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Proposition of Pseudo-Cylindrical Concave Polyhedral Shells

By

Koryo MIURA

Summary: A proposition of a new shell form, which is cylindrical in a macroscopic sense and is concave polyhedral in a microscopic sense, is the purpose of this paper. It is shown that the inextensional post-buckling configurations of general cylindrical shells subjected to axial loading have peculiar geometrical characteristics, and that these configurations compose a general group of surfaces which may be designated as the pseudo-cylindrical concave polyhedral surface. Then the fixed idea that these surfaces are essentially failed forms is abandoned and is replaced by the idea that these are the basic forms of a new shell which could function superbly as the structure under some loading conditions. It is shown that the new shell, which may be called for convenience, the pseudo-cylindrical concave polyhedral shell and the PCCP shell for its abbreviation, has many useful characteristics as follows; inclusion of an arbitrary curvature distribution, developability of its midsurface, intrinsically high circumferential bending rigidity, and simplicity of elementary faces. The application of PCCP shells to large span shell structures, reservoirs, expansion joints, and others is suggested.

1. INTRODUCTION

There is no doubt that the cylindrical shell form is one of the most important structural forms both in nature and in artificial creations. The cylindrical vault in architecture is probably one of the oldest cylindrical shell forms, if not thin shell, in the history of artificial creations. The invention of it is usually credited to Democritus, but there is the evidence that it had been used many hundreds of years earlier. The splendid accomplishments both in technical and in aesthetic sense is the medieval, Romanesque church roofs. There we can see the cylindrical vault made of thick bricks is spanning between the parallel walls. The vault may be considered as a succession of arches assembled together, and thus the structural principle is in essence the arch action. By this reason the shape of the vault is in general chosen to be the caternary curve; but it is inevitable that additional loads due to wind or snow produce a certain amount of bending in the vault. The tradition of cylindrical vault has been succeeded to our times, such as cylindrical hangars and halls. At the same time, the brick as the structural

The author wishes to thank Tomoko Miura for her assistance in preparing experimental models; the author is also indebted to Masamori Sakamaki and Hisayoshi Ohiso for preparing illustrating models.

material has been replaced by better materials, such as the reinforced concrete, thus the characteristics as the thin shell is becoming clear.

Due to Salvadori [1], the thin shells are form-resistant structure thin enough not to develop appreciable bending stress, but thick enough to carry loads by compression, shear and tension. This condition, however, is not always satisfied, thus the occurrence of a certain amount of bending moment is usually unavoidable. If we consider the case of short shells in which the ratio of axial length to free span length is not large, such bending moment is principally in the circumferential direction. Therefore, it is usual to devise some means to increase the bending stiffness in that direction. Either the circumferentially stiffened rib structure or the axially corrugated structure is the representative answer in case of thin concrete shells. It goes without saying that the similar examples can be found not only in the architectural field but also in machine structures, aircraft structures, and others. Conclusively, the contemporary engineering technique to increase the circumferential bending rigidity of thin cylindrical shells depends primarily on the circumferentially stiffened structure or the axially corrugated structure.

If, however, we could find a shell form having following characteristics; the form is cylindrical at least macroscopically; the circumferential bending rigidity is essentially very large; the midsurface is, if possible, developable: then this engineering problem can be solved rather easily. This is the very purpose of the present paper, and the new shell form having such characteristics as mentioned above is presented herein. The new form was discovered accidentally by the present author in an effort to study the inextensional buckling deformation of general cylindrical shells. As the study has been advanced, it is becoming clear that the shell has many extraordinary characteristics, some of which are beyond our expectations. These additional features of the shell will also be described in the paper. In short, the author proposes herein a new shell form in answer to various engineering problemes.

2. Introduction of Pseudo-Cylindrical Concave Polyhedral Surfaces

Before getting at the kernel of the subject of this paper, we must have the recollections of the studies on the classical problem of axial buckling of thin cylindrical shells. Because the subject owes its origin to the forms of deformations encountered in the buckling process. That problem has been a target of many researchers for almost half a century and yet it has not been conclusively explored. Fortunately, such situation does by no means bother our discussion here, since the chief concern about the subject is not buckling criterion but the geometrical forms of deformations.

Now let us consider the post-buckling deformation of axially compressed circular cylindrical shells. As Kármán and Tsien [2] assumed a buckled shape based on the observation, it consists of three trigonometric terms (Fig. 1):

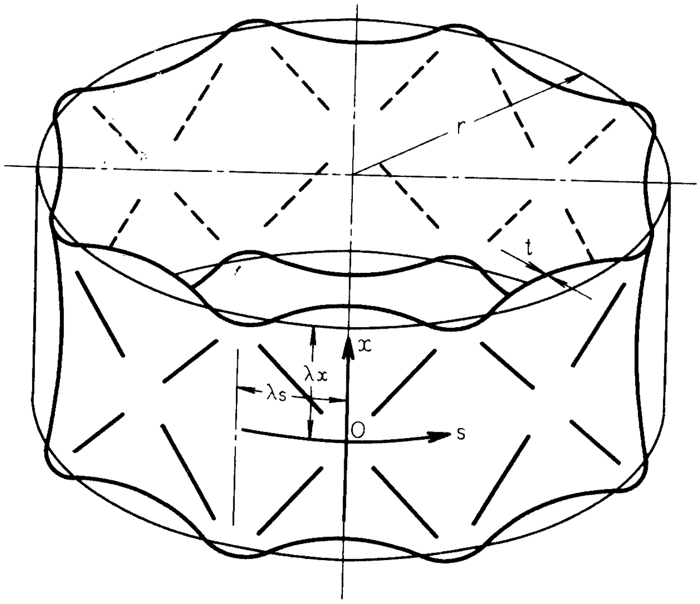

FIG. 1. Geometrical definitions of buckling deformation of circular cylindrical shell.

$$w = A_{00} + A_{11}\cos(\xi x/r)\cos(\zeta s/r) + A_{20}\cos(2\xi x/r) + A_{02}\cos(2\zeta s/r) \quad (1)$$

Here w is the radial inward displacement, r the radius of the midsurface of the shell, x and s the axial and circumferential coordinates, ξ and ζ numbers, and A's unknown coefficients. It goes without saying that the three trigonometric terms expression is not accurate enough to represent the buckle. Therefore, as the high speed digital computer became available, the computation which can include more terms representing the shape of the buckle has been tried by, for example, Almroth [3], Hoff and his collaborators [4]; though the computation essentially follows the Kármán-Tsien procedure. In the latter's paper, it was shown that such a procedure results in displacement patterns that approach a kind of concave polyhedral surface more and more closely as the number of terms representing the shape of the buckle is increased. Since that polyhedral surface had been predicted by Yoshimura [5] in 1951, it is called Yoshimura-pattern (Fig. 2). Indeed a typical experimental result as shown in Fig. 3 certainly resembles the Yoshimura-pattern. Moreover, it has been shown by the present author [6] that through a certain modification of Yoshimura-pattern, the so-called local buckling pattern appeared in Fig. 3 can be explained by Yoshimura-like inextensional deformation as shown in Fig. 4. It appears, therefore, very likely that an infinite number of terms would yield the exact Yoshimura-pattern if a means could be found to include an infinite numbers of terms in the calculation.

It should be noted that this pattern can be obtained from the original circular cylindrical shape through an inextensional process. In other words, any line element of midsurface of the circular cylindrical shell does not change its length after deformation. In exact geometrical terms, these two configurations are

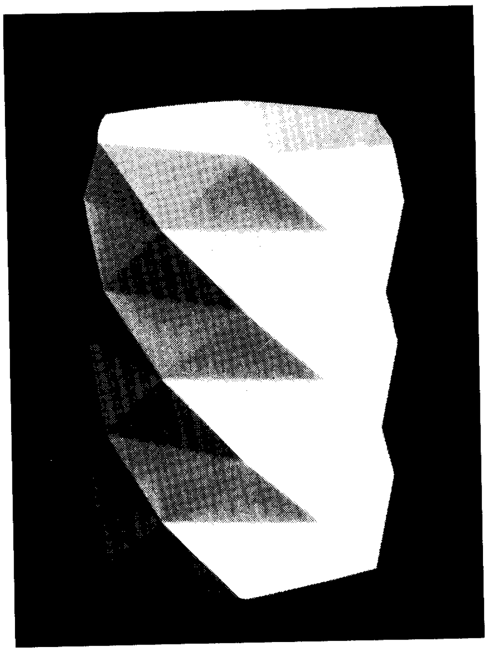

FIG. 2. Yoshimura-pattern.

