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Elastic Stability of Spherical Shells Subjected to External Pressure

By

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Summary: The nonlinear fundamental equations of a spherical shell for thermoelastic problems, subjected to the pressure and the change in temperature, are derived first of all, taking a large deformation into account. Then, the equations are solved approximately assuming the deformation pattern to be the axisymmetric damped wave. The relations between the external pressure and the deflection at the center of the shell are obtained by use of the Galerkin method and the critical pressures are discussed based on the author's viewpoint that the buckling phenomenon of spherical shell is of a local one at the initial stage of its occurrence and the localized deformation has the triggering effect in inducing the much larger dynamic deformation and that the buckling process of spherical shell under external pressure is of a "snapping through". The result obtained seems to be one of the promising contributions to bridge the gap between the theoretical and experimental values of the problem.

NOMENCLATURE

h	:	thickness of the spherical shell.
$k_{n,\xi}$	:	Eq. (3.5).
m	:	damping coefficient, Eq. (3.1).
n	:	number of half waves of deflection, Eq. (3.1); $\bar{n}=n\pi$.
p	:	normal pressure (positive for the external pressure).
u, w	:	non-dimensional U and W with respect to h.
x, y, z	:	rectangular coordinates, Fig. 2.1.
A	:	Eqs. (3.1) and (3.3).
C_1, C_2	:	integral constants, Eqs. (3.9) and (3.10).
D	:	flexural rigidity; $D=Eh^3/12(1-\nu^2)$.
E, G	:	moduli of elasticity and rigidity, respectively.
F	:	free energy in a unit volume.
P	:	pressure parameter, $P=\frac{1}{2}\left(\frac{R}{h}\right)^2\left(\frac{p}{E}\right)$.
R	:	radius of curvature of the spherical shell.
T	:	change in temperature from the initial state.
$\bar{T}, \tilde{T}$	:	Eqs. (2.10).
U, W	:	displacement components in the middle plane in the x- and z-directions, respectively.

α	: coefficient of linear thermal expansion.
$\delta=-w(0)$	: non-dimensional deflection at the center of the spherical shell.
$\varepsilon_{11}, \varepsilon_{22}$	: extensional strains in the x- and y-directions, respectively.
ε_{12}	: shearing strain in the xy-plane.
σ_{11}, σ_{22}	: extensional stresses in the x- and y-directions, respectively.
σ_{12}	: shearing stress in the xy-plane.
κ_1, κ_2	: change of curvatures of the middle plane about x- and y-axes, respectively.
κ_{12}	: change of twist of the middle plane.
θ	: Fig. 2.1.
θ_0	: reference semi-apex angle of the shell.
$\vartheta=\theta/\theta_0$.	
ν	: Poisson's ratio.
$\xi=m/\bar{n}$	: Eq. (3.4).
Π	: total potential energy.
χ	: stress function.
ϕ	: geometrical parameter, $\phi=\theta_0^2\left(\frac{R}{h}\right)$.

Subscript $\theta(\vartheta)$ denotes the differentiation with respect to $\theta(\vartheta)$.
Bar over letter refers to the middle plane.

1. Introduction

The existence of large discrepancy between the classical theoretical values and experimental ones of the critical loads of shells has resulted in the development of approaches from many viewpoints. Using the nonlinear or finite deformation theory, von Kármán and Tsien [*1*] have analyzed the buckling of complete spherical shell subjected to external pressure and gave a light in attacking these problems. Yoshimura and Uemura [*2*] have analyzed the same problem using the similar approach and obtained a more detailed result. Investigations on partial spherical shells have also been proceeded in accordance with the practical importance. The analyses carried out in the early stage of nonlinear research have adopted the axisymmetric deformation patterns only. The discrepancy mentioned above has still existed in spite of much efforts which have been concentrated on these problems including also the effects of initial geometrical imperfections, and the asymmetric modes of buckling have been introduced lately. The post buckling behaviors have been analyzed parallel to the prebuckling and buckling analyses. Some of the analyses followed are referred in References [*3*]–[*26*] at the end of paper.

On the other hand, there exist some doubts on the experimental results [*27*]–[*33*] reported so far. It is very difficult or rather impossible to make the perfect shells. The initial geometrical imperfections and residual stresses are necessarily introduced in fabricating and supporting shells. And the practical boundary conditions do not necessarily coincide with the conditions used in the theoretical analyses. Another important factor is the technique in carrying out the experiments. So it seems that the discrepancy between the theoretical and experimental values has to be reduced from both the theoretical and experimental approaches.

The spherical shell is one of the most important structural components, and the fields of its utilization have a wide range of variety. The loads which the shells have to carry are also various, and the most important one is the normally distributed load. The thermal load also cannot be ignored in many fields of engineering, especially in the field of aerospace engineering.

In the present paper, the nonlinear fundamental equations of a spherical shell for thermoelastic problems, subjected to the pressure and the change in temperature, are derived first of all. Then, a tentative solution for the shell subjected to external pressure is presented, where special attentions are paid to the damped deformation pattern and to the discussion on the initiation of the buckling based on the author's opinion that the buckling is a local phenomenon at the initial stage of its occurrence and that the buckling process of spherical shell under external pressure is of a "snapping through".

2. Derivation of Fundamental Equations

A spherical shell as shown in Fig. 2.1 is considered, and it is assumed that the shell is subjected to external pressure and is heated at the outer or inner surface and there exist the temperature gradients in the x-direction on the middle plane and through the thickness of the shell.

The two-dimensional stress-strain law can be given by

$$\left.\begin{aligned}\varepsilon_{11}&=\frac{1}{E}(\sigma_{11}-\nu\sigma_{22})+\alpha T,\\ \varepsilon_{22}&=\frac{1}{E}(\sigma_{22}-\nu\sigma_{11})+\alpha T,\\ \varepsilon_{12}&=\frac{1}{G}\sigma_{12}=\frac{2(1+\nu)}{E}\sigma_{12},\end{aligned}\right\}\qquad(2.1)$$

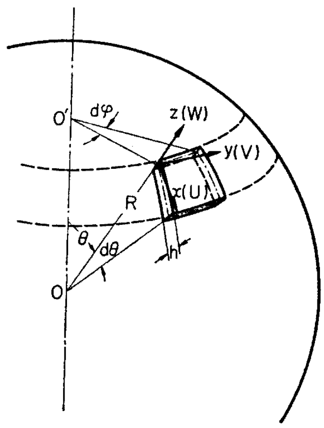

Fig. 2.1. Spherical shell.

where, using the hypothesis of Kirchhoff-Love, the strain components can be given as follows:

$$\left.\begin{aligned}\varepsilon_{11}&=\bar{\varepsilon}_{11}-z\kappa_{1},\\ \varepsilon_{22}&=\bar{\varepsilon}_{22}-z\kappa_{2},\\ \varepsilon_{12}&=\bar{\varepsilon}_{12}-2z\kappa_{12}\,.\end{aligned}\right\}\qquad(2.2)$$

Now, the following assumptions are introduced.

(i) the deformation of the shell is rotationally symmetrical with respect to the axis of symmetry,

(ii) the semi-apex angle of the spherical shell considered is small, and

(iii) the temperature distribution in the shell is also rotationally symmetrical with respect to the axis of symmetry.

Then the strain components and the change of curvatures of the middle plane for the case of axisymmetric deformation can be expressed [34] as in Eqs. (2.3) and (2.4), where the terms of the first order of infinitesimal only have been taken into account.

$$\left.\begin{aligned}\bar{\varepsilon}_{11}&=\frac{1}{R}(U_{,\theta}+W)+\frac{1}{2R^2}W_{,\theta}^2,\\ \bar{\varepsilon}_{22}&=\frac{1}{R}(U\cot\theta+W),\\ \bar{\varepsilon}_{12}&=0,\end{aligned}\right\}\tag{2.3}$$

$$\left.\begin{aligned}\kappa_1&=\frac{1}{R^2}W_{,\theta\theta},\\ \kappa_2&=\frac{1}{R^2}W_{,\theta}\cot\theta,\\ \kappa_{12}&=0.\end{aligned}\right\}\tag{2.4}$$

And so Eqs. (2.2) reduce to the following expressions.

$$\left.\begin{aligned}\varepsilon_{11}&=\frac{1}{R}(U_{,\theta}+W)+\frac{1}{2R^2}W_{,\theta}^2-\frac{z}{R^2}W_{,\theta\theta},\\ \varepsilon_{22}&=\frac{1}{R}(U\cot\theta+W)-\frac{z}{R^2}W_{,\theta}\cot\theta,\\ \varepsilon_{12}&=0.\end{aligned}\right\}\tag{2.5}$$

The equilibrium equations can be derived by the variational method with the aid of the well-known theorem of the stationary potential energy. The process is the same as that in the case of rectangular plates [35], and so its detailed description will not be given here.

The total potential energy, Π of an element of the spherical shell which is bounded by two conical surfaces normal to the meridians and inclined to the axis of symmetry by the angle θ_1 and θ_2 and by the outer and inner surfaces is expressed as

$$\begin{aligned}\Pi&=\int_V FdV+\int_S pWdS\\ &=2\pi R^2\left(\int_{\theta_1}^{\theta_2}\int_{-h/2}^{h/2}F\sin\theta dzd\theta+\int_{\theta_1}^{\theta_2}pW\sin\theta d\theta\right),\end{aligned}\tag{2.6}$$

where, the free energy, F can be given as

$$\begin{aligned}F=&\frac{E}{2(1-\nu^2)}\left[\varepsilon_{11}^2+2\nu\varepsilon_{11}\varepsilon_{22}+\varepsilon_{22}^2+\frac{1}{2}(1-\nu)\varepsilon_{12}^2\right]\\ &-\frac{E\alpha T}{(1-\nu)}(\varepsilon_{11}+\varepsilon_{22})+C_T(T).\end{aligned}\tag{2.7}$$

Operating the variational process on Π with respect to the displacement components, U and W, and using the usual theorem of the stationary potential energy, that is, $\delta\Pi=0$, the following equilibrium equations in the x- and z-directions can be obtained.

