^{-\frac{5}{2}}\right.\\&\left.-\frac{1}{8}K_1^{(0)}(M_a^2-1)^{-\frac{1}{2}}\eta^{-\frac{3}{2}}+K_2^{(0)}\eta^{-\frac{1}{2}}\right]f_1(\xi-\sqrt{M_a^2-1}\,\eta)+\cdots,\end{aligned} \tag{5.53a}$$

$$\begin{aligned}v_r'l=\phi_{\eta}=&-K^{(0)}\eta^{-\frac{1}{2}}(M_a^2-1)^{\frac{1}{2}}f'(\xi-\sqrt{M_a^2-1}\,\eta)-\left[\frac{3}{8}K^{(0)}\eta^{-\frac{3}{2}}\right.\\&\left.+K_1^{(0)}(M_a^2-1)^{\frac{1}{2}}\eta^{\frac{1}{2}}\right]f(\xi-\sqrt{M_a^2-1}\,\eta)+\left[\frac{15}{128}K^{(0)}(M_a^2-1)^{-\frac{1}{2}}\eta^{-\frac{5}{2}}\right.\\&\left.+\frac{1}{8}K_1^{(0)}\eta^{-\frac{3}{2}}-K_2^{(0)}(M_a^2-1)^{\frac{1}{2}}\eta^{-\frac{1}{2}}\right]f_1(\xi-\sqrt{M_a^2-1}\,\eta)+\cdots.\end{aligned} \tag{5.53b}$$

On the other hand the gasdynamics equation in case of a uniform atmosphere is

$$(M_a^2-1)\phi_{\xi\xi}-\phi_{\eta\eta}-\frac{1}{\eta}\phi_{\eta}=0. \tag{5.54}$$

From its well known solution, we get

$$\phi_\xi=-\int_0^{\xi-\sqrt{M_a^2-1}\,\eta}\frac{\sigma'(t)dt}{[(\xi-t)^2-(M_a^2-1)\eta^2]^{\frac{1}{2}}}, \tag{5.55a}$$

$$\phi_\eta=\frac{1}{\eta}\int_0^{\xi-\sqrt{M_a^2-1}\,\eta}\frac{(\xi-t)\sigma'(t)dt}{[(\xi-t)^2-(M_a^2-1)\eta^2]^{\frac{1}{2}}} \tag{5.55b}$$

where $\sigma(\xi)$ is related to the distribution function, $r_b(\xi)$, of the body radius as

$$\sigma(\xi)=lr_br_b'. \tag{5.56}$$

Eqs. (5.55) can be expanded into the following forms:

$$\begin{aligned}\phi_\xi=&-\frac{1}{(2\sqrt{M_a^2-1}\,\eta)^{\frac{1}{2}}}\int_0^{\xi-\sqrt{M_a^2-1}\,\eta}\frac{\sigma'(t)}{(\xi-\sqrt{M_a^2-1}\,\eta-t)^{\frac{1}{2}}}\\&\times\left[1-\frac{\xi-\sqrt{M_a^2-1}\,\eta-t}{4(M_a^2-1)^{\frac{1}{2}}\eta}+\frac{3(\xi-\sqrt{M_a^2-1}\,\eta-t)^2}{32(M_a^2-1)\eta^2}\right.\\&\left.-\frac{5(\xi-\sqrt{M_a^2-1}\,\eta-t)^3}{128(M_a^2-1)^{\frac{3}{2}}\eta^3}+\cdots\right]dt,\end{aligned} \tag{5.57a}$$

$$\begin{aligned}\phi_\eta=&\frac{(M_a^2-1)^{\frac{1}{2}}}{(2\sqrt{M_a^2-1}\,\eta)^{\frac{1}{2}}}\int_0^{\xi-\sqrt{M_a^2-1}\,\eta}\frac{\sigma'(t)}{(\xi-\sqrt{M_a^2-1}\,\eta-t)^{\frac{1}{2}}}\\&\times\left[1+\frac{3(\xi-\sqrt{M_a^2-1}\,\eta-t)}{4(M_a^2-1)^{\frac{1}{2}}\eta}-\frac{5(\xi-\sqrt{M_a^2-1}\,\eta-t)^2}{32(M_a^2-1)\eta^2}\right.\\&\left.+\frac{7(\xi-\sqrt{M_a^2-1}\,\eta-t)^3}{128(M_a^2-1)^{\frac{3}{2}}\eta^3}\cdots\right]dt.\end{aligned} \tag{5.57b}$$

In order that Eqs. (5.53) coincide with Eqs. (5.57) respectively, the following relations must be satisfied:

$$K^{(0)}=-\frac{1}{(2\sqrt{M_a^2-1})^{\frac{1}{2}}},\quad K_1^{(0)}=K_2^{(0)}\cdots=0, \tag{5.58}$$

$$f'(\xi-\sqrt{M_a^2-1}\,\eta)=\int_0^{\xi-\sqrt{M_a^2-1}\,\eta}\frac{\sigma'(t)dt}{(\xi-\sqrt{M_a^2-1}\,\eta-t)^{\frac{1}{2}}}. \tag{5.59}$$

Thus, in the plane of $\theta=0$, the disturbance velocity components in the linearized theory are expressed as

$$\bar{\phi}_\xi=\chi^{(0)}f'(\xi-\omega^{(0)})+\chi_1^{(0)}f(\xi-\omega^{(0)})+\chi_2^{(0)}f_1(\zeta-\omega^{(0)})+\cdots, \tag{5.60a}$$

$$\begin{aligned}\bar{\phi}_\eta=&-\omega_\eta^{(0)}\chi^{(0)}f'(\xi-\omega^{(0)})+(\chi_\eta^{(0)}-\omega_\eta^{(0)}\chi_1^{(0)})f(\xi-\omega^{(0)})+(\chi_{1\eta}^{(0)}-\omega_\eta^{(0)}\chi_2^{(0)})\\&\times f_1(\xi-\omega^{(0)})+\cdots.\end{aligned} \tag{5.60b}$$

A detailed investigation of each term in Eqs. (5.60) indicates that those terms higher than the second are small compared to the first ones at a large distance from the body. Neglecting those terms, we finally obtain

$$\bar{\phi}_\xi=\chi^{(0)}f'(\xi-\omega^{(0)}), \tag{5.61a}$$

$$\bar{\phi}_\eta = -(\bar{M}_\infty^2 - 1)^{\frac{1}{2}}\bar{\phi}_\xi. \tag{5.61b}$$