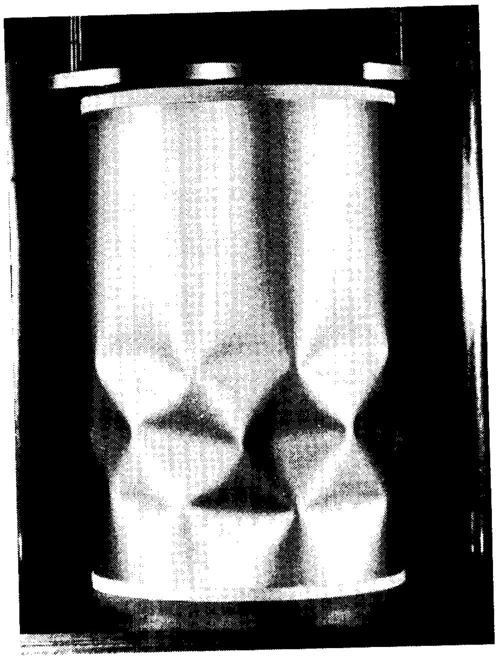

FIG. 3. Buckling pattern of a circular cylindrical shell subjected to axial loading.

isometric. The significance of this fact with regard to the elastic stability can easily be understood through energy consideration, but their consideration is outside the scope of the present paper. In short, the purely geometrical feature of the polyhedral surface representing the Yoshimura-pattern is the exactly isometric, axially and uniformly shortened surface that exists indefinitely close to an arbitrary circular cylindrical surface.

Now, as a natural extension of the above discussion, the existence of such isometric surfaces for general cylindrical surface is asked. This question was answered by the present author in his recent paper [*6*]. On purpose of revealing the important features of such isometric surfaces, the essential part of that paper is presented in the following.

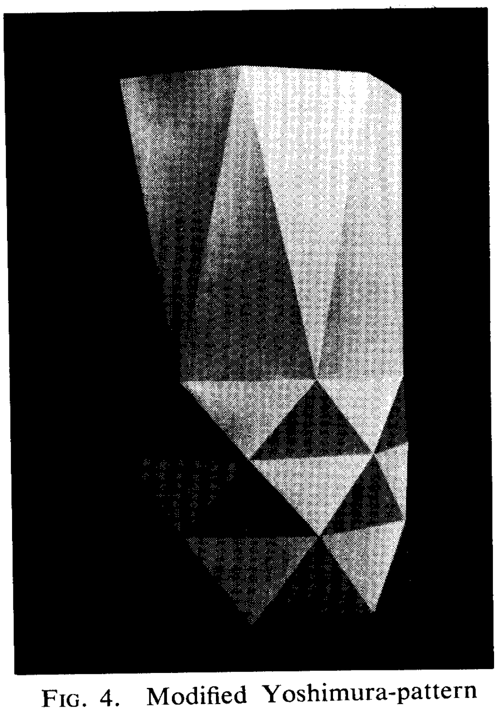

FIG. 4. Modified Yoshimura-pattern for local buckling.

Let us prove the presence of a surface which has the following characteristics: it is isometric with a given arbitrary cylindrical surface S; it has a uniform axial shortening; and it is indefinitely close to S. S can be either the closed or open cylindrical surface.

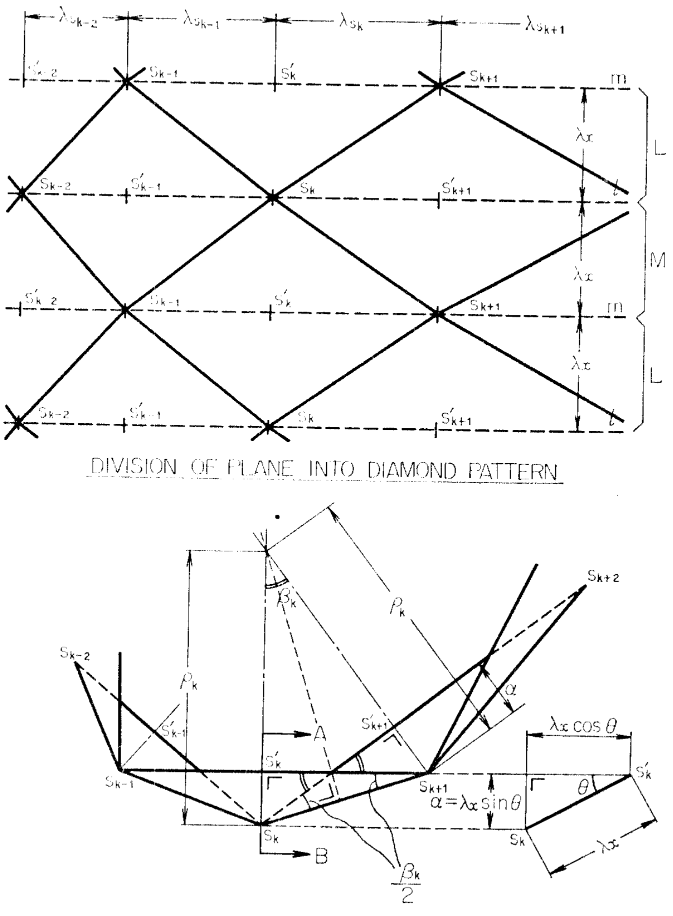

Fig. 5. Construction of a polyhedral surface from a plane.

As shown in Fig. 5, the parallel lines $l, m, l, m, \cdots$ with a constant interval λ_x and the parallel zones $L, M, L, M, \cdots$ constructed by the former are considered on a plane. The arbitrary points on the lines l and m are designated by $s_j(j=\cdots, k-2, k, k+2, \cdots)$ and s_{j+1}, respectively, with the condition $s_j < s_{j+1}$. The orthogonal projections of these points on the other group of the lines are distinguished by the superscript of dash. Connecting those points as shown, the rows of many triangles with an identical height are formed. Let us fold along the oblique side of those triangles, for example on the L zone, so that every triangular plane may be inclined to the x axis by an angle of θ. In order to add clarity to the matter, the solid and dashed lines refer to the folding lines which are convex and concave to the reader's side, respectively. A part of the L zone formed by such a process is seen in Fig. 5 as $s_k s'_k s_{k+1} s'_{k+1}$. If the line segment $\overline{s_k s'_{k+1}}$ is assumed to be on the plane l^* normal to the x axis, the line segment $\overline{s'_k s_{k+1}}$ must be on the plane m^* parallel to l^* and as much as $\lambda_x \cos\theta$ distant from l^*. This relation holds everywhere between corresponding line segments on l and m. Thus the broken lines l and m are formed on the parallel planes l^* and m^*, respectively; and these planes, apart as much as $\lambda_x \cos\theta$, are evidently normal to the x axis.

By the same process troughout the M's and L's zones, each zone constructs a polyhedral surface that is symmetrical with the adjacencies about the planes l^* and m^*.

In order to study the characteristics of this cylinder-like polyhedral surface, the quantity which in a broad sense may be called "curvature" is the best conceivable means at present. In a general sense, the mean curvature is defined as the ratio of the angle between the tangents at the edges of a curve to the length of it. By the same token, it may be possible to define the "quasi-curvature" $\langle\kappa_1\rangle$ of the circumference $\cdots s'_{k-1}s_k s'_{k+1} \cdots$ as

$$\langle\kappa_1\rangle_k=(\pi-\angle s'_{k-1}s_k s'_{k+1})/(\overline{s'_{k-1}s_k}+\overline{s_k s'_{k+1}}) \tag{2}$$

It is, however, more convenient for the following analysis to use the angular variation between the line segments $\overline{s_{k-1}s'_k s_{k+1}}$ and $\overline{s_k s'_{k+1} s_{k+2}}$, though these are in different parallel planes. Thus

$$\langle\kappa_2\rangle_k=\beta_k/\overline{s'_k s_{k+1}}=\beta_k/\lambda_{s_k} \tag{3}$$

Evidently, when the broken lines converge to a curve through an infinitesimal division, these quasi-curvatures represent exactly the curvature in a general sense. By the help of Fig. 5, the following relation can easily be obtained:

$$\beta_k=2\tan^{-1}(\lambda_x \sin\theta/\lambda_{s_k}) \tag{4}$$

Thus, the quasi-curvature $\langle\kappa_2\rangle$ is written as

$$\langle\kappa_2\rangle_k=(2/\lambda_{s_k})\tan^{-1}(\lambda_x \sin\theta/\lambda_{s_k}) \tag{5}$$

Also, the unit shortening in the x axis direction ε, and the amplitude of the wave α are given as follows:

$$\varepsilon=1-\cos\theta \tag{6}$$

$$\alpha=\lambda_x \sin\theta \tag{7}$$

Next, let us consider the limiting case where the line segments of l and m are infinitesimally small while s_j and s_{j+1} are following a given arbitrary smooth continuous function, and at the same time the amplitude of the wave is infinitesimally small such as $\alpha/\lambda_{s_k}\ll 1$. In this case, the quasi-curvature $\langle\kappa_2\rangle$, and therefore $\langle\kappa_1\rangle$, almost coincide with the curvature in a general sense and it is given by

$$\kappa(s)=2\lambda_x(2\varepsilon)^{1/2}/[\lambda_s(s)]^2=2\alpha/[\lambda_s(s)]^2 \tag{8}$$

Simultaneously, the broken lines l's and m's converge to a curve; as an inevitable consequence, the zones M's and L's and then the whole surface converge indefinitely close to a cylindrical surface S, whose curvature is given by $\kappa(s)$. The limiting case of this polyhedral surface is now denoted as S_p.