$$\left(\frac{Eh}{(1-\nu^2)}\left\{\nu\left[\frac{1}{R}(U_\theta+W)+\frac{1}{2R^2}W_\theta^2\right]+\frac{1}{R}(U\cot\theta+W)\right\}-\frac{Eh\alpha}{(1-\nu)}\bar{T}\right)\cos\theta$$
$$-\frac{d}{d\theta}\left(\left\{\frac{Eh}{(1-\nu^2)}\left[\frac{1}{R}(U_\theta+W)+\frac{1}{2R^2}W_\theta^2+\frac{\nu}{R}(U\cot\theta+W)\right]\right.\right.$$
$$\left.\left.-\frac{Eh\alpha}{(1-\nu)}\bar{T}\right\}\sin\theta\right)=0, \tag{2.8}$$

$$\left\{\frac{Eh}{(1-\nu^2)}(1+\nu)\left[\frac{1}{R}(U_\theta+W)+\frac{1}{2R^2}W_\theta^2+\frac{1}{R}(U\cot\theta+W)\right]-2\frac{Eh\alpha}{(1-\nu)}\bar{T}\right\}\sin\theta$$
$$-\frac{d}{d\theta}\left(\left\{\frac{Eh}{(1-\nu^2)}\left[\frac{1}{R}(U_\theta+W)+\frac{1}{2R^2}W_\theta^2+\frac{\nu}{R}(U\cot\theta+W)\right]\right.\right.$$
$$\left.\left.-\frac{Eh\alpha}{(1-\nu)}\bar{T}\right\}\frac{1}{R}W_\theta\sin\theta\right)+\frac{1}{R}\frac{d}{d\theta}\left(\frac{d}{d\theta}\left\{\left[\frac{Eh^3}{12(1-\nu^2)}\left(\frac{1}{R^2}W_{\theta\theta}+\frac{\nu}{R^2}W_\theta\cot\theta\right)\right.\right.\right.$$
$$\left.\left.+\frac{Eh^2\alpha}{(1-\nu)}\bar{\bar{T}}\right]\sin\theta\right\}-\left[\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\nu}{R^2}W_{\theta\theta}+\frac{1}{R^2}W_\theta\cot\theta\right)+\frac{Eh^2\alpha}{(1-\nu)}\bar{\bar{T}}\right]\cos\theta\right)$$
$$+pR\sin\theta=0, \tag{2.9}$$

where

$$\left.\begin{aligned}\bar{T}&=\frac{1}{h}\int_{-h/2}^{h/2}T(\theta,z)dz,\\ \bar{\bar{T}}&=\frac{1}{h^2}\int_{-h/2}^{h/2}zT(\theta,z)dz.\end{aligned}\right\} \tag{2.10}$$

In the followings, some approximations to the trigonometric functions, say $\sin\theta\doteqdot\theta$ and $\cos\theta\doteqdot1$ and so on, are used in accordance with the assumption that the semi-apex angle of the spherical shell is small.

Introducing the extensional stresses $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$ and the shearing stress $\bar{\sigma}_{12}$ in the middle plane corresponding to $\bar{\varepsilon}_{11}$, $\bar{\varepsilon}_{22}$ and $\bar{\varepsilon}_{12}$ respectively with the following relations:

$$\left.\begin{aligned}\bar{\varepsilon}_{11}&=\frac{1}{E}(\bar{\sigma}_{11}-\nu\bar{\sigma}_{22})+\alpha\bar{T},\\ \bar{\varepsilon}_{22}&=\frac{1}{E}(\bar{\sigma}_{22}-\nu\bar{\sigma}_{11})+\alpha\bar{T},\\ \bar{\varepsilon}_{12}&=\frac{2(1+\nu)}{E}\bar{\sigma}_{12},\end{aligned}\right\} \tag{2.11}$$

Eq. (2.8) is reduced to

$$\vartheta\frac{d\bar{\sigma}_{11}}{d\vartheta}+(\bar{\sigma}_{11}-\bar{\sigma}_{22})=0, \tag{2.12}$$

where

$$\vartheta=\frac{\theta}{\theta_0}. \tag{2.13}$$

Eq. (2.12) can be satisfied by introducing Airy's stress function, χ defined by the relations,

$$\frac{\bar{\sigma}_{11}}{E}=\frac{\chi}{\vartheta}, \qquad \frac{\bar{\sigma}_{22}}{E}=\frac{d\chi}{d\vartheta}. \tag{2.14}$$

Then, the equilibrium equation in the z-direction, Eq. (2.9) is expressed by

$$\frac{1}{12(1-\nu^2)\phi}\frac{1}{\vartheta}\frac{d}{d\vartheta}\left\{\vartheta\frac{d}{d\vartheta}\left[\frac{1}{\vartheta}\frac{d}{d\vartheta}\left(\vartheta\frac{dw}{d\vartheta}\right)\right]\right\}$$

$$=\frac{1}{\vartheta}\frac{d}{d\vartheta}\left[\left(\frac{dw}{d\vartheta}-\phi\vartheta\right)\chi\left(\frac{R}{h}\right)\right]-\frac{\alpha}{(1-\nu)}\left(\frac{R}{h}\right)\frac{1}{\vartheta}\frac{d}{d\vartheta}\left(\vartheta\frac{d\bar{T}}{d\vartheta}\right)-2\phi P, \tag{2.15}$$

where

$$w=\frac{W}{h}, \tag{2.16}$$

$$\phi=\theta_0^2\left(\frac{R}{h}\right), \tag{2.17}$$

$$P=\frac{1}{2}\left(\frac{R}{h}\right)^2\left(\frac{p}{E}\right). \tag{2.18}$$

Eliminating U from Eqs. (2.3) and using Eqs. (2.11) and (2.14), the following compatibility equation can be obtained.

$$\vartheta\frac{d}{d\vartheta}\left[\frac{1}{\vartheta}\frac{d}{d\vartheta}(\vartheta\chi)\right]\left(\frac{R}{h}\right)=\vartheta\frac{dw}{d\vartheta}-\frac{1}{2\phi}\left(\frac{dw}{d\vartheta}\right)^2-\alpha\left(\frac{R}{h}\right)\vartheta\frac{d\bar{T}}{d\vartheta}. \tag{2.19}$$

Equations (2.15) and (2.19) are the fundamental equations for a spherical shell subjected to normal pressure and heating. They coincide naturally with the fundamental equations for a heated circular plate [*36*] of radius r_0, by putting $R\theta_0=r$ and the radius of curvature, R infinitely large.

All the natural boundary conditions obtained are summarized in Eqs. (2.20).
at $\theta=\theta_1$ and θ_2

$$\left.\begin{aligned}
&\frac{Eh}{(1-\nu^2)}\left[\frac{1}{R}(U_\theta+W)+\frac{1}{2R^2}W_\theta^2+\frac{\nu}{R}(U\cot\theta+W)\right]-\frac{Eh\alpha}{(1-\nu)}\bar{T}=0,\\
&\left\{\frac{Eh}{(1-\nu^2)}\left[\frac{1}{R}(U_\theta+W)+\frac{1}{2R^2}W_\theta^2+\frac{\nu}{R}(U\cot\theta+W)\right]-\frac{Eh\alpha}{(1-\nu)}\bar{T}\right\}W_\theta\sin\theta\\
&\quad-\frac{d}{d\vartheta}\left\{\left[D\left(\frac{1}{R^2}W_{\theta\theta}+\frac{\nu}{R^2}W_\theta\cot\theta\right)+\frac{Eh^2\alpha}{(1-\nu)}\bar{T}\right]\sin\theta\right\}\\
&\quad+\left[D\left(\frac{\nu}{R^2}W_{\theta\theta}+\frac{1}{R^2}W_\theta\cot\theta\right)+\frac{Eh^2\alpha}{(1-\nu)}\bar{T}\right]\cos\theta=0,\\
&D\left(\frac{1}{R^2}W_{\theta\theta}+\frac{\nu}{R^2}W_\theta\cot\theta\right)+\frac{Eh^2\alpha}{(1-\nu)}\bar{T}=0.
\end{aligned}\right\} \tag{2.20}$$

As seen in Eq. (2.15), the existence of the term of $\frac{d}{d\vartheta}\left(\vartheta\frac{d\bar{T}}{d\vartheta}\right)$ is equivalent to that of the normal pressure p. That is to say, when there exists a temperature gradient through the thickness of the shell and $\frac{d}{d\vartheta}\left(\vartheta\frac{d\bar{T}}{d\vartheta}\right)$ is not equal to zero, the shell has to carry the normal load due to thermal gradient in addition to the normal pressure, not to speak of another thermal effects, and the former load is considered to be apt to be non-uniform over the surface of the shell and its effect on the stability of the shell must be examined carefully. Furthermore, where the edge is simply-supported, one of its boundary conditions, $\left[D\left(\frac{1}{R^2}W_{\theta\theta}+\frac{\nu}{R^2}W_\theta\cot\theta\right)+\frac{Eh^2\alpha}{(1-\nu)}\bar{T}\right]_{\theta=\theta_1,\theta_2}$ means that the existence of $\bar{T}$ on the boundary is equivalent to that of edge moment on the boundary, even if $\frac{d}{d\vartheta}\left(\vartheta\frac{d\bar{T}}{d\vartheta}\right)=0$. And, for the case of free edge condition, $\left(\frac{d\bar{T}}{d\vartheta}\right)_{\theta=\theta_1,\theta_2}$ has the same action as that of the shearing force at the edge. These are obvious from Eqs. (2.20).

The spherical shell is one of the most important structural components, especially in the field of the aerospace engineering. The field of its application has a wide range of variety, and so the shell has to carry frequently the pressure and thermal loads as well as another loads. Furthermore, the difficulty in fabricating shells in the shape desired and in clamping the edge necessarily result in the additional problems such as that of the initial geometrical and dynamical imperfections, that is, initial deformations and residual stresses.

The behavior of the perfect spherical shell subjected to external pressure is discussed in the proceeding sections.

3. Theoretical Analysis

Before studying the effects of initial geometrical imperfections, residual stresses and thermal gradients on the deformation behavior of spherical shells subjected to external pressure, it is necessary to know in advance the exact behavior of perfect shells under external pressure where there exists no other effect. Unfortunately, however, the definite solution has not yet been found out in spite of the fact that much researches have been carried out and many papers have been published [*1*]–[*33*].

It is very difficult to solve exactly the fundamental equations (2.15) and (2.19) because they are simultaneous nonlinear differential equations, so an approximate approach is presented here to solve the fundamental equations.

In the present analysis, a special attention is paid to the local damped deformation because the local deformation can be considered to have a triggering effect to induce the much larger deformation, for some cases, accompanied with the snap-through buckling.