Higher approximation can be obtained by replacing the Mach line, $\xi - \omega^{(0)} = \text{const}$, in the linearized theory by the second approximation of the Mach line, $\tau(\xi,\eta) = \text{const}$, in Eqs. (5.61). Thus, the higher approximation of the disturbance velocity is given in the following forms from Eqs. (5.59) and (5.61),

$$\bar{\phi}_\xi = -\frac{l}{(2\sqrt{M_a^2-1})^{\frac{1}{2}}}\left\{\int_0^\eta \left(\frac{M_a^2-1}{\bar{M}_\infty^2-1}\right)^{\frac{1}{2}} d\eta\right\}^{-\frac{1}{2}} \left(\frac{M_a^2-1}{\bar{M}_\infty^2-1}\right)^{\frac{1}{4}} \left(\frac{\bar{M}_\infty}{M_a}\right)^m F(\tau), \tag{5.62a}$$

$$\bar{\phi}_\eta = -(\bar{M}_\infty^2 - 1)^{\frac{1}{2}}\bar{\phi}_\xi \tag{5.62b}$$

where

$$F(\tau) = \frac{1}{l}\int_0^\tau \frac{\sigma'(t)dt}{(\tau - t)^{\frac{1}{2}}}. \tag{5.63}$$

(ii) Pressure Jump across the Shock Wave in the Far-Field

The analysis of the pressure jump across the shock wave in the far-field can be made completely parallel to the case of the two-dimensional body. If the Mach angle and the flow deflection angle in the plane $\theta = 0$ are denoted by $\bar{\mu}$ and $\bar{\delta}$ respectively, they are given by

$$\bar{\mu} = \bar{\mu}_\infty - (\bar{M}_\infty^2 - 1)^{-\frac{1}{2}}\left(1 + \frac{\gamma - 1}{2}\bar{M}_\infty^2\right)\bar{v}_x' + O(\bar{v}_x^{2\prime} + \bar{v}_r^{\prime 2}), \tag{5.64}$$

$$\bar{\delta} = \bar{v}_r' + O(\bar{v}_x^2 + \bar{v}_r^{\prime 2}). \tag{5.65}$$

Hence, the direction of Mach line at a point in the disturbed field becomes

$$\begin{aligned}\frac{d\xi}{d\eta} &= \cot(\bar{\mu} + \bar{\delta}) = (\bar{M}_\infty^2 - 1)^{\frac{1}{2}} + \frac{(\gamma + 1)\bar{M}_\infty^4}{2(\bar{M}_\infty^2 - 1)^{\frac{1}{2}}}\bar{v}_x' \\ &\quad - \bar{M}_\infty^2[\bar{v}_r' + (\bar{M}_\infty^2 - 1)^{\frac{1}{2}}\bar{v}_x'] + O(\bar{v}_x^{\prime 2} + \bar{v}_r^{\prime 2})\end{aligned} \tag{5.66}$$

which gives the equation of the Mach line behind the shock wave as

$$\xi = \int_0^\eta (\bar{M}_\infty^2 - 1)^{\frac{1}{2}} d\eta - kF(\tau)\cdot I(\eta) + \tau \tag{5.67}$$

where

$$k = 2^{-\frac{3}{4}}(\gamma + 1)M_a^{-m},$$

$$I(\eta) = \int_0^\eta \frac{\bar{M}_\infty^{m+4}}{(\bar{M}_\infty^2 - 1)^{\frac{3}{4}}}\left\{\int_0^\eta \left(\frac{M_a^2 - 1}{\bar{M}_\infty^2 - 1}\right)^{\frac{1}{2}} d\eta\right\}^{-\frac{1}{2}} d\eta.$$

Finally, the pressure jump across the shock is obtained in the following form:

$$\begin{aligned}\frac{\Delta p}{\bar{p}_\infty} &= \frac{4\gamma}{\gamma + 1}\,\frac{(\bar{M}_\infty^2 - 1)^{\frac{1}{4}}}{\bar{M}_\infty^2}\left(\frac{k}{2}\right)^{\frac{1}{2}}\frac{I'(\eta)}{[I(\eta)]^{\frac{1}{2}}}\left[\int_0^{\tau_0} F(\nu)d\nu\right]^{\frac{1}{2}} \\ &= \frac{2^{\frac{1}{4}}\gamma}{(\gamma + 1)^{\frac{1}{2}}}(M_a^2 - 1)^{\frac{1}{8}}\eta^{-\frac{3}{4}}\left[\int_0^{\tau_0} F(\nu)d\nu\right]^{\frac{1}{2}}\left(\frac{M_a^2 - 1}{\bar{M}_\infty^2 - 1}\right)^{\frac{1}{4}}\left(\frac{\bar{M}_\infty}{M_a}\right)^{m+2}\end{aligned}$$

$$\times\left\{1+\frac{B_1}{2}\alpha\eta+\frac{B_2}{3}(\alpha\eta)^2+\frac{B_3}{4}(\alpha\eta)^3+\cdots\right\}^{-\frac{1}{2}}$$
$$\times\left\{1+\frac{1}{3}(A_1+D_1)\alpha\eta+\frac{1}{5}(A_2+A_1D_1+D_2)(\alpha\eta)^2\right.$$
$$\left.+\frac{1}{7}(A_3+A_2D_1+A_1D_2+D_3)(\alpha\eta)^3+\cdots\right\}^{-\frac{1}{2}} \tag{5.68}$$

where A_1, A_2, A_3,$\cdots$are the same as those given in Eq. (5.39), and

$$B_1=\frac{M_a^2}{2}(M_a^2-1)^{-1},$$
$$B_2=\frac{3}{8}M_a^4(M_a^2-1)^{-2}-\frac{M_a^2}{2}(M_a^2-1)^{-1},$$
$$B_3=\frac{5}{16}M_a^6(M_a^2-1)^{-3}-\frac{3}{4}M_a^4(M_a^2-1)^{-2}+\frac{M_a^2}{2}(M_a^2-1)^{-1},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$D_1=-\frac{1}{4}B_1,$$
$$D_2=\frac{1}{2}\left(\frac{3}{16}B_1^2-\frac{1}{3}B_2\right),$$
$$D_3=-\frac{1}{8}\left(\frac{5}{16}B_1^3-B_1B_2+B_3\right),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots.$$

The ratio, R, of the pressure jump in an adiabatic atmosphere to that in a uniform one is given as

$$R=\left(\frac{M_a^2-1}{\overline{M}_\infty^2-1}\right)^{\frac{1}{4}}\left(\frac{\overline{M}_\infty}{M_a}\right)^{m+2}\left\{1+\frac{B_1}{2}\alpha\eta+\frac{B_2}{3}(\alpha\eta)^2\right.$$
$$\left.+\frac{B_3}{4}(\alpha\eta)^3+\cdots\right\}^{-\frac{1}{2}}\left\{1+\frac{1}{3}(A_1+D_1)\alpha\eta\right.$$
$$+\frac{1}{5}(A_2+A_1D_1+D_2)(\alpha\eta)^2+\frac{1}{7}(A_3+A_2D_1$$
$$\left.+A_1D_2+D_3)(\alpha\eta)^3+\cdots\right\}^{-\frac{1}{2}}. \tag{5.69}$$

