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A Method of Minimum Weight Design with Requirements Imposed on Stresses and Natural Frequencies

By

Tetsuo, YASAKA

Summary: A numerical method is presented to obtain the minimum weight design of structures whose stresses caused by several alternative loads and the first few natural frequencies are required to be within certain limits. The requirements may include other types of eigen-values such as buckling loads. Thicknesses of panels and cross sectional areas of trusses are taken to be independent design variables, and geometrical configuration is assumed to be given. Finite element method is used for structural analysis. A simple procedure to obtain derivatives of eigen-values with respect to the variables is first shown. The objective function which is a sum of the weight and the penalty terms is minimized using the derivatives. Steepest descent method is modified to speed up convergence. The method is applied to obtain optimum thickness of a plate beam, a two storied truss and a simply supported beam.

INTRODUCTION

In today's aerospace applications reduction of structural weight is of great significance to attain effective and satisfactory mission. A systematic study of minimum weight design is necessary to attain the allowable lowest value of the structural weight, still satisfying the requirements imposed on the structure.

In meeting this need the loading index concept was fully developed by Shanley, Gerard and others [*1–3*], which has proved very useful in efficiency analysis and in efficient design of compression members. Although this concept must be greatly evaluated, it does not seem to present the sole and the final guide to designing.

The concept basically assumes that optimum proportions result when the possible forms of failure occur simultaneously. But it has been pointed out that this concept does not always lead to the most efficient design. Cohen discussed on the subject and showed an example of truss core sandwich cylinder whose minimum weight occurs when the critical stresses of two modes are not equal [*9*].

It is a concept essentially applicable to each structural members whose external loads are precisely known. In case of redundant structures, load to be carried by each member is a function of member proportioning itself, and therefore it is a quite difficult matter to optimize all members simultaneously. Even after that task has succeeded, there exists no guarantee of optimality of the whole structure.

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In the mean time introduction of high speed digital computers made matrix structural analysis and programming techniques very powerful tools for designers and they have been successfully incorporated into the problem [10–26]. Although present technique is limited within elastic range, fully automatic design is possible in case of some simple structures such as trusses under multiple loadings.

When weight is reduced and as a result stiffness of the structure is lowered, often dynamic properties beome critical to successful attainment of the mission. For examples, Turner [35] states that, "In spacecraft design it may be required that the fundamental cantilever bending frequency of a deployed appendage shall not fall below a specified minimum value in order to avoid undesirable control coupling effects." It is expected that a structure in orbit is subjected to rather small external loads, and that frequency criterion is of preference to that of strength. In aeroelastic applications frequency requirements plays a major role and a technique to obtain a minimum weight structure with specified frequencies is fundamental to optimize a structure satisfying flutter requirements [34]. Minimization of weight with consideration of vibration and flutter requirements has been discussed by Turner and others [35–38].

It seems that there exists a need to develop a method of minimum weight design with due consideration to static and dynamic behavior, since a design is often influenced by both. The difficulty exists in the fundamental difference of governing equations of those behaviors; one the simultaneous equations and the other eigen-value equations.

As the first step to this the author tried to develop a method to obtain minimum weight design where stresses caused by several alternative loadings are required to be within certain limits and lower boundaries of natural frequencies exist.

The structure is replaced by thin panels and trusses, which are analyzed by finite element method. Thicknesses of panels and cross sectional areas of trusses are taken to be independent design variables. The degree of influence of variations of each design variables on stresses and frequencies are first calculated, then the point of the lowest possible weight is searched using techniques of non-linear programming.

The method is first developed to deal with structures whose stiffnesses and weight are linearly related to design variables, but as the extension, treatment of those with non-linear relationship is briefly discussed. Feasibility of including buckling behavior into requirements is also discussed.

1. Foundations to Minimum Weight Design

1.1 Present Status

1.1.1 Minimum Weight Design

While there has been an effort to specifically determine member sizes in terms of the load and the basic dimension, e.g., loading index [1–8], there is an iterative scheme of minimum weight design under development which basically is design and redesign processes applied to the whole structure each time modifying in such

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a manner as to reduce the total weight and satisfying requirements. A design is represented by a point in design parameter space or by a vector whose components are design variables, and modification by movement of the point or the vector. Requirements become constraints and divide the space into acceptable region and the region of violation.

Schmit [*10*] was the first to present this concept which was applied to redundant trusses. In succeeding works [*11–14*] the concept was furthered through introduction of non-linear programming techniques into redesign cycles. For example the steepest descent method was used in Ref. 14 in analysis and synthesis of trusses. What was of particular notice in those works was the feasibility of optimizing a structure under multiple loading conditions. It was also shown that fully stressed design in which the stress in each member is equal to the allowable stress in at least one loading condition may be inefficient. Fully stressed design was studied in more detail by Razani [*29*], Dayaratnam et al. [*30*] and Kitcher [*31*].

Gellatly and Gallagher [*15,16*] used a procedure made up of three different modes: (i) Initial step, (ii) Steepest descent and (iii) Side-step. In the initial step a fully stressed design is obtained revising the design repeatedly. Then in the second step the weight of the total structure is reduced in the most rapid possible manner until limitations on stresses etc. are reached. The side step move from the constraint surface is directed toward the "optimum vector" which is determined using vectors normal to the weight plane and constraint plane.

Kicher [*17*] used gradient projection technique with modification for systematic synthesis of stiffened cylinders. In gradient projection technique a new design immediately violates constraints when the feasible region is convex which is commonly so, therefore the direction of move is tilted toward inside of the acceptable region. Gradients to yield and buckling constraints are obtained by finite difference scheme and perturbation of the design variables.

Among other significant investigations [*18–28*] usage of linear programming is of special attention. It is attractive from the view point of computer economy, but when the optimum proportion is not at the vertex at which the number of constraints active is the same as that of independent design variables, convergence seems to be unsatisfactory.

A desirable direction was pointed out by Moses et al. [*24*] who introduced probability concept into constraints.

The existing techniques were recently reviewed by Maruyasu et al. [*33*] and evaluated both qualitatively and quantitatively.

Weight minimization with vibrational behavior of the structure considered is a comparatively new problem. Niordson [*39*] has treated the cantilever beam with a solid cross section of constant shape. After Turner's presentation [*35*], Taylor [*36,37*] and Prager [*38*] showed the methods of optimum shape determination of a bar for axial vibration at specified or maximum frequency. Taylor made use of a function related to the energy of the system which was subjected to variation, while Prager used Rayleigh's quotient.

The method presented by Turner [*35*] shows feasibility of application to more

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complex structures utilizing numerical procedures. Basically a functional

$$\Phi[m, u]=\int_0^L \{m+\lambda g\}dx+\lambda_1 g_1 \tag{1.1.1}$$

is subjected to variation, where the first term in the integral shows mass distribution, and the second term in the integral and the last term correspond to vibrational requirements and boundary conditions respectively with Lagrange multipliers. After introducing lumped parameter system into the equation, a set of second order simultaneous equations for determination of modal displacements is derived.

Turner's succeeding work [*34*] put another important step toward aerospace vehicle designing with due consideration to flutter requirements. Based on the similar principle as used in the former the structure is optimized for a succession of values of the flutter frequency. The frequency and mass distribution for minimum total mass are then determined graphically. The technique of minimum mass design with specified natural frequencies was used determining the initial guess of the structure.

Lagrange multiplier function and variational principles are widely used in optimum shape determination for buckling properties of bars [*41–44*].

1.1.2 Finite Element Method

Matrix structural analysis became a very powerful tool for designers and especially finite element displacement method is widely admitted to be the most efficient to automated analysis due to its simplicity of formulating stiffness matrix [*45–48*].

Levy [*46*] introduced the idea of replacing a continuous structure by pieces. An aircraft wing was replaced by elementary beams and torque boxes generating a stiffness matrix for each element, and stiffnesses are summed up. Turner et al. [*50*] refined this application by reducing the torque box to assemblies of triangular or rectangular slices. This finite element idealization was extended by Clough [*51*], Klein [*52–55*], Melosh [*56–57*], Pian [*58*] and others.

The method is successfully used in stress analysis of complex structures [*59,60*]. Kapur et al. [*61*] and others [*62,63*] have shown that the method may be used to find critical buckling stresses of thin plates. Applications into dynamic problems are also successful. Vibration problems were solved first by Leckie and Lindberg [*64*] and Archer [*65*] followed by Dawe [*66*] and Guyan [*67*]. To dynamic stability [*68*] and panel flutter problems [*69*] it is also shown to be applicable. Techniques to avoid processing very large matrices have been presented both in static [*70*] and dynamic problems [*71–74*].

1.2 Structural Analysis Using Finite Element Method

1.2.1 Introduction

The solution of stress and strain distributions in elastic continua is obtained first by deriving stiffness matrices of finitely discretised elements which are interconnected at a dicrete number of nodal points on their boundaries. Element

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stiffnesses are assembled to yield the stiffness matrix of the whole structure, and the following procedures are straight forward.

In deriving element stiffness matrices a series of approximations are assumed. Firstly a function is chosen to define the state of displacement within each finite element in terms of its nodal displacements which will be the basic unknown parameters. It is not always easy to ensure displacement continuity between adjacent elements. Secondly forces acting on the element are concentrated or replaced by equivalent forces at its nodes, thus local violation of equilibrium usually arises. Equilibrium conditions are satisfied in the overall sense only.

Although care must be paid to the degree of accuracy, in most cases it is assured that as the number of elements is large and their sizes are sufficiently small, the solution is very close to the actual situation. Since it seems that there exist very few questions left to be discussed except those mentioned above, at least in case of one dimensional and plane stress problems used in the followings, we consider here the finite element method to be one well established.

Here notations by Zienkiewicz [45] are widely used.

1.2.2 Stiffness Matrix

A typical triangular element of a two dimensional continuum shown in Fig. 1.2.1 is considered, with nodes numbered anti-clockwise order.

A six components element displacement vector is defined

$$\{\delta\}^e=\{u_i, v_i, u_j, v_j, u_m, v_m\}$$

where { } symbolizes a column vector, and the superfix e signifies the quantities concerning the element. u_i, v_i are displacements of the node i in the direction of x and y respectively.

Assuming linear elastic relations, strains $\{\varepsilon\}=\{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}$ and stresses $\{\sigma\}=\{\sigma_x, \sigma_y, \tau_{xy}\}$ are defined in terms of element displacements, as

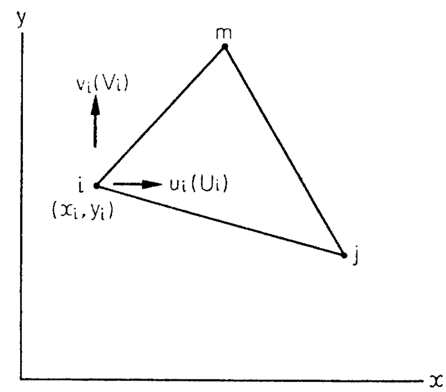

FIG. 1.2.1. Triangular Element.

$$\{\varepsilon\}=[B]\{\delta\}^e \tag{1.2.1}$$

$$\{\sigma\}=[D]\{\varepsilon\} \tag{1.2.2}$$

Element nodal forces $\{F\}^e=\{U_i, V_i, U_j, V_j, U_m, V_m\}$ are

$$\{F\}^e=[k]^e\{\delta\}^e \tag{1.2.3}$$

Here, $[k]^e$ is the element stiffness matrix, and

$$[k]^e=t\Delta A[B]^T[D][B]\{\delta\}^e \tag{1.2.4}$$

t and ΔA are the thickness and the area of the triangle, respectively. Both

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matrices $[B]$ and $[D]$ do not contain the thickness t.

The stiffness matrix of a truss element is also expressed by the similar form and in this case t and ΔA represent the cross sectional area and the length, respectively of the element.

1.2.3 Total Stiffness Matrix

The formation of the total stiffness matrix of the structure in two dimensional is straight forward.

First displacement of the structure is represented by $2N$-vector $\{u\}$ composed of displacements of all N nodes. Element nodal displacements and forces are expressed in terms of $\{u\}$:

$$\{\delta\}^e=[N]\{u\} \tag{1.2.5}$$

$$\{F\}^e=[k]^e[N]\{u\} \tag{1.2.6}$$

where $[N]$ is a $6\times 2N$ or $4\times 2N$ matrix depending on the type of the element and its components are either O or 1.

Then corresponding components of $\{F\}^e$ are added, or in matrix expression

$$\sum[N]^T\{F\}^e \tag{1.2.7}$$

which is equated to $2N$-vector of external loadings

$$\{F\}=\sum[N]^T\{F\}^e=(\sum[N]^T[k]^e[N])\{u\} \tag{1.2.8}$$

Stiffness matrix of the whole structure is

$$[K]=\sum[N]^T[k]^e[N] \tag{1.2.9}$$

When the structure is 3-dimensional composed by 2-dimensional elements, direction cosines of the local co-ordinate axis $0\text{–}x,y$ with respect to the global co-ordinates must be taken into consideration.

(1.2.5) and (1.2.6) are replaced by

$$\{\delta\}^e=[T][N]\{u\} \tag{1.2.10}$$

$$\{F\}^e=[k]^e[T][N]\{u\} \tag{1.2.11}$$

where $\{u\}$ is a $3N$-vector and $[N]$ is a $9\times 3N$ or $6\times 3N$ matrix. $[T]$ is a 6×9 or 4×6 matrix composed of direction cosines.

Before nodal forces are summed up and equated to the external loads they have to be transformed into global co-ordinates again.

$$\{F\}=\sum[N]^T[T]^T\{F\}^e=(\sum[N]^T[T]^T[k]^e[T][N])\{u\} \tag{1.2.12}$$

Stiffness matrix of the whole structure is

$$[K]=\sum[N]^T[T]^T[k]^e[T][N] \tag{1.2.13}$$

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Care must be taken when all the elements joining at a particular node are in one plane. Force normal to those elements is not originated, thus three global components of the force at the node are not independent. Due to the similar reason one of the displacement components of that particular node is not uniquely defined. Therefore it becomes necessary to eliminate the particular column and row from $[K]$.

1.2.4 Displacement and Stress

Stiffness matrix (1.2.9) or (1.2.13) generally is still singular since rigid body movements are permitted. Inserting boundary conditions, another new matrix is made. In the followings $[K]$ denotes this new matrix.

Thus

$$\{F\}=[K]\{u\} \tag{1.2.14}$$

which is solved and

$$\{u\}=[K]^{-1}\{F\} \tag{1.2.15}$$

Once $\{u\}$ is obtained, stresses in all elements are then obtained.

$$\{\sigma\}=[DBN]\{u\}=[DBN][K]^{-1}\{F\} \tag{1.2.16}$$

where

$$[DBN]=[D][B][N] \quad \text{or} \quad [D][B][T][N] \tag{1.2.17}$$

1.2.5 Substructure Analysis

Analysis of large complex structure with sufficient accuracy involves treatment of large matrices. There is a substructure concept which effectively avoids this situation [*70*].

The structure is divided into substructures by introducing interior boundaries, and boundary displacements for a substructure is denoted by $\{u_b\}$ and interior displacements by $\{u_i\}$. If the corresponding external forces are denoted by $\{F_b\}$ and $\{F_i\}$, then equation of equilibrium may be written in partitioned form as

$$\begin{bmatrix} K_{bb} & K_{bi} \\ K_{ib} & K_{ii} \end{bmatrix} \begin{Bmatrix} u_b \\ u_i \end{Bmatrix} = \begin{Bmatrix} F_b \\ F_i \end{Bmatrix} \tag{1.2.18}$$

It is now assumed that displacements may be calculated from the superposition of two vectors, $\{u^\alpha\}$ and $\{u^\beta\}$, where $\{u^\alpha\}$ denotes the displacements due to $\{F_i\}$ with $\{u_b\}=0$, while $\{u^\beta\}$ represents the necessary corrections to the displacements $\{u^\alpha\}$ to allow for boundary displacements $\{u_b\}$ with $\{F_i\}=0$. Thus,

$$\{u\}=\begin{Bmatrix} u_b \\ u_i \end{Bmatrix}=\begin{Bmatrix} 0 \\ u_i^\alpha \end{Bmatrix}+\begin{Bmatrix} u_b^\beta \\ u_i^\beta \end{Bmatrix} \tag{1.2.19}$$

Similarly, corresponding to the displacements thus defined, the external forces $\{F\}$

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can be separated into

$$\{F\}=\begin{Bmatrix}F_b\\F_i\end{Bmatrix}=\begin{Bmatrix}F_b^\alpha\\F_i^\alpha\end{Bmatrix}+\begin{Bmatrix}F_b^\beta\\0\end{Bmatrix} \tag{1.2.20}$$

When the structure boundaries are fixed, expanding (1.2.18),

$$\begin{aligned}&[K_{bi}]\{u_i^\alpha\}=\{F_b^\alpha\}\\&[K_{ii}]\{u_i^\alpha\}=\{F_i^\alpha\}=\{F_i\}\end{aligned} \tag{1.2.21}$$

Therefore

$$\{u_i^\alpha\}=[K_{ii}]^{-1}\{F_i\} \tag{1.2.22}$$

$$\{F_b^\alpha\}=[K_{bi}][K_{ii}]^{-1}\{F_i\}=\{R_b\} \tag{1.2.23}$$

When the boundary is then relaxed,

$$\begin{aligned}&[K_{bb}]\{u_b^\beta\}+[K_{bi}]\{u_i^\beta\}=\{F_b^\beta\}\\&[K_{ib}]\{u_b^\beta\}+[K_{ii}]\{u_i^\beta\}=0\end{aligned} \tag{1.2.24}$$

and

$$\{u_i^\beta\}=-[K_{ii}]^{-1}[K_{ib}]\{u_b^\beta\} \tag{1.2.25}$$

$$\{F_b^\beta\}=[K_b]\{u_b^\beta\} \tag{1.2.26}$$

Where

$$[K_b]=[K_{bb}]-[K_{bi}][K_{ii}]^{-1}[K_{ib}] \tag{1.2.27}$$

represents the boundary stiffness matrix.

Combining (1.2.20), (1.2.23) and (1.2.26), we obtain

$$[K_b]\{u_b^\beta\}=\{F_b\}-\{R_b\}=\{Q_b\} \tag{1.2.28}$$

which is equivalent to (1.2.3) if this substructure is considered to be a sort of an element. The following process is carried out with $\{u_b^\beta\}$ as the basic parameter. $\{u_b^\beta\}$ is generally much smaller in the number of components than $\{u\}$, therefore considerable computational advantage is derived. Behavior of the whole structure is then obtained by simple matrix algebra.

1.3 Vibration Analysis

1.3.1 Method of Analysis

Vibration analysis using finite element method is presented in Refs. 64–67, in which consistent mass matrix is used. Considerable accuracy has been reported in vibration analysis; for instance, the deviations of the bending frequencies of the first three modes of a three segmented beam from the exact are 0.024, 0.373 and 1.20 per cent [*65*] and those of nine-element cantilever plate are 0.571, 2.3 and 1.1 per cent [*66*].

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In the present analysis as shown in 1.2, bending term is not included for simplicity. Bending effect is obtained in overall sense only, i.e., by tesion-compression of plates and trusses. Therefore to be consistent with this situation, mass is lumped at certain convenient nodes. In the author's previous work [*79*] in which Levy type stiffness matrix [*49*] and simple lumped mass system were incorporated, acceptable accuracy in frequencies was obtained. A low aspect ratio wing was analysed and the agreement of the results with experiment was within a few per cent deviation.

Equation of dynamic equilibrium when damping is neglected is written,

$$[M]\{\ddot{u}\}+[K]\{u\}=0 \tag{1.3.1}$$

where $[M]$ is a diagonal mass matrix, and $\{\ddot{\ }\}$ represents time derivative, $\dfrac{d^2}{d\tau^2}\{\ \}$.

Assuming harmonic motion

$$\{u\}=e^{i\omega\tau}\{\bar{u}\}$$

$$-\omega^2[M]\{\bar{u}\}+[K]\{\bar{u}\}=0 \tag{1.3.2}$$

Eigen-value problems thus defined may be effectively solved by iterative procedure. In the present work, so called power method is used through-out to solve (1.3.2). This method is known to be particularly effective when only the first few modes of vibration are concerned.

1.4 Optimum Design: Non-Linear Programming

1.4.1 "Is it optimum?"

