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Gas-Liquid Laminar Boundary-Layer Flows with a Phase-Changing Interface and its Hydrodynamic Instability

By

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Summary: Laminar boundary-layer flows of gas and liquid having a phase-changing interface at their common boundary are studied theoretically to predict their aspect of flow and thermal fields first in the steady state. By using these results and examining their perturbed fields with small wavy disturbance, the disturbance flow fields and the hydrodynamic instability of the system are investigated.

The phase-change at the interface has a considerable effect on the velocity and temperature profiles, thus on the coefficients of skin-friction and heat-transfer at the interface. Intense evaporation decreases the velocity and temperature gradients at the interface.

The wavy disturbances on the interface have influence on the heat-transfer at the interface by one order of magnitude less than on the skin-friction. The small wavy disturbance fields show a hydrodynamic instability similar to that of boundary-layer flows on a rigid wall.

CONTENTS

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NOMENCLATURE

c,	wave velocity (U_∞)
c_p,	specific heat
F,	undisturbed steady stream function
f,	amplitude of disturbance stream function
g,	amplitude of disturbance temperature
g,	gravitational acceleration (U_∞^2/l_r)
H_t	$=W_0L(T_{2\infty}-T_{1\infty})^2/(RT_0^2)$, $H_p=W_0/P_{10}$
h,	amplitude of disturbance concentration
k,	amplitude of disturbance pressure
L,	latent heat of vaporization
M, m,	rate of phase-change at the interface and its disturbance (ρ_1U_∞)
l_r,	characteristic length $=\sqrt{\nu x/U_\infty}$
P, p,	pressure and its disturbance ($\rho_1U_\infty^2$)
R,	gas constant of vapor (c_{p1})
T,	temperature
T_b,	boiling temperature of liquid
t,	time (l_r/U_∞)
U, V,	main velocity components (U_∞)
u, v,	disturbance velocity components (U_∞)
W, w,	vapor concentration and its disturbance
x, y,	co-ordinates (l_r)

Greek symbols

α,	wave number $\alpha_n=\alpha l_{rn}$ ($n=1, 2$)
β_n	$=(-1)^n i\alpha_n U'_{n0}/\nu_n$, $\beta_{\kappa n}=(-1)^n i\alpha_n U'_{n0}/\kappa_n$, $\beta_\varepsilon=-i\alpha_1 U'_{10}/\varepsilon$
Δ	$=F_{10}-(F'_{10})^2/(2F''_{10})$
δ,	disturbance of the interface position (l_r)
ε,	diffusion coefficient of vapor ($U_\infty l_r$)
Θ, θ,	non-dimensional temperature and its disturbance
κ,	thermal diffusivity ($U_\infty l_r$)
Λ	$=\rho_2/\rho_1\sqrt{\nu_2/\nu_1}$
λ,	heat conductivity ($\rho_1 c_{p1} U_\infty l_r$)
ν,	kinematic viscosity ($U_\infty l_r$)
ρ,	density
σ,	surface tension ($\rho_1 U_\infty^2 l_{r1}$)
Ω, ω,	vorticity and its disturbance (U_∞/l_r)

Subscripts

0,	interface of gas and liquid
1,	gas side
2,	liquid side
∞,	infinity

Values in ( ) indicate the reference unit.

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1. Introduction

Concerning transpiration cooling, mist cooling, ablation etc., a number of works have been published because of their great promise for the maintenance of tolerable surface temperature on high speed aircraft, turbine blades or rocket-motor nozzle. In these engineering problems of the heat transfer of two-phase flow, transpiration or mass-transfer cooling, drying process etc. or in geophysical problems of water evaporation at the surface of ocean or land, the basic and essential features are simultaneous heat and mass transfer at the interface between gas and liquid flows where evaporation or condensation takes place. A preliminary and fundamental approach to such problems is the investigation of the boundary-layer flows under the condition that the quantities (velocity, temperature etc.) are specified explicitly at the interface or boundary as in most cases of published works [*1*~*5*]. For two-layers flows of gas and liquid which have an appreciable velocity at the interface to force the liquid into motion, or a heat flux by conduction through the liquid layer, however, the boundary conditions at the interface have to be provided implicitly by the continuity relationships of mass, momentum and energy flows through the surface between two layers and to be determined as an eigen value problem of the whole system.

The first part of the present study is concerned with the theoretical investigation of the flow and thermal fields of laminar boundary-layer flows of gas and liquid having a phase-changing (evaporating or condensing) interface at their common boundary and also with the aspect of the heat and mass transfer and the skin friction of the interface under such a condition. The fields of velocity, temperature and concentration are interacted each other at the interface so that their solutions should be asked as ëigen functions corresponding to the boundary conditions at the interface. A mathematical system of the problem is presented and solved both numerically and analytically.

In cases of practical interest in these problems, the hydrodynamic instability of the system comes to be of important features such as the transition from laminar to turbulent flow, the wave generation on the surface of a liquid by wind blowing over it, the detachment and entrainment of liquid parcels from the liquid layer into the gas stream etc. [*6*~*16*]. By Benjamin [7], for flows over a flexible wall, there are three essentially different types of instabilities possible, denoted by A, B and C, which can be identified with Tollmien-Schlichting waves (A), free surface waves (B) and Kelvin-Helmholtz waves (C), respectively, in their forms. Lock [*19*] also recognized that for air flows over a water surface two essentially different classes of waves are possible, 'air waves' and 'water waves', which correspond to the Tollmien-Schlichting waves and Benjamin's class B waves, respectively. The experimental investigation of a laminar air flow over water by Gupta et al. [*11*] shows these distinct modes of unstable oscillations. The class A waves are observed at low speeds only slightly modified from that of Tollmien-Schlichting waves over a rigid wall. At higher speeds, the 'water waves' (class B) appear, moving at speeds close to their free surface propagation speed. At still higher speeds, there is some evidence of Kelvin-Helmholtz (class C) waves.