6. Intensities of Sonic Booms in a Standard Atmoshere

(i) The Solution of the Linearized Equation

As already mentioned in Chapter 5, the potential flow is no more valid in this case, so that the fundamental equations are Eqs. (4.21), (4.22) and (4.23). Now, the solutions for disturbances in velocity, pressure and density are assumed as

$$v'_x = v'_{x1} f_1(\xi-\omega) + v'_{x2} f_2(\xi-\omega) + v'_{x3} f_3(\xi-\omega) + \cdots, \tag{6.1a}$$

$$v'_r = v'_{r1} f_1(\xi-\omega) + v'_{r2} f_2(\xi-\omega) + v'_{r3} f_3(\xi-\omega) + \cdots, \tag{6.1b}$$

$$v'_\theta = v'_{\theta 1} f_1(\xi-\omega) + v'_{\theta 2} f_2(\xi-\omega) + v'_{\theta 3} f_3(\xi-\omega) + \cdots, \tag{6.1c}$$

$$p' = p'_1 f_1(\xi-\omega) + p'_2 f_2(\xi-\omega) + p'_3 f_3(\xi-\omega) + \cdots, \tag{6.1d}$$

$$\rho' = \rho'_1 f_1(\xi-\omega) + \rho'_2 f_2(\xi-\omega) + \rho'_3 f_3(\xi-\omega) + \cdots, \tag{6.1e}$$

where again the relations (5.42) hold. Substituting these expansions in Eqs. (4.21) to (4.23) and equating the coeficients of f'_1 to zero, we get

$$\rho'_1 - \frac{\rho_\infty}{\omega_\eta}\left(1 + \omega_\eta^2 + \frac{\omega_\theta^2}{\eta^2}\right) v'_{r1} = 0, \tag{6.2}$$

$$v'_{x1} + \frac{v'_{r1}}{\omega_\eta} = 0, \tag{6.3}$$

$$p'_1 - \frac{\rho_\infty U^2 v'_{r1}}{\omega_\eta} = 0, \tag{6.4}$$

$$v'_{\theta 1} - \frac{\omega_\theta}{\omega_\eta} \frac{1}{\eta} v'_{r1} = 0, \tag{6.5}$$

$$p'_1 + \gamma p_\infty \left(v'_{x1} - \omega_\eta v'_{r1} - \frac{1}{\eta}\omega_\theta v'_{\theta 1}\right) = 0. \tag{6.6}$$

From Eqs. (6.3) to (6.6), the differential equation by which Mach cones are determined is deduced as

$$(M_\infty^2 - 1) - \omega_\eta^2 - \frac{1}{\eta^2}\omega_\theta^2 = 0. \tag{6.7}$$

By putting the coefficient f_1 equal to zero, the following equations are obtained:

$$\rho_\infty v'_{x2} + \rho'_2 + \rho_\infty (v'_{r1\eta} - \omega_\eta v'_{r2}) + v'_{r1}\rho_{\infty\eta} + \frac{1}{\eta}\{\rho_\infty v'_{r1} + \rho_\infty (v'_{\theta 1\theta} - \omega_\theta v'_{\theta 2}) + v'_{\theta 1}\rho_{\infty\theta}\} = 0, \tag{6.8}$$

$$\rho_\infty U^2 v'_{x2} = -p'_2, \tag{6.9}$$

$$\rho_\infty U^2 v'_{r2} = \rho'_1 g l \cos\theta - (p'_{1\eta} - \omega_\eta p'_2), \tag{6.10}$$

$$\rho_\infty U^2 v'_{\theta 2} = -\rho'_1 g l \sin\theta - \frac{1}{\eta}(p'_{1\theta} - \omega_\theta p'_2), \tag{6.11}$$

$$p'_2 + v'_{r1} p_{\infty\eta} + \frac{v'_{\theta 1}}{\eta} p_{\infty\theta} + \gamma p_\infty \left\{ v'_{x2} + (v'_{r1\eta} - \omega_\eta v'_{r2}) + \frac{v'_{r1}}{\eta} + \frac{1}{\eta}(v'_{\theta 1\theta} - \omega_\theta v'_{\theta 2}) \right\} = 0. \tag{6.12}$$

From Eqs. (6.9) and (6.10) we get

$$v'_{x2}=\frac{\rho'_1 gl\cos\theta-p'_{1\eta}-\rho_\infty U^2 v'_{r2}}{\rho_\infty U^2\omega_\eta}\,. \tag{6.13}$$

From Eqs. (6.10) and (6.11)

$$v'_{\theta 2}=\frac{1}{\eta}\frac{\omega_\theta}{\omega_\eta}v'_{r2}+\frac{1}{\rho_\infty U^2}\left\{\frac{1}{\eta}\left(\frac{\omega_\theta}{\omega_\eta}p'_{1\eta}-p'_{1\theta}\right)\right.$$
$$\left.-\rho'_1 gl\left(\frac{1}{\eta}\frac{\omega_\theta}{\omega_\eta}\cos\theta+\sin\theta\right)\right\}. \tag{6.14}$$

From Eqs. (6.10) and (6.12)

$$(\rho_\infty U^2-\gamma p_\infty-\gamma p_\infty\omega_\eta^2)v'_{r2}-\rho'_1 gl\cos\theta+p'_{1\eta}+\omega_\eta v'_{r1}p_{\infty\eta}+\omega_\eta\frac{v'_{\theta 1}}{\eta}p_{\infty\theta}$$
$$+\gamma p_\infty\omega_\eta\left\{v'_{x2}+v'_{r1\eta}+\frac{v'_{r1}}{\eta}+\frac{1}{\eta}(v'_{\theta 1\theta}-\omega_\theta v'_{\theta 2})\right\}=0. \tag{6.15}$$

When Eqs. (6.13) and (6.14) are substituted in Eq. (6.15), the coefficient of v'_{r2} vanishes automatically using Eq. (6.7). As a result we get

$$\frac{c_\infty^2}{U^2}\left[\rho'_1 gl\cos\theta-p'_{1\eta}-\frac{1}{\eta}\omega_\theta\left\{\frac{1}{\eta}(\omega_\theta p'_{1\eta}-\omega_\eta p'_{1\theta})\right.\right.$$
$$\left.\left.-\rho'_1 gl\left(\frac{1}{\eta}\omega_\theta\cos\theta+\omega_\eta\sin\theta\right)\right\}\right]+\gamma p_\infty(\omega_\eta v'_{r1\eta}+\omega_\eta\frac{v'_{r1}}{\eta}$$
$$+\omega_\eta\frac{1}{\eta}v'_{\theta 1\theta})+\omega_\eta v'_{r1}p_{\infty\eta}+\omega_\eta\frac{v'_{\theta 1}}{\eta}p_{\infty\theta}-\rho'_1 gl\cos\theta+p'_{1\eta}=0. \tag{6.16}$$

