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Approximate Solutions of Laminar Flow Between Two Rotating Coaxial Disks

By

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Summary: On the basis of boundary-layer approximations, an analysis is made for the laminar flow between two rotating coaxial disks in the following cases:
(A) Both disks are rotating with different velocities.
(B) One disk is held fixed, and the other disk is rotating at a constant velocity, together with uniform discharge of fluid.
(C) A source at the centre produces small radial net flow in Case A.
Some numerical results are obtained for the case in which one disk is at rest.

1. INTRODUCTION

The steady rotationally-symmetric laminar flow between two rotating coaxial disks has been investigated by Batchelor [*1*] and by Stewartson [*2*], and recently by Kreith and Viviand [*3*] for the case with a source in the centre.

This problem is also important in engineering in connection with friction of turbine disks [*4*], thrust bearings [*5*], viscosity pumps [*6*], and so forth. The foregoing studies, however, lead to the limited information pertinent to those phenomena. In order to obtain a better understanding of the phenomena associated with an enclosed disk, Soo [7] presented a simplified analysis introducing the usual boundary-layer approximations for the case of small Reynolds number and small radial net flow. He obtained, however, erroneous results by considering that the friction on two disks must be equal.

In the present paper, introducing the boundary-layer approximations as Soo did, an analysis will be made for the laminar flow between two rotating coaxial disks in the following cases:

(A) Both disks are rotating with different velocities.
(B) One disk is held fixed, and the other disk is rotating at a constant velocity, together with uniform discharge of fluid.
(C) A source at the centre produces small radial net flow in Case A.

2. SYSTEM AND THE GOVERNING EQUATIONS

As shown in Fig. 1, the system consists of two disks, which occupy the planes $z=0$ and $z=d$ and are rotating with constant angular velocities $s\Omega$ and Ω respec-

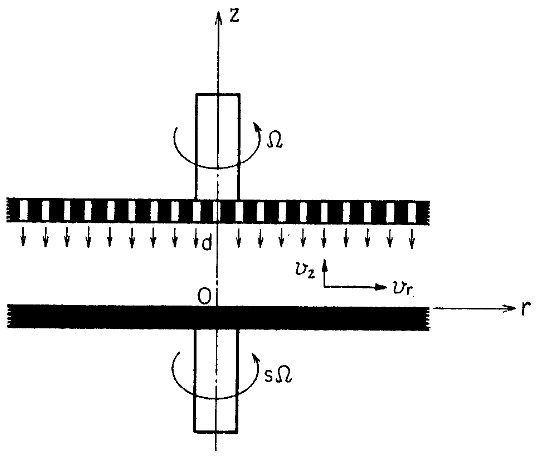


FIG. 1. Coordinate System.

tively ($|s|\leqq 1$), about the z-axis, in an incompressible viscous fluid.

Introducing the usual boundary-layer approximations, and restricting discussion to the case of the steady rotationally-symmetric flow, the momentum equations and the equation of continuity for the present case may be expressed in cylindrical polar coordinates as

$$\left.\begin{aligned} v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}&=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\frac{\partial^2 v_r}{\partial z^2}\\ v_r\frac{\partial v_\theta}{\partial r}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}&=\nu\frac{\partial^2 v_\theta}{\partial z^2}\\ \frac{\partial p}{\partial z}&=0 \end{aligned}\right\} \quad (1)$$

$$\frac{\partial(rv_r)}{\partial r}+\frac{\partial(rv_z)}{\partial z}=0 \quad (2)$$

where v_r, v_θ, v_z are the radial, peripheral, axial components of velocity, and p is the pressure, ρ the density of fluid, and ν the kinematic viscosity.

3. CASE A; BOTH DISKS ROTATING WITHUOT DISCHARGE.

Eqs. (1) and (2) are now solved with the boundary conditions

$$\left.\begin{aligned} v_r=v_z=0,\quad v_\theta=s\Omega r \qquad \text{at } z=0\\ v_r=v_z=0,\quad v_\theta=\Omega r \qquad \text{at } z=d \end{aligned}\right\} \quad (3)$$

As was first pointed out by Kármán [8], a solution of the following form exists.

$$\left.\begin{aligned} &v_r=\Omega rF'(\eta),\ v_\theta=\Omega rG(\eta),\ v_z=-2\Omega dF(\eta);\ \eta\equiv z/d\\ \text{and}\qquad &\frac{p}{\rho}=\frac{1}{2}\lambda\Omega^2r^2+\text{constant} \end{aligned}\right\} \quad (4)$$

where primes denote differentiation with respect to η and λ is a constant to be found. Here, F and G are the functions of η only, and the equation of continuity (2) is automatically satisfied.