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The mechanism by which energy may be transfered from a sheared gas flow to a wavy liquid surface without phase-change was first theoretically explained by Miles [*13*] and Benjamin [*6*] with a linearized theory. It was solved by the usual asymptotic methods to the Orr-Sommerfeld equation to estimate the perturbation stresses on the wavy surface. In cases to allow the wave-train to travel in the direction of flow on the boundary as the interface of two flows, the wave velocity may equal the fluid velocity at a certain distance from the boundary, being of an important factor, called 'critical point', in the problem of wave generation by flows over a mobile boundary. Such problems may be treated by considering a simple wave-train of arbitrary wave length and speed superposed on the base field and then examining whether or not the features of stresses at the boundary supply energy to the wave—the effect of sheltering on the leeward slopes of the wave in which the shearing stress has the maxima in the rear of the wave crest and the normal stress (pressure) has the minima in the front of the wave crest as if they made the flow separate on the downstream side. These features of laminar boundary-layer flows of gas and liquid with a phase-changing interface of small amplitude are examined analytically in an approximate method assuming linear profiles of the base flow. The aspects of the heat transfer or the rate of phase change at the wavy interface are also investigated.

An alternative approach to such a problem of infinitesimally disturbed two-phase system is to solve directly and numerically a complete set of linearized equations in which full account is taken of the interaction of two flows at the interface. Such treatments of the problem without phase-changing at the interface were made by Wuest [*22*] for velocity profiles roughly approximated actual boundary-layer ones, by Lock [*19*] for exact profiles of infinite air-water laminar boundary-layer flows, and by Feldman [*17*] for a horizontal liquid film in contact with a uniform shearing air stream. In cases that both air and liquid motions are separately unstable, the stability analyses of this kind are extremely complicated as shown in Lock's results. There are also certain mathematical restrictions necessary in order to make the treatment manageable in the way usual for problems of boundary-layer stability (e.g. Lin [*18*]). The numerical solution of the problem with a high speed computer may be thus expected to throw some light on these complicated stability problems without serious mathematical restrictions. In the last part of the present study, the stability of the wavy disturbances of gas-liquid laminar boundary-layer flows with a phase-changing interface superposed upon the base fields is investigated with a high speed computer by solving numerically linearized equations of the Orr-Sommerfeld type and the secular equation corresponding to the boundary condition at the interface to obtain eigen values of the system.

2. Formulation of the Problem

We consider two viscous incompressible fluids of gas and liquid in laminar motion, parallel to the interface at which the state of the fluid is subjected to change in the phase, that is, evaporation or condensation takes place corresponding to the thermal condition of the system. Let x denote the coordinate parallel to the interface, y the

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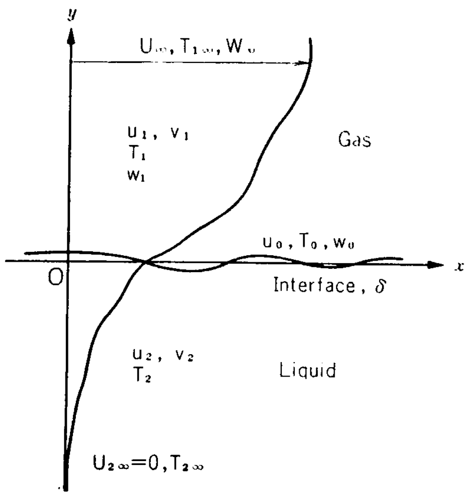

FIG. 2.1. Laminar boundary-layer flows of gas and liquid having a phase-changing interface.

coordinate perpendicular to it, (u, v) the corresponding velocity components, T the temperature and w the mass concentration of the vapor contained in the gas stream (Fig. 2.1).

For two-dimensional laminar flows with constant properties and negligible dissipation of the kinematic energy, the boundary-layer equations of momentum, energy and concentration in each region of fluids can be written as follows in vector form;

$$\frac{\partial}{\partial t}\begin{pmatrix}\omega\\ T\\ w\end{pmatrix}+\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\begin{pmatrix}\omega\\ T\\ w\end{pmatrix}=(\nu, \kappa, \varepsilon)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\begin{pmatrix}\omega\\ T\\ w\end{pmatrix}, \tag{2.1}$$

where ν is the kinematic viscosity, κ the thermal diffusivity and ε the diffusion coefficient of the vapor, being assumed to be constant throughout the layers. The vorticity ω is defined by

$$\omega=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}.$$

The equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{2.2}$$

At large distances from the interface, the velocity, temperature and vapor-concentration are kept constant;

$$\left.\begin{array}{ll}(u_1)_{y=\infty}=u_\infty\equiv U_\infty & (u_2)_{y=-\infty}=0\\ (T_1)_{y=\infty}=T_{1\infty} & (T_2)_{y=-\infty}=T_{2\infty}\\ (w)_{y=\infty}=w_\infty\equiv W_\infty, & \end{array}\right\} \tag{2.3}$$

where the liquid at infinity is assumed to be at rest since only the relative motion between the two fluids is a matter of consequence. At the interface of gas and liquid layers, the following conditions must hold. Let subscripts 0, 1 and 2 denote, respec-