One distinction between the polyhedral surface S_p and the corresponding cylindrical surface S with an identical curvature distribution is that the former has a

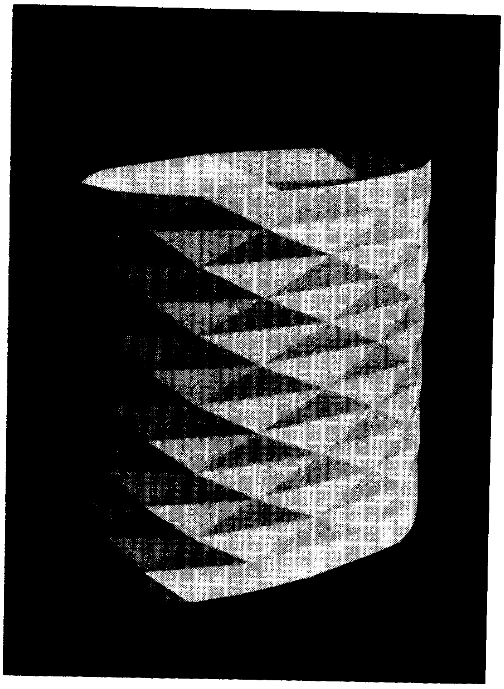

FIG. 6. Exaggerated view of an isometric, axially shortened surface corresponding to an elliptic cylindrical surface (major-to-minor axis ratio=2).

uniform axial shortening ε everywhere. Also it is evident that the surface S_p is developable as it is constructed by folding a plane. Therefore, it can be said that the surface S and S_p are isometric, or S is developable on S_p.

In conclusion, the presence of a developable surface S_p is verified in connection with an arbitrary cylindrical surface S with the following characteristics: S and S_p are isometric, S_p has a uniform axial shortening ε, and S_p is indefinitely close to S. The surface S_p is the concave polyhedral surface and the basic geometrical parameters are governed by Eq. (8). Fig. 6 shows an exaggerated (since the real surface is constructed by the infinite number of infinitesimal waves) view of such a surface corresponding to an elliptical cylindrical surface. It appears also well established that if the transfer from a surface S to the corresponding surface S_p is taken place, it should be done in the inextensional process. If the surface S is referred to the midplane of a thin arbitary general cylindrical shell, then it is found that the surface S_p satisfies probable conditions for a surface being an inextensional buckling deformation.

In common parlance, this transfer means "the folding of cylindrical surface in the axial direction". In addition, it is indeed infinite in number of the combination of axial and circumferential wave numbers and thus resulting axial shortenings. It is interesting to note that any cylindrical surface can be shortened by an arbitrary amount through such a folding.

There is certainly something in common between this fact and the geometrical paradox that the surface area of a circular cylinder can not be obtained as the simple upper bound of the surface area of the concave polyhedron inscribing the cylinder [7]. Let us consider a two-dimensional Euclidean complex $K(p, q)$ inscribing a circular cylinder and denote $F(p, q)$ as the

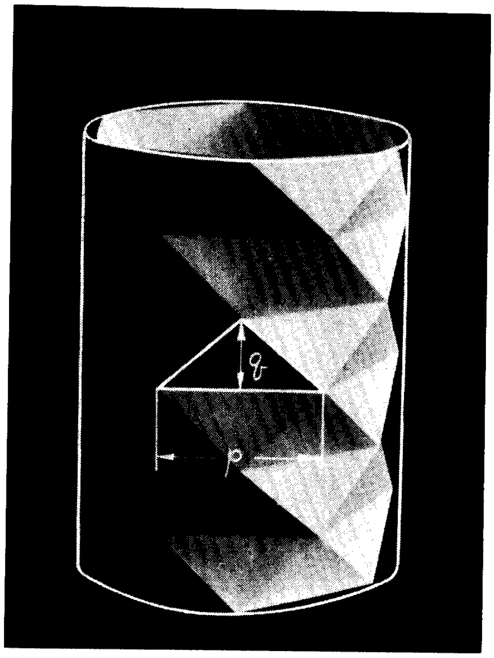

FIG. 7. Two-dimensional Euclidean complex $K(p, q)$ inscribing a circular cylinder.

sum of area of two-dimensional Euclidean simplexes (*Fig.* 7). Converging p and q to zero, we will have $K(p, q)$ indefinitely close to the cylindrical surface. On the other hand, however, the limiting value of $F(p, q)$ depends on a process of convergence of p and q to zero and the value can take from the finite to the infinite.

But to return to the point to our subject, we assumed in the preceding analysis the uniformity of the axial shortening ε with regard to both circumferential and axial coordinates. It is, however, possible to relaxe the latter assumption and give a wide definition of polyhedral surface S_p. Then Eq. (8) is valid for x-dependent λ_x and ε, provided that the amplitude of the wave $\alpha=\lambda_x(2\varepsilon)^{1/2}$ is kept constant; thus

$$\left.\begin{aligned}&\kappa(s)=2\lambda_{x_i}(2\varepsilon_i)^{1/2}/[\lambda_s(s)]^2\\&[\alpha=\lambda_{x_i}(2\varepsilon_i)^{1/2}\text{: constant}]\end{aligned}\right\} \tag{9}$$

where the subscript i indicates the i-th zone in the axial direction. This enables us to make a pattern that resembles the local buckling pattern of the axially compressed circular cylinder shown in Fig. 4.

Furthermore the triangular division appeared in the preceding analysis is not mandatory, and instead, the trapezoidal division yields the similar conclusion. Denoting the bisections of the upper and the lower bases of a trapezoid as λ^s and λ_s, respectively (Fig. 8a), the correspondents to Eqs. (5) and (8) can be written as follows:

$$\langle\kappa_2\rangle=[2/(\lambda_s+\lambda^s)]\tan^{-1}[\lambda_x\sin\theta/(\lambda_s-\lambda^s)] \tag{10}$$

$$\left.\begin{aligned}&\kappa(s)=2\lambda_{x_i}(2\varepsilon_i)^{1/2}/[\lambda_s(s)^2-\lambda^s(s)^2]\\&[\alpha=\lambda_{x_i}(2\varepsilon_i)^{1/2}\text{: constant}]\end{aligned}\right\} \tag{11}$$

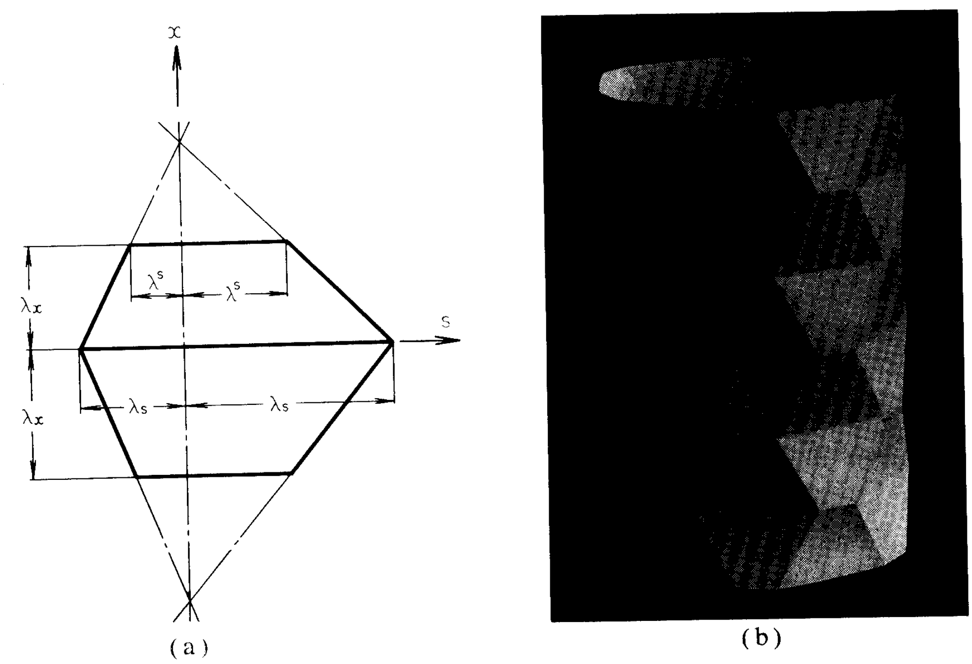

FIG. 8. Hexagonal pattern.