Substituting (4) into (1), and eliminating the pressure, reduce the momentum equations to the ordinary differential equations

$$\left.\begin{aligned} F^{\mathrm{iv}} &= -k(FF'''+GG') \\ G'' &= k(F'G-FG') \end{aligned}\right\} \quad (5)$$

where $k=2\left(\dfrac{\Omega d^2}{\nu}\right)$ is twice the Reynolds number based on the gap between two disks. The boundary conditions are now

$$\left.\begin{aligned} &F=F'=0, \quad G=s \qquad \text{at } \eta=0 \\ &F=F'=0, \quad G=1 \qquad \text{at } \eta=1 \end{aligned}\right\} \quad (6)$$

The solution of the equations may be represented by a power series of the Reynolds number as

$$\left.\begin{aligned} F(\eta) &= F_0(\eta)+kF_1(\eta)+k^2F_2(\eta)+\cdots \\ G(\eta) &= G_0(\eta)+kG_1(\eta)+k^2G_2(\eta)+\cdots \end{aligned}\right\} \quad (7)$$

By substituting (7) into (5), it is found that the first functions are given by

$$F_0^{\mathrm{iv}}=0, \qquad G_0''=0$$

with the boundary conditions

$$\left.\begin{aligned} &F_0=F_0'=0, \quad G_0=s \qquad \text{at } \eta=0 \\ &F_0=F_0'=0, \quad G=1 \qquad \text{at } \eta=1 \end{aligned}\right\} \quad (8)$$

Thus

$$F_0=0, \qquad G_0=s+(1-s)\eta \quad (9)$$

The succeeding functions satisfy

$$F_i^{\mathrm{iv}}=-\sum_{j=0}^{i-1}(F_jF'''_{i-j-1}+G_jG'_{i-j-1}) \qquad (i=1,\ 2,\ 3,\ \cdots)$$

$$G_i''=\sum_{j=0}^{i-1}(F'_jG_{i-j-1}-F_jG'_{i-j-1}) \quad (10)$$

with the boundary conditions

$$F_i=F_i'=G_i=0 \qquad \text{at } \eta=0 \text{ and } 1$$

The differential equations (10) may be solved in turn, using the first functions (9). The second and succeeding few functions are then

$$\left.\begin{aligned} &F_1=\sum_{n=2}^{5} A_{1n}\eta^n,\ F_2=0,\ F_3=\sum_{n=2}^{11} A_{3n}\eta^n,\cdots \\ &G_1=0,\ G_2=\sum_{n=1}^{7} B_{2n}\eta^n,\ G_3=0,\cdots \end{aligned}\right\} \quad (11)$$

The coefficients of these functions are presented in the Appendix.

As pointed out by Stewartson [2], the foregoing series solution is apparently rapidly convergent when k is less than 20, although it would need about 80 to be applied for separate boundary layers.

The pressure coefficient is

$$\lambda=-(F')^2+2FF''+G^2+\frac{2}{k}F'''=\text{constant for all } \eta \quad (12)$$

and then the pressure may be given by

$$\left.\begin{aligned} \frac{p}{\rho}&=\frac{1}{2}\Omega^2r^2\left[s^2+\frac{2}{k}F'''(0)\right]+\text{constant} \\ &=\frac{1}{2}\Omega^2r^2[s^2+12(A_{13}+A_{33}k^2+O(k^4))]+\text{constant} \end{aligned}\right\} \quad (13)$$

A_{33} is negative for any value of s, so that the radial pressure rise would be small at large Reynolds number in any case. This trend was also found by Stewartson for the case of $s=0$ and -1.