The shell clamped at the edge is considered (Fig. 3.1), and the variation of the slope of the deflection with respect to the distance from the apex and corresponding to the rotationally symmetric deformation of the shell is assumed as

$$\frac{dw(\vartheta)}{d\vartheta}=A\vartheta e^{-m\vartheta}\sin\bar{n}\vartheta\,, \tag{3.1}$$

where

m: numeral,
$\bar{n}=n\pi$, n: positive integer,

and, integrating Eq. (3.1) so as to satisfy the boundary conditions, that is, $w(0)=-\delta$ and $w(1)=0$, the deflection pattern and A in Eq. (3.1) are given as

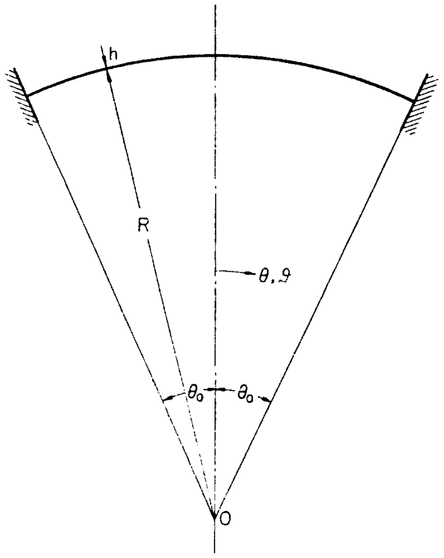

FIG. 3-1. Geometry of partial spherical shell.

$$w(\vartheta)=\frac{\delta}{(2\xi-k_{n,\xi})}[k_{n,\xi}-e^{-\bar{n}\xi\vartheta}\{[2\xi+\bar{n}(\xi^2+1)\vartheta]\cos\bar{n}\vartheta$$
$$+[(\xi^2-1)+\bar{n}\xi(\xi^2+1)\vartheta]\sin\bar{n}\vartheta\}]\,, \tag{3.2}$$

$$A=\frac{\bar{n}^2(\xi^2+1)^2\delta}{(2\xi-k_{n,\xi})}\,, \tag{3.3}$$

where

$$\xi=\frac{m}{\bar{n}}\,, \tag{3.4}$$

$$k_{n,\xi}=(-1)^n e^{-\bar{n}\xi}[2\xi+\bar{n}(\xi^2+1)]\,, \tag{3.5}$$

and δ is the non-dimensional deflection at the center of the spherical shell.

Integrating the compatibility equation (2.19) by use of Eq. (3.1), the stress function, χ is given as follows:

$$\chi\left(\frac{R}{h}\right)=\frac{C_1}{\vartheta}+C_2\vartheta$$
$$+\frac{Ae^{-\bar{n}\xi\vartheta}}{\bar{n}^4(\xi^2+1)^2}\Big\{\bar{n}^2\vartheta[2\xi\cos\bar{n}\vartheta+(\xi^2-1)\sin\bar{n}\vartheta]$$
$$+\frac{3\bar{n}}{(\xi^2+1)}[(3\xi^2-1)\cos\bar{n}\vartheta+\xi(\xi^2-3)\sin\bar{n}\vartheta]$$
$$+\frac{3}{(\xi^2+1)^2}\frac{1}{\vartheta}[4\xi(\xi^2-1)\cos\bar{n}\vartheta+(\xi^4-6\xi^2+1)\sin\bar{n}\vartheta]\Big\}$$
$$-\frac{A^2e^{-2\bar{n}\xi\vartheta}}{16\bar{n}^2\xi^2\phi}\left(\vartheta+\frac{3}{2\bar{n}\xi}+\frac{3}{4\bar{n}^2\xi^2}\frac{1}{\vartheta}\right)$$
$$+\frac{A^2e^{-2\bar{n}\xi\vartheta}}{16\bar{n}^4(\xi^2+1)^2\phi}\Big\{\bar{n}^2\vartheta[(\xi^2-1)\cos 2\bar{n}\vartheta-2\xi\sin 2\bar{n}\vartheta]$$

$$+\frac{3\bar{n}}{2(\xi^2+1)}[\xi(\xi^2-3)\cos 2\bar{n}\vartheta-(3\xi^2-1)\sin 2\bar{n}\vartheta]$$
$$+\frac{3}{4(\xi^2+1)^2}\frac{1}{\vartheta}[(\xi^4-6\xi^2+1)\cos 2\bar{n}\vartheta-4\xi(\xi^2-1)\sin 2\bar{n}\vartheta]\Big\}. \tag{3.6}$$

Then the stress distribution in the middle plane of the shell is given in the following equations.

$$\frac{\bar{\sigma}_{11}}{E}\left(\frac{R}{h}\right)=\frac{C_1}{\vartheta^2}+C_2$$
$$+\frac{Ae^{-\bar{n}\xi\vartheta}}{\bar{n}^4(\xi^2+1)^2}\Big\{\bar{n}^2[2\xi\cos\bar{n}\vartheta+(\xi^2-1)\sin\bar{n}\vartheta]$$
$$+\frac{3\bar{n}}{(\xi^2+1)}\frac{1}{\vartheta}[(3\xi^2-1)\cos\bar{n}\vartheta+\xi(\xi^2-3)\sin\bar{n}\vartheta]$$
$$+\frac{3}{(\xi^2+1)^2}\frac{1}{\vartheta^2}[4\xi(\xi^2-1)\cos\bar{n}\vartheta+(\xi^4-6\xi^2+1)\sin\bar{n}\vartheta]\Big\}$$
$$-\frac{A^2e^{-2\bar{n}\xi\vartheta}}{16\bar{n}^2\xi^2\phi}\left(1+\frac{3}{2\bar{n}\xi}\frac{1}{\vartheta}+\frac{3}{4\bar{n}^2\xi^2}\frac{1}{\vartheta^2}\right)$$
$$+\frac{A^2e^{-2\bar{n}\xi\vartheta}}{16\bar{n}^4(\xi^2+1)^2\phi}\Big\{\bar{n}^2[(\xi^2-1)\cos 2\bar{n}\vartheta-2\xi\sin 2\bar{n}\vartheta]$$
$$+\frac{3\bar{n}}{2(\xi^2+1)}\frac{1}{\vartheta}[\xi(\xi^2-3)\cos 2\bar{n}\vartheta-(3\xi^2-1)\sin 2\bar{n}\vartheta]$$
$$+\frac{3}{4(\xi^2+1)^2}\frac{1}{\vartheta^2}[(\xi^4-6\xi^2+1)\cos 2\bar{n}\vartheta-4\xi(\xi^2-1)\sin 2\bar{n}\vartheta]\Big\}, \tag{3.7}$$

$$\frac{\bar{\sigma}_{22}}{E}\left(\frac{R}{h}\right)=-\frac{C_1}{\vartheta^2}+C_2$$
$$-\frac{Ae^{-\bar{n}\xi\vartheta}}{\bar{n}^4(\xi^2+1)^2}\Big\{\bar{n}^3(\xi^2+1)\vartheta(\cos\bar{n}\vartheta+\xi\sin\bar{n}\vartheta)$$
$$+2\bar{n}^2[2\xi\cos\bar{n}\vartheta+(\xi^2-1)\sin\bar{n}\vartheta]$$
$$+\frac{3\bar{n}}{(\xi^2+1)}\frac{1}{\vartheta}[(3\xi^2-1)\cos\bar{n}\vartheta+\xi(\xi^2-3)\sin\bar{n}\vartheta]$$
$$+\frac{3}{(\xi^2+1)^2}\frac{1}{\vartheta^2}[4\xi(\xi^2-1)\cos\bar{n}\vartheta+(\xi^4-6\xi^2+1)\sin\bar{n}\vartheta]\Big\}$$
$$+\frac{A^2e^{-2\bar{n}\xi\vartheta}}{16\bar{n}^2\xi^2\phi}\left(2\bar{n}\xi\vartheta+2+\frac{3}{2\bar{n}\xi}\frac{1}{\vartheta}+\frac{3}{4\bar{n}^2\xi^2}\frac{1}{\vartheta^2}\right)$$
$$-\frac{A^2e^{-2\bar{n}\xi\vartheta}}{16\bar{n}^4(\xi^2+1)^2\phi}\Big\{2\bar{n}^3(\xi^2+1)\vartheta(\xi\cos 2\bar{n}\vartheta-\sin 2\bar{n}\vartheta)$$
$$+2\bar{n}^2[(\xi^2-1)\cos 2\bar{n}\vartheta-2\xi\sin 2\bar{n}\vartheta]$$
$$+\frac{3\bar{n}}{2(\xi^2+1)}\frac{1}{\vartheta}[\xi(\xi^2-3)\cos 2\bar{n}\vartheta-(3\xi^2-1)\sin 2\bar{n}\vartheta]$$
$$+\frac{3}{4(\xi^2+1)^2}\frac{1}{\vartheta^2}[(\xi^4-6\xi^2+1)\cos 2\bar{n}\vartheta-4\xi(\xi^2-1)\sin 2\bar{n}\vartheta]\Big\}. \tag{3.8}$$

The integral constants, C_1 and C_2 are determined, from the conditions that the stresses must be finite at $\vartheta=0$ and the displacement component u vanishes at $\vartheta=0$ and 1, respectively, as follows:

$$C_1=-\frac{12\xi(\xi^2-1)A}{\bar{n}^4(\xi^2+1)^4}+\frac{3(10\xi^6+5\xi^4+4\xi^2+1)A^2}{64\bar{n}^4\xi^4(\xi^2+1)^4\phi}, \tag{3.9}$$

$$\begin{aligned}C_2=&\left(\frac{1+\nu}{1-\nu}\right)C_1\\&+\frac{(-1)^n Ae^{-\bar{n}\xi}}{(1-\nu)\bar{n}^3(\xi^2+1)^2}\left\{\bar{n}^2(\xi^2+1)+2(2+\nu)\bar{n}\xi+\frac{3(1+\nu)}{(\xi^2+1)}\left[(3\xi^2-1)+\frac{4\xi(\xi^2-1)}{\bar{n}(\xi^2+1)}\right]\right\}\\&-\frac{A^2e^{-2\bar{n}\xi}}{16(1-\nu)\phi}\left(\frac{1}{\bar{n}^2\xi^2}\left[2\bar{n}\xi+(2+\nu)+\frac{3(1+\nu)}{2\bar{n}\xi}\left(1+\frac{1}{2\bar{n}\xi}\right)\right]\right.\\&-\frac{1}{\bar{n}^4(\xi^2+1)^2}\left\{2\bar{n}^3\xi(\xi^2+1)+(2+\nu)\bar{n}^2(\xi^2-1)\right.\\&\left.\left.+\frac{3(1+\nu)}{2(\xi^2+1)}\left[\bar{n}\xi(\xi^2-3)+\frac{(\xi^4-6\xi^2+1)}{2(\xi^2+1)}\right]\right\}\right).\end{aligned} \tag{3.10}$$

And, the damping parameter, ξ has to satisfy the following relation to assure the condition that the stresses must be finite at $\vartheta=0$.

$$\frac{128\xi^5(\xi^2-1)(2\xi-k_{n,\xi})}{\bar{n}^2(\xi^2+1)^2(10\xi^6+5\xi^4+4\xi^2+1)}=\frac{\delta}{\phi}. \tag{3.11}$$

Eq. (3.11) shows that the damping factor cannot be arbitrarily specified but it is to be given as the function of the deflection at the center, δ, the number of half waves of deflection, n and the geometrical parameter, ϕ in a way to remove the stress singularity at the center of the shell.