The resulting "hexagonal-pattern" can be seen in a typical example (Fig. 8b). Indeed one will sometimes observe the hexagonal-pattern instead of the Yoshimura-pattern in the collapsing shapes of circular cylindrical shells. In Figs. 9 and 10, the general case of diamond and hexagonal type polyhedral surfaces S_p corresponding to Eqs. (9) and (11), respectively, are shown. It is also possible to furnish a sign change in curvature of polyhedral surface S_p as shown in Fig. 11.

The above argument raises a new question whether a surface with another kind of pattern might exist that also belongs to the category of the polyhedral surface S_p. In this respect, the author will not attempt to prove in an exact manner the existence or nonexistence of such a surface, but he simply points out in the following the analogous characteristics between this problem and the classical problem of regular tessellation.

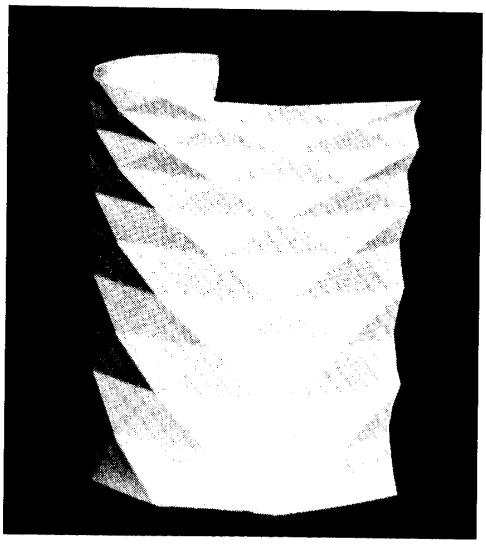

FIG. 9. Diamond type polyhedral surface S_p [general case; see Eq. (9)].

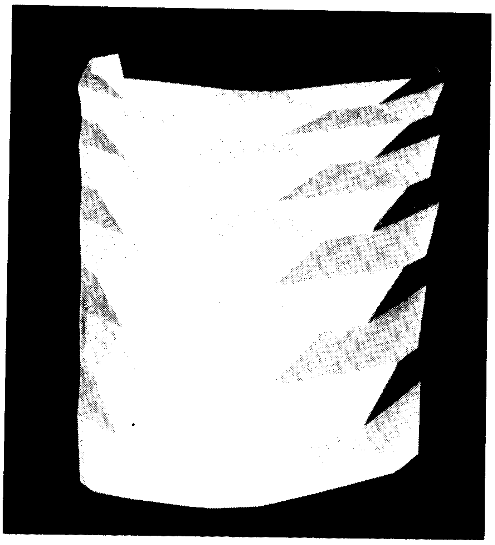

FIG. 10. Hexagonal type polyhedral surface S_p [general case; see Eq. (11)].

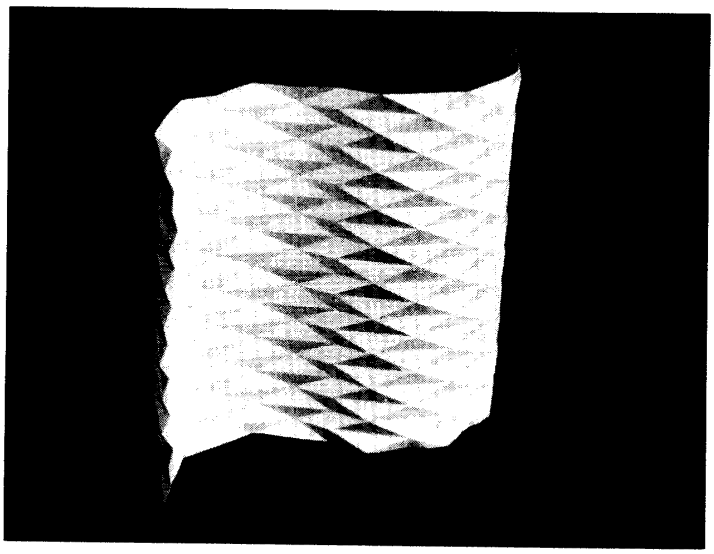

FIG. 11. Sign change in curvature of polyhedral surface S_p.

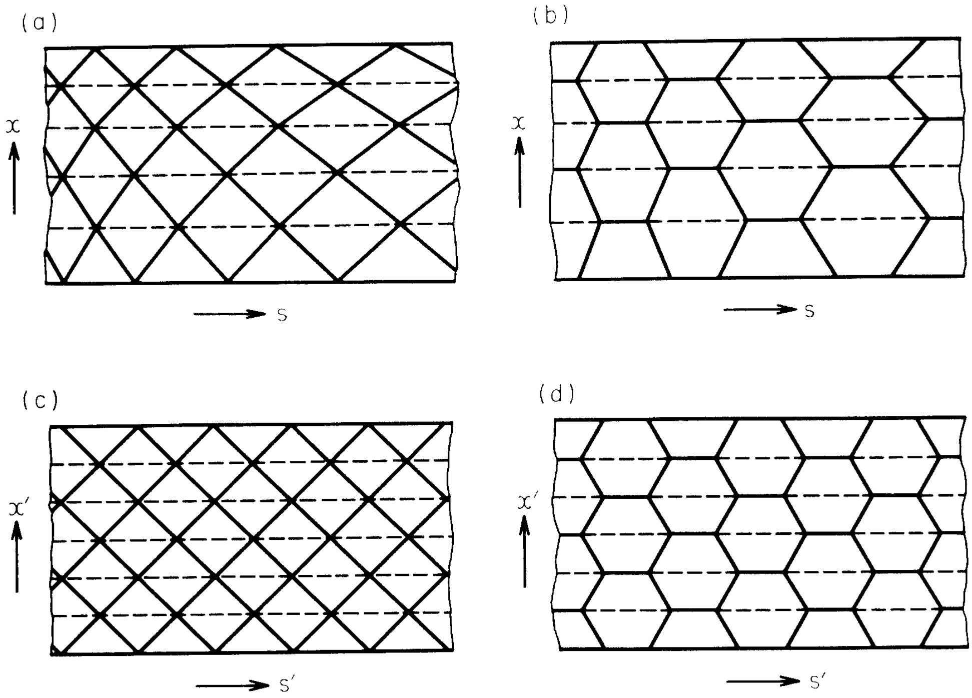

FIG. 12. Coordinate transfer of patterns to regular tessellation.

The concave polyhedral surface S_p with either the diamond or the hexagonal pattern can be developed into the plane, as it is the fundamental premise. If the edges of these patterns were "printed" on the surface, while a edge insides of each pattern is excluded, the planes printed with either the diamond or the hexagonal patterns will be obtained after the deployment (Figs. 12a and 12b). Then, it is always possible to find the appropriate coordinate transfer as to x and s, by which the pattern is "regularized", that is, each pattern is transferred to the same-sized regular polygon. Such coordinate transfer can be expressed formally as follows:

$$x'=f(x), \qquad s'=g(s) \tag{12}$$

The resulting regular patterns are shown in Figs. 12c and 12d.

These patterns remind us the classical problem of regular tessellation studied first by Kepler [8]. For a formal definition, we may say that a tessellation is regular if it has regular faces and a regular vertex figure at each vertex. It has been proved that the triangular, diamond, and hexagonal regular tessellations are possible, and these are the only regular tessellations. Since the diamond and hexagonal tessellations have their counterparts in the patterns of the polyhedral surfaces S_p, the possibility of triangular pattern of S_p is asked now. At present, however, we are unable to find a mechanism by which a cylindrical surface with triangular pattern can be transferred to a polyhedral surface S_p. Therefore, it is most likely that the group of polyhedral surface S_p is characterized by either the diamond or the hexagonal pattern. The mixture of these patterns is possible, but it is not the fundamental pattern.