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tively, the interface, gas (upper fluid) and liquid (lower fluid). The no-slip condition of the temperature and u-velocity gives

$$T_{10}=T_{20} \tag{2.4$_t$}$$

$$u_{10}=u_{20}. \tag{2.4$_u$}$$

The continuity of mass is

$$\{\rho(v-\dot{\delta})\}_{10}=\{\rho(v-\dot{\delta})\}_{20}\equiv m \tag{2.4$_c$}$$

$$\left\{\rho w(v-\dot{\delta})-\rho\varepsilon\frac{\partial w}{\partial y}\right\}_{10}=\{\rho(v-\dot{\delta})\}_{20}, \tag{2.4$_w$}$$

where ρ is the density, δ the elevation of the interface, $\dot{\delta}$ the rate of the change of the elevation with respect to time so that m means the rate of phase-change per unit area of the interface. When δ is a function of x, $\dot{\delta}$ should be expressed as

$$\dot{\delta}=\frac{\partial\delta}{\partial t}-u_0\frac{\partial\delta}{\partial x}.$$

The stresses tangential and normal to the interface must be contineous;

$$\rho_1\nu_1\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)_{10}=\rho_2\nu_2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)_{20} \tag{2.4$_s$}$$

$$\left(p-\rho g\delta+mv-2\rho\nu\frac{\partial v}{\partial y}\right)_{10}=\left(p-\rho g\delta+mv-2\rho\nu\frac{\partial v}{\partial y}\right)_{20}+\sigma\frac{\partial^2\delta}{\partial x^2}, \tag{2.4$_p$}$$

where g is the acceleration of gravity and σ the surface tension, $1/(\partial^2\delta/\partial x^2)$ being the radius of curvature of the interface. The pressure may be given by the equation of motion

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right). \tag{2.5}$$

The heat flux into the interface through the layers should be exhausted entirely for the phase-change

$$Lm=\left(-\lambda\frac{\partial T}{\partial y}\right)_{20}-\left(-\lambda\frac{\partial T}{\partial y}\right)_{10}, \tag{2.4$_g$}$$

where L is the latent heat of vaporization and λ the heat conductivity.

The number of the necessary boundary conditions for solving the system of equations (2.1) and (2.2) with respect to $u, T, w,$ and δ in the y-direction is $2\times(3_u+2_t+1_v)+2_w+1_\delta=15$. There are above conditions imposed seven at infinity including the pressure condition and seven at the interface. If it is assumed that the pressure is constant throughout the layers, the equation of motion becomes the second-order differential equation so that the system requires thirteen boundary conditions, while the imposed conditions are five at infinity and seven at the inter-

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face. This implies that one condition should be further imposed on the system. As for the condition, the vapor at the interface may be assumed to have the state of saturation corresponding to the pressure and temperature at the interface, so that Clausius-Clapeyron's relation holds

$$\frac{dp_s}{dT_s} \approx \frac{L}{R_v} \frac{p_s}{T_0^2}.$$

This yields the mass fraction, that is, the partial pressure of the vapor at the interface;

$$w_0 = \frac{p_s}{p_0} = \exp\left\{\left(\frac{L}{R_v}\left(\frac{1}{T_b} - \frac{1}{T_0}\right)\right)\right\}, \tag{2.4$_e$}$$

where R_v is the gas constant of the vapor, T_b the boiling point at the pressure p_0.

We shall express all variables in non-dimensional form, implying that the units of length and of velocity are to be taken as a certain length l_r and velocity U_∞ characteristic of the problem. Time is made dimensionless on the understanding that l_r/U_∞ is the unit. Stresses have the unit of ρU_∞^2. The temperature of gas and liquid are non-dimensionalized by $(T_{1\infty} - T_{2\infty})$ as

$$\theta_1 = \frac{T_1 - T_{1\infty}}{T_{2\infty} - T_{1\infty}} \qquad \theta_2 = \frac{T_2 - T_{2\infty}}{T_{1\infty} - T_{2\infty}}. \tag{2.6}$$

In the non-dimensional form, the governing equations are rewritten in the same form as equations (2.1) and (2.2), although ν, κ and ε have the unit of $U_\infty l_r$, that is,

$$\left.\begin{array}{lll} \dfrac{\nu_1}{U_\infty l_{r1}} \to \nu_1 & \dfrac{\nu_2}{U_\infty l_{r2}} \to \nu_2 & \\ \dfrac{\kappa_1}{U_\infty l_{r1}} \to \kappa_1 & \dfrac{\kappa_2}{U_\infty l_{r2}} \to \kappa_2 & \dfrac{\varepsilon}{U_\infty l_{r1}} \to \varepsilon \end{array}\right\} \tag{2.7}$$

where non-dimensional parameters ν_1^{-1} and ν_2^{-1} imply the boundary-layer Reynolds numbers. As for the boundary conditions, the following understanding must be taken;

$$\left.\begin{array}{llll} \dfrac{m}{\rho_1 U_\infty} \to m & \dfrac{\rho_1}{\rho_1} \to \rho_1 = 1 & \dfrac{\rho_2}{\rho_1} \to \rho_2 & \dfrac{R_v}{c_{p1}} \to R \\ \dfrac{g l_{r1}}{U_\infty^2} \to g_1 & \dfrac{g l_{r2}}{U_\infty^2} \to g_2 & \dfrac{\sigma}{\rho_1 U_\infty^2 l_{r1}} \to \sigma & \\ \dfrac{L}{c_{p1}(T_{2\infty} - T_{1\infty})} \to L & \dfrac{\lambda_1}{\rho_1 c_{p1} U_\infty l_{r1}} \to \lambda_1 & \dfrac{\lambda_2}{\rho_1 c_{p1} U_\infty l_{r2}} \to \lambda_2 & \end{array}\right\} \tag{2.8}$$

where g_1^{-1} and g_2^{-1} represent the boundary-layer Froude numbers and σ^{-1} the boundary-layer Weber number.