Eq. (3.9) is simplified as follows by use of Eq. (3.11).

$$C_1=-\frac{6\xi(\xi^2-1)A}{\bar{n}^4(\xi^2+1)^4}. \tag{3.9a}$$

To obtain the relation between the external pressure and the deflection at the center of the spherical shell, the Galerkin method is used. In applying the Galerkin method to the equilibrium equation, Eq. (3.1) rather than Eq. (3.2) is more convenient to be used as the multiplier. And the integration of the Galerkin method must represent the work done, so the equilibrium equation with respect to the shearing force is necessary to be used when Eq. (3.1), that is, the slope of the deflection is used as the multiplier.

By integrating for once Eq. (2.15), which is the equation of equilibrium with respect to the normal force, the equilibrium equation with respect to the shearing force is obtained as Eq. (3.12), where the integral constant is chosen to be zero from the condition that the shearing force must be finite, or zero for the symmetric deformation, at $\vartheta=0$.

$$\frac{1}{12(1-\nu^2)\phi}\frac{d}{d\vartheta}\left[\frac{1}{\vartheta}\frac{d}{d\vartheta}\left(\vartheta\frac{dw}{d\vartheta}\right)\right]-\frac{1}{\vartheta}\left(\frac{dw}{d\vartheta}-\phi\vartheta\right)\chi\left(\frac{R}{h}\right)+\phi P\vartheta=0. \quad (3.12)$$

Then, the Galerkin method is applied to Eq. (3.12) as

$$\int_0^1\left\{\frac{1}{12(1-\nu^2)\phi}\frac{d}{d\vartheta}\left[\frac{1}{\vartheta}\frac{d}{d\vartheta}\left(\vartheta\frac{dw}{d\vartheta}\right)\right]-\frac{1}{\vartheta}\left(\frac{dw}{d\vartheta}-\phi\vartheta\right)\chi\left(\frac{R}{h}\right)\right.$$
$$\left.+\phi P\vartheta\right\}\frac{dw(\vartheta)}{d\vartheta}\,\vartheta d\vartheta=0. \quad (3.13)$$

Integrating Eq. (3.13) by use of Eqs. (3.1) and (3.6), and substituting Eqs. (3.9*a*) and (3.10) into the result, the relation between the external pressure and the deflection at the center of the spherical shell is given as follows after some tedious integrations and calculations.

$$F_0(n,\xi)P=\left[F_{11}(n,\xi)+\frac{F_{12}(n,\xi)}{\phi^2}\right]\frac{\delta}{(2\xi-k_{n,\xi})}$$
$$-F_2(n,\xi)\frac{\delta^2}{(2\xi-k_{n,\xi})^2\phi}+F_3(n,\xi)\frac{\delta^3}{(2\xi-k_{n,\xi})^3\phi^2}, \quad (3.14)$$

where

$$F_0(n,\xi)=f_{0,0}(\xi)+f_{0,1}(n,\xi)e^{-\bar{n}\xi}, \quad (3.15)$$
$$F_{11}(n,\xi)=f_{11,0}(n,\xi)+f_{11,1}(n,\xi)e^{-\bar{n}\xi}$$
$$+f_{11,2}(n,\xi)e^{-2\bar{n}\xi}, \quad (3.16)$$
$$F_{12}(n,\xi)=f_{12,0}(n,\xi)+f_{12,2}(n,\xi)e^{-2\bar{n}\xi}, \quad (3.17)$$
$$F_2(n,\xi)=f_{2,0}(n,\xi)+f_{2,1}(n,\xi)e^{-\bar{n}\xi}$$
$$+f_{2,2}(n,\xi)e^{-2\bar{n}\xi}+f_{2,3}(n,\xi)e^{-3\bar{n}\xi}, \quad (3.18)$$
$$F_3(n,\xi)=f_{3,0}(n,\xi)+f_{3,2}(n,\xi)e^{-2\bar{n}\xi}$$
$$+f_{3,4}(n,\xi)e^{-4\bar{n}\xi}, \quad (3.19)$$

$$f_{0,0}(\xi)=\frac{24\xi(\xi^2-1)}{(\xi^2+1)^4}, \quad (3.15.1)$$

$$f_{0,1}(n,\xi)=-(-1)^n\frac{\bar{n}^3}{(\xi^2+1)}\left[1+\frac{6\xi}{\bar{n}(\xi^2+1)}+\frac{6(3\xi^2-1)}{\bar{n}^2(\xi^2+1)^2}+\frac{24\xi(\xi^2-1)}{\bar{n}^3(\xi^2+1)^3}\right], \quad (3.15.2)$$

$$f_{11,0}(n,\xi)=\frac{3(\xi^2-1)}{(\xi^2+1)^4}\left[\frac{(5\xi^4-6\xi^2+5)}{16}+\frac{48(1+\nu)\xi^2(\xi^2-1)}{(1-\nu)\bar{n}^2(\xi^2+1)^2}\right]$$
$$-\frac{3}{4\xi^2}\left[\frac{(\xi^4-4\xi^2-1)}{(\xi^2+1)^2}+\frac{(\xi^2-1)}{4\xi^2}\right], \quad (3.16.1)$$

$$f_{11,1}(n,\xi)=-(-1)^n\frac{36\bar{n}\xi(\xi^2-1)}{(1-\nu)(\xi^2+1)^3}\left[1+\frac{2(2+\nu)\xi}{\bar{n}(\xi^2+1)}+\frac{3(1+\nu)(3\xi^2-1)}{\bar{n}^2(\xi^2+1)^2}\right.$$
$$\left.+\frac{12(1+\nu)\xi(\xi^2-1)}{\bar{n}^3(\xi^2+1)^3}\right], \quad (3.16.2)$$

$$
\begin{aligned}
f_{11,2}(n,\ \xi)=\bar{n}^2\Bigg\{&\frac{\bar{n}^2}{(1-\nu)}-\frac{\bar{n}\xi[(1-\nu)\xi^2-(43+5\nu)]}{4(1-\nu)(\xi^2+1)}\\
&-\frac{3[3(1-\nu)\xi^4-14(11+\nu)\xi^2+3(9-\nu)]}{8(1-\nu)(\xi^2+1)^2}\\
&-\frac{3\xi[5(1-\nu)\xi^4-2(241+111\nu)\xi^2+(233+87\nu)]}{8(1-\nu)\bar{n}(\xi^2+1)^3}\\
&-\frac{3[5(1-\nu)\xi^6-(1835-1429\nu)\xi^4+(1547+1141\nu)\xi^2-(101+91\nu)]}{16(1-\nu)\bar{n}^2(\xi^2+1)^4}\\
&+\frac{144(1+\nu)\xi(\xi^2-1)(3\xi^2-1)}{(1-\nu)\bar{n}^3(\xi^2+1)^5}+\frac{288(1+\nu)\xi^2(\xi^2-1)^2}{(1-\nu)\bar{n}^4(\xi^2+1)^6}\\
&+\frac{\bar{n}(\xi^2-1)}{4\xi}\left(1+\frac{3}{2\bar{n}\xi}+\frac{3}{2\bar{n}^2\xi^2}+\frac{3}{4\bar{n}^3\xi^3}\right)\\
&+\frac{3(\xi^2-3)}{4(\xi^2+1)}\left(1+\frac{1}{\bar{n}\xi}+\frac{1}{2\bar{n}^2\xi^2}\right)+\frac{3(\xi^4-6\xi^2+1)}{4\bar{n}\xi(\xi^2+1)^2}\left(1+\frac{1}{2\bar{n}\xi}\right)\Bigg\},
\end{aligned}
\tag{3.16.3}
$$

$$
f_{12,0}(n,\ \xi)=\frac{\bar{n}^4(4\xi^4+3\xi^2+1)}{64(1-\nu^2)\xi^4},
\tag{3.17.1}
$$

$$
\begin{aligned}
f_{12,2}(n,\ \xi)=-\frac{\bar{n}^7(\xi^2+1)^2}{48(1-\nu^2)\xi}\Bigg[&1+\frac{3}{2\bar{n}\xi}+\frac{3(2\xi^2+1)}{2\bar{n}^2\xi^2(\xi^2+1)}\\
&+\frac{3(4\xi^4+3\xi^2+1)}{4\bar{n}^3\xi^3(\xi^2+1)^2}\Bigg],
\end{aligned}
\tag{3.17.2}
$$

$$
\begin{aligned}
f_{2,0}(n,\ \xi)=\frac{\bar{n}^2\xi}{8}\Bigg\{&\frac{1}{(\xi^2+1)^2}\Bigg[\frac{191(\xi^2-3)(3\xi^2-1)}{72}+\frac{6(\xi^2-1)(3\xi^2+1)}{\xi^2}\\
&+\frac{9(1+\nu)(\xi^2-1)(10\xi^6+5\xi^4+4\xi^2+1)}{(1-\nu)\bar{n}^2\xi^4(\xi^2+1)^2}\Bigg]-\frac{3(\xi^2+1)^2}{(9\xi^2+1)^2}\Bigg[\frac{3(295\xi^4-802\xi^2+119)}{8(\xi^2+1)^2}\\
&+\frac{(675\xi^4-1234\xi^2+75)}{(\xi^2+1)(9\xi^2+1)}+\frac{3(693\xi^4-662\xi^2+53)}{(9\xi^2+1)^2}\Bigg]\\
&-\frac{36(\xi^2+1)^4}{\xi^2(9\xi^2+1)^3}\left[\frac{(81\xi^2+1)}{48\xi^2}+\frac{(9\xi^2-1)}{(9\xi^2+1)}\right]\Bigg\},
\end{aligned}
\tag{3.18.1}
$$

$$
\begin{aligned}
f_{2,1}(n,\ \xi)=-(-1)^n\frac{3\bar{n}^3(10\xi^6+5\xi^4+4\xi^2+1)}{16(1-\nu)\xi^4(\xi^2+1)}\Bigg[&1+\frac{2(2+\nu)\xi}{\bar{n}(\xi^2+1)}\\
&+\frac{3(1+\nu)(3\xi^2-1)}{\bar{n}^2(\xi^2+1)^2}+\frac{12(1+\nu)\xi(\xi^2-1)}{\bar{n}^3(\xi^2+1)^3}\Bigg],
\end{aligned}
\tag{3.18.2}
$$