A vast amount of information will be needed to answer the question, "Is it optimum?" You must consider many aspects of it: costs and efficiency in manufacturing and usage, its strength, durability, etc., etc. Among those the most benefitial or the most desirable is chosen, and is called the objective, the others called constraints [*33*]. In our problem the objective is weight of the structure, while constraints are static and dynamic behaviours. The objective f_1 which is to be minimized is a function of design variables $\{t\}=\{t_1, \cdots, t_n\}$, and behavior functions $g_1(\{t\})$ which are commonly highly non-linear must meet certain requirements which guarantee satisfactory operational activities of the structure.

The traditional statement of the non-linear problem by Wolfe [*75*] is:

"Given the continuous functions $g_1, g_2, \cdots, g_m$ and f_1, it is required to find the point $\{t\}$ which minimizes f_1 under the constraints $g_i \leqq 0$, $i=1, 2, \cdots, m$."

1.4.2 Convexity [*75*]

It has been stated that an answer obtained is global if f_1 is convex and the set S of $\{t\}$ satisfying the constraints $g_i \leqq 0$ is also convex. Otherwise there is a possibility that it is only a local minimum and true minimum or optimum exists somewhere else.

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Convexity of a set S is defined as follows:

$$\{t\}=\theta\{t\}_1+\bar{\theta}\{t\}_2\in S \quad \text{for all} \quad \{t\}_1, \{t\}_2\in S$$

and

$$0\leqq\theta\leqq 1, \quad \bar{\theta}=1-\theta$$

which means that any point on a line segment between $\{t\}_1$ and $\{t\}_2$ belongs to S whenever $\{t\}_1$ and $\{t\}_2$ do.

Definition of convexity of a function f_1 may be stated in a number of ways. For instance:

$$f_1(\theta\{t\}_1+\bar{\theta}\{t\}_2)\leqq\theta f_1(\{t\}_1)+\bar{\theta}f_1(\{t\}_2) \tag{1.4.1}$$

for all

$$\{t\}_1, \{t\}_2\in S \quad \text{and} \quad 0\leqq\theta\leqq 1, \bar{\theta}=1-\theta$$

In our case f_1 is confined to be a linear function of $\{t\}$, therefore it is convex.

In general it is quite difficult to prove convexity of S. Fortunately it is known to be generally convex at least locally [*15*, *17*] and in the followings convexity of S is assumed.

1.4.3 Unconstrained Optimization: Method of Steepest Descent [*76*]

Although unconstrained problems are not common, it is useful clarifying the techniques of non-linear programming, and also as is seen in Reference 14 by Schmit, et al., an inequality constrained minimization problem is transformed into one unconstrained. In the present section the objective f_1 is considered to be a non-linear function of $\{t\}$.

Initially $\{t\}^0=\{t_1^0, t_2^0, \cdots, t_n^0\}$ is selected. A new position $\{t\}^1$ which makes f_1 smaller most rapidly is

$$\{t\}^1=\{t\}^0-\delta\,\mathrm{grad}\,f_1 \tag{1.4.2}$$

where δ is a positive scalar, and grad f_1 is a column vector. δ is determined after several trial and error processes in such a way that it makes f_1 smallest. Denoting the value to be δ_m, the new point is

$$\{t\}^1=\{t\}^0-\delta_m\,\mathrm{grad}\,f_1 \tag{1.4.3}$$

The cycle is continued until any larger value of δ than some prescribed valuc does not make f_1 smaller.

In practical applications it tends to make a zigzag toward the true minimum. The higher the non-linearity is the more number of zigzags is necessary and convergence is slower. There are several methods to improve convergence.

1) Subrelaxation

A constant θ, $0<\theta<1$ is introduced and the step is

$$\{t\}^k=\{t\}^{k-1}-\theta\delta_m\,\mathrm{grad}\,f_1 \tag{1.4.4}$$

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2) Diagonal Step

After several trials and $\{t\}^k$, $k=1, 2, \cdots k$ are determined, the next step is not along the gradient of f_1, but is toward the third side of the triangle $\{t\}^{k-2}$, $\{t\}^{k-1}$ and $\{t\}^k$.

$$\{t\}^{k+1}=\{t\}^k+\delta_m(\{t\}^k-\{t\}^{k-2}) \tag{1.4.5}$$

3) Parallel Tangents

Based on the same principle as that of the former, the idea is more expanded.

$$\{t\}^{2k}=\{t\}^{2k-1}-\delta_m \operatorname{grad} f_1$$
$$\{t\}^{2k+1}=\{t\}^{2k}+\delta_m(\{t\}^{2k}-\{t\}^{2k-2}) \quad (1.4.6)$$

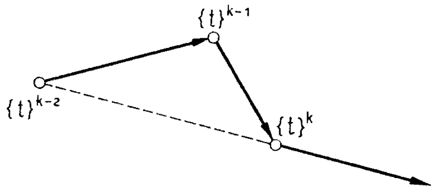

FIG. 1.4.1. Diagonal Step.

Application of diagonal step to structural analysis is seen in Ref. 77.

1.4.4 Constrained Optimization

When the acceptable region of $\{t\}$ is bounded by constraints $g_i \leqq 0$, $(i=1, \cdots, n)$ the problem becomes extremely complicated. The degree of difficulty mainly depends on the qualities of g_i and when all g_i are linear as well as the objective function f_1, the problem falls into the realm of linear programming which can be solved very effectively using Simplex Method. Some of the existing methods of non-linear programming such as Cutting Plane Method and Reduced Gradient Method are based on the same principle as that of the Simplex Method. In those methods non-linear constraints are approximated by successive sets of linear constraints. It is widely admitted that the use of the Simplex Method saves computational time compared to other methods outlined in the present section, but it requires large computer memory capacity, which makes it intolerable for today's computer to handle problems with many constraints and variables.

Application of the Steepest Descent Method:

There is an idea of using the constraints of the problem to modify its objective, rendering a constrained optimization problem into an unconstrained. The modification of the objective is to be such that, for points satisfying the constraints, the original objective is changed only little while for points outside the acceptable region the modified objective has very large values.

Consider the modified function f proposed by Wolfe [75]

$$f=f_1+\sum_{i=1}^{m} h_i(\{t\})g_i(\{t\}) \tag{1.4.6}$$

where

$$h_i(\{t\})=\begin{cases}0 & \text{if} \quad g_i(\{t\})\leqq 0\\ K & \text{if} \quad g_i(\{t\})>0\end{cases}$$

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and K is a suitable large positive number. f is sought to minimize, using the steepest descent method, in which

$$\text{grad}\, f = \text{grad}\, f_1 + \sum_{i=1}^{m} h_i \,\text{grad}\, g_i \tag{1.4.7}$$

Since grad g_i is normal to the boundary surface $g_i=0$ and points away from the acceptable region S the effect of the latter term is to kick the point $\{t\}$ back into S if it tends to leave it under the influence of the first term.

Schmit and Fox [14] presented as the "penalty functions"

$$\sum \langle g_i \rangle^2 \tag{1.4.8}$$

where the value of the function with angular bracket is,

$$\langle z\rangle^0 = \begin{cases} 0 & z \leqq 0 \\ 1 & z > 0 \end{cases}$$
$$\langle z\rangle^n = z^n \langle z\rangle^0 \tag{1.4.9}$$

Introduction of the penalty function in this way does not affect the continuity of f or its first derivatives, thus insuring smooth convergence.

Gradient Projection Method:

In the methods presented previously it is unavoidable for the point to go back and forth across the actual boundary $g_i=0$. In this method once the point $\{t\}$ reaches at a boundary of a constraint, it travels along it until another new constraint becomes active. Then the new direction of travel will be along the intersection of the two constraint surfaces.

Minimizing process using this method is realized in projecting the vector $-\text{grad}\, f_1$, onto the boundary surface to determine the direction of the next move $-\{p\}$. Since boundary surfaces are generally curved, the gradient vector is projected on local tangent planes.

As can be seen in the Fig. 1.4.2,

$$\{p\} = \text{grad}\, f_1 + \alpha \,\text{grad}\, g_i \tag{1.4.10}$$

Since $\{p\}$ is perpendicular to grad g_i

$$(\text{grad}\, g_i)^T\{p\} = 0 = (\text{grad}\, g_i)^T \text{grad}\, f_1 + \alpha (\text{grad}\, g_i)^T \text{grad}\, g_i \tag{1.4.11}$$

Thus,

$$\alpha = -(\text{grad}\, g_i)^T \text{grad}\, f_1 / (\text{grad}\, g_i)^T \text{grad}\, g_i \tag{1.4.12}$$

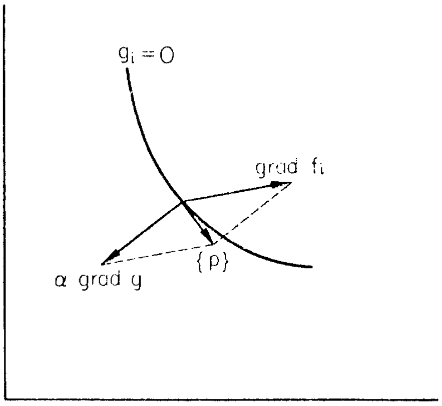

FIG. 1.4.2. Desirable Direction.

When more than one constraint, $g_i, (i=1, 2, \cdots, m)$ become active, the same principles

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are used.

$$\{p\}=\text{grad}\, f_1+\sum\alpha_i\, \text{grad}\, g_i \tag{1.4.13}$$

Let $[G]$ be the m by n matrix consisting of the corresponding m rows of grad g_i

$$\{p\}=\text{grad}\, f_1+[G]^T\{\alpha\} \tag{1.4.14}$$

Supposing all grad g_i are linearly independent, $[G][G]^T$ is nonsingular. A vector $\{\alpha\}$ consisting of α_i and $\{p\}$ are determined.

$$[G]\{p\}=[G]\,\text{grad}\, f_1+[G][G]^T\{\alpha\} \tag{1.4.15}$$

$$\{\alpha\}=-([G][G]^T)^{-1}[G]\,\text{grad}\, f_1 \tag{1.4.16}$$

$$\{p\}=\text{grad}\, f_1-[G]^T([G][G]^T)^{-1}[G]\,\text{grad}\, f_1 \tag{1.4.17}$$

Although this technique speeds up convergence considerably, it is apt to lead the point to vertex of n constraint surfaces which some time is not the optimal point. At the point of convergence, it is always necessary to be testified against Kuhn-Tucker condition for optimality [*32*].

Rosen [*81*] derived a conclusion which is equivalent to the Kuhn-Tucker theorem, that if

$$\{p\}=0 \tag{1.4.18}$$

$$\{\alpha\}>0 \tag{1.4.19}$$

are satisfied at a point $\{t\}$, it offeres the global minimum of f_1 within the acceptable region. If $\alpha_i<0$, the corresponding constraint g_i is dropped from the formation of $[G]$ and the process continues.

2. Minimum Weight Design

2.1 General Description

2.1.1 Object and Assumptions

The present chapter proposes a general method to obtain the minimum weight design of structures built up of thin panels and trusses.

It is required that both static and vibrational behaviors are considered simultaneously. In considering static behaviors the structure must be safe against several alternative loading conditions.

Assumptions made on the structure are:

1) Geometrical configurations and materials are already given and design variables are thicknesses of panels and cross sectional areas of trusses.

2) The structure is composed of several sections of panels or trusses and each section contains only one design variable holding thickness or cross sectional area constant within the section.

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3) Bending stiffness of panels and trusses are either negligible or not considered.

4) The structure is within elastic limit and initial stresses and thermal effects are not present.

Requirements which may be imposed on the structural behaviors are:

1) Stresses must satisfy certain requirements.
2) The first few natural frequencies must satisfy certain requirements.
3) Design variables must have values between some prescribed realistic limits.

2.1.2 Structural Idealization

Consider for example a box beam shown in Fig. 2.1.1. It is considered to be composed of two panel sections and a truss section which are shown in Fig. 2.1.2. Thickness or cross sectional area of each section is constant, which is the variable to be determined.

Each section is again divided into an appropriate number of finite elements shown in Fig. 2.1.3 in order to obtain stiffness matrices and stresses using the method outlined in 1.2.

Due to the properties of the elements and variables employed, both stiffness and mass matrices are expressed as

$$[K]=t_1[K_1]+t_2[K_2]+\cdots+t_n[K_n]+[K_0] \tag{2.1.1}$$

$$[M]=t_1[M_1]+t_2[M_2]+\cdots+t_n[M_n]+[M_0] \tag{2.1.2}$$

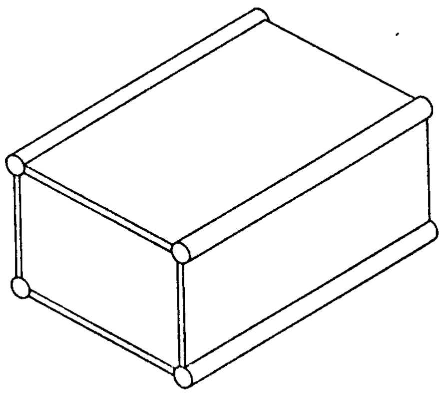

FIG. 2.1.1. Box Beam.

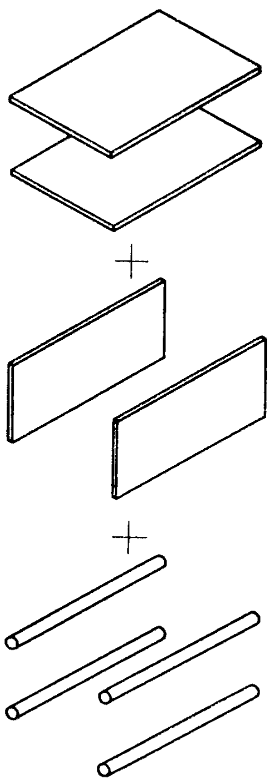

FIG. 2.1.2. Idealized Sections of the Structure.

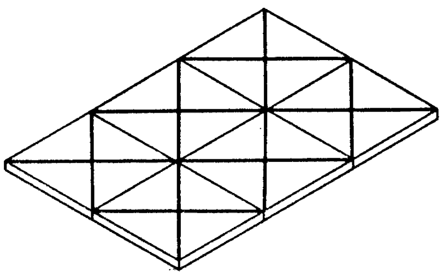

FIG. 2.1.3. Finite Elements.

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where $t_1, t_2, \cdots, t_n$ are design variables, and $[K_0]$ and $[M_0]$ are stiffness and mass matrices of the non-structural or predetermined members.

Substructure method is applied to panel sections for computational economy. Displacements of nodes common to other sections are taken to be $\{u_b\}$ defined in *1.2.5*. Because of linearity of the stiffness with respect to the thickness of the panel, reaction $\{R_b\}$ is independent of the variable and the boundary stiffness matrix is linear of the variable. General form of the stiffness matrix (2.1.1) still holds validity. Mass is generally lumped at those boundary nodes.

2.2 Mathematical Model

2.2.1 Mathematical Presentation of the Problem

The given problem is to minimize weight of the structure, while requirements on stresses and frequencies are satisfied. We now know that the problem can be stated as one of the non-linear programming whose presentation is in *1.4.1*, where f_1 represents weight of the structure and g_i requirements imposed.

2.2.2 Objective Function

The objective function f_1 is a linear combination of design variables $t_1, t_2, \cdots, t_n$ which are components of a vector

$$\{t\}=\{t_1, t_2, \cdots, t_n\}$$

f_1 is defined as

$$f_1=\sum_{i=1}^{n} A_i t_i=\{A\}^T\{t\} \tag{2.2.1}$$

where A_i is weight of the i-th section when t_i is unity. If some non-structural or predetermined members are incorporated, f_1 does not give actual weight of the structure, but still f_1 defined by Eq. (2.2.1) holds validity as the objective.

2.2.3 Constraints

Constraints may be formulated in various forms depending on the type of requirements they represent.

Requirements on stresses may be given in two-folds; from view points of elastic limit and stability. To be elastic, the following must be satisfied,

$$\begin{aligned} &(\sigma_x^2+\sigma_y^2-\sigma_x\sigma_y+3\tau_{xy}^2)^{1/2}-\sigma_Y\leqq 0 \text{ (plane stress)} \\ &\sigma-\sigma_y\leqq 0 \qquad\qquad\qquad\qquad\qquad \text{(truss)} \end{aligned} \tag{2.2.2}$$

where σ_Y is yield stress and σ_x, σ_y etc. are stress components calculated following the procedure in 1.2. The relations must be held for all finite elements and for all loading conditions.

To avoid elastic buckling

$$\sigma_k-\sigma_{cr,k}\leqq 0 \tag{2.2.3}$$

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must be held for appropriate stress component of some elements σ_k and for all loading conditions. $\sigma_{cr,k}$ is a critical stress of the k-th buckling mode considered and may be a function of design variables.

Frequency of the k-th vibration mode ω_k must satisfy

$$\omega_{cr,k} - \omega_k \leqq 0 \tag{2.2.4}$$

where $\omega_{cr,k}$ is a prescribed critical frequency.

Design variable t_i may have upper and lower boundaries, and

$$\begin{aligned} t_i - t_{U,i} &\leqq 0 \\ t_{L,i} - t_i &\leqq 0 \end{aligned} \tag{2.2.5}$$

where $t_{U,i}$ and $t_{L,i}$ are upper and lower boundaries respectively of the variable t_i.

2.3 Derivatives

2.3.1 "Why Derivatives?"

In applying programming techniques we have seen that derivation of gradients of the objective and constraints is necessary. Apart from that immediate need, recognition of the role of derivatives of stresses and frequencies with respect to design variables is important in designing structures especially indeterminate. There is a "hybrid action" defined by Cross [*78*] as "structural action in which two or more parts participate in carrying loads to such an extent that if the strength of one part is changed the forces acting on other part are largely affected." If this action is profound, one member cannot be designed without due consideration for its effect on other members. Design and redesign process in a usual way in which an over stressed member is increased in strength and so on will be greatly simplified and be made systematic if gradients are found and incorporated in. This is true in dynamic behaviors as well. Also when the mission is changed a little or a member is necessitated to alter a little, evaluation of the behavior is straight forward with the knowledge of derivatives.

One way of obtaining those derivatives is a finite difference approach. Early works of Schmit et al. used this approach to find out tangent planes of constraints. Generally speaking it would involve a quite tedeous computation, since basically it is a series of complete reanalyses and considerable accuracy is demanded in each computation.

Direct method of deriving derivatives is shown in the following sections.

2.3.2 Derivatives of Stresses

Evaluation of derivatives of stresses with respect to design variables is widely done by those discussing structural optimization [*15,19*]. This essentially is to find out the rate of change of solution of simultaneous equations with respect to parameters included in coefficients. Or it may be reduced to the problem of obtaining derivatives of an inversed matrix.

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Suppose that a non-singular square matrix $[K]$ includes a parameter t_i. When t_i is increased by a small amount Δt_i, $[K]$ and its inverse $[K]^{-1}$ will be altered by small quantities $\Delta[K]$ and $\Delta[K]^{-1}$ respectively. Thus

$$([K]+\Delta[K])([K]^{-1}+\Delta[K]^{-1})=[I] \tag{2.3.1}$$

where $[I]$ is a unit matrix. Expanding and substituting the relation $[K][K]^{-1}=[I]$, (2.3.1) becomes after neglecting second-order term,

$$\begin{aligned} &\Delta[K][K]^{-1}+[K]\Delta[K]^{-1}=0 \\ \text{or} \quad &\Delta[K]^{-1}=-[K]^{-1}\Delta[K][K]^{-1} \end{aligned} \tag{2.3.2}$$

Reducing Δt to an infinitesimally small quantity, we obtain the relation

$$\frac{\partial[K]^{-1}}{\partial t_i}=-[K]^{-1}\frac{\partial[K]}{\partial t_i}[K]^{-1} \tag{2.3.3}$$

Stresses are calculated by (1.2.16) and (1.2.15).

$$\{\sigma\}=[DBN]\{u\} \tag{1.2.16}$$

$$\{u\}=[K]^{-1}\{F\} \tag{1.2.15}$$

Design variables are found in $[K]^{-1}$ only, so differentiating (1.2.15) we have

$$\begin{aligned} \frac{\partial}{\partial t_i}\{u\}&=\frac{\partial}{\partial t_i}[K]^{-1}\{F\} \\ &=-[K]^{-1}\frac{\partial}{\partial t_i}[K][K]^{-1}\{F\} \\ &=-[K]^{-1}\frac{\partial}{\partial t_i}[K]\{u\} \end{aligned} \tag{2.3.4}$$

and

$$\begin{aligned} \frac{\partial}{\partial t_i}\{\sigma\}&=[DBN]\frac{\partial}{\partial t_i}\{u\} \\ &=-[DBN][K]^{-1}\frac{\partial}{\partial t_i}[K]\{u\} \end{aligned} \tag{2.3.5}$$

Since stiffness matrix $[K]$ is expressed as a linear combination of t_i in (2.1.1), finally we have

$$\frac{\partial}{\partial t_i}\{u\}=-[K]^{-1}[K_i][K]^{-1}\{F\}=-[K]^{-1}[K_i]\{u\} \tag{2.3.6}$$

$$\frac{\partial}{\partial t_i}\{\sigma\}=-[DBN][K]^{-1}[K_i]\{u\} \tag{2.3.7}$$

These relations are equivalent to those derived by Gellatly et al. [*15*] and Romstad et al. [*19*].