The peripheral component of the shearing stress at the disk is

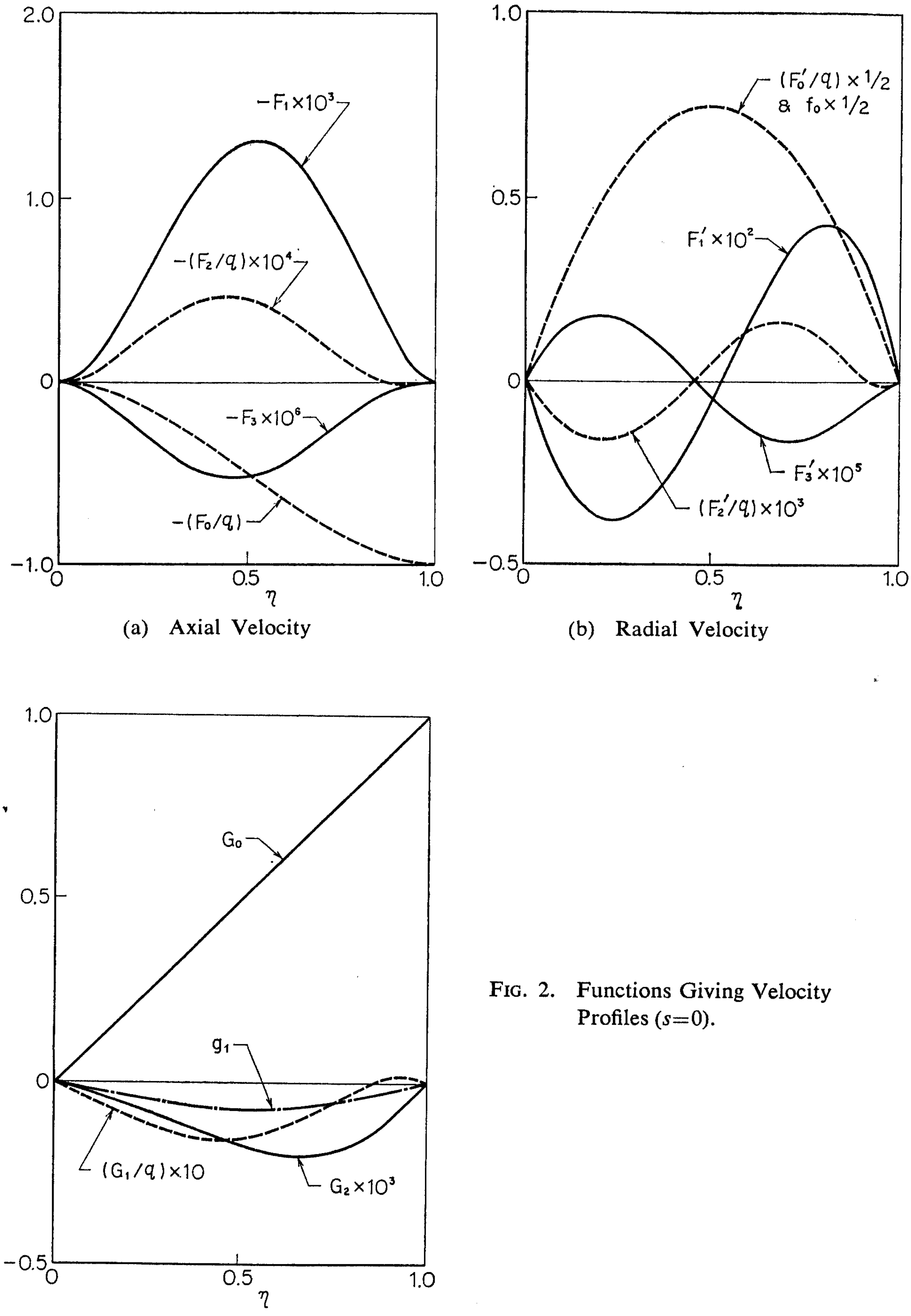
$$\left.\begin{aligned} \tau_\theta&=\mu\left(\frac{\partial v_\theta}{\partial z}\right)_{\text{disk}} \\ &=\mu\left(\frac{\Omega r}{d}\right)\begin{cases}(1-s)+B_{21}k^2+O(k^4) & \text{at } \eta=0 \\ (1-s)+\left(\sum_{n=1}^{7} nB_{2n}\right)k^2+O(k^4) & \text{at } \eta=1\end{cases} \end{aligned}\right\} \quad (14)$$

where μ is the viscosity. Integrating the second equation of (1) with respect to z gives

$$\tau_\theta(d)-\tau_\theta(0)=\frac{\rho}{r^2}\frac{\partial}{\partial r}\left(r^2\int_0^d v_r v_\theta dz\right)$$

that is to say, the difference of skin friction on two disks is directly related to the excess of angular momentum of radial outflow over that of radial inflow.