$$
\begin{aligned}
f_{2,2}(n,\ \xi)=-\frac{6\bar{n}^3(\xi^2-1)}{(1-\nu)(\xi^2+1)}\Bigg[&1+\frac{(2+\nu)(3\xi^2+1)}{2\bar{n}\xi(\xi^2+1)}+\frac{3(1+\nu)(6\xi^4+3\xi^2+1)}{4\bar{n}^2\xi^2(\xi^2+1)^2}\\
&+\frac{3(1+\nu)(10\xi^6+5\xi^4+4\xi^2+1)}{8\bar{n}^3\xi^3(\xi^2+1)^3}\Bigg],
\end{aligned}
\tag{3.18.3}
$$

$$
f_{2,3}(n,\ \xi)=-(-1)^n\Bigg(-\frac{\bar{n}^6(\xi^2+1)^2}{4(1-\nu)\xi}\Bigg[1+\frac{2(2+\nu)\xi}{\bar{n}(\xi^2+1)}+\frac{3(1+\nu)(3\xi^2-1)}{\bar{n}^2(\xi^2+1)^2}
$$

$$
\begin{aligned}
&+\frac{12(1+\nu)\xi(\xi^2-1)}{\bar{n}^3(\xi^2+1)^3}\Big]\Big[1+\frac{3(3\xi^2+1)}{2\bar{n}\xi(\xi^2+1)}+\frac{3(6\xi^4+3\xi^2+1)}{2\bar{n}^2\xi^2(\xi^2+1)^2}\\
&+\frac{3(10\xi^6+5\xi^4+4\xi^2+1)}{4\bar{n}^3\xi^3(\xi^2+1)^3}\Big]\\
&+\frac{\bar{n}^6(\xi^2+1)^3}{8(1-\nu)\xi}\Big\{\frac{\xi^2}{(\xi^2+1)}\Big[1+\frac{(2+\nu)(\xi^2-1)}{2\bar{n}\xi(\xi^2+1)}+\frac{3(1+\nu)(\xi^2-3)}{4\bar{n}^2(\xi^2+1)^2}\\
&+\frac{3(1+\nu)(\xi^4-6\xi^2+1)}{8\bar{n}^3\xi(\xi^2+1)^3}\Big]-\Big[1+\frac{(2+\nu)}{2\bar{n}\xi}+\frac{3(1+\nu)}{4\bar{n}^2\xi^2}+\frac{3(1+\nu)}{8\bar{n}^3\xi^3}\Big]\Big\}\\
&\times\Big[1+\frac{6\xi}{\bar{n}(\xi^2+1)}+\frac{6(3\xi^2-1)}{\bar{n}^2(\xi^2+1)^2}+\frac{24\xi(\xi^2-1)}{\bar{n}^3(\xi^2+1)^3}\Big]\\
&+\frac{3\bar{n}^5(\xi^2+1)^2}{4}\Big\{\frac{1}{9\bar{n}^2(\xi^2+1)}\Big[\bar{n}^2(3\xi^2-1)+\frac{16\bar{n}\xi(\xi^2-1)}{(\xi^2+1)}+\frac{17(5\xi^4-10\xi^2+1)}{3(\xi^2+1)^2}\\
&+\frac{34\xi(\xi^2-3)(3\xi^2-1)}{9\bar{n}(\xi^2+1)^3}\Big]-\frac{1}{\bar{n}^2(9\xi^2+1)}\Big[\bar{n}^2(3\xi^2-1)+\frac{4\bar{n}\xi(9\xi^2-5)}{(9\xi^2+1)}\\
&+\frac{6(45\xi^4-34\xi^2+1)}{(9\xi^2+1)^2}+\frac{4\xi(243\xi^4-234\xi^2+19)}{\bar{n}(9\xi^2+1)^3}\Big]\\
&-\frac{1}{\bar{n}^3(\xi^2+1)(9\xi^2+1)}\Big[12\bar{n}^2\xi(\xi^2-1)+\frac{\bar{n}(225\xi^6-297\xi^4-133\xi^2+5)}{(\xi^2+1)(9\xi^2+1)}\\
&+\frac{2\xi(405\xi^6-777\xi^4-257\xi^2+29)}{(\xi^2+1)(9\xi^2+1)^2}\Big]\Big\}\\
&-\frac{\bar{n}^5(\xi^2+1)^4}{16\xi^2(9\xi^2+1)}\Big[1+\frac{3(21\xi^2+1)}{2\bar{n}\xi(9\xi^2+1)}+\frac{3(513\xi^4+34\xi^2+1)}{4\bar{n}^2\xi^2(9\xi^2+1)^2}\\
&+\frac{3(1161\xi^4+42\xi^2+1)}{2\bar{n}^3\xi(9\xi^2+1)^3}\Big]\\
&+\frac{\bar{n}^5(\xi^2+1)^2}{32}\Big\{\frac{1}{3\bar{n}^2(\xi^2+1)}\Big[\bar{n}^2(3\xi^2-1)+\frac{10\bar{n}\xi(\xi^2-1)}{(\xi^2+1)}+\frac{29(5\xi^4-10\xi^2+1)}{12(\xi^2+1)^2}\\
&+\frac{29\xi(\xi^2-3)(3\xi^2-1)}{18\bar{n}(\xi^2+1)^3}\Big]-\frac{1}{\bar{n}^2(9\xi^2+1)}\Big[\bar{n}^2(7\xi^2-1)+\frac{24\bar{n}\xi(3\xi^2-1)}{(9\xi^2+1)}\\
&+\frac{6(81\xi^4-46\xi^2+1)}{(9\xi^2+1)^2}+\frac{12\xi(135\xi^4-114\xi^2+7)}{\bar{n}(9\xi^2+1)^3}\Big]\\
&-\frac{1}{\bar{n}^3(\xi^2+1)(9\xi^2+1)}\Big[3\bar{n}^2\xi(5\xi^2-3)+\frac{3\bar{n}(249\xi^6-137\xi^4-125\xi^2+5)}{4(\xi^2+1)(9\xi^2+1)}\\
&+\frac{3\xi(405\xi^6-377\xi^4-241\xi^2+29)}{2(\xi^2+1)(9\xi^2+1)^2}\Big]\Big\}\Big),
\end{aligned}
\tag{3.18.4}
$$

$$
\begin{aligned}
f_{3,0}(n,\xi)=&\frac{3\bar{n}^4(\xi^2+1)^4}{256}\Big\{\frac{(\xi^2+1)^2}{\xi^2}\Big[\frac{(48\xi^4-40\xi^2-5)}{(4\xi^2+1)^4}+\frac{(4\xi^2-1)}{2\xi^2(4\xi^2+1)^2}-\frac{5}{16\xi^4}\Big]\\
&-\frac{5(\xi^2-1)(\xi^4-14\xi^2+1)}{32(\xi^2+1)^4}+\frac{1}{(4\xi^2+1)^2}\Big[\frac{(4\xi^6-41\xi^4+26\xi^2-1)}{2(\xi^2+1)^2}
\end{aligned}
$$

$$+\frac{(8\xi^6-66\xi^4+33\xi^2-1)}{(\xi^2+1)(4\xi^2+1)}+\frac{(16\xi^6-104\xi^4+41\xi^2-1)}{(4\xi^2+1)^2}\Bigg]$$

$$-\frac{1}{8\xi^2}\left[\frac{(\xi^2-1)}{4\xi^2}+\frac{(\xi^4-4\xi^2-1)}{(\xi^2+1)^2}\right]\Bigg\}, \qquad (3.19.1)$$

$$f_{3,2}(n,\xi)=-\frac{3\bar{n}^5(\xi^2+1)(10\xi^6+5\xi^4+4\xi^2+1)}{128(1-\nu)\xi^5}\left[1+\frac{(2+\nu)(3\xi^2+1)}{2\bar{n}\xi(\xi^2+1)}\right.$$

$$\left.+\frac{3(1+\nu)(6\xi^4+3\xi^2+1)}{4\bar{n}^2\xi^2(\xi^2+1)^2}+\frac{3(1+\nu)(10\xi^6+5\xi^4+4\xi^2+1)}{8\bar{n}^3\xi^3(\xi^2+1)^3}\right], \qquad (3.19.2)$$

$$f_{3,4}(n,\xi)=-\frac{\bar{n}^8(\xi^2+1)^4}{32}\Bigg(\frac{(\xi^2+1)}{(1-\nu)\xi^2}\Bigg\{\frac{\xi^2}{(\xi^2+1)}\left[1+\frac{(2+\nu)(\xi^2-1)}{2\bar{n}\xi(\xi^2+1)}\right.$$

$$\left.+\frac{3(1+\nu)(\xi^2-3)}{4\bar{n}^2(\xi^2+1)^2}+\frac{3(1+\nu)(\xi^4-6\xi^2+1)}{8\bar{n}^3\xi(\xi^2+1)^3}\right]$$

$$-\left[1+\frac{(2+\nu)}{2\bar{n}\xi}+\frac{3(1+\nu)}{4\bar{n}^2\xi^2}+\frac{3(1+\nu)}{8\bar{n}^3\xi^3}\right]\Bigg\}$$

$$\times\left[1+\frac{3(3\xi^2+1)}{2\bar{n}\xi(\xi^2+1)}+\frac{3(6\xi^4+3\xi^2+1)}{2\bar{n}^2\xi^2(\xi^2+1)^2}+\frac{3(10\xi^6+5\xi^4+4\xi^2+1)}{4\bar{n}^3\xi^3(\xi^2+1)^3}\right]$$

$$+\frac{(\xi^2+1)^2}{4\bar{n}\xi^3}\Bigg\{\frac{4\xi^2}{(4\xi^2+1)}\left[1+\frac{3(12\xi^2+1)}{4\bar{n}\xi(4\xi^2+1)}+\frac{3(20\xi^2+1)}{2\bar{n}^2(4\xi^2+1)^2}+\frac{3(48\xi^4-40\xi^2-5)}{8\bar{n}^3\xi(4\xi^2+1)^3}\right]$$

$$-\left[1+\frac{9}{4\bar{n}\xi}+\frac{15}{8\bar{n}^2\xi^2}+\frac{15}{32\bar{n}^3\xi^3}\right]+\frac{3(4\xi^2-1)}{4\bar{n}^3\xi(4\xi^2+1)^2}\Bigg\}$$

$$-\frac{1}{2\bar{n}^3}\Bigg\{\frac{\xi}{4(\xi^2+1)}\left[\bar{n}^2(\xi^2-3)+\frac{9\bar{n}(\xi^4-6\xi^2+1)}{4\xi(\xi^2+1)}+\frac{15(\xi^4-10\xi^2+5)}{8(\xi^2+1)^2}\right.$$

$$\left.+\frac{15(\xi^2-1)(\xi^4-14\xi^2+1)}{32\bar{n}\xi(\xi^2+1)^3}\right]-\frac{2\xi}{(4\xi^2+1)}\left[\bar{n}^2(\xi^2-2)+\frac{3\bar{n}(4\xi^4-13\xi^2+1)}{4\xi(4\xi^2+1)}\right.$$

$$\left.+\frac{3(4\xi^4-19\xi^2+4)}{2(4\xi^2+1)^2}+\frac{3(16\xi^6-104\xi^4+41\xi^2-1)}{8\bar{n}\xi(4\xi^2+1)^3}\right]$$

$$+\frac{\bar{n}^2(\xi^2-1)}{4\xi}\left[1+\frac{3}{4\bar{n}\xi}+\frac{3}{8\bar{n}^2\xi^2}+\frac{3}{32\bar{n}^3\xi^3}\right]$$

$$-\frac{3}{8\bar{n}(\xi^2+1)(4\xi^2+1)}\left[\bar{n}^2(4\xi^4-25\xi^2+7)+\frac{8\bar{n}\xi(4\xi^6-26\xi^4-7\xi^2+5)}{(\xi^2+1)(4\xi^2+1)}\right.$$

$$\left.+\frac{(32\xi^8-276\xi^6-3\xi^4+86\xi^2-3)}{(\xi^2+1)(4\xi^2+1)^2}\right]+\frac{3(\xi^4-4\xi^2-1)}{8\xi(\xi^2+1)^2}\left[1+\frac{1}{4\bar{n}\xi}\right]\Bigg\}\Bigg). \qquad (3.19.3)$$