From Eqs. (6.2) and (6.7)

$$\rho'_1=\frac{\rho_\infty}{\omega_\eta}M_\infty^2 v'_{r1}. \tag{6.17}$$

Differenciations of Eqs. (6.4) and (6.5) with respect to η and θ yield

$$p'_{1\eta}=\left(\frac{\rho_\infty U^2}{\omega_\eta}\right)_\eta v'_{r1}+\frac{\rho_\infty U^2}{\omega_\eta}v'_{r1\eta},$$

$$p'_{1\theta}=\left(\frac{\rho_\infty U^2}{\omega_\eta}\right)_\theta v'_{r1}+\frac{\rho_\infty U^2}{\omega_\eta}v'_{r1\theta},$$

$$v'_{\theta 1\theta}=\left(\frac{\omega_\theta}{\omega_\eta}\frac{1}{\eta}\right)_\theta v'_{r1}+\frac{\omega_\theta}{\omega_\eta}\frac{1}{\eta}v'_{r1\theta}.$$

Using these expressions together with Eq. (6.5), Eq. (6.16) finally can be written as

$$v'_{r1\eta}+\frac{1}{\eta^2}\left(\frac{\omega_\theta}{\omega_\eta}\right)v'_{r1\theta}+\frac{1}{2}\frac{\omega_\eta}{\rho_\infty}\left(\frac{\rho_\infty}{\omega_\eta}\right)_\eta v'_{r1}+\frac{1}{2\eta}\left\{1+\left(\frac{\omega_\theta}{\omega_\eta}\right)_\theta\frac{1}{\eta}\right\}v'_{r1}$$
$$+\frac{1}{2\rho_\infty\eta^2}\frac{\omega_\theta}{\omega_\eta}\left(\frac{\rho_\infty}{\omega_\eta}\right)_\theta v'_{r1}-\frac{1}{\eta^2}\left(\frac{\omega_\theta}{\omega_\eta}\right)^2\frac{gl}{c_\infty^2}v'_{r1}\cos\theta$$
$$-\frac{\omega_\theta}{\omega_\eta}\frac{gl}{c_\infty^2}v'_{r1}\sin\theta=0. \tag{6.18}$$

Further we assume v'_{r1} and ω can be expanded in the form

$$v'_r = v'^{(0)}_{r1} + \theta^2 v'^{(1)}_{r1} + \theta^4 v'^{(2)}_{r1} + \cdots, \tag{6.19}$$

$$\omega = \omega^{(0)} + \theta^2 \omega^{(1)} + \theta^4 \omega^{(2)} + \cdots. \tag{6.20}$$

Substituting these expressions in Eq. (6.18) and gathering the coefficients of θ^0, we have

$$v'^{(0)}_{r1\eta} + \frac{1}{2}\frac{\omega^{(0)}_\eta}{\bar\rho_\infty}\left(\frac{\bar\rho_\infty}{\omega^{(0)}_\eta}\right)_\eta v'^{(0)}_{r1} + \frac{1}{2\eta}\left(1 + 2\frac{\omega^{(1)}}{\omega^{(0)}_\eta}\frac{1}{\eta}\right) v'^{(0)}_{r1} = 0 \tag{6.21}$$

where from Eq. (6.7)

$$\frac{\omega^{(1)}}{\omega^{(0)}_\eta} = -\frac{\eta}{2} + \frac{\eta^2}{(\bar M^2_\infty - 1)^{\frac{1}{2}}}\left\{\int \frac{2}{(\bar M^2_\infty - 1)^{\frac{1}{2}}} d\eta\right\}^{-1}. \tag{6.22}$$

Hence, Eq. (6.21) becomes

$$v'^{(0)}_{r1\eta} + \frac{1}{2}\left\{\frac{\omega^{(0)}_\eta}{\bar\rho_\infty}\left(\frac{\bar\rho_\infty}{\omega^{(0)}_\eta}\right)_\eta + \frac{1}{(\bar M^2_\infty - 1)^{\frac{1}{2}}\int_0^\eta (\bar M^2_\infty - 1)^{-\frac{1}{2}} d\eta}\right\} v'^{(0)}_{r1} = 0. \tag{6.23}$$

Integrating this, we find

$$v'^{(0)}_{r1} = K^{(0)} \exp\left[-\frac{1}{2}\int \left\{\frac{\omega^{(0)}_\eta}{\bar\rho_\infty}\left(\frac{\bar\rho_\infty}{\omega^{(0)}_\eta}\right)_\eta + \frac{1}{(\bar M^2_\infty - 1)^{\frac{1}{2}}\int_0^\eta (\bar M^2_\infty - 1)^{-\frac{1}{2}} d\eta}\right\} d\eta\right] \tag{6.24}$$

$$= -K^{(0)}(\bar M^2_\infty - 1)^{\frac{1}{2}}\sqrt{\frac{\rho_a}{\bar\rho_\infty}}\left(\frac{M^2_a - 1}{\bar M^2_\infty - 1}\right)^{\frac{1}{4}}\left\{\int_0^\eta \left(\frac{M^2_a - 1}{\bar M^2_\infty - 1}\right)^{\frac{1}{2}} d\eta\right\}^{-\frac{1}{2}}. \tag{6.25}$$

We must chose the integration constant as

$$K^{(0)} = -\frac{1}{(2\sqrt{M^2_a - 1})^{\frac{1}{2}}}$$

because of the same reason as in case of an adiabatic atmosphere. Hence we have

$$\bar v'_r \approx (\bar M^2_\infty - 1)^{\frac{1}{2}}\frac{1}{(2\sqrt{M^2_a - 1})^{\frac{1}{2}}}\sqrt{\frac{\rho_a}{\bar\rho_\infty}}\left(\frac{M^2_a - 1}{\bar M^2_\infty - 1}\right)^{\frac{1}{4}}$$
$$\times\left\{\int_0^\eta \left(\frac{M^2_a - 1}{\bar M^2_\infty - 1}\right)^{\frac{1}{2}} d\eta\right\}^{-\frac{1}{2}} f_1(\xi - \omega^{(0)}), \tag{6.26}$$

where

$$f_1(\xi - \sqrt{M^2_a - 1}\,\eta) = \frac{1}{l}\int_0^{\xi - \sqrt{M^2_a - 1}\,\eta} \frac{\sigma'(t)dt}{(\xi - \sqrt{M^2_a - 1}\,\eta - t)^{\frac{1}{2}}}.$$