Some numerical results will be shown in the case when one disk is at rest, or $s=0$, which would be one of the special interest in practice. In Fig. 2 are illustrated the graphs of the functions $-F_1$, $-F_3$, F_1', F_3', G_0, G_2 (full lines), giving the axial, radial, and peripheral components of the velocity induced by a rotating disk. The radial velocity is inward near the stationary disk and outward near the rotating one. Consequently the angular velocity decreases throughout the gap, and



(a) Axial Velocity

(b) Radial Velocity

(c) Peripheral Velocity

FIG. 2. Functions Giving Velocity Profiles ($s=0$).

the boundary layer is beginning to form only near the rotating disk at large Reynolds number.

In this case ($s=0$), the pressure is given by

$$\frac{p}{\rho} \sim \frac{1}{2}\Omega^2 r^2 (0.3-0.000191k^2)+\text{constant} \tag{15}$$

and the friction moment for the rotating disk of radius a is

$$M=2\pi\int_0^a \tau_0 r^2 dr \sim \mu\frac{\pi}{2}\frac{\Omega a^4}{d}(1+0.00107k^2)$$

Introducing a non-dimensional moment coefficient C_M and the Reynolds number based on the radius of disk $R_e=\dfrac{\Omega a^2}{\nu}$,

$$C_M=\frac{M}{(1/2)\rho(\Omega a)^2\cdot\pi a^2\cdot a} \sim \frac{1}{R_e}\frac{a}{d}(1+0.00107k^2) \tag{16}$$

Hence the increase of the Reynolds number (k) would tend to reduce the rate of pressure rise, while it raises the frictional torque for the rotating disk.

4. Case B; one disk at rest, and the other rotating with uniform discharge of fluid.

Here will be considered the case in which one disk is held fixed and the other disk is rotating at a constant velocity with uniform discharge of fluid, as shown in Fig. 1. The quantity of fluid discharging from the rotating disk is assumed to be sufficiently small, so that the system can be represented by Eq. (1). The boundary conditions are

$$\left.\begin{aligned} &v_r=v_z=v_\theta=0 && \text{at } z=0 \\ &v_r=0, \quad v_z=-2\Omega dq, \quad v_\theta=\Omega r && \text{at } z=d \end{aligned}\right\} \tag{17}$$

where q is a constant ($\ll 1$).

A solution of the same form as Eq. (4) for the previous case exists, and then the momentum equations (5) holds for the present case, together with the boundary conditions

$$\left.\begin{aligned} &F=F'=0, \quad G=0 && \text{at } \eta=0 \\ &F=q, \quad F'=0, \quad G=1 && \text{at } \eta=1 \end{aligned}\right\} \tag{18}$$

Representing again the solution by a power series of the Reynolds number as Eq. (7), the first functions are given by

$$\left.\begin{array}{l} F_0^{\mathrm{iv}}=0, \qquad G_0''=0 \\ \text{with the boundary conditions} \\ F_0=F_0'=0, \qquad G_0=0 \qquad \text{at } \eta=0 \\ F_0=q, \quad F_0'=0, \quad G_0=1 \qquad \text{at } \eta=1 \end{array}\right\} \tag{19}$$

Thus

$$F_0=(3\eta^2-2\eta^3)q, \qquad G_0=\eta \tag{20}$$

The succeeding functions satisfy Eq. (10). Solving Eq. (10) in turn with Eq. (20), the first few functions are then given as follows:

$$\left.\begin{array}{l} F_1=\dfrac{1}{120}(-2\eta^2+3\eta^3-\eta^5)+O(q^2) \\ F_2=\dfrac{q}{10!}(-2844\eta^2+4632\eta^3+1512\eta^5-6552\eta^6+2592\eta^7 \\ \qquad +540\eta^8+120\eta^9)+O(q^3) \\ \cdots\cdots\cdots\cdots \\ G_1=\dfrac{q}{20}(-\eta+5\eta^4-4\eta^5) \\ G_2=\dfrac{1}{5\cdot 7!}(-8\eta-35\eta^4+63\eta^5-20\eta^7)+O(q^2) \\ \cdots\cdots\cdots\cdots \end{array}\right\} \tag{21}$$

The effects of the fluid discharge can be seen primarily in the functions F_0, F_2, G_1, G_3, ..., first few of which are shown in Fig. 2 by dash lines.