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It should be noted as mentioned by Gellatly that, "this method is particularly attrative since it only requires the matrix multiplications" of (2.3.6) and (2.3.7) and "$[K]^{-1}$ and $\{u\}$ are calculated only once". It should be also mentioned that the similarity of (1.2.16) and (2.3.7) allows the usage of the same computation coding to obtain as that used to obtain $\{\sigma\}$ by only inserting a vector $-[K_i]\{u\}$ instead of the last vector of the external force $\{F\}$ in (1.2.16).

2.3.3 *Derivatives of Frequencies*

It has been an uncommon practice in structural optimization to use derivatives of frequencies discussing vibrational behaviors; rather the method incorporating Lagrange multipliers has been extensively used by Turner [35], Prager et al. [38] and Taylor [36,37]. Though this method may be efficient dealing with simple columns or beams as has been applied, ingenuity would be needed for it to be applied to more complex structures.

A numerical method is developed to obtain derivatives of frequencies, which may be effectively incorporated into the existing methods of structural optimization.

Frequencies are determined by the eigen-value problem (1.3.2) of order N, or

$$\begin{gathered}[K]\{u\}=\lambda[M]\{u\}\\ \lambda=\omega^2\end{gathered} \tag{2.3.8}$$

where $[K]$ and $[M]$ are expressed as

$$[K]=\sum t_i[K_i] \tag{2.1.1}$$

$$[M]=\sum t_i[M_i] \tag{2.1.2}$$

in terms of design parameters t_i.

Following the similar procedure as in *2.3.2*, a parameter t_i is altered by a small amount Δt_i.

Stiffness and mass matrices $[K]$ and $[M]$ of the modified structure become

$$[K']=[K]+\Delta[K] \tag{2.3.9}$$

$$[M']=[M]+\Delta[M] \tag{2.3.10}$$

The eigen-value and the eigen-vector of the new system are consequently altered by small quantities

$$\lambda'=\lambda+\Delta\lambda \tag{2.3.11}$$

$$\{u'\}=\{u\}+\Delta\{u\} \tag{2.3.12}$$

(2.3.8) may now be written as

$$[K']\{u'\}=\lambda'[M']\{u'\}$$

or

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$$
\begin{aligned}
&[K]\{u\}+\Delta[K]\{u\}+[K]\Delta\{u\}+\Delta[K]\Delta\{u\} \\
&\quad =\lambda[M]\{u\}+\Delta\lambda[M]\{u\}+\lambda\Delta[M]\{u\}+\lambda[M]\Delta\{u\} \\
&\qquad +\Delta\lambda\Delta[M]\{u\}+\Delta\lambda[M]\Delta\{u\}+\lambda\Delta[M]\Delta\{u\}+\Delta\lambda\Delta[M]\Delta\{u\}
\end{aligned} \tag{2.3.13}
$$

Substituting (2.3.8) and neglecting higher-order terms (2.3.13) becomes

$$
\Delta[K]\{u\}+[K]\Delta\{u\}=\Delta\lambda[M]\{u\}+\lambda\Delta[M]\{u\}+\lambda[M]\Delta\{u\} \tag{2.3.14}
$$

or dividing it by Δt_i,

$$
[P]\frac{\Delta\{u\}}{\Delta t_i}-\frac{\Delta\lambda}{\Delta t_i}\{Q\}=\{R\} \tag{2.3.15}
$$

and

$$
\begin{aligned}
&[P]=[K]-\lambda[M] \\
&\{Q\}=[M]\{u\} \\
&\{R\}=(-\Delta[K]+\lambda\Delta[M])\{u\}/\Delta t_i
\end{aligned} \tag{2.3.16}
$$

The number of unknowns appearing in (2.3.15) is $N+1$, i.e., N components of $\Delta\{u\}$ and $\Delta\lambda$, while there are N equations contained.

It is always possible to eliminate one component of $\Delta\{u\}$, since essentially an eigen-vector $\{u\}$, hence $\{u\}+\Delta\{u\}$ is subject to an appropriate way of normalization. The simplest of all would be to set one component permanently to unity. Supposing the r-th component u_r is set to unity, the r-th component of $\Delta\{u\}$ must vanish in order that $\{u'\}$ is also normalized in the same manner as $\{u\}$.

Now we form a new unknown vector $\{X\}$ of the order N which is composed of $\Delta\{u\}$ with its r-th component replaced by $\Delta\lambda$. Corresponding coefficient matrix $[P']$ is basically $[P]$ with its r-th column replaced by $-\{Q\}$. Then (2.3.15) becomes

$$
[P']\{X\}=\{R\} \tag{2.3.17}
$$

where

$$
\{X\}=\{\Delta u_1, \Delta u_2, \cdots, \Delta u_{r-1}, \Delta\lambda, \Delta u_{r+1}, \cdots\}/\Delta t_i
$$

and

$$
[P']=[P_1, P_2, \cdots, P_{r-1}, -Q, P_{r+1}, \cdots]
$$

P_i representing i-th column of $[P]$. Although $[P]=[K]-\lambda[M]$ is singular with the rank $N-1$ in most vibrational problems, $[P']$ is generally not singular since one column is replaced by a completely new vector. Then,

$$
\{X\}=[P']^{-1}\{R\} \tag{2.3.18}
$$

When Δt_i tends to infinitesimally small quantity dt_i,

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$$\Delta[K]=\frac{\partial}{\partial t_i}[K]dt_i=[K_i]dt_i \tag{2.3.19}$$

$$\Delta[M]=\frac{\partial}{\partial t_i}[M]dt_i=[M_i]dt_i \tag{2.3.20}$$

$\{X\}$ and $\{R\}$ in (2.3.18) become with relations (2.3.16) taken into consideration

$$\{X\}=\left\{\frac{\partial u_1}{\partial t_i},\frac{\partial u_2}{\partial t_i},\ldots,\frac{\partial u_{r-1}}{\partial t_i},\frac{\partial \lambda}{\partial t_i},\frac{\partial u_{r+1}}{\partial t_i},\ldots\right\} \tag{2.3.21}$$

$$\{R\}=(-[K_i]+\lambda[M_i])\{u\} \tag{2.3.22}$$

The r-th component of $\{X\}$ is the derivative of the eigen-value.

Care must be payed to the choice of r which is the number of component equated to unity. If $|u_r|$ is small compared to the absolute values of other components of $\{u\}$, $\Delta\{u\}$ would be large and susceptive of computational error. Therefore it would be safe always to choose the component of the largest absolute value as one to be equated to unity.

2.3.4 Extension into Non-Linear Relationships

In the preceding sections it has been assumed that linear relationships are held between design variables and stiffness and mass matrices. These assumptions are not essential in deriving derivatives of either stresses or frequencies.

When the linear relationships (2.1.1) and (2.1.2) are not assumed, it is clear that

$$\frac{\partial}{\partial t_i}\{\sigma\}=-[DBN][K]^{-1}\frac{\partial}{\partial t_i}[K]\{u\} \tag{2.3.5}$$

still holds validity to obtain derivatives of stresses. If the stiffness matrix $[K]$ is differenciable with respect to t_i, evaluation of (2.3.5) is straight forward. When derivatives of frequencies are concerned, (2.3.22) must be replaced by

$$\{R\}=\left(-\frac{\partial}{\partial t_i}[K]+\lambda\frac{\partial}{\partial t_i}[M]\right)\{u\} \tag{2.3.23}$$

and

$$\{X\}=[P']^{-1}\{R\} \tag{2.3.18}$$

is now valid, where $\{X\}$ and $[P']$ are unchanged.

2.4 Minimum Weight Design by Steepest Descent Method

2.4.1 A New Objective

Based on the concept shown in the earlier part of *1.4.4*, constraints are incorporated into the objective function in such a way that they make the objective very

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large when they are not satisfied, while they do not affect it much when they are satisfied. Among various types of penalty terms, one proposed by Schmit and Fox [*14*] which may be stated as

$$\langle g_i \rangle^2$$

where

$$\langle g_i \rangle^2 = \begin{cases} 0 & \text{when} \quad g_i \leqq 0 \\ g_i^2 & \text{when} \quad g_i > 0 \end{cases}$$

is chosen from the stand point of smoothness of the function.

The new function becomes

$$\begin{aligned} f &= f_1 + C \sum_{i=1}^{M} \langle g_i \rangle^2 \\ &= \sum_{i=1}^{n} A_i t_i + C \sum_{i=1}^{M} \langle g_i \rangle^2 \end{aligned} \qquad (2.4.1)$$

where C is an appropriately chosen positive constant and M is the number of constraints.

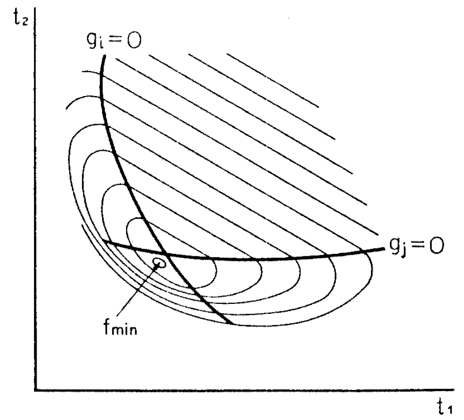

FIG. 2.4.1. Contour Lines of Fully Stressed.

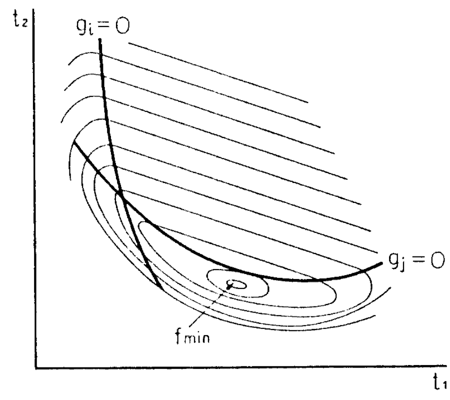

FIG. 2.4.2. Contour Lines of Non-Fully stressed.

Contour lines and minimum of f of two parameters are shown in Figs. 2.4.1 and 2.4.2. Fig. 2.4.1 indicates the case in which the fully stressed design is the optimum, and Fig. 2.4.2 indicates the case in which it is not the optimum. In either case steepest descent process will lead to the point of $f_{\min}$ which is near the optimum. It should be noted however that the point of $f_{\min}$ lies always in the region of violation. But it is expected that when the constant C is increased the point moves toward the boundary until finally the violations of constraints become well within the tolerable limit.

2.4.2 Steepest Descent Move

First of all an arbitrary point $\{t\}^1 = \{t_1^1, t_2^1, \cdots, t_n^1\}$ is chosen in n-dimensional design parameter space and the value of f is evaluated. Then the design point

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is moved toward the direction of the steepest descent, $-\mathrm{grad}\, f$ and the new point will be

$$\{t\}^2=\{t\}^1-\delta\,\mathrm{grad}\, f \qquad (2.4.2)$$

where δ determines the distance of travel. It is a general practice to determine δ such that the value of $f(\{t\}^2)$ is minimum, which will be denoted as δ_m, and

$$\{t\}^2=\{t\}^1-\delta_m\,\mathrm{grad}\, f \qquad (2.4.3)$$

There is no "royal road" to the determination of δ_m and hence a trial and error process is adopted. First ρ_0 is determined such that the largest component of $\rho_0\,\mathrm{grad}\, f$ which is denoted as $(\rho_0\,\mathrm{grad}\, f)_m$ is 10% of the corresponding component of $\{t\}^1$. If the value of f evaluated at that point which is denoted as $f^{(1)}$ is smaller than the former, the same distance is again traveled. This is repeated until at last at the $(k+1)$-th move $f^{(k+1)}$ is no more smaller than $f^{(k)}$, the value just one before, then the point traces back to the (k)-th point. The distance of travel ρ_0 is then halved and swayed back and forth evaluating f each time. Finding the point of the smallest f, ρ_0 is halved again and swayed. The process continues diminishing f continuously until some satisfactory convergence is obtained. The process is shown in Fig. 2.4.3. In actual application very high accuracy is not needed, and halving process is conducted three or four times. If in the first place $f^{(1)}$ is not smaller than the original, ρ_0 is divided by 5 and the same process follows from the original point.

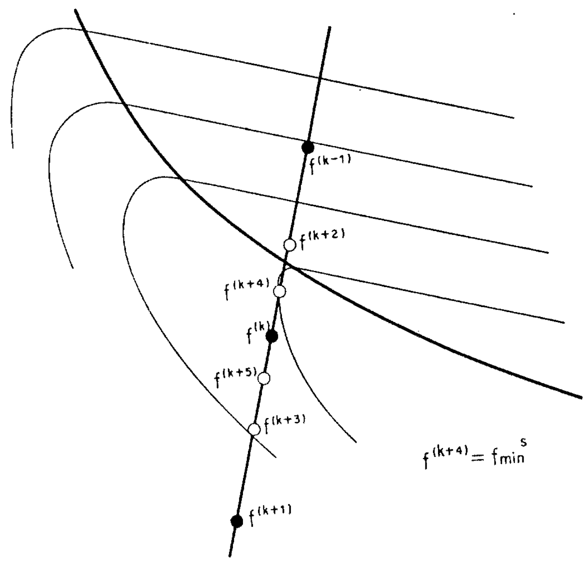

FIG. 2.4.3. Halving Process.

Succeeding points $\{t\}^1, \{t\}^2, \cdots, \{t\}^s$ approaches toward the point of $f_{\min}$. When the one-fifthing processes do not succeed in finding smaller value of f after several trials until the largest component of ρ_0 grad f is smaller than the prescribed value t_c, the series of points is considered to have converged. Fig. 2.4.4 shows the flow chart for the whole process.

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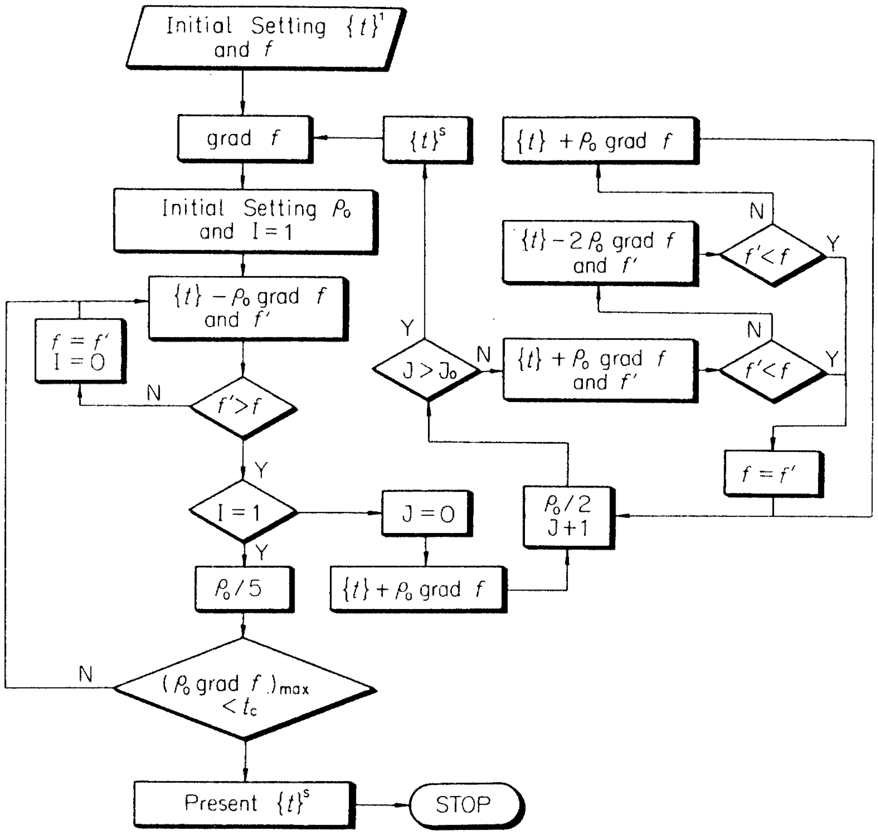

FIG. 2.4.4. Seeking the Minimum Point with the Steepest Descent Method.

2.4.3 Convergence Speed and Modification

As stated in section 1.4, the process tends to make a zigzag toward the minimum point. Generally it can be said that as the value of the constant C is increased the zigzag becomes sharp and convergence slow. It is intended that during the earlier part of the process C is fixed at a relatively small value until a fairly rough first guess is obtained and then is gradually increased to improve accuracy. Still it is not rare to find that a few hundred redesign processes are needed to obtain a satisfactory design subject to variables of only four or so. Speaking in terms of computer economy, stress and vibration analysis which includes inversion of total stiffness matrix is the main source of computing time consumption. Therefore linearization concept of stresses and frequencies is first introduced to reduce the number of complete analyses done during the optimization procedure:

$$\begin{aligned}\{\sigma\} &= \{\sigma\}_0 + (\text{grad}\,\{\sigma\})^T(\{t\} - \{t\}_0)\\ \omega &= \omega_0 + (\text{grad}\,\omega)^T(\{t\} - \{t\}_0)\end{aligned} \tag{2.4.4}$$

where grad $\{\sigma\}$ and grad ω are evaluated at $\{t\}_0$. Then f is minimized and have a design $\{t'\}_0$. Derivatives are evaluated at this new point and the process continues until satisfactory convergence is obtained.

The basic philosophy is the same as that in Ref. 18 and 19 except the process to obtain $\{t'\}_0$ from $\{t\}_0$. Therefore the underlying difficulty encountered in those references is also the difficulty on the present method.

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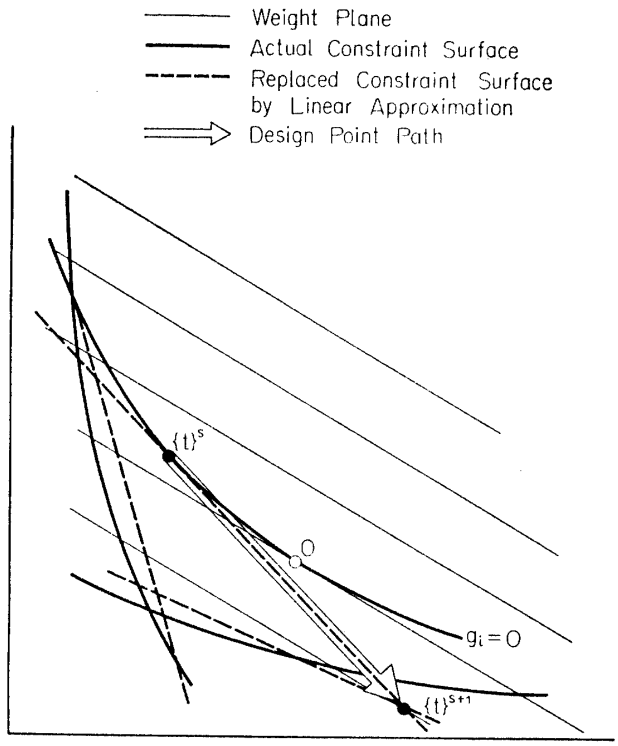

FIG. 2.4.5. Linearized Constraints and Path.

In linear programming with n variables, the optimum always lies at a vertex of n constraint surfaces; in our problem it corresponds to a so called fully stressed design. A structure which is optimum at non-fully stressed region encounters a difficulty. This situation is shown in Fig. 2.4.5. The optimum is obtained at a point O where the constraint surface $g_i=0$ is tangent to the weight-constant plane, but when the constraint surface is replaced by a plane, the point travels along the plane until another constraint becomes active. Romstad [*19*] avoided this circumstance by imposing a limit to the maximum distance of travel. Examples using this concept are shown in the next chapter.