(ii) Pressure Jump across the shock wave at large distance

From Eq. (3.7) we have

$$\frac{U^2}{2}\left\{(1 + \bar v'_x)^2 + \bar v'^2_r\right\} - g(z^* + \Delta z^*) + \frac{1}{\gamma - 1}\bar c^2 = \frac{U^2}{2} - gz^* + \frac{1}{\gamma - 1}\bar c^2_\infty \tag{6.27}$$

along any given streamline passing through the point z^*, the distance from the upper limit of an atmosphere. The deflection of the streamline, Δz^*, is considered small, or $\Delta z^*/z^* \ll 1$. Neglecting $\Delta z^*/z^*$, the local speed of sound becomes as

$$\bar{c}=\bar{c}_\infty\left\{1-\frac{\gamma-1}{2}\bar{M}_\infty^2\bar{v}_x'+O(\bar{v}_x'^2+\bar{v}_r'^2)\right\} \tag{6.28}$$

which is the same form as in the case of an adiabatic atmosphere. Therefore, the equation for the direction of the Mach line is given by

$$\xi=\int_0^\eta(\bar{M}_\infty^2-1)^{\frac{1}{2}}d\eta-kF(\tau)\cdot I(\eta)+\tau \tag{6.29}$$

where

$$k=2^{-\frac{3}{2}}(\gamma+1)\sqrt{\rho_a}\,,$$

$$F(\tau)=\frac{1}{l}\int_0^\eta\frac{\sigma'(t)dt}{(\tau-t)^{\frac{1}{2}}},$$

$$I(\eta)=\int_0^\eta\frac{\bar{M}_\infty^4}{\sqrt{\bar{\rho}_\infty}(\bar{M}_\infty^2-1)^{\frac{3}{4}}}\left\{\int_0^\eta\left(\frac{M_a^2-1}{\bar{M}_\infty^2-1}\right)^{\frac{1}{2}}d\eta\right\}^{-\frac{1}{2}}d\eta$$

Using Eq. (6.7) and the results obtained above, the pressure jump across the shock in the far-field can be derived by the parallel process to the adiabatic atmosphere case. The final result becomes

$$\begin{aligned}\frac{\Delta p}{\bar{p}_\infty}=&\frac{2^{\frac{3}{4}}\gamma}{(\gamma+1)^{\frac{1}{2}}}\left(\frac{\bar{M}_\infty}{M_a}\right)^2\left(\frac{M_a^2-1}{\bar{M}_\infty^2-1}\right)^{\frac{1}{4}}\sqrt{\frac{\rho_a}{\bar{\rho}_\infty}}(M_a^2-1)^{\frac{1}{8}}\\&\times\left\{\int_0^\eta\left(\frac{M_a^2-1}{\bar{M}_\infty^2-1}\right)^{\frac{1}{2}}d\eta\right\}^{-\frac{1}{2}}\left[\int_0^\eta\sqrt{\frac{\rho_a}{\bar{\rho}_\infty}}\left(\frac{\bar{M}_\infty}{M_a}\right)^4\left(\frac{M_a^2-1}{\bar{M}_\infty^2-1}\right)^{\frac{3}{4}}\right.\\&\left.\times\left\{\int_0^\eta\left(\frac{M_a^2-1}{\bar{M}_\infty^2-1}\right)^{\frac{1}{2}}d\eta\right\}^{-\frac{1}{2}}d\eta\right]^{-\frac{1}{2}}\left[\int_0^{\tau_0}F(\nu)d\nu\right]^{\frac{1}{2}}\end{aligned} \tag{6.30}$$

which is capable to describe pressure jumps in any kind of atmospheres. For instance, in the cases of isothermal and uniform atmospheres, Eq. (6.30) becomes, respectively,

$$\frac{\Delta p}{\bar{p}_\infty}=\frac{2^{\frac{3}{4}}\gamma}{(\gamma+1)^{\frac{1}{2}}}(M_a^2-1)^{\frac{1}{8}}\sqrt{\frac{\rho_a}{\bar{\rho}_\infty}}\eta^{-\frac{1}{2}}\left[\int_0^\eta\sqrt{\frac{\rho_a}{\bar{\rho}_\infty}}\eta^{-\frac{1}{2}}d\eta\right]^{-\frac{1}{2}}\left[\int_0^{\tau_0}F(\nu)d\nu\right]^{\frac{1}{2}}, \tag{6.31}$$

and

$$\frac{\Delta p}{p_\infty}=\frac{2^{\frac{1}{4}}\gamma}{(\gamma+1)^{\frac{1}{2}}}(M_a^2-1)^{\frac{1}{8}}\eta^{-\frac{3}{4}}\left[\int_0^{\tau_0}F(\nu)d\nu\right]^{\frac{1}{2}}. \tag{6.32}$$

Of course, Eq. (6.32) coincide fully with Whitham's formula. In case of a polytropic atmosphere, we can write as

$$\sqrt{\frac{\rho_a}{\bar{\rho}_\infty}}=\left(\frac{\bar{M}_\infty}{M_a}\right)^{m'},\qquad m'=\frac{1}{n-1}. \tag{6.33}$$

In case of an adiabatic atmosphere which is a special case of a polytropic one, we can get again Eq. (5.68) by putting $n=\gamma$ in Eq. (6.30).

Next, the case of a standard atmosphere will be treated. As already stated in Chapter 2, the atmosphere has a discontinuity of temperature gradient at altitude of about 11 km, and under this boundary its temperature gradient is constant, while above the boundary it is zero, namely, the temperature is constant. The treatment for the flight under 11 km in altitude can be done by use of Eqs. (6.30) and (6.33), but in case of the flight above 11 km it is necessary to divide the region into two parts which are of nondimensional breadth of η_c from the aircraft and of $\eta-\eta_c$ from η_c to the ground as shown in Fig. 3.

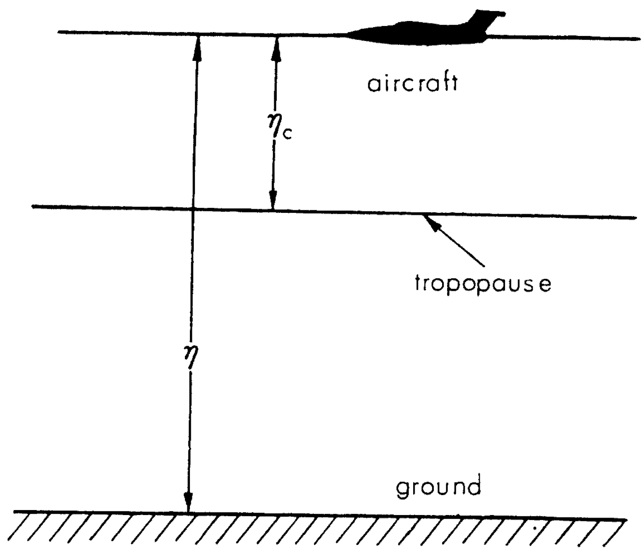

FIG. 3. Division of the region for flight in a stratosphere.