Eqs.(3.14) and (3.11) give the relation between the external pressure and the deflection at the center of the perfect spherical shell where there exists no geometrical imperfection, residual stress or thermal effect.

There are several methods to obtain the critical pressure from the above-mentioned equations, but it is so difficult to derive analytically the general expression of critical pressure that some numerical examples will be shown in the next section to show the behavior of spherical shells subjected to external pressure.

4. NUMERICAL EXAMPLES

The relations between the deflection at the center of the spherical shell and the corresponding values of the damping parameter are calculated from Eq. (3.11) and shown in Fig. 4.1.

The relations between the external pressure and the deflection at the center of the spherical shell, which are given by Eq. (3.14) accompanied with Eq. (3.11), are shown in Figs. 4.2, 4.3, 4.4 and 4.5 for the cases of $n=1$, 2, 3 and 4, respectively, where ν is put to be $\nu=1/3$. The relations for the cases of $\phi=10$ and 30 are obtained as shown in Figs. 4.6 and 4.7 from the above figures, where the values of damping parameter corresponding to each deformed state are also shown. These figures can be characterized by Fig. 4.8.

The present analysis is concerned with the points where the snap-through buckling of shells is considered to be possible to take place, and the post-buckling state of shells is not taken into consideration. In Fig. 4.8, the following points are considered as the critical ones. (1) point $A_{U1}(A_{U2})$: the deformation process follows the path $OA_{U1}(OA_{U2})$ and the shell snaps from the point $A_{U1}(A_{U2})$ to the another deformed state. (2) point B_U: the deformation process follows the path OB_U and the shell snaps from the point B_U to the another deformed state. (3) point C_{U1}: this point corresponds to the so-called upper buckling pressure for $n=3$. (4) points $A_{L1}(A_{L2})$, B_L, C_{L1}: these points correspond to the so-called lower buckling pressures for $n=1$, 2 and 3, respectively. (5) point K: on this point it is possible to exist two different deformation patterns whose damping factors are different

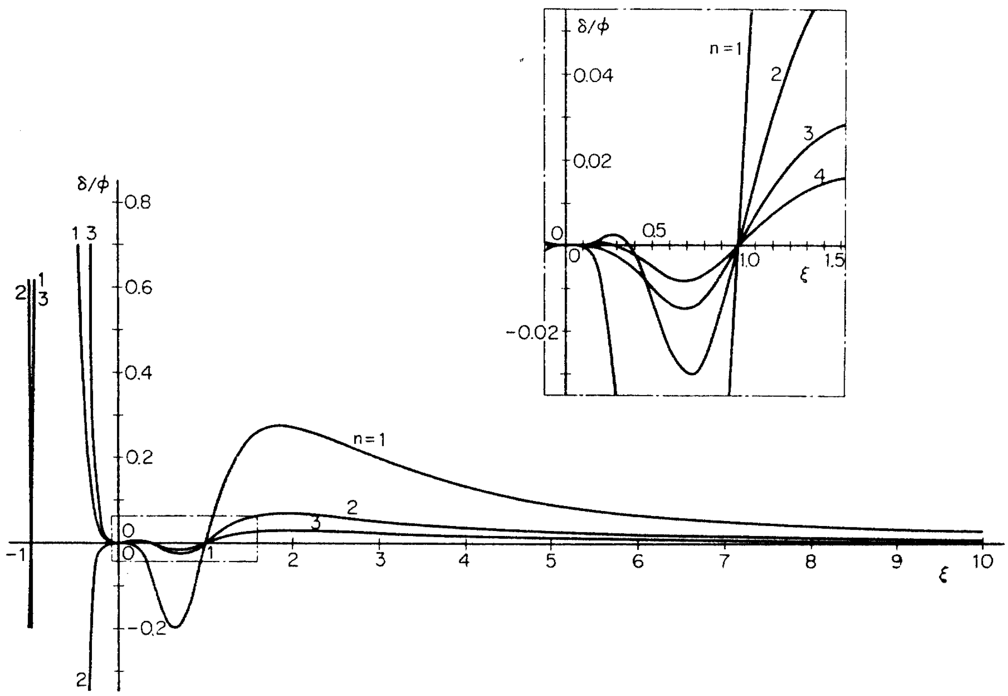

FIG. 4.1. Relation between the deflection at the center of spherical shell and the damping parameter.

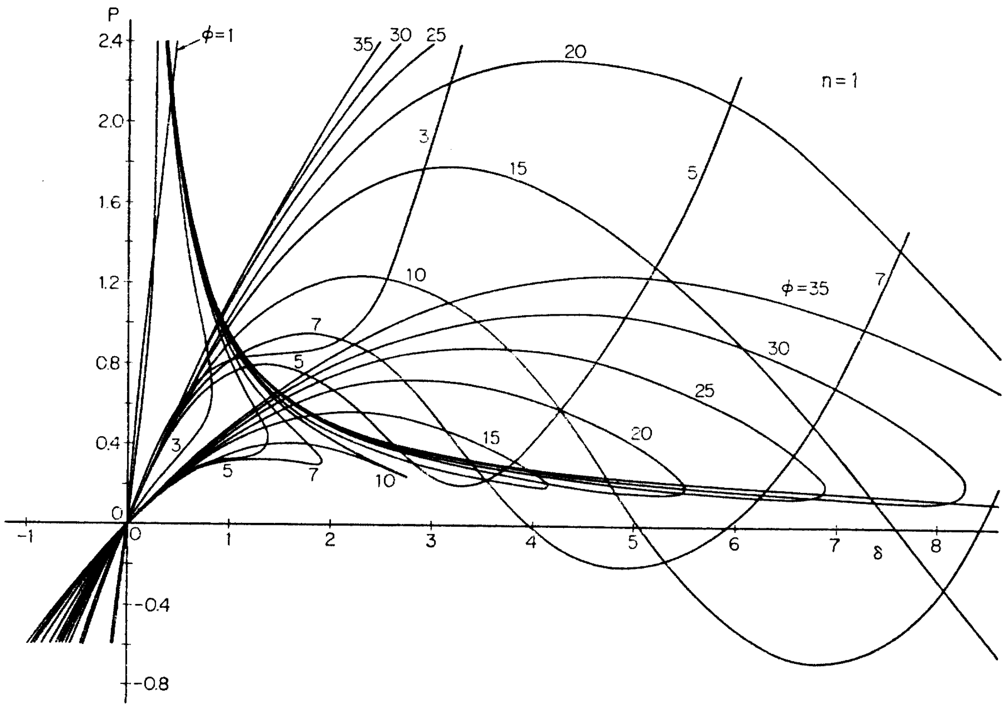

FIG. 4.2. Relation between the external pressure and the deflection at the center of spherical shell, $n=1$.

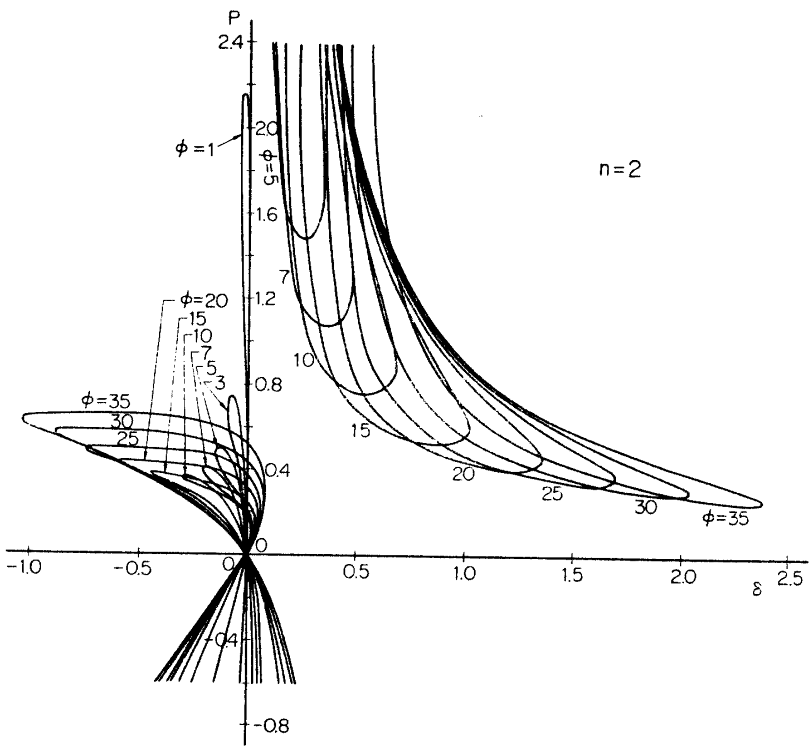

FIG. 4.3. Relation between the external pressure and the deflection at the center of spherical shell, $n=2$.