Since the group of these surfaces has the distinct characteristics, it will be convenient to give them an appropriate designation. The author suggests the word "pseudo-cylindrical concave polyhedral surface", and PCCP surface as an abbreviation, because this surface is not only macroscopically cylindrical but also it has a cylindrical surface as its limit. It is also assumed that this designation of PCCP surface includes the case where the fundamental parameters, λ_x, λ_s, λ^s, and α, are finite. We can say, there is two kinds of PCCP surfaces, one is the PCCP surface with diamond pattern and the other is the PCCP surface with hexagonal pattern. The geometrical characteristics of PCCP surfaces are summarized in the following.

PCCP surface (diamond pattern)

(1) The developable concave polyhedral surface composed of triangular faces.

(2) The relation between an arbitrary triangular face 1–2–3 and the three adjacent triangular faces (Fig. 13):

a) The one particular sides of every triangle are on the mutually parallel planes. Let us call them the bases of triangles. A line which is vertical to these planes is denoted as x.

b) The two triangular faces, which own a side jointly, make the identical angle to x.

c) The two triangles, which own a base jointly, have the identical orthogonal projection to a plane vertical to x. This orthogonal projection is also a triangle with the same base. The height of it is denoted as the amplitude α. The amplitude α characterizes the depth of the wavy concave polyhedral surface and is constant throughout the whole surface.

d) The x-coordinate of the base, jointly owned by two triangles, is in-between the x-coordinates of vertexes facing to the base.

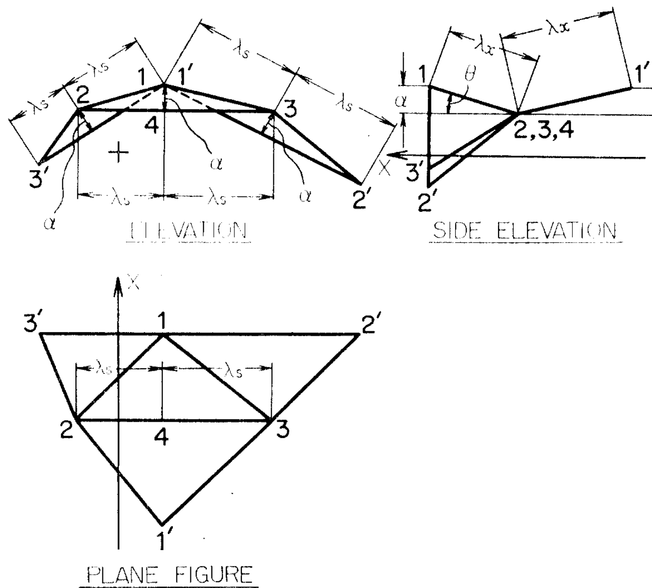

FIG. 13. Geometrical relation between elementary faces of PCCP surface (diamond pattern).

(3) The macroscopical configuration composed by this surface is cylindrical, thus the quasi-curvature of the surface is approximately given by

$$\kappa(s)=2\alpha/\lambda_s(s)^2$$

PCCP surface (hexagonal pattern)

(1) The developable concave polyhedral surface composed of non-rectangular trapezoidal faces.

(2) The relation between an arbitrary trapezoidal face 1–2–3–4 and the four adjacent trapezoidal faces (Fig. 14):

a) The bases of every trapezoidal face are on the mutually parallel planes. A line which is vertical to these planes is denoted by x.

b) The two trapezoidal faces, which own a side jointly, make the identical angle to x.

c) The two trapezoidal faces, which own a base jointly, have the identical orthogonal projection to a plane vertical to x. This orthogonal projeciton is also a trapezoid and the height of it is denoted as the amplitude α. The amplitude α characterizes the depth of the wavy concave polyhedral surface and is constant throughout the whole surface.

d) The x-coordinate of the base, jointly owned by two trapezoids, is in-between the x-coordinates of the other two bases.

(3) The macroscopic configuration composed by this surface is cylindrical, thus the quasi-curvature of the surface is approximately given by

$$\kappa(s)=2\alpha/[\lambda_s(s)^2-\lambda^s(s)^2]$$

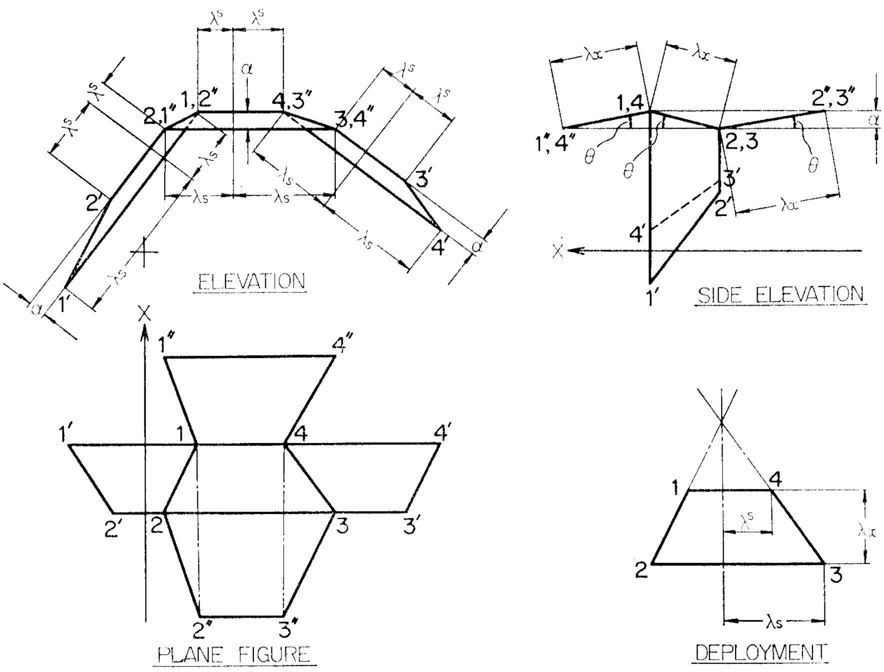

FIG. 14. Geometrical relation between elementary faces of PCCP surface (hexagonal pattern).

3. Proposition of Pseudo-Cylindrical Concave Polyhedral Shells

A new idea is not necessarily conceived as a logical conclusion by a certain rational mathematical analysis, but rather it is frequently obtained by chance by a process transcending a logical way. This is exactly the process by which the new form of shell structure is introduced from the PCCP surface discussed in the preceding chapter.

For the purpose of illustration of PCCP surface, the author has made a model using 0.25 mm thick Kent paper. This is the one shown in Fig. 6, and is representing an elliptic PCCP surface with diamond pattern. Playing unconsciously with the model, the author has noticed the considerable supporting capability of it in the axial direction. This phenomenon may be explained in an approximate manner as follows. The form of this model represents indeed a stable post-buckling equilibrium of a very thin cylindrical shell subjected to an axial load, that is, in other words, a failure configuration under such loading. But the typical load-shortening curves as shown in Fig. 15 indicate the positive gradient in the post-buckling region for a fixed combination of axial and circumferential wave numbers. So it is obvious that the model should be capable of supporting substantial loads though it is not as stiff as the original cylindrical form. Therefore, if a thin structure is designed from the first in a form of the PCCP surface, the elastically stable region will be assured until the next form of failure, possibly either the material yield or the local instability, will occur. The author believes that there is indeed a great potentiality of such structures in practice.

The conversion of thought from the failed form of a structure to the potential new form of a structure is really a drastic turn. But an even more drastic turn has been done by the turn, in its literal meaning, as much as ninety degrees of the loading direction. The author has noticed immediately that the model exhibits quite a large rigidity against the load normal to the surface, in macroscopic sense,

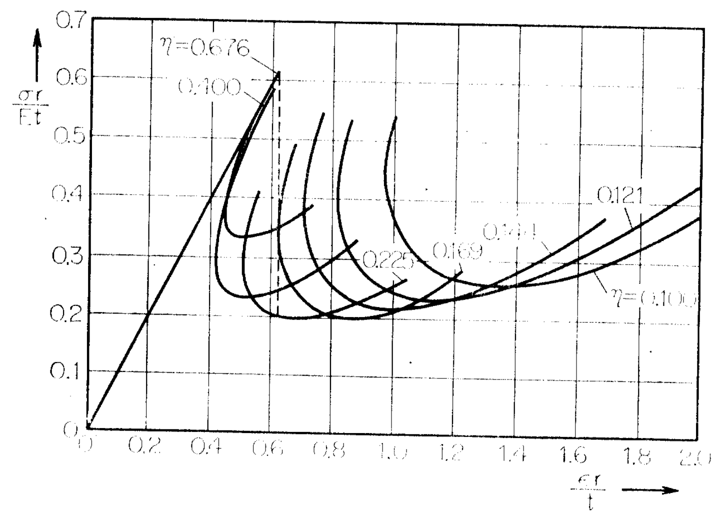

FIG. 15. Reduced compression stress $\sigma r/Et$ against unit end shortening $\varepsilon r/t$ for $\mu=1.00$ and different number of waves in circumferential direction (Kármán-Tsien).

of it. In spite of the minute thickness of the paper, the model can actually support a substantial load without indicating a sign of collapsing. On the contrary, an elliptic cylindrical model made of the same paper collapses by its own weight. In short, the former has much larger circumferential bending rigidity than the latter.