Using Eq. (12), the pressure is given by

$$\frac{p}{\rho}\sim \text{R.H.S. of Eq. (15)}+\frac{1}{2}\Omega^2 r^2\left(-\frac{24}{k}+0.0153k\right)q \tag{22}$$

and the friction moment for the rotating disk of radius a is

$$C_M\sim \text{R.H.S. of Eq. (16)}-0.1\left(\frac{d}{a}\right)q \tag{23}$$

Hence, the discharge of fluid would reduce both the rate of pressure rise and the frictional torque for the rotating disk as well as for the stationary one.

5. Case C; source flow between two rotating disks.

In the following there will be considered the case in which a source situated at the centre produces small radial net flow between two rotating disks, when the condition for mass conservation is given by

$$m=2\pi r\int_0^d v_r dz=\text{constant} \tag{24}$$

A solution is assumed to have a form

$$\left.\begin{aligned}
v_r&=\Omega rF'(\eta)+\bar{m}\frac{d}{r}f(\eta)\\
v_\theta&=\Omega rG(\eta)+\bar{m}\frac{d}{r}g(\eta)\\
v_z&=-2\Omega dF(\eta); \qquad \bar{m}\equiv\frac{m}{2\pi d^2}
\end{aligned}\right\} \tag{25}$$

where the second term on the right-hand side of each equation is small enough as compared with the first one, which is the solution for the case with no radial net flow. Here again f and g are the functions of η alone, and the equation of continuity is automatically satisfied.

Substituting equations (25) into (1), and neglecting the terms of higher order in small quantities, lead to

$$\left.\begin{aligned}
&f'''=-k[(Ff')'+(Gg)']\\
&g''=k(Gf-Fg')\\
&\text{with the boundary conditions}\\
&f=g=0 \qquad \text{at } \eta=0 \text{ and } 1
\end{aligned}\right\} \tag{26}$$

In a similar manner as in the previous case, equations may be solved by means of a power series

$$\left.\begin{aligned}
f(\eta)&=f_0(\eta)+kf_1(\eta)+k^2f_2(\eta)+\cdots\\
g(\eta)&=g_0(\eta)+kg_1(\eta)+k^2g_2(\eta)+\cdots
\end{aligned}\right\} \tag{27}$$

Substituting equations (27) into (26), the functions are given by

$$\left.\begin{aligned}
&f_i'''=-\sum_{j=0}^{i-1}[(F_jf'_{i-j-1})'+(G_jg_{i-j-1})'] \qquad (i=0, 1, \cdots)\\
&g_i''=\sum_{j=0}^{i-1}(G_jf_{i-j-1}-F_jg'_{i-j-1})\\
&\text{with the boundary conditions}\\
&f_i=g_i=0 \qquad \text{at } \eta=0 \text{ and } 1
\end{aligned}\right\} \tag{28}$$

These differential equations may be solved in turn, and the first few solutions are (see Appendix)

$$\left.\begin{aligned}
&f_0=6(\eta-\eta^2), \qquad f_1=0, \cdots\\
&g_0=0, \qquad g_1=\sum_{n=1}^{5}b_{1n}\eta^n, \cdots
\end{aligned}\right\} \tag{29}$$

The pressure and the shearing stress at the disks are

$$\left.\begin{aligned}\frac{p}{\rho}&=\text{R.H.S. of Eq. (13)}+2\bar{m}\frac{\Omega d}{k}f''(0)\log r\\&=\text{R.H.S. of Eq. (13)}+2\bar{m}\Omega d\left[-\frac{12}{k}+2a_{22}k+O(k^3)\right]\log r\end{aligned}\right\} \quad (30)$$

$$\tau_\theta=\text{R.H.S. of Eq. (14)}+\mu\frac{\bar{m}}{r}\begin{Bmatrix}b_{11}k+O(k^3) & \text{at } \eta=0\\ \left(\sum_{n=1}^{5} nb_{1n}\right)k+O(k^3) & \text{at } \eta=1\end{Bmatrix} \quad (31)$$