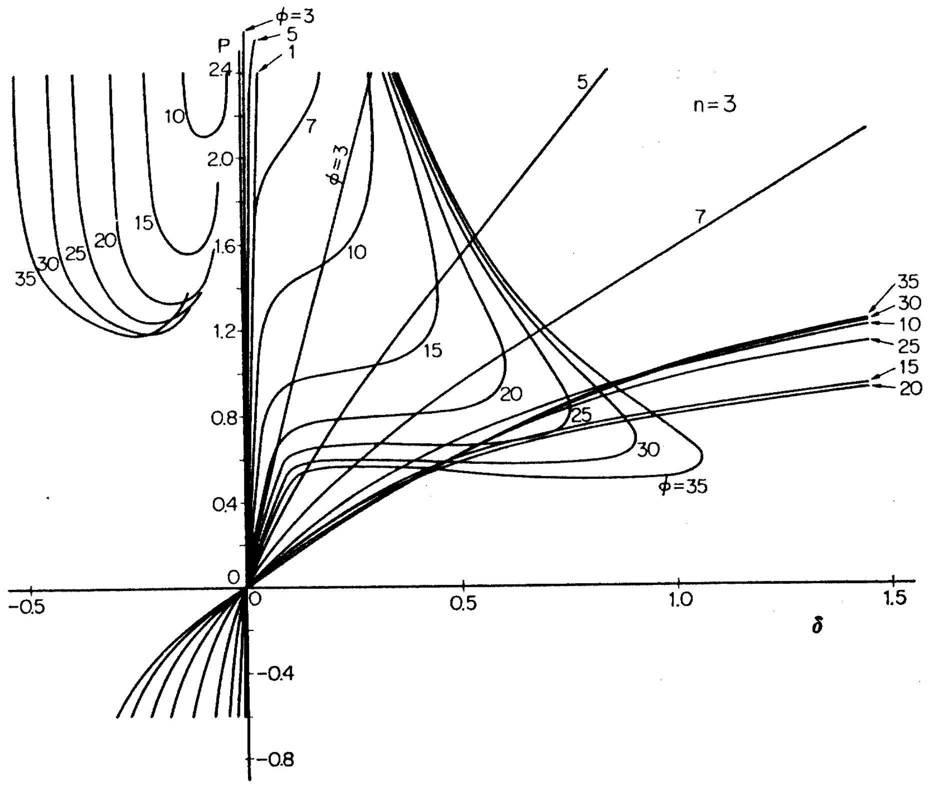

FIG. 4.4. Relation between the external pressure and the deflection at the center of spherical shell, $n=3$.

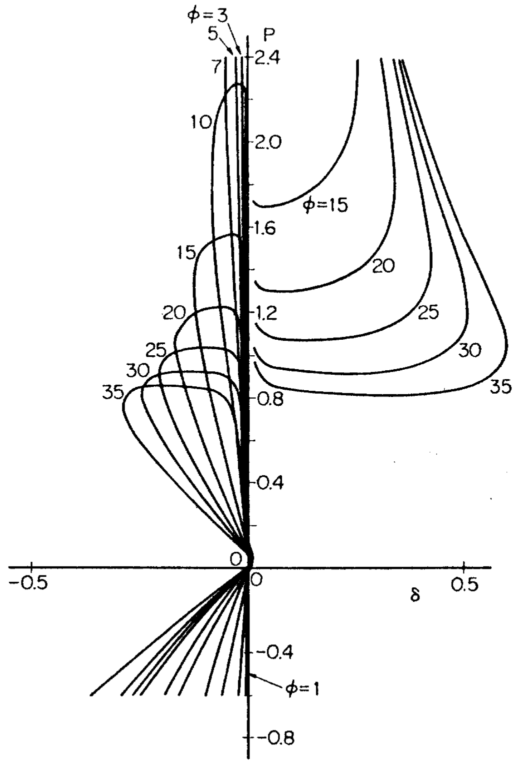

FIG. 4.5. Relation between the external pressure and the deflection at the center of spherical shell, $n=4$.

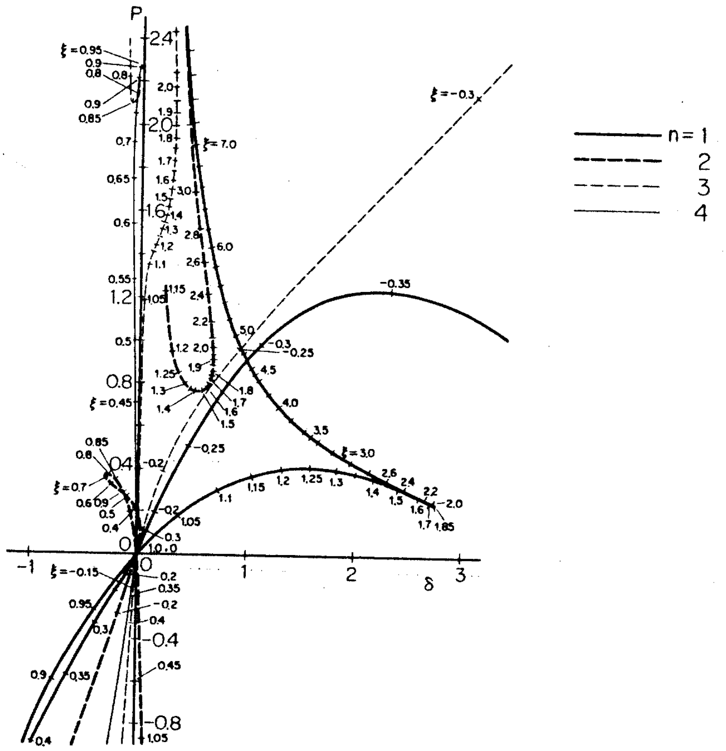

FIG. 4.6. Relation between the external pressure and the deflection at the center of spherical shell, $\phi=10$.

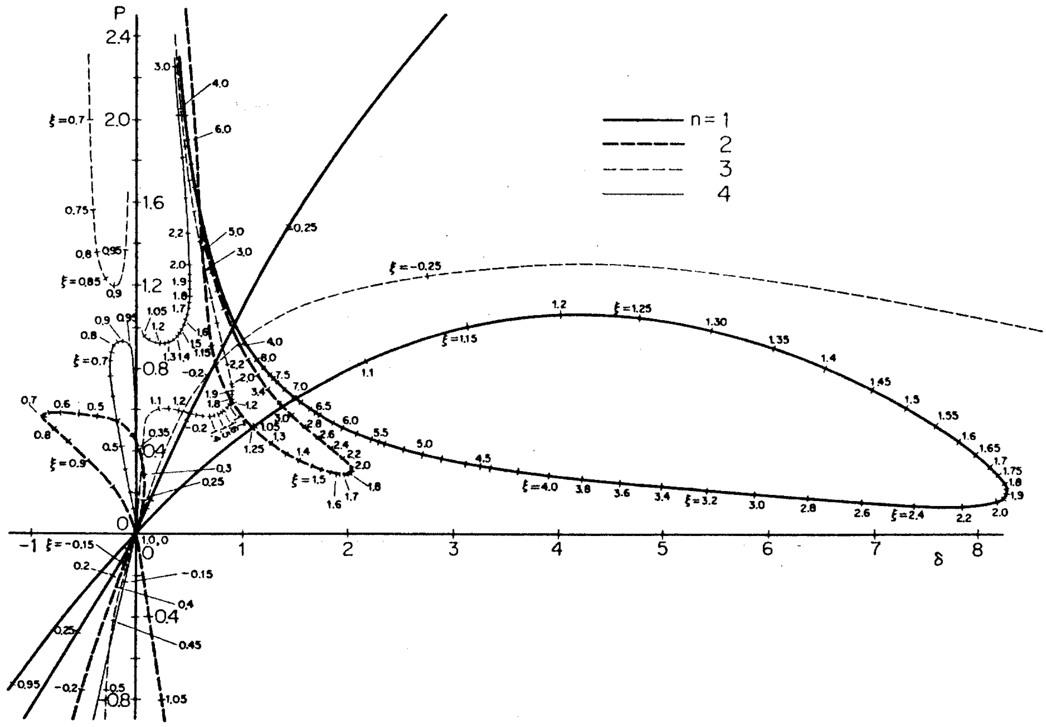

FIG. 4.7. Relation between the external pressure and the deflection at the center of spherical shell, $\phi=30$.

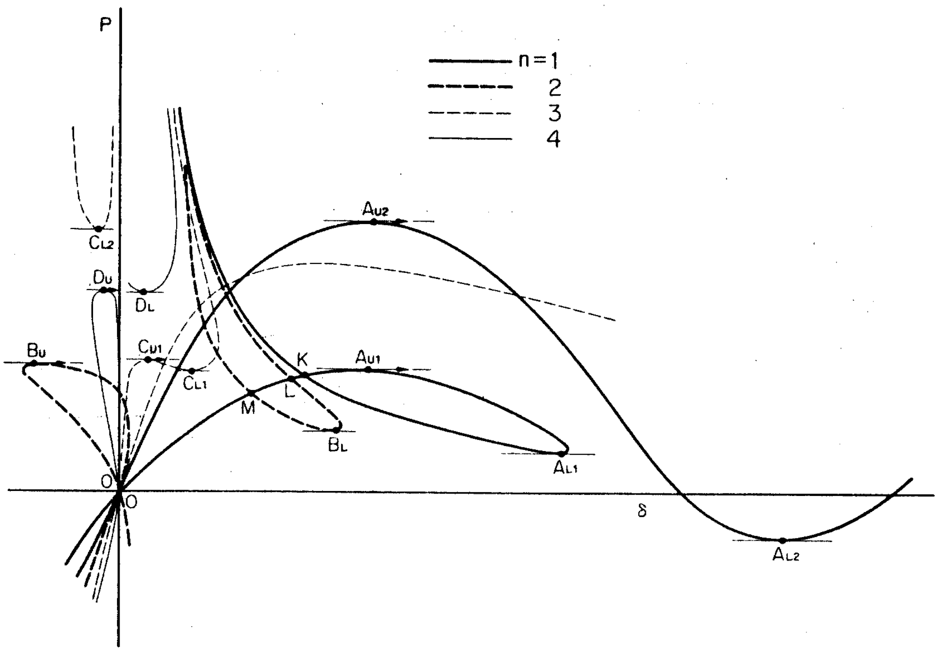

FIG. 4.8. Schematic drawing of the relation between the external pressure and the deflection at the center of spherical shell.