2.4.4 Constrained Path

The path of the design point in the parameter space is zigzag as long as the steepest descent method is used throughout: linearization of stresses etc. does not change this fundamental property, since f is non-linear any way. Careful study of the zigzag path tells us that it is along the constraints, that is, the point "vibrates" across the constraint surface being active. It is of practical advantage to smooth the path in the whole ignoring small fluctuations, thus reducing the number of analyses considerably.

This fact resembles to physical phenomena of a small rubber ball rolling down along a U or V-ditch. If the initial starting point is improper, the path is zigzag until the friction takes it over. Subrelaxation in *1.4.3* corresponds to the effect of the friction. But if there is a force to push the ball along the ditch the path will be smoothened to the greatest extent.

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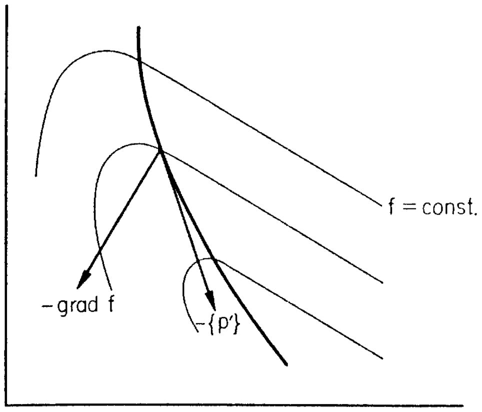

FIG. 2.4.6. Gradient Projection.

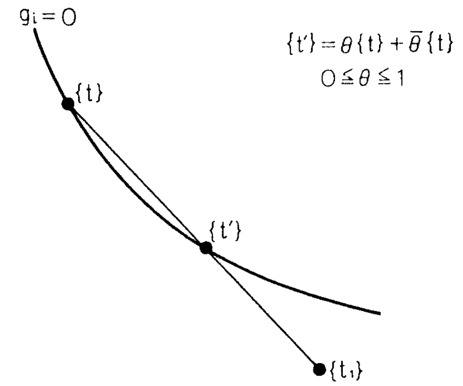

FIG. 2.4.7. Violation of the Constraint.

Suppose a constraint g_i is active, that is, the point $\{t\}$ satisfies $g_i(\{t\})=0$.

The direction of the steepest descent shown in Fig. 2.4.6 does not generally coincide with the constraint surface. A move in the direction of $-\{p'\}$ which is in a plane tangent to $g_i=0$ is the most desirable. $\{p'\}$ may be obtained readily applying the concept of gradient projection outlined in *1.4.4*, where f_1 is replaced by f, and $\{p\}$ replaced by $\{p'\}$.

When $\{t\}$ lies near the constraint surface $g_i=0$ to a certain admissible degree, that is

$$g_i(\{t\})<\varepsilon \tag{2.4.5}$$

where ε is a small positive scalar, the next move is constrained to the direction of $-\{p'\}$. When some other constraints also become active, calculation of $\{p'\}$ is straight forward following the procedure in *1.4.4*.

Now we have as an assumption that g_i is convex and from the definition of convexity (*1.4.1*), we can prove that g_i increases in the direction of $\{p'\}$.

$\{p'\}$ is considered to coincide with $\{t_1\}-\{t\}$, where $\{t_1\}$ lies on the extension of $\{t'\}-\{t\}$ and $\{t'\}$ which is also on the surface $g_i=0$ tends to $\{t\}$. $\{t'\}$ may be expressed as

$$\{t'\}=[\theta\{t_1\}+\bar{\theta}\{t\}]_{\theta\to 0} \tag{2.4.6}$$

where $\bar{\theta}=1-\theta$. From the definition (1.4.1)

$$g_i(\theta\{t_1\}+\bar{\theta}\{t\})\leqq\theta g_i(\{t_1\})+\bar{\theta}g_i(\{t\}) \tag{2.4.7}$$

therefore

$$g_i(\{t_1\})\geqq 0 \tag{2.4.8}$$

Now taking $\{t_2\}$ on the further extension of $\{t_1\}-\{t\}$ and applying (1.4.1) again with finite value of θ, it may be concluded

$$g_i(\{t_2\})\geqq g_i(\{t_1\}) \tag{2.4.9}$$

On the consequence it can be expected that g_i will be increased by more than ε in time.

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The constrained path concept was developed basically under the intension of improving convergence in the steepest descent method. There is little tendency that $\{t\}$ severely violates the actual constraint, since f must monotonously decrease. Once, after a series of the application of the concept, g becomes larger than ε, the point $\{t\}$ is free of the particular constraint and the normal type of the steepest descent move restored will bring it back to the boundary of the acceptable region.

In gradient projection method presented by Rosen [*81,82*], the point $\{t\}$ must always be in the neighbourhood of the boundary of the acceptable region. After each movement toward $-\{p\}$ defined by (1.4.17), another process must be employed such that

$$\{t\}^{(k+1)}=\{t\}^{(k)}-[G]^T([G][G])^{-1}[G]\{g\} \quad k=0,1,\cdots$$

where $[G]$ is a matrix composed of grad g_i's defined in *1.4.4* and $\{g\}$ is a vector composed of $g_i(\{t\}^k)$, until $g_i<\varepsilon$ is attained.

In the steepest descent method presented in *2.4.2*, whether the constrained path is employed or not, the initial point may exist in the region of violation as well as in the acceptable region. The Rosen's method requires it to be in the acceptable. Ref. 82 proposes a method to find an acceptable point, which is another maximization problem.

2.4.5 Optimality of the Vertex

Design parameter space with n co-ordinates is divided into acceptable region and the region of violation by constraint hyper surfaces. Generally movement along the constrained path will lead the point $\{t\}$ toward a vertex of constraint surfaces, which does not always offer the minimum of f.

The similar situation is encountered in the gradient projection method, in which f_1 is not the global minimum at the vertex. Since, in the neighbourhood of the vertex, behaviors of the true objective f_1 and the modified objective f are almost identical, the optimality of the vertex reached by the constrained path may be examined by the criteria (1.4.18) and (1.4.19) developed by Rosen. In the present section the Rosen's criteria are demonstrated in a more direct way.

If a vertex is not the optimal point, the optimum lies on a surface of the acceptable region which is locally tangent to f_1-constant surface as shown in Fig. 2.4.8. At the optimal, it is clear that $\{p\}$ which is the projection of grad f_1 on to the local tangent of the surface of the acceptable region must vanish.

Suppose we have a point $\{t\}_0$ at which m constraints g_i, $(i=1,\cdots m)$ are active, i.e.,

$$g_i(\{t\}_0)=0 \quad i=1,\cdots,m \tag{2.4.10}$$

and j-th component of $\{\alpha\}$ $(1\leqq j\leqq m)$ determined by (1.4.16) is negative. It is always possible to find a vector of a small magnitude such that

$$g_i(\{t\}_0+\varDelta\{t\}_j)\begin{cases}=0 & (i\neq j)\\ <0 & (i=j)\end{cases} \tag{2.4.11}$$

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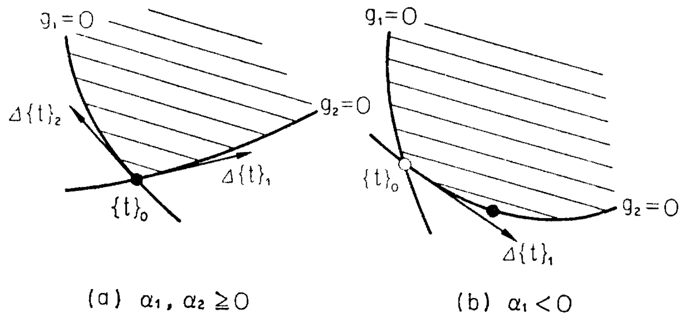

FIG. 2.4.8.

which may be restated incorporating (2.4.10) as

$$g_i(\{t\}_0+\Delta\{t\}_j)=g_i(\{t\}_0)+(\mathrm{grad}\, g_i)^T\Delta\{t\}_j$$
$$=(\mathrm{grad}\, g_i)^T\Delta\{t\}_j\begin{cases}=0 & (i\neq j)\\ <0 & (i=j)\end{cases} \tag{2.4.12}$$

Increment of f_1 due to the movement from $\{t\}_0$ to $\{t\}_0+\Delta\{t\}_j$

$$\Delta f_1=f_1(\{t\}_0+\Delta\{t\}_j)-f_1(\{t\}_0)=(\mathrm{grad}\, f_1)^T\Delta\{t\}_j \tag{2.4.13}$$

which is rewritten using the relations (1.4.13) and (2.4.12) as

$$\Delta f_1=\left(\{p\}-\sum_{i=1}^{m}\alpha_i\,\mathrm{grad}\, g_i\right)^T\Delta\{t\}_j$$
$$=(\{p\})^T\Delta\{t\}_j-\sum_{i=1}^{m}\alpha_i(\mathrm{grad}\, g_i)^T\{t\}_j$$
$$=(\{p\})^T\Delta\{t\}_j-\alpha_j(\mathrm{grad}\, g^j)^T\Delta\{t\}_j \tag{2.4.14}$$

When $\{p\}$ vanishes, that is, when $\{t\}_0$ is a vertex ($m=n$) or grad f_1 is perpendicular to all constraint surfaces active, f_1 can be reduced in its value by moving $\{t\}_0$ in the direction of $\Delta\{t\}_j$.

The physical meaning of $\Delta\{t\}_j$ is shown in Fig. 2.4.8-b in which $\Delta\{t\}_1$ satisfies the relations

$$(\mathrm{grad}\, g_2(\{t\}_0))^T\Delta\{t\}_1=0$$
$$(\mathrm{grad}\, g_1(\{t\}_0))^T\Delta\{t\}_1<0 \tag{2.4.15}$$

and further reduction of the value of f_1 is possible by moving away from the vertex in the direction of $\Delta\{t\}_1$.

Conversely if $\{\alpha\}\geqq 0$ it can be proved that $\{t\}_0$ is the optimum, since in Fig. 2.4.8-a,

$$f_1(\{t\}_0+\Delta\{t\}_1),\quad f_1(\{t\}_0+\Delta\{t\}_2)\geqq f_1(\{t\}_0)$$

and

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$$f_1(\{t\}_0+\theta\Delta\{t\}_1+\bar{\theta}\Delta\{t\}_2)=\theta f_1(\{t\}_0+\Delta\{t\}_1)+\bar{\theta}f_1(\{t\}_0+\Delta\{t\}_2)$$
$$\geqq f_1(\{t\}_0) \qquad (2.4.16)$$

because of linearlity of f_1, where $0\leqq\theta\leqq 1$ and $\bar{\theta}=1-\theta$. $\theta\Delta\{t\}_1+\bar{\theta}\Delta\{t\}_2$ expresses all possible directions within the cone spanned by $\Delta\{t\}_1$ and $\Delta\{t\}_2$, and along this direction f_1 is increased in its value.

Consequently, when $\{p\}=0$ is reached, if all components of $\{\alpha\}$ are non-negative, the global optimum is assured to have been obtained, and if a component of $\{\alpha\}$ is negative, the corresponding constraint is dropped to make the movement free of the constraint.

3. Examples

3.1 Cantilevered Beam

3.1.1 The Model

As a preliminary study to find the best suited method, a cantilevered beam made of steel with the length of 400 mm and the depth of 100 mm is chosen. The beam is partitioned into four sections of equal size along its span and their thicknesses are design variables to be determined (See Fig. 3.1.1).

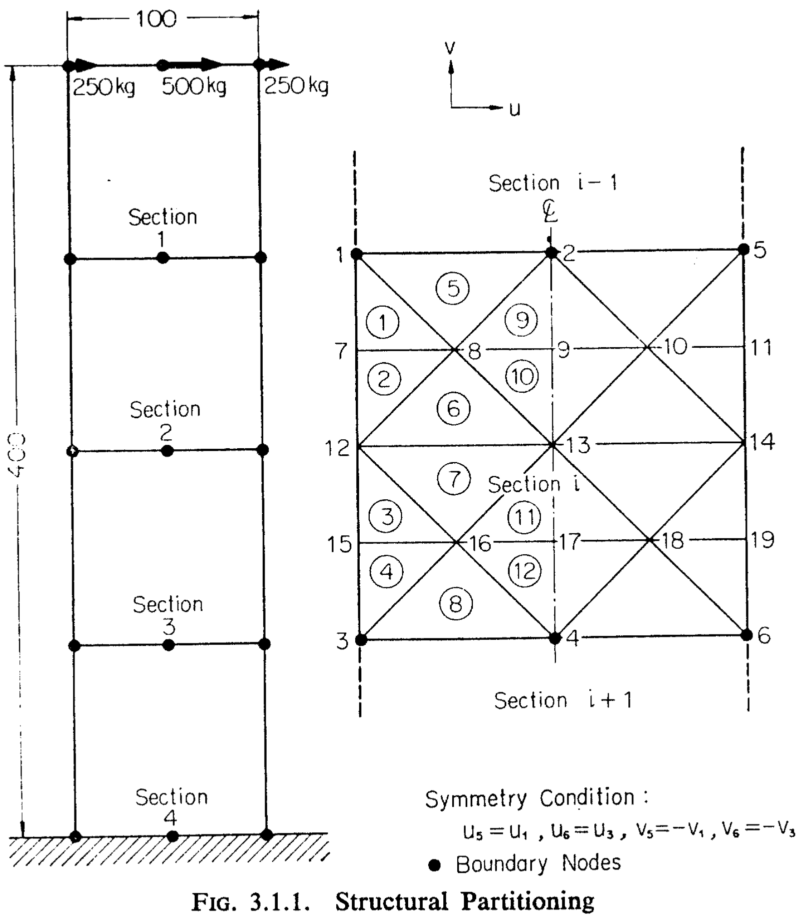

FIG. 3.1.1. Structural Partitioning

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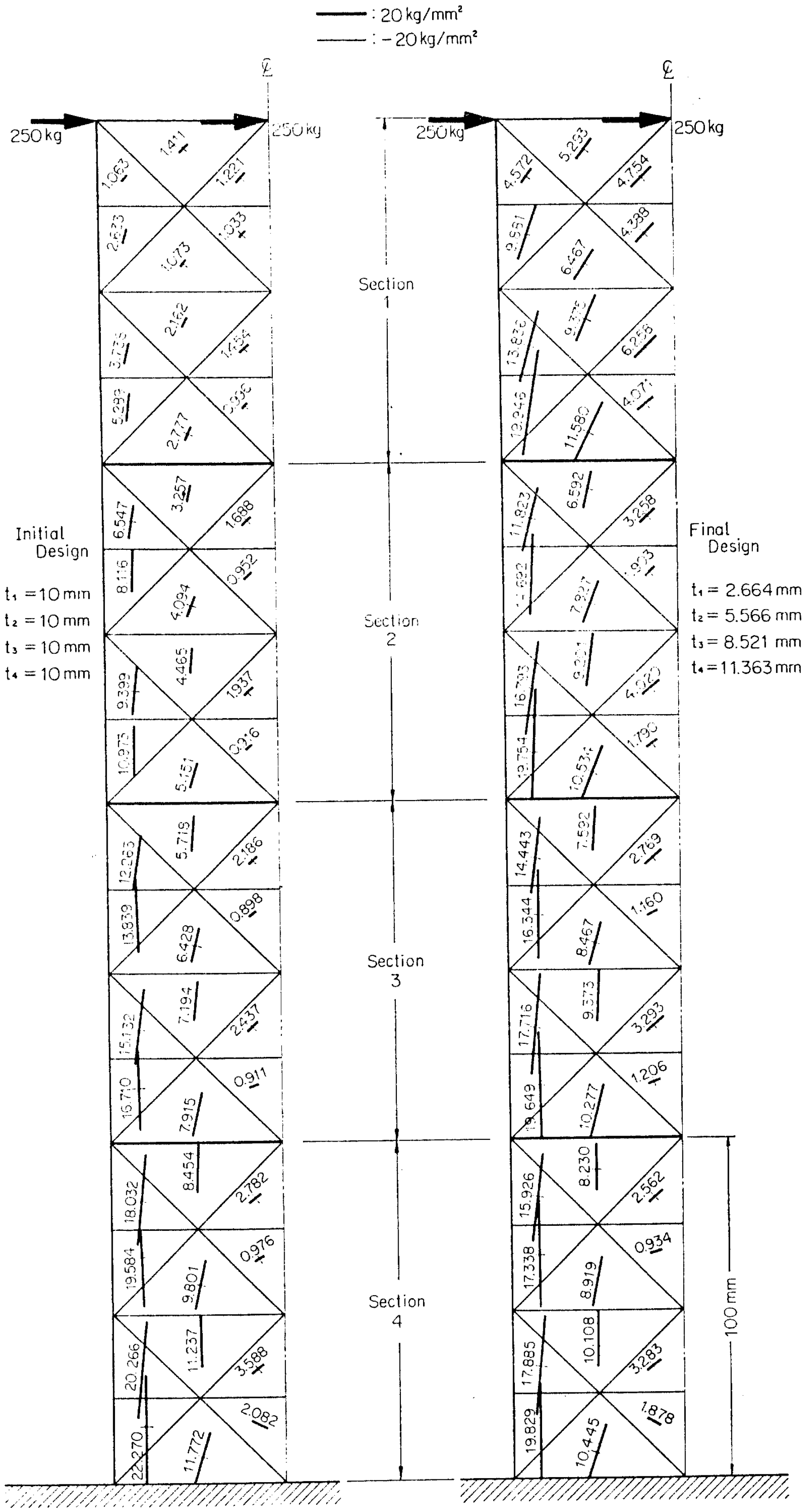

FIG. 3.1.2. Principal Stresses.

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It is subjected to the lateral tip load of 1000 kg. The requirements to be met are: 1) Any part of the beam is within the elastic limit, 2) The fundamental bending frequency is above a prescribed value. Young's modulus $E=2.1\times10^4$ kg/mm^2, Poisson's ratio $\nu=0.3$, yield stress $\sigma_Y=20$ kg/mm^2 and specific weight of 7.8×10^{-6} kg/mm^3 are assumed.

3.1.2 *Finite Element Analysis*

Assuming plane stress condition, each section is now divided into 24 triangular finite elements with 19 nodes (See Fig. 3.1.1).

The section stiffness matrix $[K_i]$ obtained is of order of 38×38. Now, incorporating substructure method outlined in *1.2.5*, $[K_i]$ may be reduced to 12×12, where the boundary displacement vector $\{u_b\}$ is defined by displacements of nodes 1 through 6. Also considering the symmetry of the structure and of the external forces, the number of independent displacements may be reduced.

$$u_5=u_1,\ v_5=-v_1,\ u_6=u_3,\ v_6=-v_3 \qquad (3.1.1)$$

The final matrix $[K_i]$ thus obtained is of order 8×8.

The total stiffness matrix $[K]$ obtained by linear summation (2.1.1) is of order of 18×18 after the geometrical boundary conditions are taken into account. If the substructure method was not used, it would be of order of 128×128, and if symmetry was not considered, it would be of 24×24.

Thicknesses $t_1=t_2=t_3=t_4=10$ mm are assumed and displacements, stresses and the first natural frequency are obtained. Principal stresses in each triangular elements are shown on the left side of Fig. 3.1.2. Comparison of the results with those obtained by the simple beam theory is shown in Table 3.1.1.

TABLE 3.1.1

	FEM	Beam Th.	Deviation (%)
Maximum Stress (kg/mm^2)	22.27	24.00	7.21
Maximum Deflection (mm)	1.209	1.219	0.82
Fundamental Frequency (cps)	521.4	518.6	0.56

Computation by HITAC 5020F took about 9 seconds for the generation of boundary sticness matrices including the compilation time, and only 1.1 seconds were needed for stress calculation followed.

3.1.3 *Linearized Version*

Stresses are assumed by the linear relation (2.4.4) and the steepest descent method with diagonal step is used to optimize thickness of each section. The objective to be minimized is defined by (2.4.1)

$$f=\sum A_i t_i + C\sum\langle g_i\rangle^2 \qquad (2.4.1)$$

Since all the sections are of the same material, A_i is chosen to represent the volume of the i-th seciton with unit thickness instead of the weight.

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There are three types of g_i's:

i) $(\sigma_{0,i}-\sigma_Y)/\sigma_Y$ (3.1.2)

$\sigma_{0,i}=(\sigma_x^2+\sigma_y^2-\sigma_x\sigma_y+3\tau_{xy}^2)^{1/2}|_{\text{max within section } i}$

ii) $(\omega_{cr}-\omega)/\omega_{cr}$ (3.1.3)

iii) $(t_{L,i}-t_i)/t_{L,i}$ (3.1.4)

These types insure that contributions of various constraints to f are of the same magnitude.