Fiom Eq. (6.30),

$$
\begin{aligned}
\frac{\Delta p}{\bar{p}_\infty} = & \frac{2^{\frac{3}{4}}\gamma}{(\gamma+1)^{\frac{1}{2}}}\left(\frac{\bar{M}_\infty}{M_a}\right)^2\left(\frac{M_a^2-1}{\bar{M}_\infty^2-1}\right)^{\frac{1}{4}}\sqrt{\frac{\rho_a}{\bar{\rho}_\infty}}(M_a^2-1)^{\frac{1}{8}} \\
& \times\left\{\int_0^{\eta_c} d\eta+\int_{\eta_c}^{\eta}\left(\frac{M_a^2-1}{\bar{M}_\infty^2-1}\right)^{\frac{1}{2}} d\eta\right\}^{-\frac{1}{2}}\left[\int_0^{\eta_c}\sqrt{\frac{\rho_a}{\bar{\rho}_\infty}}\left(\int_0^{\eta} d\eta\right)^{-\frac{1}{2}} d\eta\right. \\
& +\int_{\eta_c}^{\eta}\sqrt{\frac{\rho_a}{\bar{\rho}_\infty}}\left(\frac{\bar{M}_\infty}{M_a}\right)^4\left(\frac{M_a^2-1}{\bar{M}_\infty^2-1}\right)^{\frac{3}{4}}\left\{\int_0^{\eta_c} d\eta\right. \\
& \left.\left.+\int_{\eta_c}^{\eta}\left(\frac{M_a^2-1}{\bar{M}_\infty^2-1}\right)^{\frac{1}{2}} d\eta\right\}^{-\frac{1}{2}}\right]^{-\frac{1}{2}}\left[\int_0^{\tau_0} F(\nu)d\nu\right]^{\frac{1}{2}}.
\end{aligned} \tag{6.34}
$$

where by simple calculation,

$$
\sqrt{\frac{\rho_a}{\bar{\rho}_\infty}} = \left(\frac{\bar{M}_\infty}{M_a}\right)^{m'} e^{-\frac{n}{2(n-1)}\alpha\eta c} \tag{6.35}
$$

$$
\left.
\begin{aligned}
& \bar{M}_\infty^2 = M_a^2 && \text{for} \quad 0\leqq\eta\leqq\eta_c, \\
& \bar{M}_\infty^2 = M_a^2\{1+\alpha(\eta-\eta_c)\}^{-1} && \text{for} \quad \eta_c\leqq\eta.
\end{aligned}
\right\} \tag{6.36}
$$

Using Eqs. (6.35) and (6.36) and reforming it to a convienient form for calculation we finally get

$$\frac{\Delta p}{\bar{p}_\infty}=R\cdot 2^{\frac{1}{4}}\gamma(\gamma+1)^{-\frac{1}{2}}(M_a^2-1)^{\frac{1}{8}}\left[\int_0^{\tau_0}F(\nu)d\nu\right]^{\frac{1}{2}}\eta^{-\frac{3}{4}}, \tag{6.37}$$

$$R=\left(\frac{\bar{M}_\infty}{M_a}\right)^{m'+2}\left(\frac{M_a^2-1}{\bar{M}_\infty^2-1}\right)^{\frac{1}{4}}e^{-\frac{k}{2}\alpha\eta_c}J(\eta)\cdot H(\eta) \tag{6.38}$$

where

$$k=\frac{n}{2(n-1)}, \tag{6.39}$$

$$J(\eta)=\{1+e_1\alpha\eta+e_2(\alpha\eta)^2+e_3(\alpha\eta)^3+\cdots\}^{-\frac{1}{2}}, \tag{6.40}$$

$$e_1=\frac{1}{4}M_a^2(M_a^2-1)^{-1}\left(1-\frac{\eta_c}{\eta}\right)^2,$$

$$e_2=\frac{1}{8}M_a^4(M_a^2-1)^{-2}\left(1-\frac{\eta_c}{\eta}\right)^3-\frac{1}{6}M_a^2(M_a^2-1)^{-1}\left(1-\frac{\eta_c}{\eta}\right)^3,$$

$$e_3=\frac{5}{64}M_a^6(M_a^2-1)^{-3}\left(1-\frac{\eta_c}{\eta}\right)^4-\frac{3}{16}M_a^4(M_a^2-1)^{-2}\left(1-\frac{\eta_c}{\eta}\right)^4$$
$$+\frac{1}{8}M_a^2(M_a^2-1)^{-1}\left(1-\frac{\eta_c}{\eta}\right)^4,$$

$$H(\eta)=\left[\sqrt{\frac{\eta_c}{\eta}}\left\{\frac{(2k\alpha\eta_c)}{1\cdot 3}+\frac{(2k\alpha\eta_c)^2}{1\cdot 3\cdot 5}+\frac{(2k\alpha\eta_c)^3}{1\cdot 3\cdot 5\cdot 7}+\cdots\right\}+\left\{1+D_1(\alpha\eta)\right.\right.$$
$$\left.\left.+D_2(\alpha\eta)^2+D_3(\alpha\eta)^3+\cdots\right\}\right]^{-\frac{1}{2}}, \tag{6.41}$$

$$D_1=\frac{1}{3}\left(A_1-\frac{1}{4}a_1\right)\left(1-\frac{\eta_c^{\frac{3}{2}}}{\eta^{\frac{3}{2}}}\right)+\left(\frac{1}{2}a_1-A_1\right)\left(\frac{\eta_c}{\eta}\right)\left(1-\frac{\eta_c^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\right)$$
$$+\frac{1}{4}a_1\left(\frac{\eta_c}{\eta}\right)^2\left(1-\frac{\eta^{\frac{1}{2}}}{\eta_c^{\frac{1}{2}}}\right),$$

$$D_2=\frac{1}{5}\left\{A_2-\left(\frac{1}{4}a_1A_1+\frac{1}{6}a_2\right)+\frac{3}{32}a_1^2\right\}\left(1-\frac{\eta_c^{\frac{5}{2}}}{\eta^{\frac{5}{2}}}\right)-\frac{1}{3}\left\{2A_2\right.$$
$$\left.-3\left(\frac{1}{4}a_1A_1+\frac{1}{6}a_2\right)+\frac{3}{8}a_1^2\right\}\frac{\eta_c}{\eta}\left(1-\frac{\eta_c^{\frac{3}{2}}}{\eta^{\frac{3}{2}}}\right)+\left\{A_3-3\left(\frac{1}{4}a_1A_1+\frac{1}{6}a_2\right)\right.$$
$$\left.+\frac{9}{16}a_1^2\right\}\left(\frac{\eta_c}{\eta}\right)^2\left(1-\frac{\eta_c^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\right)-\left\{\left(\frac{1}{4}a_1A_1+\frac{1}{6}a_2\right)-\frac{3}{8}a_1^2\right\}\left(\frac{\eta_c}{\eta}\right)^3\left(1-\frac{\eta^{\frac{1}{2}}}{\eta_c^{\frac{1}{2}}}\right)$$
$$-\frac{1}{32}a_1^2\left(\frac{\eta_c}{\eta}\right)^4\left(1-\frac{\eta^{\frac{3}{2}}}{\eta_c^{\frac{3}{2}}}\right),$$