This odd experience and the consideration of the intrinsic geometrical characteristics of this developable surface, both has convinced the author of that the thin structure in the form of the PCCP surface should have great possibilities for engineering applications [9]. This kind of structure has more characteristics of the shell than those of the so-called folded plate structure, because it has more than anything else the curvature in a macroscopic sense and indeed it can be converged into the cylindrical shell as has already mentioned. Therefore, it should be allowed to call such a category of structures as the pseudo-cylindrical concave polyhedral shell and use the PCCP shell (or CP shell) for its abbreviation.

As a matter of fact, a folded plate structure which certainly belongs to the family of PCCP shells has been appeared before. Salvadori discussed such a structure in his excellent book entitled "Structure in Architecture" [1]. The example shown in the book is certainly a circular PCCP shell roof with diamond pattern, though its rather deep folding does not immediately indicate the characteristics of shells.

The author insists in this paper that the PCCP shell should be recognized as the shell and that it has a great versatility not known before in its forms, characteristics, and accordingly applications. As for its forms, as has already mentioned, it can be designed for an arbitrary curvature distribution with either the diamond or the hexagonal pattern. Moreover, a rather wide selection of independent parameters, which define the form both macroscopically and microscopically, is possible. As for its characteristics and applications, the following two chapters are devoted to their description, respectively.

4. Principal Characteristics of PCCP Shells

In this chapter we discuss in details the characteristics inherent in PCCP shells. These can be conveniently divided into the following six principal items, though these are closely connected with each other.

1. *Versatility in forms*

It goes without saying that the cylindrical shell form is the most versatile and thereby the widely used shell form. Since the PCCP shell is also in the cylindrical form at least macroscopically, the merit of the cylindrical form over other types of the shell is retained in this case.

The independent form parameters which define the form of PCCP surface are as follows;

$\alpha, \quad \lambda_s(s), \quad \lambda_x(x)$ (diamond pattern)

$\alpha, \quad \lambda_s(s), \quad \lambda^s(s), \quad \lambda_x(x)$ (hexagonal pattern) where the amplitude α of the wave is constant throughout the surface, while the wave-length parameters can vary stepwise with the coordinates indicated in parentheses.

The sole quantity which determines the macroscopic form of the shell is the curvature. The quasi-curvature of the PCCP surface is given approximately as follows;

$$\kappa(s)=2\alpha/\lambda_s(s)^2 \qquad \text{(diamond pattern)}$$
$$\kappa(s)=2\alpha/[\lambda_s(s)^2-\lambda^s(s)^2] \qquad \text{(hexagonal pattern)}$$

As the amplitude α is constant throughout the surface, the arbitrary curvature distribution can be gained by λ_s and λ^s variations along the circumferential coordinate s.

If both α and $\lambda_s(s)^2$, or $\lambda_s(s)^2-\lambda^s(s)^2$, are infinitesimal numbers of the same order, that is, by Landau's symbol

$$\alpha=0[\lambda_s(s)^2] \tag{13}$$

or

$$\alpha=0[\lambda_s(s)^2-\lambda^s(s)^2] \tag{14}$$

a PCCP surface converges to a cylindrical surface. At the same time, the quasi-curvature represents exactly the curvature in general sense. In regard to the convergence of surface area, however, the above condition is not sufficient; the reason has been mentioned before in chapter 1. Then, the following condition must be satisfied simultaneously.

$$\alpha/\lambda_x \to 0 \tag{15}$$

Generally speaking, when the quantity α/λ_x is small, the PCCP shell has almost an identical surface area with the corresponding cylindrical shell; this fact might have technical as well as economical meanings.

Excluding an aesthetic problem of design, considerable differences in technical meaning will be encountered by selecting either of two patterns of PCCP shells. For instance, let us consider the case of designing a PCCP shell when the curvature distribution and the quantity α, on which the circumferential bending rigidity primarily depends, are given in advance. Since the two conditions are given in advance, the rest of free parameters are one in case of the diamond pattern, and two in case of the hexagonal pattern; thus the latter provides more freedom in designing the shell.

2. *Circumferential bending rigidity*

Now the comparison is made between a circumferential strip of a circular PCCP shell with the diamond pattern and that of a circular cylindrical shell as shown in Fig. 16. The width of both strips is as much as $2\lambda_x$, that is, a single pattern width. It is also assumed that both have the identical uniform thickness. The moment of inertia of a cylindrical shell strip is

$$I=(1/6)\lambda_x t^3 \qquad (16)$$

The moment of inertia of a PCCP shell strip, I_p, is variable along the circumferential coordinate s. It takes the maximum value at nodal position and the minimum at the middle of nodal position. That is

$$(1/24)\lambda_x^3 t \sin^2\theta \leq I_p \leq (1/6)\lambda_x^3 t \sin^2\theta \qquad (17)$$

The similar relation holds also in the case of the hexagonal pattern. Therefore, the relative magnitude of the moment of inertia can be written as follows;

$$I_p/I \geq (\alpha/2t)^2 \qquad (18)$$

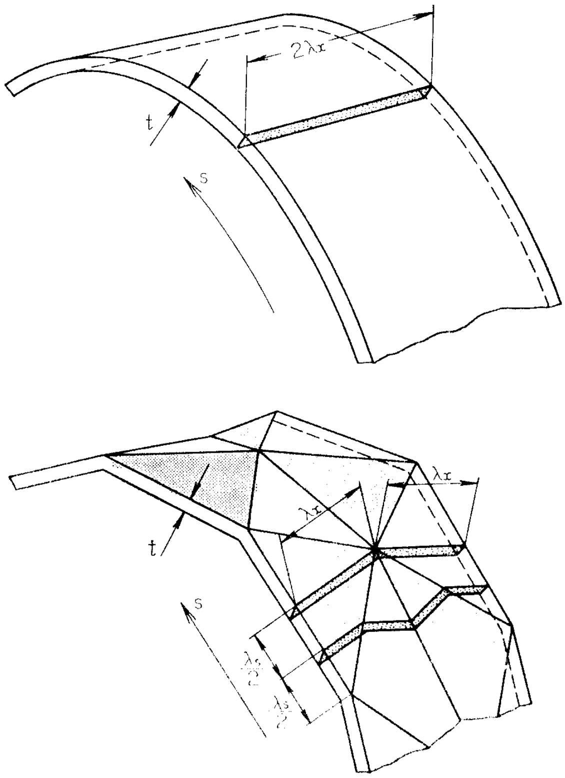

FIG. 16. Comparison of moment of inertia between cylindrical shell and PCCP shell.

This comparison on the moment of inertia can not directly be transferred to that of the circumferential bending rigidity of two shells, because the section of the PCCP shell changes periodically along the circumferential direction. The period of its variation is apparently λ_s or $\lambda_s+\lambda^s$ depending on patterns. Also an important fact we should know is that there is a completely different type of the large deformation not familiar with us. If the face angles are changeable, the large deformation is possible even without the elastic deformation of elementary faces. This somewhat peculiar characteristic is in fact essential for a new deployable structure, which will be discussed in the later part of this chapter. In the context of discussion developed here, the face angles are assumed to be constant. Under these circumstances, and as far as the rough qualitative comparison is concerned, the comparison of bending rigidities of two shells by using Eqs. (16), (17), and (18) should be justified.

Eq. (18) clearly shows that the potential of having greater circumferential bending rigidity is almost essential in case of PCCP shells. For example, if the amplitude is taken as much as several times of the thickness of the shell, the bending rigidity of a PCCP shell could be approximately 10 times of a regular cylindrical shell. Due to the definition of the shell in general, the amplitude several times of the thickness is still a small amount comparing with other dimensions. Therefore, the PCCP shell, being cylindrical in macroscopic sense, can be designed for a structure with greater circumferential bending rigidity by a relatively small amount of the amplitude.