The value of C is first chosen to be

$$C=V_0\equiv\sum A_i(t_i)_{\text{initial}} \tag{3.1.11}$$

which is the initial volume of the structure, and then f is minimized under the assumptions of (2.4.4) where gradients are evaluated at $\{t\}_0=\{t\}_{\text{initial}}$. Con-

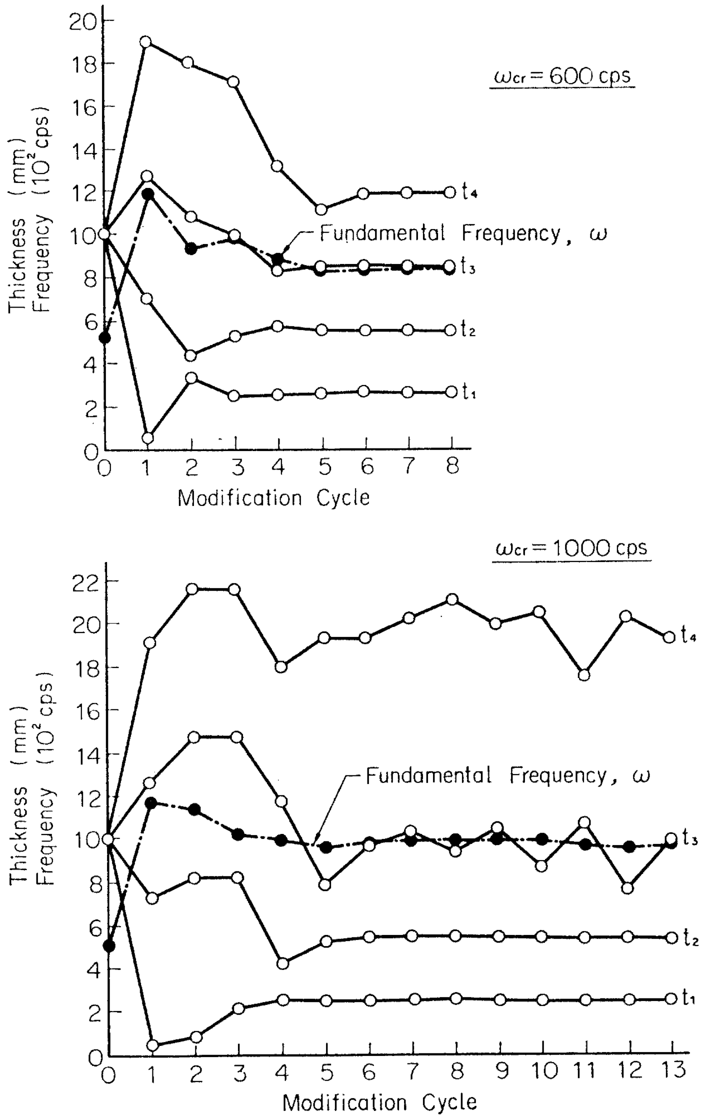

FIG. 3.1.3. Design Modifications. Linearized.

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vergence condition t_c defined in *2.4.2* is chosen to be 10^{-4} mm.

Then as the second step, C is doubled and f is minimized under the same assumptions, starting from the final values of $\{t\}$ in the first step. At the termination of the K-th step, step, C is multiplied by $2K$ and the $(K+1)$-th step enters. Steps are conducted up to the 6-th and the final value of C equals to $3840\,V_0$.

The final values of $\{t\}$ of the 6-th step are denoted as $\{t'\}_0$, and if the difference of the total volume evaluated at $\{t\}_0$ and $\{t'\}_0$ is less than 0.001%, convergence is assumed to have been achieved. If not, $\{t'\}_0$ become the starting values $\{t\}_0$ of the next modification cycle.

If the variables $\{t\}$ are too far removed from the starting values $\{t\}_0$ at which derivatives are evaluated, linearized assumptions (2.4.4) would not hold validity anymore. Before calculating the direction of the steepest descent $-\text{grad } f$, each variable is tested against its starting value to find the deviation, and if it is more than 20%, the modification cycle ceases and the new cycle enters evaluating new

TABLE 3.1.2. Design modification of four sectioned cantilevered beam. Linearized concept.

$\omega_{cr}=600$ cps, $\sigma_Y=20$ kg/mm²

CYCLE	SECT 1 mm kg/mm²	SECT 2 mm kg/mm²	SECT 3 mm kg/mm²	SECT 4 mm kg/mm²	FREQ cps	VOL 10^4mm³
1 t_i	10.0000	10.0000	10.0000	10.0000	521.41	40.0000
σ_0	5.320	11.141	17.047	22.685		
2	0.5000	7.0011	12.6685	19.0653	1181.88	39.2348
	106.754	15.888	13.449	11.926		
3	3.2996	4.3776	10.7825	18.0111	929.56	36.4708
	16.135	25.383	15.783	12.619		
4	2.4805	5.2578	9.9668	17.1913	968.41	34.8964
	21.475.	21.154	17.075	13.321		
5	2.5718	5.7281	8.3319	13.1744	867.15	29.8062
	20.713	19.436	20.434	17.249		
6	2.6607	5.5627	8.5176	11.0739	823.24	27.8149
	20.002	20.009	20.010	20.521		
7	2.6634	5.5656	8.5210	11.3551	823.24	28.1051
	20.000	19.999	20.000	20.012		
8	2.6637	5.5682	8.5211	11.3626	829.29	28.1156
	19.997	19.999	20.000	20.000		
9	2.6640	5.5655	8.5213	11.3626	829.34	28.1134
	19.995	20.000	19.999	20.000		
CYCLE NO. 8						
C	SECT 1	SECT 2	SECT 3	SECT 4	FREQ	ZIG NO
V_0	2.6565	5.5376	8.4560	11.2495	827.58	28
	20.100	20.049	20.098	20.199		
$2V_0$	2.6603	5.5512	8.4884	11.3012	828.36	23
	2.0023	20.051	20.077	20.108		
$8V_0$	2.6623	5.5615	8.5120	11.3480	829.17	5
	20.008	20.014	20.021	20.025		
$48V_0$	2.6632	5.5650	8.5187	11.4594	829.32	5
	20.001	20.001	20.004	20.005		
$384V_0$	2.6640	5.5655	8.5214	11.3615	829.31	2
	19.995	20.000	19.999	20.002		
$3840V_0$	2.6640	5.5655	8.5213	11.3626	829.33	1
	19.995	20.000	19.999	20.000		

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TABLE 3.1.2. (Continued)

$\omega_{cr}=1000$ cps, $\sigma_Y=20$ kg/mm²

CYCLE		SECT 1 mm kg/mm²	SECT 2 mm kg/mm²	SECT 3 mm kg/mm²	SECT 4 mm kg/mm²	FREQ cps	VOL 10^4mm³
1	t_i	10.0000	10.0000	10.0000	10.0000	521.41	40.0000
	σ_0	5.320	11.141	17.047	22.685		
2		0.4900	7.0286	12.7288	19.1353	1195.15	39.3827
		108.934	15.826	13.385	11.882		
3		0.9922	7.9423	14.3835	21.6229	1145.05	44.9409
		53.771	14.006	11.844	10.515		
4		2.2502	7.9425	14.3831	21.6226	1028.78	46.1984
		23.688	14.005	11.843	10.513		
5		2.5876	4.3296	10.7714	18.0107	1005.49	35.6993
		20.581	25.665	15.801	12.620		
6		2.5992	5.2945	7.8738	19.2871	963.45	35.0546
		20.493	21.028	21.576	11.782		
7		2.6324	5.5051	9.8433	18.2671	985.45	37.2479
		20.235	20.308	17.278	11.796		
8		2.6645	5.5607	10.4862	20.2303	1000.52	38.9416
		19.992	20.002	16.220	11.235		
9		2.6631	5.5634	9.4267	21.0394	996.63	38.6926
		20.003	20.001	18.030	10.802		
10		2.6635	5.5611	10.5287	19.9671	997.11	38.7204
		19.999	20.001	16.156	11.383		
11		2.6197	5.5256	8.3664	20.4833	978.84	36.9950
		20.334	20.148	20.307	11.095		
12		2.5964	5.5740	10.8917	17.5705	974.30	36.6326
		20.513	19.951	15.631	12.936		
13		2.5260	5.5263	7.7543	20.3250	964.32	36.1316
		20.272	20.151	21.903	11.181		

CYCLE NO. 7

C	SECT 1	SECT 2	SECT 3	SECT 4	FREQ	ZIG NO
V_0	2.6224	5.5048	9.8688	19.3801	988.02	17
	20.313	20.210	17.233	11.727		
$2V_0$	2.6432	5.5333	10.1560	19.8330	994.24	86
	20.153	20.103	16.730	11.450		
$8V_0$	2.6568	5.5524	10.3676	20.0903	998.07	37
	20.049	20.032	16.359	11.293		
$48V_0$	2.6632	5.5606	10.5076	20.2516	1000.81	10
	20.001	20.000	16.114	11.194		
$384V_0$	2.6645	5.5607	10.4862	20.2303	1000.23	11
	19.999	20.000	16.151	11.207		
$3840V_0$	2.6645	5.5607	10.4862	20.2303	1000.23	1
	19.999	20.000	16.151	11.207		

CYCLE NO. 8

C	SECT 1	SECT 2	SECT 3	SECT 4	FREQ	ZIG NO
V_0	2.6177	5.5018	8.8030	20.0539	987.29	70
	20.344	20.227	18.812	11.332		
$2V_0$	2.6411	5.5343	9.0933	20.5880	993.73	61
	20.169	20.108	18.363	11.036		
$8V_0$	2.6570	5.5558	9.3329	20.9137	998.09	43
	20.049	20.029	17.993	10.855		
$48V_0$	2.6631	5.5634	9.4267	21.0394	999.83	17
	20.005	20.005	17.848	10.785		
$384V_0$	2.6631	5.5634	9.4267	21.0394	999.83	2
	20.005	20.005	17.848	10.785		
$3840V_0$	2.6631	5.5634	9.4267	21.0394	999.83	1
	20.005	20.005	17.848	10.785		

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derivatives.

The critical frequency ω_{cr} is first assumed to be 600 cps and then increased by 100 cps successively. At lower critical frequencies, convergence is satisfactory, and as the final result, obtained is a fully stressed design which is identical below 800 cps and the results are shown on the right of Fig. 3.1.2. The fundamental frequency at this proportioning is 829 cps. At higher values of critical frequencies than 1000 cps, succession of modification tends to oscillate. This is due to the fundamental property of the linearized concept which is pointed out in *2.4.3*. Examples of convergence processes are shown in Fig. 3.1.3 and Table 3.1.2. In the table shown are designs of each modification cycle $\{t\}_0$, corresponding maximum equivalent stresses $\sigma_0=(\sigma_x^2+\sigma_y^2-\sigma_x\sigma_y-3\tau_{xy}^2)^{1/2}$ of each section, fundamental frequencies and the volume. Also shown are examples of processes within a cycle. Converged values at each step are listed with the number of zigzags experienced and, here, stresses and frequencies are assumed values by (2.4.4).

Average of about 200 zigzags has been experienced in one modification cycle which has not been stopped due to more than 20% deviation of the variable. Computation time was about 180 seconds for a case with the critical frequency

TABLE 3.1.3. Design modification of four sectioned cantilevered beam. Constrained path concept. * signifies a quantity constrained. The numeral in ( ) is a caracteristic value α.

$\omega_{cr}=600$ cps, $\sigma_Y=20$ kg/mm²

NZG	SECT 1 mm kg/mm²	SECT 2 mm kg/mm²	SECT 3 mm kg/mm²	SECT 4 mm kg/mm²	FREQ cps	VOL 10^4 mm³	f 10^4
$C=$	V_0, $\varepsilon=0.05$						
$1 t_i$	10.0000	10.0000	10.0000	10.0000	521.41	40.0000	120.913
σ_0	5.320	11.141	17.047	22.685			
2	8.2302	9.4262	10.4690	12.5625	600.06*	40.6879	40.6879
	6.466	11.819	16.275	18.071			
3	7.0844	8.0261	8.8710	10.8352	600.04*	34.8167	35.7217
	7.512	13.881	19.205*	20.952*			
4	7.7733	5.5681	8.8693	10.8351	596.22*	33.0458	34.1053
	6.845	19.974*	19.203*	20.949*			
		(2.40)	(5.89)	(8.08)	(−0.36)		
5	2.6721	5.5721	8.8788	10.8473	819.34	27.9703	28.8742
	19.935*	19.973*	19.204*	20.951*			
	(1.33)	(2.80)	(4.64)	(5.15)			
$C=$	$2V_0$, $\varepsilon=0.025$						
1	2.6721	5.5721	8.8788	10.8473	819.34	27.9703	29.7783
	19.935*	19.973*	19.204	20.951			
2	2.6721	5.5724	8.8090	11.2879	828.98	28.3414	28.3767
	19.935*	19.972*	19.350	20.133*			
3	2.6721	5.5732	8.4787	11.2876	826.62	28.0116	28.0668
	19.935*	19.972*	20.100*	20.133*			
	(1.33)	(2.80)	(4.23)	(5.59)			
$C=$	$8V_0$, $\varepsilon=0.00625$						
1	2.6721	5.5732	8.4787	11.2876	826.62	28.0116	28.2319
	19.935*	19.972*	20.100*	20.133			
2	2.6721	5.5733	8.4785	11.3441	827.86	28.0680	28.1550
	19.936*	19.972*	20.100*	20.032*			
	(1.33)	(2.80)	(4.24)	(5.64)			

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TABLE 3.1.3. (Continued)

$\omega_{cr}=1000$ cps, $\sigma_Y=20$ kg/mm²

NZG	SECT 1 mm kg/mm²	SECT 2 mm kg/mm²	SECT 3 mm kg/mm²	SECT 4 mm kg/mm²	FREQ cps	VOL 10⁴ mm³	f 10⁴
$C=$	V_0, $\varepsilon=0.05$						
$1t_i$ σ_0	10.0000 5.320	10.0000 11.141	10.0000 17.047	10.0000 22.685	521.41	40.0000	973.346
2	2.4613 21.654	7.6436 14.561	12.1627 14.001	19.1250 11.885	959.49*	41.3926	50.6947
3	2.6551 20.069*	7.1659 15.527	12.7178 13.937	19.3443 11.750	959.15*	41.8831	48.0626
4	2.6544 20.069*	5.4494 20.410*	10.4268 16.324	17.5309 12.965	956.95*	36.0615	43.6468
5	2.6544 20.068*	5.4502 20.410*	10.0749 16.890	17.7245 12.823	956.73*	35.9040	43.5644
6	2.6544 20.068*	5.4508 20.410*	9.7727 17.409	17.9175 12.685	956.56*	35.7954	43.5145
7	2.6543 20.069*	5.4512 20.410*	9.5894 17.739	18.0506 12.591	956.49*	35.7455	43.4909
8	2.6543 20.069*	5.4515 20.410*	9.4576 17.984	18.1543 12.519	956.46*	35.7177	43.4744
9	2.6542 20.070*	5.4517 20.410*	9.3748 18.142	18.2233 12.471	956.45*	35.7040	43.4622
10	2.6542 20.070*	5.4517 20.410*	9.3722 18.147	18.2256 12.470	956.44*	35.7037	43.4610
11	2.6540 20.071*	5.4520 20.410*	9.2410 18.403	18.3393 12.392	956.43*	35.6863	43.4534
$C=$	$2V_0$, $\varepsilon=0.025$						
1	2.6540 20.071*	5.4520 20.410*	9.2410 18.403	18.3393 12.392	956.43	35.6863	51.2205
2	2.6541 20.071*	5.4484 20.411*	10.8031 15.747	20.0586 11.331	994.36*	38.9642	39.5675
3	2.6539 20.072*	5.4497 20.412*	10.0941 16.846	20.5554 11.057	993.42*	38.7531	39.4489
4	2.6540 20.072*	5.4499 20.412*	10.0184 16.972	20.6258 11.020	993.40*	38.7481	39.4457
5	2.6539 20.073* (11.74)	5.4501 20.412* (7.40)	9.9057 17.164	20.7342 10.962	993.39* (0.87)	38.7439	39.4422
$C=$	$8V_0$, $\varepsilon=0.00625$						
$1t_i$ σ_0	2.6539 20.073*	5.4501 20.412	9.9057 17.164	20.7342 10.962	993.39	38.7439	41.5385
2	2.6539 20.073*	5.5268 20.130	9.9258 17.129	20.7539 10.951	992.27	38.8604	40.9483
3	2.6539 20.073*	5.4818 20.293	9.9848 17.028	20.8125 10.920	994.55*	38.9330	40.6143
4	2.6539 20.073* (11.77)	5.5504 20.044* (7.77)	10.0457 16.925	20.8737 10.089	994.54* (0.88)	39.1237	40.1362

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of lower than 900 cps, and for cases with higher ones, convergence was not attained.

3.1.4 Constrained Path

It is found that linearized concept is not applicable to some types of problems. It cannot be avoided to evaluate stresses and frequencies at each redesign, but it is intolerable, at the same time, to do so from the stand point of computation time.

The constrained path concept is now used to reduce the number of redesigns. Since the behaviors of the structure are not assumed but exactly calculated anytime, only a single cycle is necessary to be carried out. Three steps to increase the value of C and to decrease ε are conducted. The processes and results are shown in Table 3.1.3. The values of equivalent stresses and frequencies with "*" signify that they meet constraints (3.1.2) and (3.1.3) within the limit of ε, i.e.,

$$(g_i)^2=((\sigma_{0,i}-\sigma_Y)/\sigma_Y)^2\leqq\varepsilon^2,\quad i=1,\cdots,4 \tag{3.1.6}$$

$$(g_5)^2=((\omega_{cr}-\omega)/\omega_{cr})^2\leqq\varepsilon^2 \tag{3.1.7}$$

In these cases, the constrained path $-\{p'\}$ is taken for the next move. As an example the case of $\omega_{cr}=600$ cps, and the second zigzag, NZG=2, where the fundamental frequency is constrained is chosen to show the proedure to generate $-\{p'\}$.

Since all g_i's are negative

$$\langle g_i\rangle=0 \tag{3.1.8}$$

and

$$\begin{aligned}\operatorname{grad} f&=\operatorname{grad} f_1\\&=\{A_1, A_2, A_3, A_4\}\\&=\{1, 1, 1, 1\}\times 10^4\end{aligned} \tag{3.1.9}$$

Also it is calculated that

$$\operatorname{grad}\omega=\{-22.81,\ -6.46,\ 6.26,\ 14.57\}$$

and

$$\operatorname{grad} g_5=-\frac{1}{\omega_{cr}}\operatorname{grad}\omega \tag{3.1.10}$$

Similar to the procedure in *1.4.4,*

$$\{p'\}=\operatorname{grad} f+\alpha'\operatorname{grad} g_5 \tag{3.1.11}$$

or

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$$\{p'\}=\begin{Bmatrix}1\\1\\1\\1\end{Bmatrix}\times 10^4-\frac{\alpha'}{\omega_{cr}}\begin{Bmatrix}-22.81\\-6.46\\6.26\\14.57\end{Bmatrix} \tag{3.1.12}$$

Since $\{p'\}$ must be perpendicular to grad g_5,

$$\begin{aligned}&(\text{grad } g_5)^T(\text{grad } f+\alpha'\text{grad } g_5)\\
&=-\frac{1}{\omega_{cr}}\begin{Bmatrix}-22.81\\-6.46\\6.26\\14.57\end{Bmatrix}^T\left(\begin{Bmatrix}1\\1\\1\\1\end{Bmatrix}\times 10^4-\frac{\alpha'}{\omega_{cr}}\begin{Bmatrix}-22.81\\-6.46\\6.26\\14.57\end{Bmatrix}\right)\\
&=-\frac{1}{\omega_{cr}}\left(-8.44\times 10^4-\frac{\alpha'}{\omega_{cr}}\cdot 813.5\right)=0\end{aligned} \tag{3.1.13}$$

$$\frac{\alpha'}{\omega_{cr}}=-\frac{8.44\times 10^4}{813.5}=-0.1037\times 10^3 \tag{3.1.14}$$

$$\{p'\}=\begin{Bmatrix}1\\1\\1\\1\end{Bmatrix}\times 10^4+0.1037\times 10^3\begin{Bmatrix}-22.81\\-6.46\\6.26\\14.57\end{Bmatrix}=\begin{Bmatrix}0.763\\0.933\\1.067\\1.151\end{Bmatrix}\times 10^4 \tag{3.1.15}$$

Now we proceed toward $-\{p'\}$, i.e., we decrease thicknesses proportionaly. It is seen in the table that, because of the movement, the frequency ω is changed from 600.063 cps to 600.037 cps. It may be concluded that this movement is almost along the constraint surface $g_5=0$ and since ω is decreased, there is a tendency that g_5 is increased in this direction which is anticipated by (2.4.9).