We now consider the fields of velocity, temperature and concentration to be made up of the steady fields with small disturbances and write

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$$\left.\begin{array}{ll} u \rightarrow U+u & v \rightarrow V+v \\ \theta \rightarrow \Theta+\theta & w \rightarrow W+w \end{array}\right\} \qquad (2.9)$$

where the lower-case symbols in the right hand side indicate small quantities so that their squares and products are to be neglected. In the steady flows of zero-pressure gradient, the interface may be taken to be of a plane at the origin of the y-coordinate. The introduction of equation (2.9) and boundary-layer approximation into the governing equations and boundary conditions leads to the following system of equations:

(i) Steady field

Equations:

$$\left(U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\right)\begin{pmatrix}\Omega\\ \Theta\\ W\end{pmatrix}=(\nu, \kappa, \varepsilon)\frac{\partial^2}{\partial y^2}\begin{pmatrix}\Omega\\ \Theta\\ W\end{pmatrix} \qquad (2.10)$$

$$\Omega\equiv\frac{\partial U}{\partial y}-\frac{\partial V}{\partial x} \qquad (2.11)$$

Boundary conditions:

$$U_{1\infty}=1 \quad U_{2\infty}=0 \quad \Theta_{1\infty}=0 \quad \Theta_{2\infty}=0 \quad W_{\infty}=W_{\infty} \qquad (2.12)$$

$$\Theta_{10}+\Theta_{20}=1 \qquad (2.13)_t$$

$$U_{10}=U_{20}=U_0 \qquad (2.13)_u$$

$$(\rho V)_{10}=(\rho V)_{20}\equiv M \qquad (2.13)_c$$

$$\left(\rho V W-\rho\varepsilon\frac{\partial W}{\partial y}\right)_{10}=(\rho V)_{20} \qquad (2.13)_w$$

$$\rho_1\nu_1\left(\frac{\partial U}{\partial y}\right)_{10}=\rho_2\nu_2\left(\frac{\partial U}{\partial y}\right)_{20} \qquad (2.13)_s$$

$$\left(P+MV-2\rho\nu\frac{\partial V}{\partial y}\right)_{10}=\left(P+MV-2\rho\nu\frac{\partial V}{\partial y}\right)_{20} \qquad (2.13)_p$$

$$LM=\lambda_1\left(\frac{\partial\Theta}{\partial y}\right)_{10}+\lambda_2\left(\frac{\partial\Theta}{\partial y}\right)_{20} \qquad (2.13)_q$$

$$W_0=\exp\left\{\frac{L}{R_v}\left(\frac{1}{T_b}-\frac{1}{T_0}\right)\right\}. \qquad (2.13)_e$$

Since the pressure varies quite gradually with y in the plane boundary-layer so that the equation in the y-direction should not be taken into account, the boundary condition of the normal stress continuity is no longer substantial.

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(ii) Disturbance field

Equations:

$$\frac{\partial}{\partial t}\begin{pmatrix}\omega\\ \theta\\ w\end{pmatrix}+\left(U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\right)\begin{pmatrix}\omega\\ \theta\\ w\end{pmatrix}+\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\begin{pmatrix}\Omega\\ \Theta\\ W\end{pmatrix}$$
$$=(\nu,\kappa,\varepsilon)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\begin{pmatrix}\omega\\ \theta\\ w\end{pmatrix} \tag{2.14}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \omega\equiv\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \tag{2.15}$$

Boundary conditions:

$$\left.\begin{matrix}u_{1\infty}=v_{1\infty}=0 \qquad u_{2\infty}=v_{2\infty}=0\\ \theta_{1\infty}=0 \quad \theta_{2\infty}=0 \quad w_{1\infty}=0\end{matrix}\right\} \tag{2.16}$$

$$\theta_{10}+\theta_{20}=0 \tag*{(2.17)$_t$}$$

$$u_{10}=u_{20}=u_0 \tag*{(2.17)$_u$}$$

$$\{\rho(v-\dot{\delta})\}_{10}=\{\rho(v-\dot{\delta})\}_{20}\equiv m \tag*{(2.17)$_c$}$$

$$\left\{\rho W(v-\dot{\delta})+\rho wV-\rho\varepsilon\frac{\partial w}{\partial y}\right\}_{10}=\{\rho(v-\dot{\delta})\}_{20} \tag*{(2.17)$_w$}$$

$$\left\{\rho\nu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}_{10}=\left\{\rho\nu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}_{20} \tag*{(2.17)$_s$}$$

$$\left(p-\rho g\delta+Mv+mV-2\rho\nu\frac{\partial v}{\partial y}\right)_{10}=\sigma\frac{\partial^2\delta}{\partial x^2}$$
$$+\left(p-\rho g\delta+Mv+mV-2\rho\nu\frac{\partial v}{\partial y}\right)_{20} \tag*{(2.17)$_p$}$$

$$Lm=\lambda_1\left(\frac{\partial\theta}{\partial y}\right)_{10}+\lambda_2\left(\frac{\partial\theta}{\partial y}\right)_{20} \tag*{(2.17)$_q$}$$

$$w_0=H_t\theta_{10}-H_p(p_{10}-g_1\delta_1) \tag*{(2.17)$_e$}$$

where

$$H_t\equiv W_0\frac{L}{R}\left(\frac{T_{2\infty}-T_{1\infty}}{T_0}\right)^2 \qquad H_p\equiv\frac{W_0}{P_{10}}.$$