$$D_3=\frac{1}{7}\left\{A_3-\left(\frac{1}{4}a_1A_2+\frac{1}{6}a_2A_1+\frac{1}{8}a_3\right)+\left(\frac{3}{32}a_1^2A_1+\frac{1}{8}a_1a_2\right)\right.$$
$$\left.-\frac{5}{128}a_1^3\right\}\left(1-\frac{\eta_c^{\frac{7}{2}}}{\eta^{\frac{7}{2}}}\right)-\frac{1}{5}\left\{3A_3-4\left(\frac{1}{4}a_1A_2+\frac{1}{6}a_2A_1+\frac{1}{8}a_3\right)\right.$$

$$+5\left(\frac{3}{32}a_1^2A_1+\frac{1}{8}a_1a_2\right)-\frac{15}{64}a_1^3\Big\}\left(\frac{\eta_c}{\eta}\right)\left(1-\frac{\eta_c^{\frac{5}{2}}}{\eta^{\frac{5}{2}}}\right)+\frac{1}{3}\Big\{3A_3$$

$$-6\left(\frac{1}{4}a_1A_2+\frac{1}{6}a_2A_1+\frac{1}{8}a_2\right)+10\left(\frac{3}{32}a_1^2A_1+\frac{1}{8}a_1a_2\right)-\frac{75}{128}a_1^3\Big\}$$

$$\times\left(\frac{\eta_c}{\eta}\right)^2\left(1-\frac{\eta_c^{\frac{3}{2}}}{\eta^{\frac{3}{2}}}\right)-\Big\{A_3-4\left(\frac{1}{4}a_1A_2+\frac{1}{6}a_2A_1+\frac{1}{8}a_3\right)$$

$$+10\left(\frac{3}{32}a_1^2A_1+\frac{1}{8}a_1a_2\right)-\frac{25}{32}a_1^3\Big\}\left(\frac{\eta_c}{\eta}\right)^3\left(1-\frac{\eta_c^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\right)+\Big\{\left(\frac{1}{4}a_1A_2\right.$$

$$\left.+\frac{1}{6}a_2A_1+\frac{1}{8}a_3\right)-5\left(\frac{3}{32}a_1^2A_1+\frac{1}{8}a_1a_2\right)+\frac{75}{128}a_1^3\Big\}\left(\frac{\eta_c}{\eta}\right)^4\left(1-\frac{\eta^{\frac{1}{2}}}{\eta_c^{\frac{1}{2}}}\right)$$

$$+\frac{1}{3}\Big\{\left(\frac{3}{32}a_1^2A_1+\frac{1}{8}a_1a_2\right)-\frac{15}{64}a_1^3\Big\}\left(\frac{\eta_c}{\eta}\right)^5\left(1-\frac{\eta^{\frac{3}{2}}}{\eta_c^{\frac{3}{2}}}\right)$$

$$+\frac{1}{128}a_1^3\left(\frac{\eta_c}{\eta}\right)^6\left(1-\frac{\eta^{\frac{5}{2}}}{\eta_c^{\frac{5}{2}}}\right).$$

In the above formula $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3 \cdots$ are the same as those given in Eq. (5.68) in which m is replaced by m'.

7. Numerical Results and Discussion

The results of numerical calculation of the pressure jump ratio, R, versus flight altitude are shown in Fig. 4 for several Mach numbers in both cases of two-

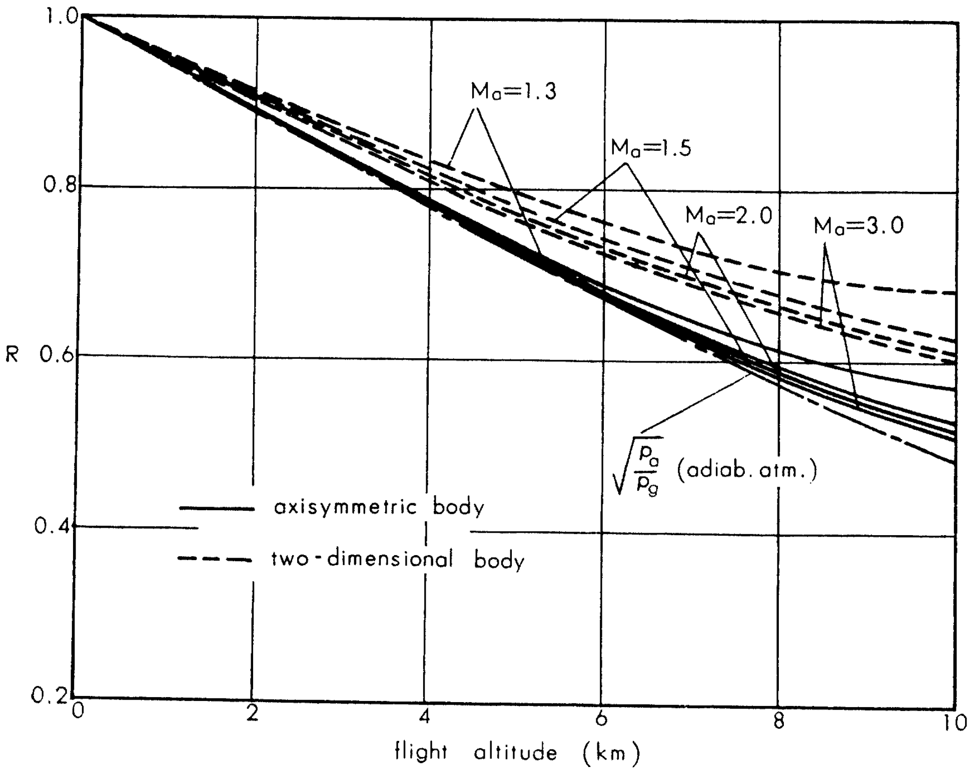

FIG. 4. Variation of R with flight altitude for an adiabatic atmosphere.

dimensional and axisymmetric bodies. The results for the axisymmetric body in a standard atmosphere are illustrated in Fig. 5. These calculations have been carried out by taking the terms up to the third power of $\alpha\zeta$ or $\alpha\eta$. In most of the existing theory of sonic boom problems, $\sqrt{p_a/p_g}$ is used instead of R, where p_a and p_g are the pressures at the flight level and on the ground respectively. This classical modification factor is shown by dotted lines in Fig. 4 and Fig. 5.