Every time $\{\alpha'\}$ and $\{p'\}$ are calculated, $\{\alpha\}$ and $\{p\}$ defined in *1.4.4*, by (1.4.10) and (1.4.12) or (1.4.16) and (1.4.17), are also calculated in order to testify optimality. At a point where $\{p\}$ vanishes, if all components of $\{\alpha\}$ are non-negative, the point is optimum, and if the i-th component α_i is negative, it is expected that the optimum does not lie on the constraint $g_i=0$ (see *2.4.5*). In actual application if $|p_i|\leqq|A_i|/100$, $(i=1,\cdots,4)$ are satisfied, it is assumed that $\{p\}$ vanished.

At termination of NZG=4, $\{p\}$ vanished, since $-$grad f_1 is projected on to the intersection of four constraint surfaces in the four dimensional design space.

The 4 by 4 matrix $[G]$ consisting of four rows of grad g_i is,

$$[G]=\begin{Bmatrix}(\text{grad }\sigma_{02})^T\\(\text{grad }\sigma_{03})^T\\(\text{grad }\sigma_{04})^T\\-(\text{grad }\omega)^T\end{Bmatrix}=\begin{bmatrix}-0.0028 & -3.6 & -0.0086 & 0.0016\\-0.0040 & 0.0014 & -2.2 & -0.0091\\-0.0046 & -0.0006 & 0.0026 & -1.9\\28 & 4.0 & -7.7 & -16\end{bmatrix} \tag{3.1.16}$$

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and

$$\{\alpha\} = -([G][G]^T)^{-1}[G]\,\mathrm{grad}\,f_1 \tag{1.4.17}$$

$$\{p\} = \mathrm{grad}\,f_1 + [G]^T\{\alpha\} \tag{1.4.18}$$

$$[G][G]^T = \begin{bmatrix} 13.0 & 0.014 & 0.019 & -14.4 \\ 0.014 & 4.484 & 0.012 & 17.0 \\ 0.019 & 0.012 & 3.61 & 30.3 \\ -14.4 & 17.0 & 30.3 & 1115 \end{bmatrix} \tag{3.1.19}$$

$$([G][G]^T)^{-1} = \begin{bmatrix} 0.0785 & -0.0049 & -0.0117 & 0.0014 \\ -0.0049 & 0.2223 & 0.2223 & -0.0045 \\ -0.0117 & 0.0371 & 0.0371 & -0.0107 \\ 0.0014 & -0.0045 & -0.0107 & 0.0013 \end{bmatrix} \tag{3.1.20}$$

$$[G]\mathrm{grad}\,f_1 = \begin{Bmatrix} -3.6 \\ -2.2 \\ -1.9 \\ 8.3 \end{Bmatrix} \times 10^4, \qquad \{\alpha\} = \begin{Bmatrix} 0.2380 \\ 0.5793 \\ 0.8828 \\ -0.0360 \end{Bmatrix} \times 10^4 \tag{3.1.21}$$

Also,

$$[G]^T\{\alpha\} = \begin{Bmatrix} -1.001 \\ -1.000 \\ -0.997 \\ -0.993 \end{Bmatrix} \times 10^4 \qquad \{p\} = \begin{Bmatrix} -0.001 \\ 0.000 \\ 0.003 \\ 0.007 \end{Bmatrix} \times 10^4 \tag{3.1.22}$$

It is seen that α_4, corresponding to the frequency requirement, is negative, i.e., the optimum does not lie on that surface. Therefore $\{p'\}$ is again calculated rearranging $[G]$, dropping the last row. In the next move, ω is increased, and instead, stresses in the section 1 are increased to the critical value and convergence is obtained, i.e., $\{\alpha\} \geqq 0$ within 5% of requirement violation.

In the following steps C is increased and ε is decreased inversely proportional to C, reducing the requirement violations.

In the case of $\omega_{cr} = 1000$ cps, convergence was rather slow, but finally the process ended in satisfactory convergence. In the first step, $C = V_0$, $\{p\}$ did not vanish, instead, the one fifthing process became critical. In the following steps, $\{p\}$ vanished and the optimality was proved. The optimum lies on the surface on which only three constraints meet. It may be seen that in the process using the linearized concept, the point of design variables was wandering around this optimum point.

3.1.5 Remarks on the Results

Comparing the two methods applied on the problem, incorporation of the constrained path concept was found to be superior to that of the linearized.

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Rosen's criteria were successfully applied to examining optimality of the vertex. In the following examples the constrained path concept is used throughout.

It is interesting to note that not all the requirements necessarily become critical at the final design. When the critical frequency is sufficiently low the maximum stress at each section reaches at the critical value while the frequency still have margin. With higher critical frequency, the margin appears at the stress side. This phenomena correspond to the non-fully stressed optimum proportioning under multiple loading conditions.

3.2 Two Storied Truss

3.2.1 Model

A two storied truss with diagonal members across the corners which may be applicable to sounding rocket instrument compartment framework is chosen as the next example with the variables of four. The height of each story is 300 mm and the base is also 300 mm in length. Effectively rigid trusses are attached to each story and they are assumed to be predetermined. The structure is subjected to two alternative external loading systems. Axial compressive load of 1000 kg and lateral load of 500 kg which may act in both directions are then assumed. Tubular cross section of trusses made of aluminum alloy with Young's modulus of 7200 kg/mm^2 is assumed.

3.2.2 Minimum Weight Design with Constant Mean Diameter

All tubular trusses are assumed to have a constant mean diameter D. Wall thickness of the i-th section $\bar{t}_i$ is expressed in terms of the area t_i which is the basic variable, as

$$\bar{t}_i = \frac{t_i}{\pi D}. \tag{3.2.1}$$

The following requirements are first imposed on stresses.

i) $|\sigma| - \sigma_Y \leqq 0$ (3.2.2)

ii) $\sigma_{cr,1} - \sigma \leqq 0$ (3.2.3)

iii) $\sigma_{cr,2} - \sigma \leqq 0$ (3.2.4)

where $\sigma_{cr,1}$ and $\sigma_{cr,2}$ are Euler buckling stress and local buckling stress, respectively. For example, assuming perfect elasticity and neglecting the effects of eccentricity of load application and initial curvature, for the section i,

$$\sigma_{cr,1} = -\pi^2 E D^2 / 8 A_i^2 \tag{3.2.5}$$

$$\sigma_{cr,2} = -0.4 E \bar{t}_i / D = -0.4 E t_i / \pi D^2 \tag{3.2.6}$$

are chosen.

Overall stability criteria is discussed by Hayashi [*83*]. The structure of this kind may be instable in shear mode when stiffness of diagonal members is in-

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sufficient. Critical axial load of single storied one with one diagonal member is

$$F_{cr}=\frac{Etb^2l}{2a^3}\times 2 \tag{3.2.7}$$

where a is the length of the diagonal member and l and b are height and base length respectively. With two diagonal members F_{cr} is doubled and inserting $F_{cr}=1000\,\mathrm{kg}$, $l=b=a/\sqrt{2}=300\,\mathrm{mm}$, the minimum cross sectional area t_L of the diagonal member becomes

$$t_L=0.196(mm^2) \tag{3.2.8}$$

In the two storied truss under consideration, coupling of the two buckling modes, those of the first and the second story, does not occur, and t_L defined by (3.2.8)

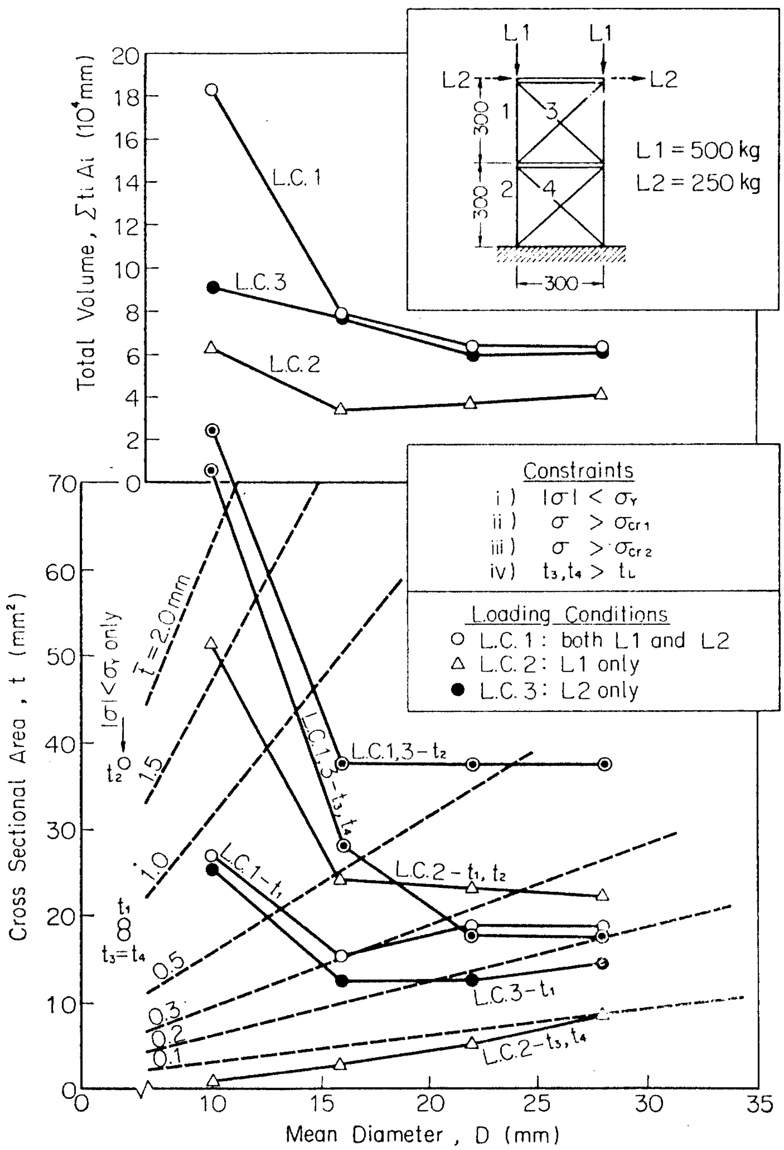

FIG. 3.2.1. Optimum Proportioning of Two Storied Truss.

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may be adopted as the lower boundary of both t_3 and t_4. Additional constraints are

$$t_L - t_3 \leqq 0 \tag{3.2.9}$$

$$t_L - t_4 \leqq 0$$

The structure is optimized in terms of the total volume of the variable sections using the constrained path concept. Derivatives of the constraint iii) becomes

$$-\frac{0.4E}{\pi D^2} - \frac{\partial \sigma_i}{\partial t_i} \quad \text{or} \quad -\frac{\partial \sigma_i}{\partial t_j} (i \neq j) \tag{3.2.10}$$

(3.2.6) and (3.2.10) are evaluated whenever the objective is evaluated. Three loading conditions are chosen: 1. the axial force L1 and lateral force L2 acting alternatively, 2. L1 only acting, 3. L2 only acting. Fig. 3.2.1 and Table 3.2.1 show the optimum proportionings for various values of D, and Table 3.2.2. shows an example of design modifications toward the optimum.

It is seen that under the loading condition 1, when both loads act alternatively, L2 almost solely determines the member sizes of 2 through 4, since values of t_2 through t_4 are almost identical with those obtained under the loading condition 3. On the other hand, the member 1 is not determined either by L1 or L2, but rather by the combined action of both. The member 3 which has a small value of t_3 under the application of L1, is required to have larger value of t_3 under L2, and

TABLE 3.2.1. Optimum Proportioning of the Two Storied Truss. Constant Mean Diameter

D mm	L.C.	t_1 mm²	t_2 mm²	t_3 mm²	t_4 mm²	VOL 10^4mm³
10	1	26.823 E_1	76.019 E_2	71.614 E_2	71.539 E_2	18.324
	2	51.536 E	51.176 E	0.527 W	0.533 W	6.253
	3	25.316 E	76.000 E	71.139 E	71.139 E	9.076
16	1	15.345 Y_1	37.500 Y_2	28.005 E_2	28.005 E_2	7.923
	2	24.034 Y	24.141 Y	2.835 W	2.774 W	3.367
	3	12.500 Y	37.490 Y	27.989 E	27.989 E	7.749
22	1	18.883 Y_1	37.500 Y_2	17.711 Y_2	17.679 Y_2	6.386
	2	23.226 Y	23.248 Y	5.237 W	5.310 W	3.683
	3	12.503 Y	37.500 Y	17.682 Y	17.682 Y	6.001
28	1	18.858 Y_1	37.500 Y_2	17.678 Y_2	17.679 Y_2	6.382
	2	22.131 Y	22.147 Y	8.447 W	8.500 W	4.095
	3	14.622 W	37.500 Y	17.601 Y	17.602 Y	6.115

Loading Condition 1=$L1$ and $L2$
2=$L1$ only
3=$L2$ only

Failure Mode E=Euler buckling
W=local wall buckling
Y=yielding
E_1, E_2=Euler buckling due to $L1$, $L2$
Y_1, Y_2=yielding due to $L1$, $L2$

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consequently, the load carried by the member 1 under L1 is reduced. Optimization of the structure under multiple loading systems is thus automatically done in the present example.

Table 3.2.1 gives the idea that Euler buckling constraint is predominant at smaller value of D and at moderate values, yielding becomes so, and still larger it becomes, local buckling tends to become critical. The total volume is small when buckling constraints are not active.

Some of the members seem to have far too small value of $\bar{t}$ than realistic. Although in the present example, since clarification of the effects of each constraint is aimed at, there were included no upper and lower boundaries of the variables except that required by the shear instability, setting realistic t_L and t_U seems necessary in actual applicaiton.

TABLE 3.2.2. Design Modification of the Two Storied Truss
Constant Diameter, $D=16$ mm
Loading Condition 1

σ_Y 20 kg/mm²

Initial Critical Stress:

Member	1	2	3	4
$\sigma_{cr,1}$	25.266	25.266	12.633	12.633
$\sigma_{cr,2}$	89.524	89.524	89.524	89.524

t_1 mm²	σ_1 kg/mm²	t_2 mm²	σ_2 kg/mm²	t_3 mm²	σ_3 kg/mm²	t_4 mm²	σ_4 kg/mm²
$C=V_0$,	$\varepsilon=0.05$						
25.000	14.876# 10.000##	25.000	14.843 30.000	25.000	7.246 14.142	25.000	7.293 14.142
24.996	14.434 10.001	66.406	6.581 11.294	27.940	7.046 12.654*	27.940	3.189 12.654*
15.450	19.919* 16.181	56.560	7.531 13.190	27.940	9.731* 12.654	27.940	3.634* 12.654
15.461	19.922* 16.170	37.670	10.577 19.910*	27.940	9.718 12.654*	27.940	5.142 12.654*
$C=2V_0$,	$\varepsilon=0.025$	CONVERGED					
$C=8V_0$,	$\varepsilon=0.00625$	CONVERGED					
$C=48V_0$,	$\varepsilon=0.00104$						
15.461	19.922 16.170	37.670	10.577 19.910	27.940	9.718 12.654	27.940	5.142 12.654
15.460	19.905 16.170	37.670	10.572 19.909	28.005	9.709 12.624*	28.005	5.139 12.624*
15.433	19.926 16.199	37.642	10.578 19.924	28.005	9.720 12.624*	28.005	5.142 12.624*
15.345	19.996* 16.292	37.554	10.598 19.971	28.005	9.755 12.624*	28.005	5.152 12.624*
15.345	19.996* 16.292	37.500	10.610 20.000*	28.005	9.755 12.624*	28.005	5.158 12.624*

Final Critical Stress:

Member	1	2	3	4
$\sigma_{cr,1}$	25.266	25.266	12.633	12.633
$\sigma_{cr,2}$	54.949	134.28	100.29	100.28

#, ##: stresses due to L1, L2

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3.3 Buckling load of a Beam

3.3.1 Finite Element Analysis of Beam Instability

References 61–63 suggest matrix displacement formulations of structural in-.tability problem. Since mathematically this is identical with vibrational problem, the same treatment may be used when the buckling load is included in one of the requirements imposed on the structure whose weight is to be minimized. A simply supported beam under axial compression is taken as an example to optimize its shape.

Gallagher and Padlog [63] formulated element force-displacement relationships including instability effects applicable to beams of constant stiffness, and excellent agreement of the result with the theoretical one is reported. Their formulations are rearranged to obtain

$$[\bar{k}_f]\{\bar{u}\}+F_{cr}[\bar{k}_h]\{\bar{u}\}=\{\bar{F}\} \tag{3.3.1}$$

$$[\bar{k}_f]=\frac{2EI_i}{\Delta l^3}\begin{bmatrix} 6 & -3\Delta l & -6 & -3\Delta l \\ -3\Delta l & 2\Delta l^2 & 3\Delta l & \Delta l^2 \\ -6 & 3\Delta l & 6 & 3\Delta l \\ -3\Delta l & \Delta l^2 & 3\Delta l & 2\Delta l^2 \end{bmatrix} \tag{3.3.2}$$

$$[\bar{k}_h]=\frac{1}{10\Delta l}\begin{bmatrix} 12 & -\Delta l & -12 & -\Delta l \\ -\Delta l & \frac{4}{3}\Delta l^2 & \Delta l & -\frac{1}{3}\Delta l^2 \\ -12 & \Delta l & 12 & \Delta l \\ -\Delta l & -\frac{1}{3}\Delta l^2 & \Delta l & \frac{4}{3}\Delta l^2 \end{bmatrix} \tag{3.3.3}$$

and

$$\begin{aligned} \{\bar{u}\}&=\{w_1, \theta_{y1}, w_2, \theta_{y2}\} \\ \{\bar{F}\}&=\{F_{z1}, M_{y1}, F_{z2}, M_{y2}\} \end{aligned} \tag{3.3.4}$$

where Δl and EI_i are length and bending stiffness of the i-th element respectively and F_{cr} is the critical load to be solved and, when tensile, is positive. The directions of the transverse and the angular displacements w, θ_y, force F_z and momet M_y are shown in Fig. 3.3.1.

Suppose a beam with the length l is divided into n elements of equal length, $\Delta l=l/n$. In order to non-dimensionalize (3.3.1), it is now divided by EI_0/l^2 or EI_0/l, where EI_0 is a reference bending stiffness.

$$[k_f]\{u\}+\lambda[k_h]\{u\}=\{F\} \tag{3.3.5}$$

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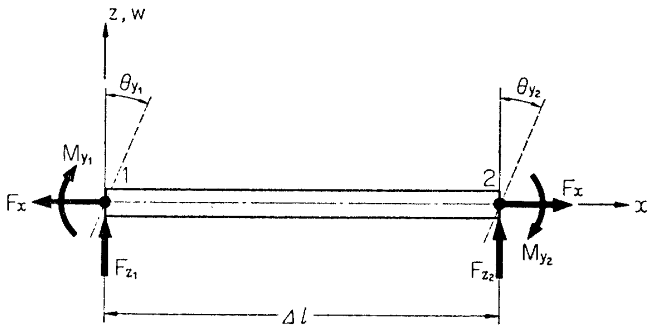

FIG. 3.3.1. Beam Element.

$$[k_f]=2n^3\frac{EI_i}{EI_0}\begin{bmatrix} 6 & -\frac{3}{n} & -6 & -\frac{3}{n} \\ -\frac{3}{n} & \frac{2}{n^2} & \frac{3}{n} & \frac{1}{n^2} \\ -6 & \frac{3}{n} & 6 & \frac{3}{n} \\ -\frac{3}{n} & \frac{1}{n^2} & \frac{3}{n^2} & \frac{2}{n^2} \end{bmatrix} \tag{3.3.6}$$

$$[k_h]=\frac{n}{10}\begin{bmatrix} 12 & -\frac{1}{n} & -12 & -\frac{1}{n} \\ -\frac{1}{n} & \frac{4}{3n^2} & \frac{1}{n} & -\frac{1}{3n^2} \\ -12 & \frac{1}{n} & 12 & \frac{1}{n} \\ -\frac{1}{n} & -\frac{1}{3n^2} & \frac{1}{n} & \frac{3}{4n^2} \end{bmatrix} \tag{3.3.7}$$

$$\begin{aligned} \{u\}&=\{w_1/l,\ \theta_{y1},\ w_2/l,\ \theta_{yz}\} \\ \{F\}&=l^2/EI_0\{F_{z1},\ M_{y1}/l,\ F_{z2},\ M_{y2}/l\} \end{aligned} \tag{3.3.8}$$

and

$$\lambda=F_{cr}l^2/EI_0 \tag{3.3.9}$$

$[k_f]$ and $[k_h]$ are assembled to yield $[K_f]$ and $[K_h]$ and

$$[K_f]\{u\}+\lambda[K_h]\{u\}=\{F\} \tag{3.3.10}$$

The vector $\{u\}$ is composed of transverse and angular displacements of all nodes, and $\{F\}$ represents external loads corresponding to $\{u\}$. Putting $\{F\}=0$, (3.3.10) becomes

$$-[K_f]\{u\}=\lambda[K_h]\{u\} \tag{3.3.11}$$

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TABLE 3.3.1. Buckling Load: Simply Supported

n	$\lambda=\frac{F_{cr}l^2}{EI_0}$	Error
1	-12.0	0.2159
2	-9.9440	0.00754
3	-9.8855	0.00161
4	-9.8748	0.00053
5	-9.8718	0.00022
6	-9.8707	0.00011
7	-9.8701	0.00005
8	-9.8698	0.00002
9	-9.8700	0.00004
10	-9.8693	-0.00003
11	-9.8701	0.00005
EXACT	-9.869607	

Assuming $EI_i/EI_0=1$, $i=1, \cdots n$, λ is solved for various values of n, and the results are shown in Table 3.3.1. It may be stated that a good agreement with theoretical value is obtained even when partioning is rough. With n larger than 8, the deviation is within the computing error.