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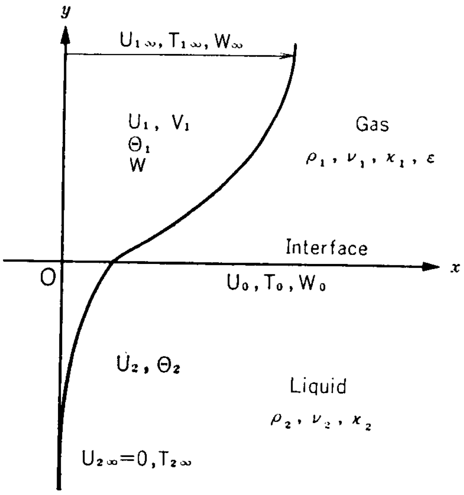

FIG. 3.1. Laminar boundary-layer flows of gas and liquid having a plane phase-changing interface.

3. STEADY LAMINAR BOUNDARY-LAYER FLOWS OF GAS AND LIQUID WITH A PHASE-CHANGING INTERFACE

3.1 *Flow fields*

The forgoing equations and boundary conditions for the undisturbed steady field (Fig. 3.1) can be transformed to a more tractable form by introducing the stream function defined by

$$U=\frac{\partial\psi}{\partial y} \qquad V=-\frac{\partial\psi}{\partial x} \tag{3.1}$$

which satisfies the continuity equation of mass. The assumption of fully developed flows allows a similarity solution in the form

$$\left.\begin{aligned} \eta_1&=\sqrt{\frac{U_\infty}{\nu_1 x}}y_1 \qquad & \eta_2&=\sqrt{\frac{U_\infty}{\nu_2 x}}y_2 \\ \psi_1&=\sqrt{\nu_1 x U_\infty}F_1(\eta_1) \qquad & \psi_2&=\sqrt{\nu_2 x U_\infty}F_2(\eta_2)\,. \end{aligned}\right\} \tag{3.2}$$

With these substitution, the governing equations can be reduced to

$$F'''+\frac{1}{2}FF''=0 \tag*{(3.3)$_u$}$$

$$\Theta''+\frac{\nu}{2\kappa}F\Theta'=0 \tag*{(3.3)$_t$}$$

$$W''+\frac{\nu}{2\varepsilon}FW'=0, \tag*{(3.3)$_w$}$$

where a prime denotes an appropriate derivative with respect to η. The boundary conditions are then rewritten as

$$F'_{1\infty}=1 \qquad F'_{2\infty}=0 \tag*{(3.4)$_u$}$$

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$$\Theta_{1\infty}=0 \qquad \Theta_{2\infty}=0 \tag{3.4$_t$}$$

$$\Theta_{10}+\Theta_{20}=1 \tag{3.5$_t$}$$

$$F'_{10}=F'_{20} \tag{3.5$_u$}$$

$$F_{10}=\Lambda F_{20} \qquad \Lambda\equiv\frac{\rho_2}{\rho_1}\sqrt{\frac{\nu_2}{\nu_2}} \tag{3.5$_c$}$$

$$\frac{\nu_1}{2\varepsilon}F_{10}(1-W_0)=W'_0 \tag{3.5$_w$}$$

$$F''_{10}=\Lambda F''_{20} \tag{3.5$_s$}$$

$$LF_{10}=\lambda_1\Theta'_{10}+\lambda_2\Theta'_{20} \tag{3.5$_q$}$$

$$W_0=\exp\left\{\frac{L}{R_v}\left(\frac{1}{T_b}-\frac{1}{T_0}\right)\right\}. \tag{3.5$_e$}$$

The formal solution of equation $(3.3)_u$ with respect to F' gives

$$F'_1=F'_{10}+F''_{10}\cdot\gamma_1(1,\eta_1) \qquad F'_2=F'_{20}-F''_{20}\cdot\gamma_2(1,\eta_2), \tag{3.6}$$

where

$$\gamma_1(a,\eta)\equiv\int_0^\eta \exp\left(-\frac{a}{2}\int_0^\eta F_1 d\eta\right)d\eta,\ \gamma_2(b,\eta)\equiv\int_\eta^0 \exp\left(\frac{b}{2}\int_\eta^0 F_2 d\eta\right)d\eta.$$

Substituting these solutions into the boundary conditions $(3.4)_u$, $(3.5)_u$ and $(3.5)_c$ leads to

$$1=F'_{10}+F''_{20}\cdot\gamma_1(1,\infty) \qquad 0=\Lambda F'_{10}-F''_{10}\cdot\gamma_2(1,-\infty),$$

from which we obtain

$$\left.\begin{aligned} F''_{10}&=\frac{\Lambda}{\Lambda\gamma_1(1,\infty)+\gamma_2(1,-\infty)}\\ F'_{10}&=\frac{\gamma_2(1,-\infty)}{\Lambda\gamma_1(1,\infty)+\gamma_2(1,-\infty)}.\end{aligned}\right\} \tag{3.7}$$

From equations $(3.3)_t$ and $(3.3)_w$, we can write similarly the temperature and concentration fields as

$$\left.\begin{aligned} \Theta_1&=\Theta_{10}\left\{1-\frac{\gamma_1(\nu_1/\kappa_1,\eta_1)}{\gamma_1(\nu_1/\kappa_1,\infty)}\right\}\\ \Theta_2&=\Theta_{20}\left\{1+\frac{\gamma_2(\nu_2/\kappa_2,\eta_2)}{\gamma_2(\nu_2/\kappa_2,-\infty)}\right\}\\ W-W_\infty&=(W_0-W_\infty)\left\{1-\frac{\gamma_1(\nu_1/\varepsilon,\eta_1)}{\gamma_1(\nu_1/\varepsilon,\infty)}\right\}.\end{aligned}\right\} \tag{3.8}$$