In the case of $s=0$, the calculated results of Eq. (29) giving the velocity profiles are also given in Fig. 2, and the pressure and the friction moment for the rotating disk of radius a are

$$\frac{p}{\rho}\sim\text{R.H.S. of Eq. (15)}-2\bar{m}\Omega d\left(\frac{12}{k}+0.026k\right)\log r \quad (32)$$

$$C_M\sim\text{R.H.S. of Eq. (16)}+0.6\frac{k}{R_e}\frac{\bar{m}}{\Omega a} \quad (33)$$

The source flow would tend to reduce the rate of pressure rise and to increase the frictional torque for the rotating disk.

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APPENDIX

The functions given by Eqs. (8), (10), and (28) are as follows:

$$\begin{bmatrix} F_i \\ G_i \end{bmatrix} = \sum_n \begin{bmatrix} A_{in} \\ B_{in} \end{bmatrix} \eta^n \quad (11); \qquad \begin{bmatrix} f_i \\ g_i \end{bmatrix} = \sum_n \begin{bmatrix} a_{in} \\ b_{in} \end{bmatrix} \eta^n \quad (29)$$

$$F_i = g_i = 0 \qquad \text{for } i = 0, 2, 4, \cdots$$
$$G_i = f_i = 0 \qquad \text{for } i = 1, 3, 5, \cdots$$

$$A_{10} = A_{11} = 0$$
$$A_{12} = -(1-s)(2+3s)/120$$
$$A_{13} = (1-s)(3+7s)/120$$
$$A_{14} = -s(1-s)/24$$
$$A_{15} = -(1-s)^2/120$$
$$A_{30} = A_{31} = 0$$
$$A_{32} = (1-s)(332+239s-34s^2+63s^3)/\boldsymbol{A}; \ \boldsymbol{A} = 10 \times 10!$$
$$A_{33} = (1-s)(-579-563s+563s^2+579s^3)/\boldsymbol{A}$$
$$A_{34} = 30s(1-s)(16-32s-54s^2)/\boldsymbol{A}$$
$$A_{35} = 24(1-s)(8-24s-11s^2+27s^3)/\boldsymbol{A}$$
$$A_{36} = 42(1-s)(6+17s+38s^2+39s^3)/\boldsymbol{A}$$
$$A_{37} = -6(1-s)(27+19s+161s^2+393s^2)/\boldsymbol{A}$$
$$A_{38} = -15(1-s)(2+17s+20s^2-99s^3)/\boldsymbol{A}$$
$$A_{39} = -5(1-s)(3+s-111s^2+107s^3)/\boldsymbol{A}$$
$$A_{3,10} = 110s(1-s)^3/\boldsymbol{A}$$
$$A_{3,11} = 10(1-s)^4/\boldsymbol{A}$$

$$B_{00} = s$$
$$B_{01} = 1-s$$
$$B_{20} = B_{22} = 0$$
$$B_{21} = -(1-s)(8-16s-27s^2)/\boldsymbol{B}; \ \boldsymbol{B} = 5 \times 7!$$
$$B_{23} = -70s(1-s)(2+3s)/\boldsymbol{B}$$
$$B_{24} = 35(1-s)(-1+4s+12s^2)/\boldsymbol{B}$$
$$B_{25} = 21(1-s)(3+4s-17s^2)/\boldsymbol{B}$$
$$B_{26} = -140s(1-s)^2/\boldsymbol{B}$$
$$B_{27} = -20(1-s)^3/\boldsymbol{B}$$

$$a_{00} = 0$$
$$a_{01} = 6$$
$$a_{02} = -6$$

$$b_{10} = b_{12} = 0$$
$$b_{11} = -(2+3s)/10$$
$$b_{13} = s$$
$$b_{14} = (1-2s)/2$$
$$b_{15} = -3(1-s)/10$$