3.3.2 Derivatives

When the stiffness EI_i is a function of t_i, the deviation rate of λ, with respect to t_i may be calculated following the procedure in *2.3.3*. When a sandwich beam or thin tubular beam with a constant mean diameter is considered, the stiffness is proportional to the variable t_i which is the thickness of face plate or tube correspondingly. When the cross-sectional shape is similar for all elements, it is proportional to t_i^2, where t_i represents cross sectional area. The total weight of the beam, in both cases, is represented by a linear combination of t_i's.

The case $n=3$ is taken as an example and $EI_i/EI_0=\bar{t}_i/\bar{t}_0=t_i$, $\bar{t}_i=\bar{t}_0$, $i=1, 2, 3$ are assumed where $\bar{t}_0$ is a reference value, and t_i the variable. The matrices $[k_f]$ and $[k_h]$ become

$$[k_f]=\begin{bmatrix} 324 & -54 & -324 & -54 \\ -54 & 12 & 54 & 6 \\ -324 & 54 & 324 & 54 \\ -54 & 6 & 54 & 12 \end{bmatrix} \tag{3.3.12}$$

$$[k_h]=\begin{bmatrix} 3.600 & -0.100 & -3.600 & -0.100 \\ -0.100 & 0.044 & 0.100 & -0.011 \\ -3.600 & 0.100 & 3.600 & 0.100 \\ -0.100 & -0.011 & 0.100 & 0.044 \end{bmatrix} \tag{3.3.13}$$

$[K_f]$ and $[K_h]$ become

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$$[K_f]=\begin{bmatrix} 12 & 54 & 6 & 0 & 0 & 0 \\ -54 & 648 & 0 & -324 & -54 & 0 \\ 6 & 0 & 24 & 54 & 6 & 0 \\ 0 & -324 & 54 & 648 & 0 & -54 \\ 0 & -54 & 6 & 0 & 24 & 6 \\ 0 & 0 & 0 & 0 & -54 & 12 \end{bmatrix} \tag{3.3.14}$$

$$[K_h]=\begin{bmatrix} 0.044 & 0.100 & -0.011 & 0.000 & 0.000 & 0.000 \\ 0.100 & 7.200 & 0.000 & -3.600 & -0.100 & 0.000 \\ -0.011 & 0.000 & 0.088 & 0.100 & -0.011 & 0.000 \\ 0.000 & -3.600 & 0.100 & 7.200 & 0.000 & -0.100 \\ 0.000 & -0.100 & -0.011 & 0.000 & 0.088 & -0.011 \\ 0.000 & 0.000 & 0.000 & -0.100 & -0.011 & 0.044 \end{bmatrix} \tag{3.3.15}$$

(3.3.11) is now solved by the power method and

$$\lambda=-9.8855 \tag{3.3.16}$$

$$\{u\}=\{1.000 \quad -0.276 \quad 0.500 \quad -0.276 \quad -0.500 \quad -1.000\} \tag{3.3.17}$$

$[P]$ matrix defined by (2.3.16) is, by using (3.3.14), (3.3.15) and (3.3.16),

$$[P]=[K_f]+\lambda[K_h]$$

$$=\begin{bmatrix} 11.561 & 53.011 & 6.110 & 0.000 & 0.000 & 0.000 \\ 53.011 & 576.82 & 0.000 & -288.41 & -53.011 & 0.000 \\ 6.110 & 0.000 & 23.121 & 53.011 & 6.110 & 0.000 \\ 0.000 & -288.41 & 53.011 & 576.82 & 0.000 & -53.011 \\ 0.000 & -53.011 & 6.110 & 0.000 & 23.121 & 6.110 \\ 0.000 & 0.000 & 0.000 & -53.011 & 6.110 & 11.561 \end{bmatrix} \tag{3.3.18}$$

The vector $\{Q\}$ defined by (2.3.16) is, by using (3.3.15) and (3.3.17)

$$\{Q\}=[K_h]\{u\}=\{0.011 \quad -0.843 \quad 0.011 \quad -0.843 \quad -0.011 \quad -0.011\} \tag{3.3.19}$$

The first column of $[P]$ is replaced by $\{Q\}$ and

$$[P']=\begin{bmatrix} 0.011 & 53.011 & 6.110 & 0.000 & 0.000 & 0.000 \\ -0.843 & 576.82 & 0.000 & -288.41 & -53.011 & 0.000 \\ 0.011 & 0.000 & 23.121 & 53.011 & 6.110 & 0.000 \\ -0.843 & -288.41 & 53.011 & 576.82 & 0.000 & -53.011 \\ -0.011 & -53.011 & 6.110 & 0.000 & 23.121 & 6.110 \\ -0.011 & 0.000 & 0.000 & -53.011 & 6.110 & 11.561 \end{bmatrix} \tag{3.3.20}$$

Derivatives with respect to t_1 is first sought to be obtained. The vector $\{R\}$ in (2.3.16) becomes simply

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$$\{R\}=\left(\frac{\partial}{\partial t_1}[K_f]\right)\{u\}$$

$$=\begin{bmatrix} 12 & 54 & 6 & 0 & 0 & 0 \\ 54 & 324 & 54 & 0 & 0 & 0 \\ 6 & 54 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 1.000 \\ -0.276 \\ 0.500 \\ -0.276 \\ -0.500 \\ -1.000 \end{Bmatrix}=\begin{Bmatrix} 0.112 \\ -8.329 \\ -2.888 \\ 0.000 \\ 0.000 \\ 0.000 \end{Bmatrix} \qquad (3.3.21)$$

And

$$\{X\}=-[P']^{-1}\{R\}=\{-1.934,\ -0.044,\ 0.364,\ -0.092,\ -0.098,\ -0.374\} \qquad (3.3.22)$$

The first component of $\{X\}$ shows the derivative of λ and the rest, derivatives of mode.

$$\partial\lambda/\partial t_1=-1.934$$
$$\partial\{u\}/\partial t_1=\{0.000,\ -0.044,\ 0.364,\ -0.092,\ -0.098,\ -0.374\}$$

TABLE 3.3.2. The Strongest Beam: Simply Supported

Case 1;

$EI_i=(t_i/t_0)EI_0=t_iEI_0$

n	λ	t_1	t_2	t_3	t_4	t_4	t_5
3	−10.6278	0.816	1.369				
4	−11.0334	0.673	1.327				
5	−11.2817	0.670	1.212	1.437			
6	−11.4450	0.494	1.095	1.411			
7	−11.5564	0.433	0.992	1.341	1.468	Simmetrical	
8	−11.6387	0.387	0.903	1.260	1.451		
9	−11.6961	0.346	0.828	1.185	1.402	1.478	
10	−11.7437	0.319	0.761	1.111	1.344	1.463	
11	−11.7787	0.291	0.706	1.043	1.288	1.433	1.481

Case 2;

$EI_i=(t_i/t_0)^2EI_0=t_i^2EI_0$

n	λ	t_1	t_2	t_3	t_4	t_5	t_6	t_7
3	−10.8525	0.882	1.236					
4	−11.4318	0.786	1.214					
5	−11.8097	0.710	1.151	1.279				
6	−12.0696	0.649	1.081	1.269				
7	−12.2569	0.600	1.020	1.231	1.299		Simmetrical	
8	−12.3967	0.558	0.963	1.186	1.293			
9	−12.5034	0.524	0.913	1.142	1.266	1.310		
10	−12.5880	0.491	0.869	1.100	1.238	1.304		
11	−12.6569	0.475	1.055	1.198	1.287	1.321		
12	−12.7158	0.449	0.792	1.018	1.161	1.261	1.319	
13	−12.7606	0.425	0.760	0.983	1.132	1.236	1.303	1.326

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In the similar manner, the rest of derivatives are obtained.

$$\partial\lambda/\partial t_2 = -6.018$$
$$\partial\{u\}/\partial t_2 = \{0.000,\ 0.033,\ -0.275,\ 0.033,\ 0.275,\ 0.000\}$$
$$\partial\lambda/\partial t_3 = -1.934$$
$$\partial\{u\}/\partial t_3 = \{0.000,\ 0.011,\ -0.088,\ 0.059,\ -0.177,\ 0.373\}$$

Total computation time is less than one second.

When the element stiffness is proportional to t_i^2, the similar process may generate derivatives. In this case the matrix in (3.3.21) is multiplied by $2t_i$. The value of λ is identical with that of the former since t_i's are unity, and derivatives are

$$\partial\lambda/\partial t_1 = -3.868$$
$$\partial\lambda/\partial t_2 = -12.035$$
$$\partial\lambda/\partial t_3 = -3.868$$

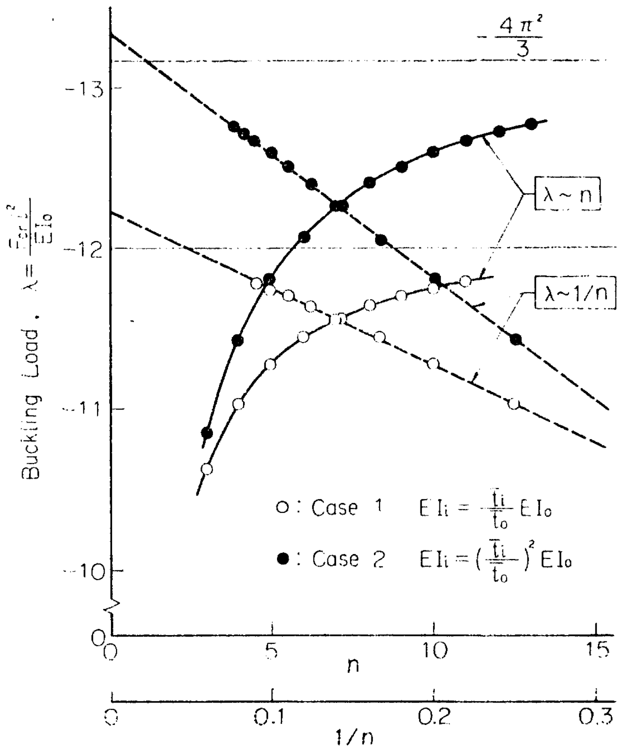

FIG. 3.3.2. Buckling Load at Optimum Shape.

3.3.3 The Strongest Beam

The shape of the beam composed of n elements are optimized so as to maximize its critical compressive load with its weight held constant. The cross sectional shape is assumed to be constant within an element, which is to be determined by the variable.

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The objective to be minimized is λ and only one constraint that is of equality

$$\sum_{i=1}^{n} t_i = \sum_{i=1}^{n} \bar{t}_i / \bar{t}_0 = \text{const.} \tag{3.3.23}$$

is incorporated.

Initially, $t_i=1$, $i=1, \ldots n$ are assumed. The gradient of the objective is composed of the derivatives of λ. $-\text{grad}\,\lambda$ is projected on to the plane defined by (3.3.23) and the direction of movement is found out. After the similar procedure as that shown in 2.4, the optimum point at which $-\text{grad}\,\lambda$ is perpendicular to the plane (3.3.23) is obtained. The optimum shape and corresponding eigen-value λ are shown in Table 3.3.2 for various values of n and for two cases;

Case 1:

$$EI_i = \frac{\bar{t}_i}{\bar{t}_0} EI_0 = t_i EI_0$$

Case 2:

$$EI_i = \left(\frac{\bar{t}_i}{\bar{t}_0}\right)^2 EI_0 = t_i^2 EI_0$$

Fig. 3.3.2 shows plots of λ against n and $1/n$. It can be seen that λ is almost linearly related to $1/n$ and the value when n tends to infinity, i.e., when the cross sectional shape varies continuously, can be extrapolated. Refs. 41 and 84 give analytical solution of both cases for continuous cross sectional shape

Case 1:

$$\lambda_{\min} = -12$$

$$t(\xi) = 6(\xi - \xi^2)$$

Case 2:

$$\lambda_{\min} = -\frac{4}{3}\pi^2$$

$$t(\xi) = \frac{4}{3} \sin^2 \theta(\xi)$$

$$\theta - \frac{1}{2} \sin 2\theta = -\pi\xi$$

where

$$0 \leqq \xi = x/l \leqq 1/2$$

Fig. 3.3.3 shows the optimum proportioning for various value of n and the analytical results. The plots of t_i at the middle of each corresponding element

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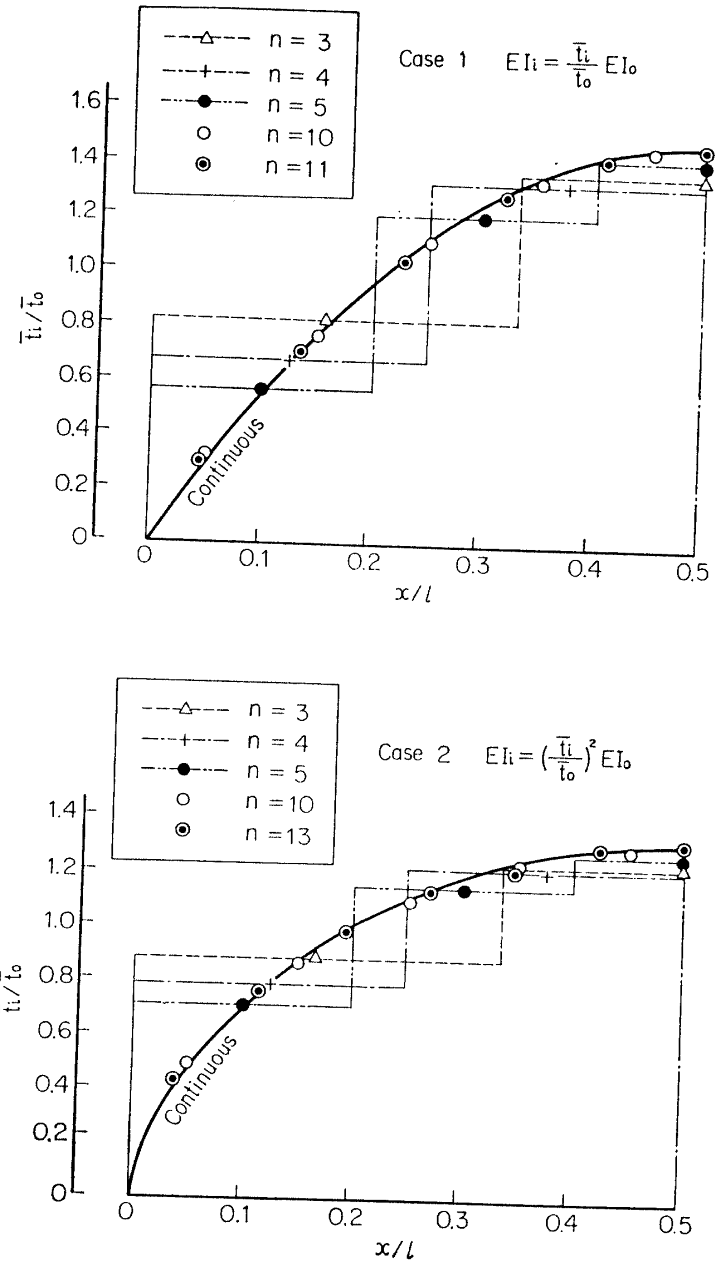

FIG. 3.3.3. Optimum Proportioning of Simply Supported Beam.

show agreement with that of continuous curve, but Fig. 3.3.2 also suggests that n must be considerably increased in order that λ becomes close to $\lambda_{\min}$. If tapered elements are used in generating the stiffness matrices $[k_f]$, λ would become closer to $\lambda_{\min}$ for relatively smaller value of n.

4. Conclusions

In reviewing recent development of techniques in minimum weight and optimum design of structures, it becomes clear that iterative means is the most adequate one from the stand point of today's computation ability using high speed digital

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computers and of needs to deal with complexity of the requirements. The proposed method herein takes full advantage of existing numerical methods of structural analysis and showed an ability to cope with complex problems expected in aerospace applications.

The existing numerical methods incorporated are:

1) Finite element method which presents a realistic basis to analyze both static and dynamic behaviors of structures and incorporation of substructure method which enables us to handle even the most complex structures.
2) Non-linear programming techniques which yield economical redesign cycles toward the most desirable.

The new concepts incorporated into the present method are:

1) Simple procedures deriving derivatives of frequencies with respect to design variables.
2) Application of the substructure method into structural minimum weight problem.

Using the above mentioned methods and concepts, the author presented a general method of minimum weight design with requirements imposed on stresses caused by several alternative loadings and natural frequencies. Although methods with strength requirement alone taken into consideration are known, incorporation of derivatives of frequencies first made it possible to optimize structures with static and vibrational behaviors considered simultaneously. It is also shown that buckling requirements can be treated in the same manner as those for vibrational behaviors. Since the substructure method can be used in analyzing a structure, a complex and realistic structure can be treated by this method.

Various techniques of non-linear programming have been considered to find the most adequate one for the present purpose, and the steepest descent method is chosen as the basic philosophy from the view point of computer storage economy. Linearization concept of behaviors is found to be inadequate to certain types of structures and requirements. Constrained path concept which is analogous to the gradient projection method is successfully applied to simple examples.

In the first example a cantilevered plate beam divided into four sections is optimized in terms of the thickness of each section. Requirements are given on stresses caused by a vertical tip load and on the first bending frequency. The linearization and constrained path concepts are applied in optimization process and the latter was found superior. Two storied plane truss is considered as the second example and optimized with the requirements that it is safe against Euler and local buckling of tubular truss members and over-all shear buckling as well as yielding under two alternative loadings. The third example shows feasibility of including buckling load obtained as an eigen-value into optimization procedure. A simply supported beam under axial compression is optimized in its shape and results are compared with analytically obtained ones. It was shown in this example that the method can be extended into the problems in which design variables are quadratically related to stiffnesses.

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The present method can be further extended without loosing the underlying philosophy. The statements of the limitations which the method is subjected to, will clarify at the same time its direct extensions which is possible.

1) Thermal effects are not taken into account.
2) Only one variable is allowed to exist within a substructure.
3) Bending terms are excluded from the plate element stiffness.
4) Modification of the geometrical configuration is not possible.

Thermal stresses can be taken into account by modifying external loads. Inability to modify geometrical configuration is probably the most significant limitation this method is subjected to. It will, without any doubt, decrease the final structural weight, and may drive out under-stressed members which otherwise are the most desirable. Although some effort is directed toward this problem [*4*], further development seems necessary.

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It must be mentioned that calculations have been carried out at Data Processing Center, ISAS, and laborious work of computation is highly appreciated.