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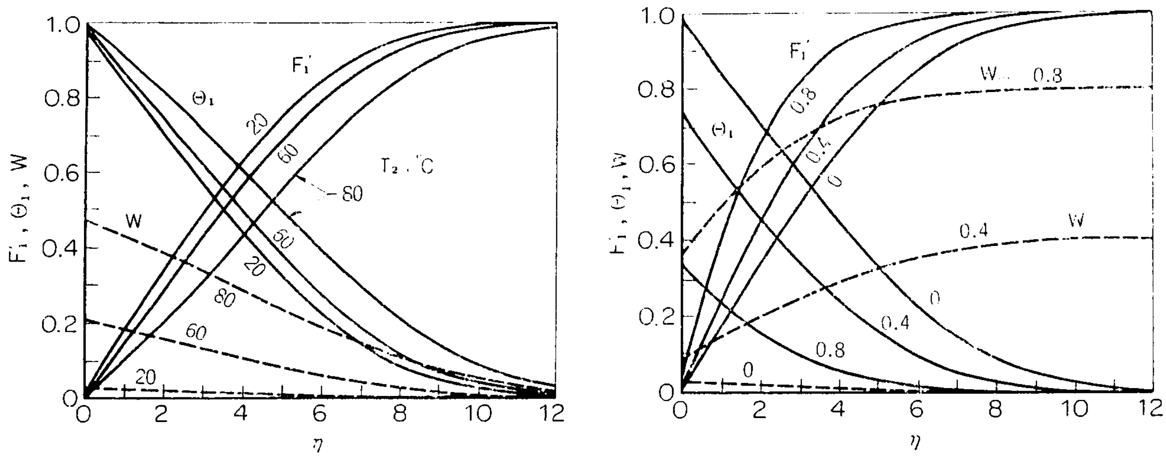

FIG. 3.2. Profiles of u-velocity, temperature and vapor-concentration.

(a) Effect of liquid temperature (Water-air, $T_{1\infty}=100°C$, $W_\infty=0$).

(b) Effect of vapor concentration at infinity (Water-air, $T_{1\infty}=100°C$, $T_{2\infty}=20°C$).

The boundary conditions $(3.5)_w$ and $(3.5)_q$ with equations (3.8) given relationships between the normal velocity, the temperature and the concentration at the interface;

$$\left.\begin{aligned} \frac{\nu_1}{2\varepsilon}F_{10} &= \frac{W_\infty - W_0}{1-W_0}\,\frac{1}{\gamma_1(\nu_1/\varepsilon,\ \infty)} \\ LF_{10} &= \frac{\lambda_1\Theta_{10}}{\gamma_1(\nu_1/\kappa_1,\ \infty)} + \frac{\lambda_2\Theta_{20}}{\gamma_2(\nu_2/\kappa_2,\ -\infty)}. \end{aligned}\right\} \quad (3.9)$$

The temperature and its gradient at the interface are then written as

$$\left.\begin{aligned} \Theta_{10} &= \frac{-LF_{10}\cdot\gamma_2(\nu_2/\kappa_2,\ -\infty)+\lambda_2}{\lambda_1\dfrac{\gamma_2(\nu_2/\kappa_2,\ -\infty)}{\gamma_1(\nu_1/\kappa_1,\ \infty)}+\lambda_2} \\ \Theta_{10}' &= -\frac{1}{\gamma_1(\nu_1/\kappa_1,\ \infty)}\,\frac{-LF_{10}\cdot\gamma_2(\nu_2/\kappa_2,\ -\infty)+\lambda_2}{\lambda_1\dfrac{\gamma_2(\nu_2/\kappa_2,\ -\infty)}{\gamma_1(\nu_1/\kappa_1,\ \infty)}+\lambda_2}. \end{aligned}\right\} \quad (3.10)$$

From equations (3.9) and (3.10) with $(3.5)_e$, the normal velocities $-F_{10}$ and $-F_{20}(=-\Lambda^{-1}F_{10})$ at the interface, that is, the rate of phase change of the fluid are thus obtainable, although equations (3.9) and (3.10) are implicit functions of F_{10} and F_{20}. The interative method may readily present their numerical solutions. It should be noted that the numerical computation with equations (3.7) to (3.10) has the superiority over the direct calculation of equations (3.3) to (3.5) with respect to the convergence of iteration for the same accuracy. In some cases, the former required only one tenth times as many iterations as the latter.

The numerical results obtained in such a way are presented in Figs. 3.2. (a), (b), which show the effects of the liquid temperature and the vapor concentration at infinity upon the fields of velocity, temperature and concentration. As the liquid

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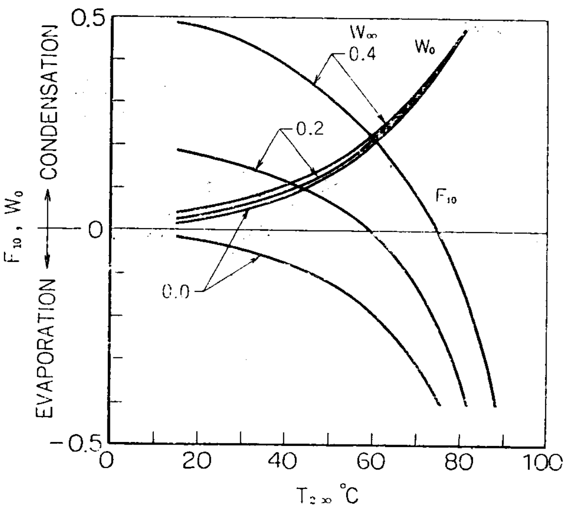

FIG. 3.3. Rate of phase-change and vapor-concentration at infinity (Water-air, $T_{1\infty}=100°C$).