It is of interest that R obtained in the present analysis, especially for the axisymmetric body case, agrees fairly well with $\sqrt{p_a/p_g}$. This seems to indicate that, for a rough estimation of sonic booms, the method used so far which modifies Whitham's theory by the factor $\sqrt{p_a/p_g}$ would give a good approximation in a practical sense. However, this factor is not a function of the flight Mach number, but R depends on the Mach number as shown in the present analysis. In this sense, R is considered as the more exact correction factor.

The comparison of R in this paper with that obtained by Friedman and his coworkers by the different method is illustrated in Fig. 6 [9]. The pressure $\tilde{p}$ is the reference pressure which is determined theoretically, and correspondes to the average pressure $\sqrt{p_a p_g}$ in the formula for the pressure jump modified empirically. Friedman's work uses the ray-tube method which rests on the basic assumption that the propagation of the disturbance down each tube may be treated separately, whereas Whitham's theory involves disturbances propagating along the shock front. Fig. 6 indicates good agreement between the two approaches.

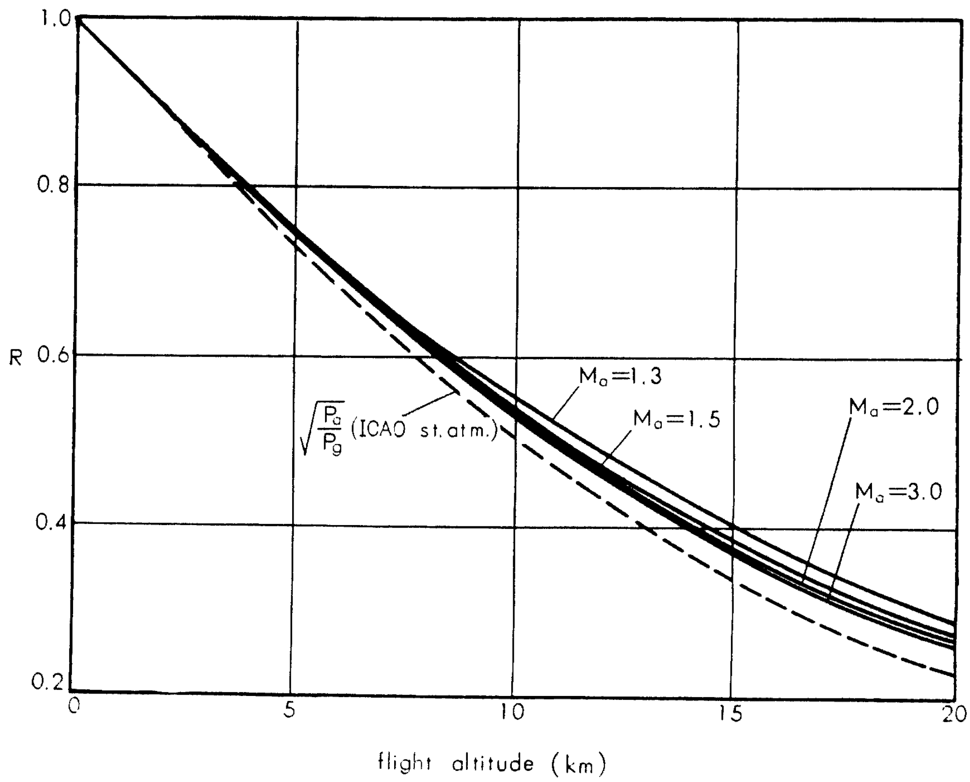

FIG. 5. Variation of R with flight altitude for a standard atmosphere.

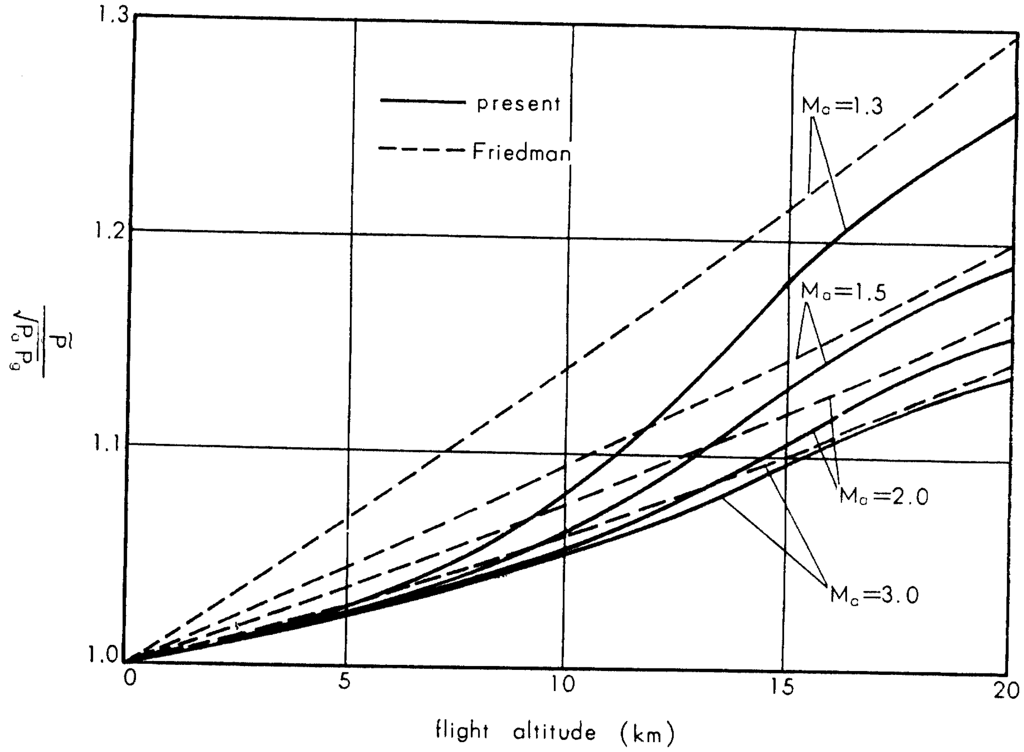

FIG. 6. Comparison with Friedman's result in case of a standard atmosphere.

From Eq. (6.31) the increase of density has an effect on decay of shock strength. The increase of atmospheric temperature with the decrease of altitude strengthens sonic boom intensities on account of ray-focusing effect. The reason why sonic booms decay towards the ground in the actual atmosphere is because the increase of atmospheric density has a greater effect on the strength of shock waves than that of temperature with the decrease of altitude. Further, the formula obtained here can be applied also to the atmospheric state with invertion, which will give us an interesting result.

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August, 22, 1967

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