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Department of Space Technology
Institute of Space and Aeronautical Science
University of Tokyo, Tokyo
May 10, 1970

Nomenclature

A_i	$=$weight of the i-th section with t_i unity
$\varDelta A$	$=$area of triangular panel element or length of truss element
$[B]$	$=$matrix defined by (1.2.1) $\{\varepsilon\}=[B]\{\delta\}^e$

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C = positive constant in the penalty term (cf. (2.4.1) $f=f_1+C\sum\langle g\rangle$)
D = mean diameter of tubular cross section
$[D]$ = matrix defined by (1.2.2) $\{\sigma\}=[D]\{\varepsilon\}$
$[DBN]$ = $[D][B][N]$ or $[D][B][T][N]$ = matrix defined by (1.2.17) $\{\sigma\}=[DBN]\{u\}$
E = Youngs' modulus
EI = bending stiffness
f = $f_1+C\sum\langle g\rangle^2$ = modified objective function
$f_{\min}$ = minimum of f
f_1 = $\sum A_i t_i$ = objective function, weight function
$\{F\}$ = force vector
F_z = transverse force
g = constraint
$[G]$ = matrix consisting of m rows of grad g_i
$[I]$ = unit matrix
$[k]^e$ = element stiffnes matrix
$[K]$ = total stiffness matrix of a structure
$[K_i]$ = stiffness matrix of the i-th section with t_i being unity
$[K_b]$ = boundary stiffness matrix
$[K_{bb}]$, $[K_{bi}]$, $[K_{ib}]$, $[K_{ii}]$ = submatrices
$[k_f]$ = bending stiffness matrix of a beam element
$[k_h]$ = additional bending stiffness matrix of a beam element due to axial elongation or compression
$[K_f]$ = total stifiness matrix corresponding to $[k_f]$
$[K_h]$ = total additional stiffness matrix corresponding to $[k_h]$
l = length
Δl = length of an element
m = number of constraints active
M = number of constraints
M_y = bending moment
$[M]$ = diagonal mass matrix
$[M_i]$ = diagonal mass matrix of the i-th section with t_i being unity
n = number of independent design variables
N = number of nodes
$[N]$ = matrix defined by (1.2.5) $\{\delta\}^e=[N]\{u\}$
$\{p\}$ = projection of f_1 on a plane locally tangent to constraint surfaces
$\{p'\}$ = projection of f on a plane locally tangent to constraint surfaces
$[P]$, $\{Q\}$, $\{R\}$ defined by (2.3.16)
$[P]=[K]-\lambda[M]$, $\{Q\}=[M]\{u\}$,
$\{R\}=(-\Delta[K]+\lambda\Delta[M])\{u\}/\Delta t$
$[P']$ = matrix $[P]$ with a column replaced by $\{Q\}$
$\{Q_b\}$ = equivalent boundary force vector
$\{R_b\}$ = boundary reaction vector
S = set of $\{t\}$ satisfying constraints

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t	=independent variable, representing thickness of panel or cross sectional area of truss and beam
$\bar{t}$	=actual thickness or cross sectional area corresponding to t
$\{t\}$	$=\{t_1\cdots t_n\}$=design point
$\{t\}^k$	=design point at the k-th zigzag
$\{t\}^{(k)}$	=design point at the k-th move in a zigzag
t_c	=convergence criterion
$t_{U,i}$, $t_{L,i}$	=upper and lower boundaries respectively of the variable t_i
$[T]$	=matrix composed of direction cosines
V_0	=initial volume of the structure
u, v	=displacement components in x and y directions respectively
$\{u\}$	=displacement vector of the structure
$\{u_b\}$, $\{u_i\}$	=boundary and interior displacement vector, respectively
$0-x, y$	=rectangular co-ordinate system
$\{X\}$	=vector composed of derivatives of mode and eigen-value
$\{\alpha\}$	=parameters defined by (1.4.14) $\{p\}=\text{grad}\, f_1+[G]\{\alpha\}$
$\{\alpha'\}$	=parameters defined by $\{p'\}=\text{grad}\, f+[G]\{\alpha'\}$
δ	=scalar
δ_m	=scalar which makes $f^1=f^0-\delta_m \text{grad}\, f_1$ minimum
$\{\delta\}^e$	=element displacement vector
ε	=small positive constant determining active constraints
$\{\varepsilon\}$	$=\{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}$ vector composed of strain components
θ	=arbitrary scalar, $0\leqq\theta\leqq 1$
$\bar{\theta}$	$=1-\theta$
λ	=eigen value
ν	=Poisson's ratio
ρ	=relaxation constant
ρ_0	=scalar defining the distance of travel
$\{\sigma\}$	$=\{\sigma_x, \sigma_y, \tau_{xy}\}$=vector composed of stress components
$\sigma_{0,i}$	=maximum equivalent stress $(\sigma_x^2+\sigma_y^2-\sigma_x\sigma_y+3\tau_{xy}^2)^{1/2}$ within the i-th section
σ_Y	=yield stress
$\sigma_{cr,k}$	=critical stress of the k-th buckling mode
ω_k	=natural frequency of the k-th vibration mode
$\omega_{cr,k}$	=critical frequency of the k-th vibration mode
[ ]	=matrix or row vector
{ }	=column vector
$(\ )^T$	=matrix transpose
$(\ )^{-1}$	=matrix inverse

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REFERENCES

[1] Gerard, G.: Minimum Weight Analysis of Compression Structures, (N. Y. U. Press, New York, 1956).

[2] Shanley, F. R.: Weight-Strength Analysis of Aircraft Structures, (McGraw-Hill Book Co., New York, 1952).

[3] Cox, H. L.: The Design of Structures of Least Weight, (Oxford and Pergamon Press 1965).

[4] Crawford, R. F. and Burns, A. B.: "Minimum Weight Potentials for Stiffened Plates and Shells, "AIAA J., Vol. 1, No. 4, April, 1963, pp. 879–886.

[5] Burns, A. B. and Almroth, B. O.: "Structural Optimization of Axially Compressed, Ring-Stringer Stiffened Cylinders", J. Spacecraft and Rocket, Vol. 3, No. 1, Jan., 1966, pp. 19–25.

[6] Burns, A. B.: "Structural Optimization of Axially Compressed Cylinders, Considering Ring-Stringer Eccentricity Effects", J. Spacecraft and Rocket, Vol. 3, No. 8, Aug., 1966, pp. 1263–1268.

[7] McWithey, R. R.: "Minimum Weight Analysis of Symmetrical-Multiweb Beam Structures Subjected to Thermal Stress", NASA TN D-104, Oct., 1959.

[8] Gerard, G.: "Optimum Structural Design Concepts for Aerospace Vehicles", J. Spacecraft and Rocket, Vol 3, No. 1, Jan., 1966, pp. 5–18.

[9] Cohen, G. A.: "Optimum Design of Truss-Core Sandwich Cylinders Under Axial Compression", AIAA J., Vol. 1, No. 7, July, 1963, pp. 1626–1630.

[10] Schmit, L. A.: "Structural Design by Systematic Synthesis", Proceedings of the 2nd National Conference of Electronic Computation, (American Society of Civil Engineers, New York, 1960).

[11] Schmit, L. A. and Kitcher, T. P.: "Structural Synthesis of Symmetric Waffle Plates", NASA TN D-1691, Dec., 1962.

[12] Schmit, L. A., Kitcher, T. P. and Morrow, W. M.: "Structural Synthesis Capability for Integrally Stiffened Waffle Plates", AIAA J., Vol. I, No. 112, Dec., 1963, pp. 2820–2836.

[13] Schmit, L. A. and Mallett, R. H.: "Structural Synthesis and Design Parameter Hierarchy", J. Str. Div., ASCE, Vol. 89, No. ST4, Aug., 1963, pp. 269–299.

[14] Schmit, L. A. and Fox, R. L.: "An Integrated Approach to Structural Synthesis and Analysis", AIAA J., Vol. 3, No. 6, June, 1965, pp. 1104–1112.

[15] Gellatley, R. A. and Gallagher, R. H.: "A Procedure for Automated Minimum Weight Structural Design, Part 1: Theoretical Basis", The Aeron. Quarterly, Aug., 1966, pp. 216–230.

[16] Gellatley, R. A. and Gallagher, R. H.: "A Procedure for Automated Minimum Weight Structural Design, Part II: Applications", The Aeron. Quarterly, Nov., 1966, pp. 332–342.

[17] Kitcher, T. P.: "Structural Synthesis of Integrally Stiffened Cylinders", J. Spacecraft and Rockets, Vol. 5, No. 1, Jan., 1968, pp. 62–67.

[18] Moses, F.: "Optimum Structural Design Using Linear Programming", Jour. Struc. Div., ASCE, Vol. 90, No. ST6, Dec., 1964.

[19] Romstad, K. M. and Wang, Chu-Kia: "Optimum Design of Framed Structures", Jour. Struc. Div. ASCE, Vol. 44, No. ST12, Dec., 1968.

[20] Brown, D. M. and Ang, A. H.: "Structural Optimization by Non-linear Programming", Jour. Struc. Div., ASCE, Vol. 92, No. ST6, Dec., 1966.

[21] Reinschmidt, K. K., Cornell, A. and Brotchie, J. R.: "Iterative Design and Structural Optimization", Jour. Struc. Div., ASCE, Vol. 92, No. ST6, Dec., 1966.

[22] Rubinstein, M. F. and Karagozian, J.: "Building Design Using Linear Programming", Jour. Struc. Div. ASCE, Vol. 92, No. ST6, Dec., 1966.

[23] Richards, D. M.: The Sequential Design of Redundant Elastic Structures", Conference on Recent Advances in Stress Analysis, Joint British Committee for Stress Analysis, March, 1968.

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[24] Moses, F. and Kinser, D. E. "Optimum Structural Design with Failure Probability Constraints", AIAA J., Vol. 5, No. 6, June, 1967, pp. 1152–1158.
[25] Zarghamee, M. S.: "Minimum Weight Design of Enclosed Antennas", Jour. Struc. Div. ASCE, No. ST6, June, 1969.
[26] Gould, P. L., "Minimum Weight Design of Hyperbolic Cooling Towers", Jour. Struc. Div. ASCE, Feb., 1969.
[27] Fox, R. L.: "Constraint Surface Normals for Structural Synthesis Techniques Based on Matrix Analysis", AIAA J., Vol. 3, No. 8, Aug., 1965, pp. 1517–1518.
[28] Best, G. A.: "A Method of Weight Minimization Suitable for High-Speed Digital Computers", AIAA J., Vol. 1, No. 3, Feb., 1963, pp. 478–479.
[29] Razani, R.: "Behavior of Fully Stressed Design of Structures and Its Relationship to Minimum-Weight Design", AIAA J., Vol. 3, No. 12, Dec. 1965, pp. 2262–2268.
[30] Dayaratnam, P. and Patnaik, S.: "Feasibility of Full Stress Design", AIAA J., Vol. 7, No. 4, April, 1969, pp. 773–774.
[31] Kitcher, T. P.: "Optimum Design Minimum Weight Versus Fully Stressed,", Jour. Struc. Div., ASCE, Vol. 92, No. ST6, Dec., 1966.
[32] Kuhn, H. W. and Tucker, A. W.: "Non-linear Programming", Proceeding of the 2nd Berkley Symposium on Math, Statistics and Probability, (Univ. of Cal. Press, Berkley, Calif., 1951), pp. 481–492.
[33] Maruyasu, T., Nakamura, H., Murai, S. and Wakabayashi, Y.: "Optimization in Civil Engineering Design—Basic Concepts and Procedures", Rept. Institute of Ind. Sci., Univ. Tokyo, Vol. 19, No. 4, Sept., 1969.
[34] Turner, M. J.: "Optimization of Structures to Satisfy Flutter Requirements", AIAA J., Vol. 7, No. 5, 1969, pp. 945–951.
[35] Turner, M. J.: "Design of Minimum Mass Structures with Specified Natural Frequencies", AIAA J., Vol. 5, No. 3, March, 1967, pp. 406–412.
[36] Taylor, J. E.: "Minimum Mass Bar for Axial Vibration at Specified Natural Frequency", AIAA J., Vol. 5, No. 10, Oct., 1967, pp. 1911–1913.
[37] Taylor, J. E.: "Optimum Design of a Vibrating Bar with Specified Minimum Cross Section", AIAA J., Vol. 6, No. 7, July, 1968, pp. 1379–1381.
[38] Prager, W.: "Problems of Optimal Structural Design", Trans. ASME, March, 1968, pp. 102–106.
[39] Niordson, F. I.: "On the Optimal Design of Vibrating Beam", Quarterly of Applied Mathematics", Vol. 23, No. 1, 1965, pp. 47–53.
[40] McIntosh, S. C. and Eastep, F. E.: "Design of Minimum-Mass Structures with Specified Stiffness Properties", AIAA J., Vol. 6, No. 5, May, 1968, pp. 962–964.
[41] Taylor, J. E.: "The Strongest Column: An Energy Approach", Trans. ASME, June, 1967, pp. 486–487.
[42] Keller, J. B. and Niordson, F. L.: "The Tallest Column", J. Math. Mech., Vol. 16, No. 5, 1966, pp. 433–446.
[43] Wang, H. C. and Worley, W. J.: "An Approach to Optimum Shape Determination for a Class of Thin Shells of Revolution", Trans. ASME, Sept., 1968, pp. 524–529.
[44] Huang, N. C. and Sheu, C. Y.: "Optimal Design of an Elastic Column of Thin Walled Cross Section", Trans. ASME, June, 1968, pp. 285–288.
[45] Zienkiewicz, O. C. and Cheung, Y. K.: The Finite Element Method in Structural and Continuum Mechanics, (McGraw-Hill Book Co., Maidenhead, 1967).
[46] Przemieniecki, J. S.: Theory of Matrix Structural Analysis, (McGraw-Hill Book Co., NY, 1968).
[47] Martin, H. C.: Introduction to Matrix Methods of Structural Analysis, (McGraw-Hill Book Co., NY, 1966).
[49] Levy, S.: "Structural Analysis and Influence Coefficients for Delta Wings", J. Aeronaut. Sci., Vol. 20, July, 1953, pp. 449–454.
[50] Tuner, M. J., Clough, R. W., Martin, H. C. and Topp, L. J.: "Stiffness and Deflection Analysis of Complex Structures", J. Aeronaut. Sci., Vol. 23, No. 9, Sept., 1956, pp. 805–823.

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[51] Clough, R. W.: "The Finite Element Method in Plane Stress Analysis", Proc. 2d Conf. Electron. Computation, ASCE, Sept., 8–9, 1960.
[52] Klein, B.: "A Simple Method of Matric Structureal Analysis", J. Aeron. Sci., Vol. 24, No. 1, Jan., 1957, pp. 39–46.
[53] Klein, B.: "A Simple Method of Matric Structural Analysis, II. Effects of Taper and a Consideration of Curvature", J. Aeron. Sci., Vol. 24, No. 11, Nov., 1957, pp. 813–820.
[54] Klein, B.: "A Simple Method of Matric Structural Analysis, III. Analysis of Flexible Frames and Stiffened Cylindrical Shells", J. Aerospace Sci., Vol. 25, No. 6, June, 1958, pp. 385–394.
[55] Klein, B.: "A Simple Method of Matric Structural Analysis, V. Structures Containing Plate Elements of Arbitrary Shape and Thickness", J. Aerospace Sci., Vol. 27, No. 11, 1960, pp. 859–866.
[56] Melosh, R. J.: "A Stiffness Matrix for the Analysis of Thin Plates in Bending", J. Aerospace Sci., Vol. 28, No. 1, Jan. 1961, pp. 34–43.
[57] Melosh, R. J.: "Basis for Derivation of Matrices for the Direct Stiffness Method", AIAA J., Vol. 1, No. 7, July, 1963, pp. 1631–1637.
[58] Pian, T. H. H.: "Derivation of Element Stiffness Matrices by Assumed Stress Distributions", AIAA J., Vol. 2, No. 7, July, 1964, pp. 1333–1336.
[59] Stoker, J. R.: "Fully Automatic Structural Analysis by the Matrix Force Method", Conference on Recent Advances in Stress Analysis, JBCSA, Mar. 27, 1968.
[60] Robinson, J.: "Practical Application of a Computerised Structural Analysis System which Adopts a Finite Element Technique", Conference on Recent Advances in Stress Analysis, JBCSA, Mar. 27, 1968.
[61] Kapur, K. K. and Hartz, B. J.: "Stability of Plates Using the Finite Element Method", J. Eng. Mech. Div., ASCE, Vol. 92, No. EM2, April, 1966, pp. 175–195.
[62] Carson, W. G. and Newton, R. E.: "Plate Buckling Analysis Using a Fully Compatible Finite Element", AIAA J., Vol. 7, No. 3, March, 1969, pp. 527–529.
[63] Gallagher, R. H. and Padlog, J.: "Diserete Element Approach to Structural Instability Analysis", AIAA J., Vol. 1, No. 6, June, 1963, pp. 1437–1439.
[64] Leckie, F. A. and Lindberg, G. M.: "The Effect of Lumped Parameters on Beam Frequencies", Aeron. Quart., Vol. 14, Aug., 1963, pp. 224–240.
[65] Archer, J. S.: "Consistent Matrix Formulation for Structural Analysis Using Finite Element Techniques", AIAA J., Vol. 3, No. 10, Oct., 1965, pp. 1910–1918.
[66] Dawe, D. J.: "A Finite Element Approach to Plate Vibration Problems", J. Mech. Eng. Sci., Vol. 7, No. 1, 1965, pp. 28–32.
[67] Guyan, R. J.: "Distributed Mass Matrix for Plate Element Bending", AIAA J., Vol. 3, No. 3, March, 1965, pp. 567–568.
[68] Brown, J. E., Hutt, J. M. and Salama, A. E.: "Finite Element Solution to Dynamic Stability of Bars", AIAA J., Vol. 6, No. 7, July, 1968, pp. 1423–1425.
[69] Olson, M. D.: "Finite Elements Applied to Panal Flutter", AIAA J., Vol. 5, No. 12, Dec., 1967, pp. 2267–2270.
[70] Przemieniecki, J. S.: "Matrix Structural Analysis of Substructures", AIAA J., Vol. 1, No. 1, Jan., 1963, pp. 138–147.
[71] Hurty, W. C.: "Dynamic Analysis of Structural Systems Using Component Modes", AIAA J., Vol. 3, No. 4, April, 1965, pp. 678–685.
[72] Gladwell, G. M. L.: "Branch Mode Analysis of Vibrating Systems", J. Sound Vib., Vol. 11, 1964, pp. 41–59.
[73] Craig, R. R. and Bampton, M. C. C.: "Coupling of Substructures for Dynamic Analysis", AIAA J., Vol. 6, No. 7, July, 1968, pp. 1313–1319.
[74] Goldman, R. L.: "Vibration Analysis by Dynamic Partitioning", AIAA J., Vol. 7, No. 6, June, 1969, pp. 1152–1154.
[75] Wolfe, P.: "Methods of Nonlinear Programming", Chap. VI, Nonlinear Programming, Abadie ed., (North-Holland Publishing Co., Amsterdam, 1967).
[76] Kelley, H. J.: "Method of Gradients", Optimization Techniques, Leitmann ed., (Academic Press, 1962).

This document is provided by JAXA.

[77] Mayers, J. and Budiansky, B.: "Analysis of Simply Supported Flat Plates Compressed Beyond the Buckling Load into the Plastic Range", NACA TN 3368, Feb., 1965.

[78] Cross, H.: "The Relation of Analysis of Structural Design", Transactions, ASCE, Vol. 62, No. 8, 1936, pp. 1363–1408.

[79] Yasaka, T.: "Vibration Analysis of Low Aspect-Ratio Wing", Master Thesis, Univ. Tokyo, March, 1967.

[80] Yasaka, T.: "Minimum Weight Problem Using Finite Element Method", Symposium on Stru. Strength, Kyoto, 1969.

[81] Rosen, J. B.: "The Gradient Projection Method for Nonlinear Programming. Part I. Linear Constraints", J. Soc. Indust. Appl. Math., Vol. 8, No. 1, March, 1960, pp. 181–217.

[82] Rosen, J. B.: "The Gradient Projection Method for Nonlinear Programming. Part II. Nonlinear Constraints", J. Soc. Indust. Appl. Math., Vol. 9, December, 1961, pp. 514–532.

[83] Hayashi, T.: "On the Shear Instability of Structures Caused by Compressive Load", Proc. 6th, Japan National Congress for Applied Mechanics, 1966, pp. 149–157.

[84] Tadjbakhsh, I. and Keller, J. B.: "Strongest Columns and Isoperimetric Inequalities for Eigenvalues", Transactions of the ASME, March, 1962, pp. 159–164.

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