3. Being developable surface

One of the most important characteristics of PCCP shells is that the midsurface is developable. This feature is inherent in the shell with zero Gaussian curvature. Whether the shell is developable or not has a vital influence on the production process of it. Because the manufacturing of an undevelopable shell from the sheet material involves the process entails the extensional deformation of the material, while a developable shell does not. The similar circumstance will occur also in case of the concrete shell, for the manufacturing of the mould involves the same problem.

4. Elements are in simple form

The whole structure of PCCP shell can be constructed by simple triangular or trapezoidal faces. This feature as well as the possibility of including a large number of same-sized elements will contribute to rationalize the production process. One typical example is the case of concrete shells, as the mould can be very simple form and especially the pre-cast concrete can make the most use of this feature.

5. Characteristic as deployable structures

The geometrical fundamental of the deployable surface structure is the inextensional transfer between two surfaces, the two occupying the different expanses in space. Among other things, the deployable surface structure with rigid plane element has important applications; those structures such as accordion type or fan type structures are examples. It is quite interesting to note that a new mechanism of deployment is possible by using a characteristic of the PCCP surface. This mechanism is most clearly exhibited by using a paper model as shown in Fig. 17. Fig. 17a shows a flat plane where the subsequent edge lines of diamond patterns are marked. In other words, the quantities λ_x and λ_s are fixed and α is left unfixed. Give each triangular element a small angle θ to x axis; this is equivalent with giving a small amount of the amplitude α. This process results in the surface shown in Fig. 17b. Increasing θ, or α, we have in sequence the surfaces shown in Figs. 17c, 17d, and 17e.

From a theoretical point of view, this mechanism is quite different from those of accordion type or fan type deployment, because this is the deployment in two-dimensional while the latter two are one-dimensional deployments. Through two-dimensional deployment mechanism, it is theoretically possible to contract a plane to a point. Then, by patterns with infinitesimal wave lengths, the following conditions are perfectly realizable, that is

$$\varepsilon \longrightarrow 1 \tag{19}$$

$$\kappa \longrightarrow \infty \tag{20}$$

Viewing the models in Fig. 17, we could imagine the ultimate result of these conditions as an infinitesimal coil with the axial length of zero.

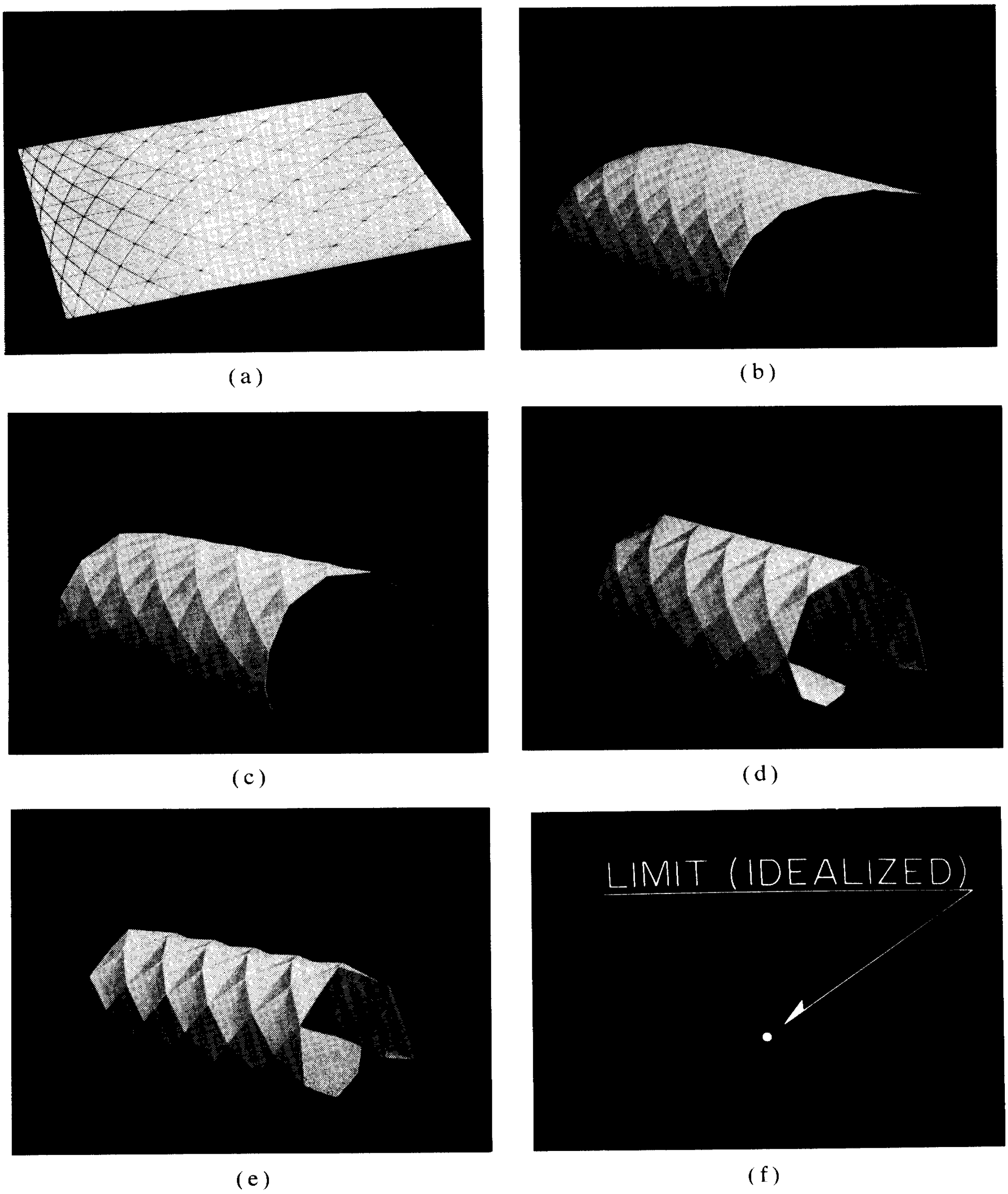

FIG. 17. Two-dimensional deployment mechanism by PCCP shell.

6. *Characteristic as bellows or expansion joints*

Another distinct characteristic of PCCP shell will be found, if we view the classical problem of axial buckling of cylindrical shells from a different standpoint. Now let us observe Fig. 15 again. In this figure the reduced compressive stress $\sigma r/Et$ against unit end shortening $\varepsilon r/t$ for $\mu=1.00$ and different number of waves in circumferential direction is plotted. Where $\eta=n^2t/r$ represents the non-dimensional circumferential wave number. What in particular catches our interest is not the domains of buckling and post-buckling as usual, but the region

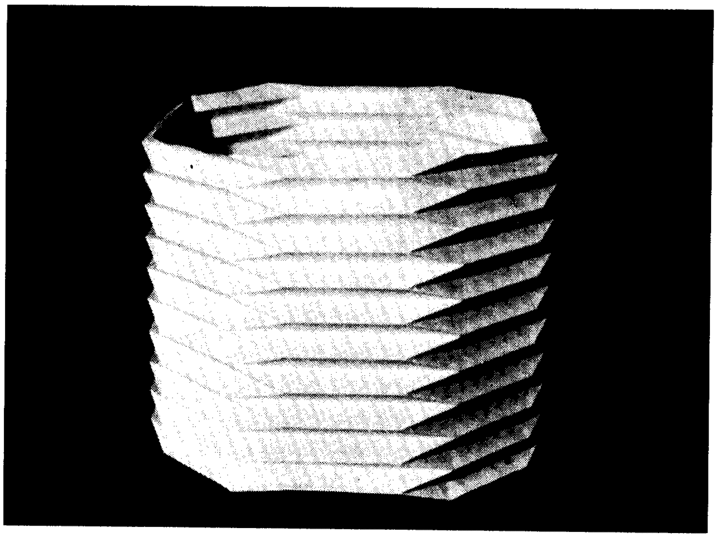

FIG. 18. Bellows-like PCCP shell.

following the post-buckling. In this region, the gradient of the curves are much smaller than in the pre-buckling region. In other words, the elastic spring constant of the shell in the axial direction is comparatively small. This tendency becomes more pronounced as the circumferential wave number decreases. Based on this fact, we may deduce that a shell designed at first in the form of post-buckling configuration should have a small spring constant in the axial direction. Indeed it can be likened to a bellows. Since the PCCP shell is in a sense the idealization of post-buckling configuration of cylindrical shells, it seems probable that this characteristic as bellows is also inheritted to the PCCP shell. Especially, the PCCP shell with large angle of inclination exhibits clearly a spring-like behavior against the axial force (Fig. 18). A noticeable feature of this bellows-like structure is that its midsurface is developable. It is also interesting to note that the bellows in usual form represents something like the axisymmetrical buckling deformation of circular cylindrical shells.