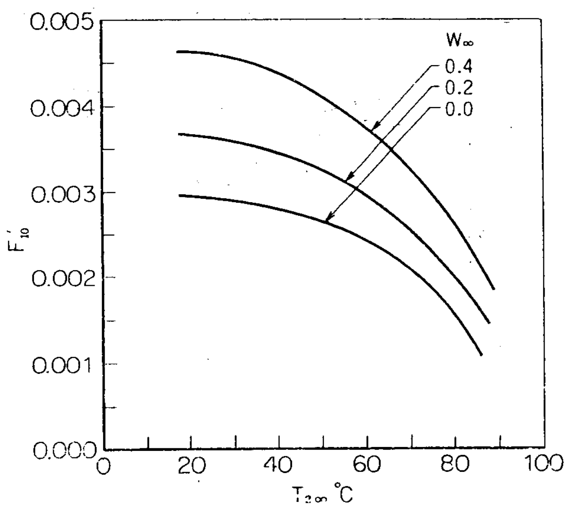

FIG. 3.4. The u-velocity at the interface (Water-air, $T_{1\infty}=100°C$).

temperature becomes higher so that more violent evaporation takes place at the interface, the velocity and temperature distributions across the layer come to be of more diffuse profiles, having an inflection point ($U''=0, \Theta''=0$) at the further distance from the interface. At higher concentrations of vapor at infinity, which result in more intense condensation at the interface, the velocity profile varies much more rapidly near the interface. These features are scarcely affected by the kind of liquid such as water, methanol or benzene.

The normal velocity or the rate of phase-change and the vapor concentration at the interface are shown in Fig. 3.3, where the positive value of F_{10} means condensation at the interface and the negative value evaporation. The u-velocity of the interface is presented in Fig. 3.4. The more intense evaporation tends to make the liquid into motion the more resitively.

The skin-friction coefficient c_f and the heat-transfer coefficient c_h of the gas side at the interface can be defined as

$$c_f \equiv \frac{\rho_1\nu_1\left(\dfrac{\partial u}{\partial y}\right)_{10}}{\rho_1(u_{1\infty}-u_{2\infty})^2} = F''_{10}\sqrt{\frac{\nu_1}{x}} \qquad c_h \equiv \frac{\lambda_1\left(\dfrac{\partial T}{\partial y}\right)_{10}}{\lambda_1(T_{1\infty}-T_{2\infty})/x} = -\Theta'_{10}\sqrt{\frac{\nu_1}{x}} \quad (3.11)$$

so that F''_{10} and $-\Theta'_{10}$ mean the dimensionless coefficients, being 0.332 for the Blausius profiles. The heat-transfer of the gas side has the meaning that for cases of intense evaporation at the interface the temperature gradient of the gas side Θ'_{10} could almost dominate the rate of phase-change because of $\Theta'_{20}\approx 0$, that is,

$$M \approx -\frac{\lambda_1}{L}\Theta'_{10}. \quad (3.12)$$

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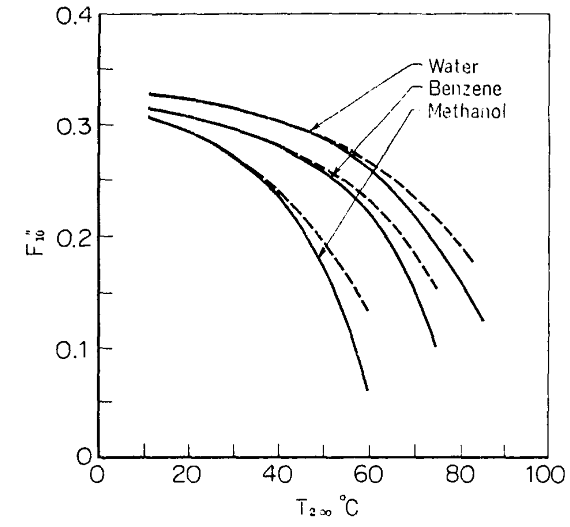

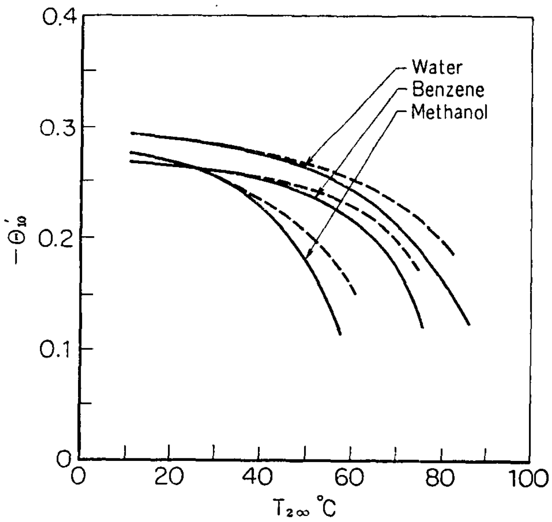

FIG. 3.5. Effect of liquid kind on coefficients of skin-friction and heat-transfer (Air, $T_{1\infty}=100°C$, $W_{\infty}=0$).

(a) Skin-friction coefficient. (b) Heat-transfer coefficient.