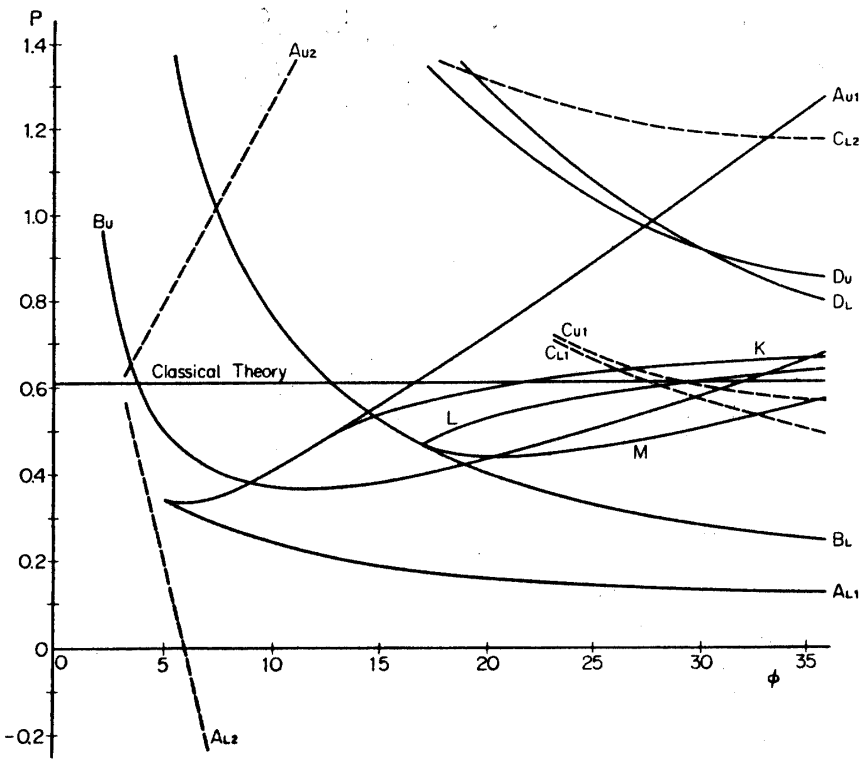

FIG. 4.9. Critical pressures for clamped spherical shells.

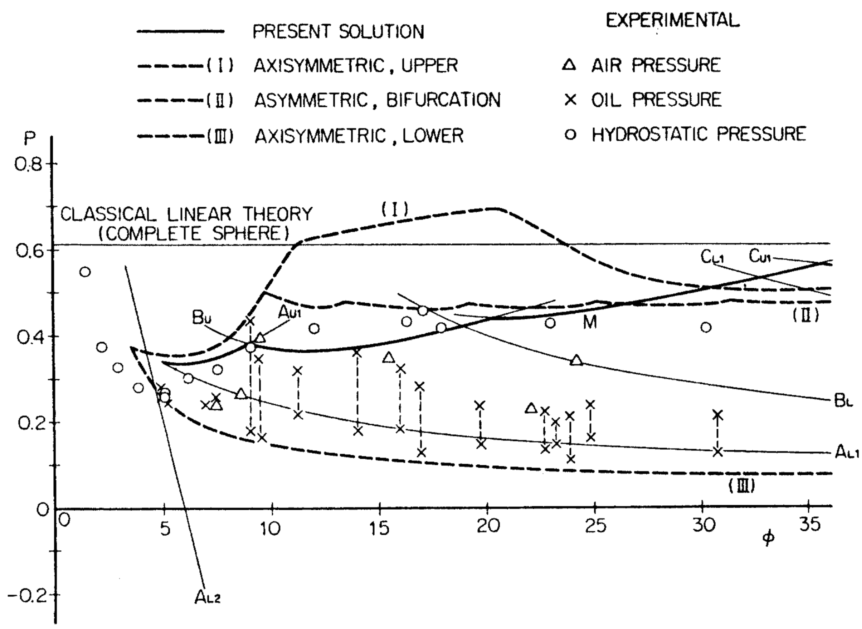

FIG. 4.10. Buckling pressures for clamped spherical shells from theories and experiments.

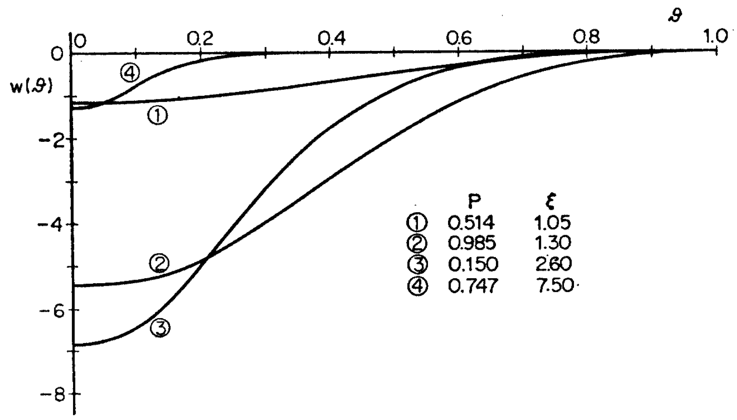

FIG. 4.11. Deflection patterns, $\phi=30$, $n=1$.

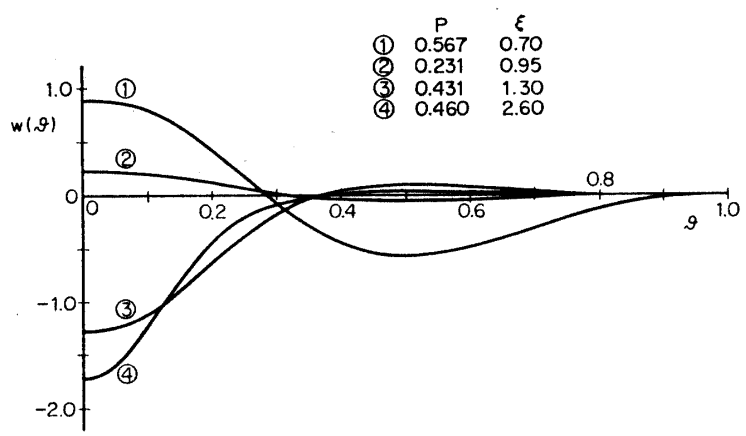

FIG. 4.12. Deflection patterns, $\phi=30$, $n=2$.

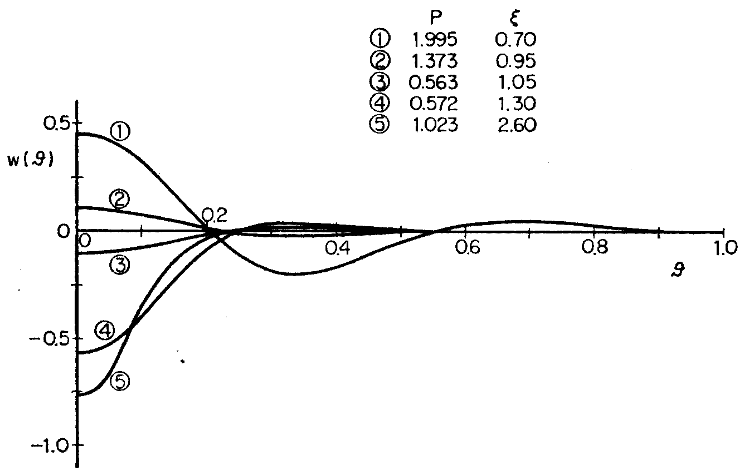

FIG. 4.13. Deflection patterns, $\phi=30$, $n=3$.

from each other for the same values of δ and $n=1$. (6) points L, M and so on: on these points it is possible to exist two different patterns whose values of n and ξ are different from each other for the same values of δ, respectively. It seems that some another points are not so important in the present analysis and so they are not described here.

The pressures corresponding to the above-mentioned points are picked up and summarized in Fig. 4.9 against the geometrical parameter, ϕ. As mentioned already, the special attention in the present analysis is paid to the initiation of snap-through buckling and the post-buckling state is not analyzed. And so the minimum snap-through buckling pressures for each value of ϕ are picked up in Fig. 4.9 and are shown in Fig. 4.10. The theoretical results for the cases of axisymmetric buckling by Keller & Reiss [*9*], Weinitschke [*10*], Budiansky [*11*], Thurston [*14*], Archer [*16*], Keller & Reiss [*18*] and Keller & Wolfe [*23*] and of asymmetric bifurcation buckling by Huang [*21*] and Weinitschke [*22*] and the experimental results by Kaplan & Fung [*28*] and Krenzke & Kiernan [*30*] are also shown in Fig. 4.10.

It seems that the present analysis based on the assumption of the damped axisymmetric deformation cannot explain the experimental results by Kaplan & Fung but agrees fairly well with those by Krenzke & Kiernan. It is shown in Fig. 4.10 that there exists no critical pressure for the shell of $\phi\leqq 5$ and that the deformation pattern which corresponds to the lowest critical pressure is the one at $A_{U1}(n=1)$ for the range of $5<\phi\leqq 9$ and the one at $B_U(n=2)$ for $9<\phi\leqq 20$ and thereafter M gives the lowest critical pressure for the range of ϕ calculated in the present analysis.

Some deformation patterns for the case of $\phi=30$ are calculated by use of Eq. (3.2) and are shown in Figs. 4.11, 4.12 and 4.13 for $n=1$, 2 and 3, respectively.

In the above discussion, the critical pressures have been discussed independently for each value of the number of waves of the deflection. The comprehensive discussions including also that of the energy level for respective deformed states are needed to decide which one of the processes of deformation is followed by the given shell, and they will be reported separately.

5. Conclusions

In the present paper, the nonlinear fundamental equations of a spherical shell for thermoelastic problems, subjected to the pressure and the change in temperature, have been derived first of all, taking a large deformation into account. Then, the equations have been solved approximately assuming the deformation pattern to be the axisymmetric damped wave, where the damping factor is not constant but a function of the external load, the number of half waves of deflection with respect to the distance from the apex and the shell geometry.

The relations between the external pressure and the deflection at the center of the shell have been obtained by use of the Galerkin method and the critical pressures have been discussed based on the author's viewpoint that the buckling phenomenon of spherical shell is of a local one at the initial stage of its occurrence and the localized deformation has the triggering effect in inducing the much larger dynamic deformation and that the buckling process of spherical shell under external pressure is of a "snapping through".

The analytical result obtained has been compared with the theoretical and experimental results reported so far, and it seems that the present result is one of the promising contributions to bridge the gap between the theoretical and experimental values of the problem.

An extensive experiment has been carried on at the author's laboratory, and the buckling pressures have not scattered so much but dropped into a narrow band of pressure and are higher than 80% of the classical value except for a few special cases, and its results will be published later on.

The present analysis is a part of the author's work which had been carried out at the Applied Mechanics Laboratory of Syracuse University, Syracuse, N.Y., during the author's stay through the 1964–65 academical year. The author would like to express his sincere thanks to Dr. R. M. Evan-Iwanowski, Professor at Syracuse University, for his successive encouragements and warm arrangement for the work. The author is grateful to Dr. M. Uemura, Professor at the University of Tokyo, for his careful discussions given in preparing the manuscript. Thanks are also given to Messrs. C.-S. Chu at Syracuse University, T. Sakurai at Mitsubishi Atomic Powers Ind., Inc. and K. Ichida at the University of Tokyo for their assistances in carrying out the numerical calculations.

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