5. APPLICATIONS OF PCCP SHELLS

As for the application of the PCCP shell, some obvious uses of its features can easily be found. But, since the shell is versatile in its form as well as its characteristics, and since the practical application may involve their own particular conditions not known to us, it is difficult to sum up these applications systematically at this stage. So only a few typical examples of applications, whose merits are quite obvious, are shown in this paper.

1. Large span shell structure in architecture

The large span shell structure in architecture is in general used for the case where a large space without pillars is required. Large markets, stadiums, gymnasiums, and hangars are such examples. In Fig. 19 and Fig. 20, the catenary PCCP shells for that purpose are illustrated. There are two principal contrivances

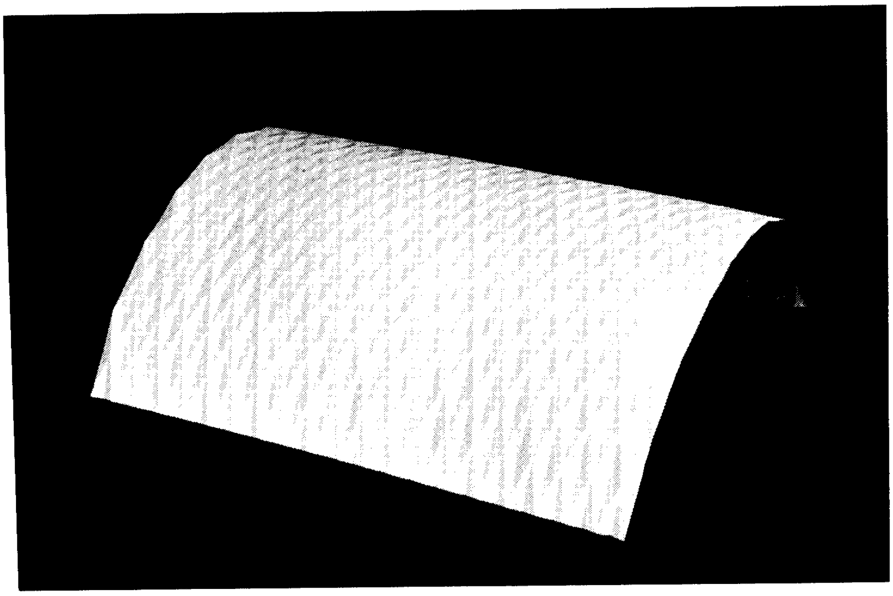

FIG. 19. Catenary PCCP shell as a large span structure (diamond pattern).

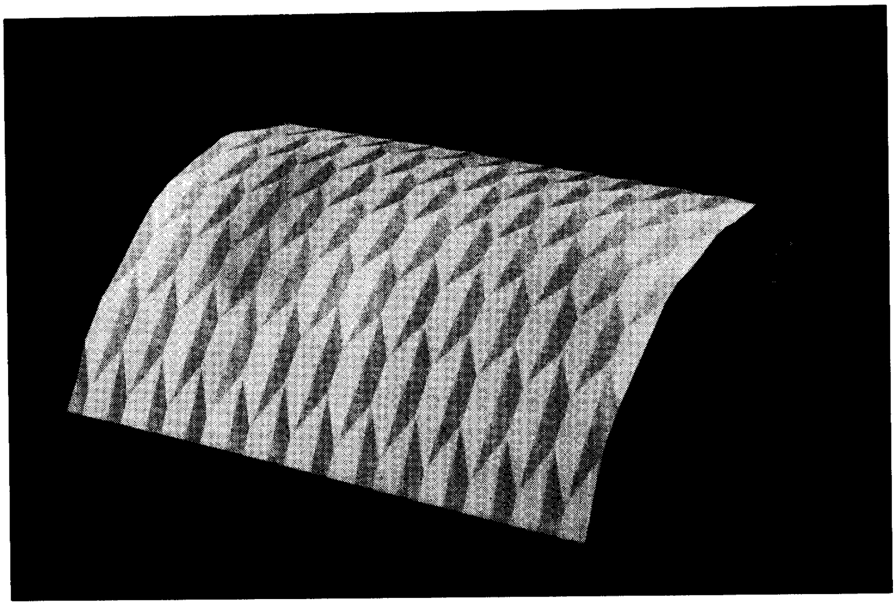

FIG. 20. Catenary PCCP shell as a large span structure (hexagonal pattern).

of designing these structures. First, the bending moments due to local disturbances, snow and wind loads can be coped with the superior circumferential bending rigidity of PCCP shells. Second, as the element of the shell is in the simplest form of all, either triangular or trapezoidal plane, the producibility of the shell is quite high. Depending on the design requirements, it is possible to design a PCCP shell with an arbitrary curvature distribution.

2. Reservoir

We can consider the model of Fig. 8b as illustrating an example of the circular cylindrical type reservoir by the PCCP shell with hexagonal pattern. It may possibly be made of either steel or reinforced concrete and may be used for a reservoir of powder or grain materials. The superior circumferential bending rigidity will show higher resistance to the bending moment due to local load normal to the shell surface; while the sufficient stiffness against axial loadings can be obtained by an appropriate selection of form parameters. It is, therefore, possible to design a reservoir without using the stringers. The simplicity of the elementary faces of the shell will largely contribute to the ease of its construction.

3. Sandwitch structure

A double layered shell made by overlapping of a cylindrical shell with a PCCP shell, both have an identical curvature distribution and are bonded together, is one possible application of PCCP shell concept.

4. Bellows or expansion joints

As shown in the chapter 4, the PCCP shell with large angle of inclination θ has a property as a bellows or a expansion joint. An important feature for the manufacturing process is that it can be formed from a flat sheet material through inextensional processes, while the usual type of bellows is made through quite a large amount of extensional deformation.

5. Deployable structure

It is clearly shown in section 5 of chapter 4.

6. CONCLUSION

It is shown that the inextensional post-buckling configurations of general cylindrical shells subjected to axial loading have peculiar geometrical characteristics, and that these configurations compose a general group of surfaces which may be designated as the pseudo-cylindrical concave polyhedral surface. Then the fixed idea that these surfaces are essentially failed forms is abandoned and is replaced by the idea that these are the basic forms of a new shell which could function superbly as the structure under some loading conditions. It is shown that the new shell, which may be called for convenience, the pseudo-cylindrical concave polyhedral shell and the PCCP shell for its abbreviation, has many useful characteristics as follows; inclusion of an arbitrary curvature distribution, developability of its midsurface, intrinsically high circumferential bending rigidity, and simplicity of elementary faces. The application of PCCP shells to large span structures, reservoirs, expansion joints, and others seems to be promising.

Department of Aerodynamics and Structures
Institute of Space and Aeronautical Science
University of Tokyo, Tokyo
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SYMBOLS

l, m = parallel lines

l^*, m^* = parallel planes

n = circumferential wave number

p, q = base and height of two-dimensional Euclidean simplex, respectively

r = radius of circular cylindrical shell

s = circumferential coordinate

$s' = g(s)$ = coordinate transfer

t = thickness of shell

w = radial inward displacement

x = axial coordinate

$x' = f(x)$ = coordinate transfer

$A_{00}, A_{11}, A_{20}, A_{02}$ = coefficients

E = Young's modulus

$F(p, q)$ = sum of area of two-dimensional Euclidean simplexes

I = moment of inertia of cylindrical shell in circumferential direction

I_p = moment of inertia of PCCP shell in circumferential direction

$K(p, q)$ = two-dimensional Euclidean complex

L, M = parallel zones

S = general cylindrical surface

S_p = pseudo-cylindrical concave polyhedral surface or PCCP surface

α = radial amplitude of concave polyhedral surface

β = angle

ε = unit shortening in axial direction

ζ, ξ = numbers

η = circumferential wave number parameter
θ = angle of inclination of elementary face to x axis
κ = curvature
$\langle\kappa_1\rangle$ = quasi-curvature of circumference, Eq. (2)
$\langle\kappa_2\rangle$ = quasi-curvature of circumference, Eq. (3)
λ_x = half wave-length of buckle in axial direction
λ_s = half wave-length of buckle in circumferential direction
λ_s, λ^s = bisections of the upper and the lower bases of trapezoids composing a hexagonal pattern, respectively
μ = aspect ratio of wave
ρ = radius
σ = average compressive stress in axial direction
s_j, s_{j+1} ($j = \cdots, k-2, k, k+2, \cdots$) = circumferential coordinate of nodes of concave polyhedral surface
1, 2, 3, $\cdots$ = nodes and semi-nodes of PCCP surface