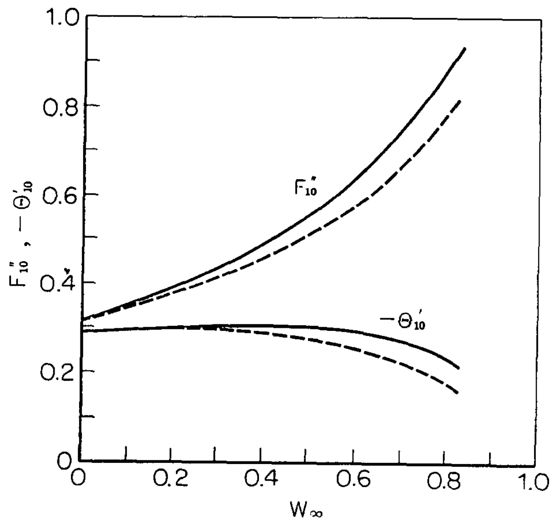

FIG. 3.6. Effect of vapor concentration on coefficients of skin-friction and heat-transfer (Water-air, $T_{1\infty}=100°C$, $T_{2\infty}=20°C$).

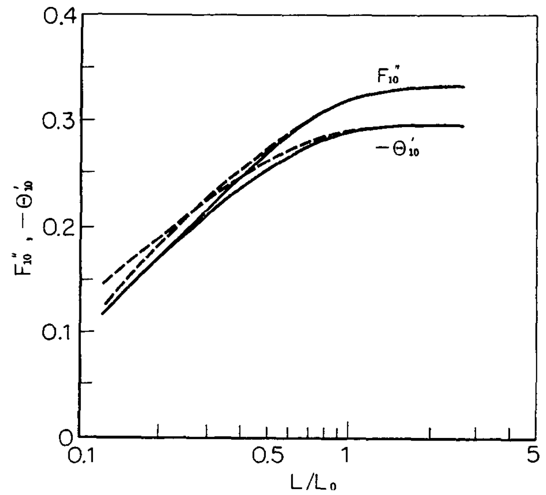

FIG. 3.7. Effect of latent heat of vaporization on coefficients of skin-friction and heat-transfer (Water-air, $T_{1\infty}=100°C$, $T_{1\infty}=20°C$, $W_{\infty}=0$, $L_0=539$ cal/g).

The numerically calculated values of F''_{10} and Θ'_{10} are shown in Figs. 3.5 to 3.7, which illustrate the effects of the liquid temperature and the vapor concentration at infinity and the latent heat of vaporization, respectively. It is shown that the coefficients are greatly influenced by the liquid temperature rather by the air temperature. At first sight to equation (3.10) the phase-change ($F_{10}\neq 0$) seems to make the heat-transfer coefficient $-\Theta'_{10}$ increase or decrease, corresponding to evaporation ($F_{10}<0$) or condensation ($F_{10}>0$), respectively, although the numerical

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results do not necessarily show such a behavior. This means that the phase-change at the interface has a considerable effect upon the velocity distribution F_{10} and F_{20} which are implicitly involved in γ_1 and γ_2.

3.2 *Heat-transfer and skin-friction at the interface*

To make the role of the phase-change at the interface more perspective, let us treat the problem analytically in an approximate method. Equations $(3.3)_{u,t,w}$ are independent each other so that the fields of velocity, temperature and concentration should not be interacted through these basic equations. Their interactions result from the boundary conditions at the gas-liquid interface (3.4). The change in the phase state of the fluid at the interface, controlled by the temperature field, causes a mass flow normal to the surface to modify the velocity and temperature distributions across the layers. Such an interaction of the fields through the boundary conditions at the interface can be represented explicitly in the basic equations with the use of an intrinsic coordinate of the stream function ψ.

Denote z the intrinsic coodinate with its origin $\bar{\psi}_0(x)$ which is to be determined later;

$$z=\psi-\bar{\psi}_0$$

and transform the (x, y) cordinates into (x, z) with the relation

$$\frac{\partial}{\partial y}=U\frac{\partial}{\partial z} \qquad \frac{\partial}{\partial x}=\frac{\partial}{\partial x}-\left(V+\frac{\partial\bar{\psi}_0}{\partial x}\right)\frac{\partial}{\partial z}, \tag{3.13}$$

then, the governing equations can be reduced to

$$\left(\frac{\partial}{\partial x}-\frac{\partial\bar{\psi}_0}{\partial x}\frac{\partial}{\partial z}\right)\begin{pmatrix}U\\ \Theta\\ W\end{pmatrix}=(\nu,\kappa,\varepsilon)\frac{\partial}{\partial z}\left\{U\frac{\partial}{\partial z}\begin{pmatrix}U\\ \Theta\\ W\end{pmatrix}\right\} \tag{3.14}$$

of which the second term on the left hand side expresses the nonlinear interaction at the interface. In the neighbourhood of the interface, we can approximate the stream function as

$$\psi-\psi_0=U_0y+\frac{1}{2}U_{y0}y^2,$$

where ψ_0 is the value of the stream function at the interface and $U_y=\partial U/\partial y$. Then, the streamwise velocity becomes

$$U=U_0+U_{y0}y \tag{3.15}$$

so that

$$\psi-\psi_0=\frac{1}{2U_{y0}}(U^2-U_0^2).$$

Here, we take the value of ψ_0 as

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$$\bar{\varphi}_0=\varphi_0-\frac{U_0^2}{2U_{y0}}, \tag{3.16}$$

then the velocity can be expressed in the form

$$U=\sqrt{2U_{y0}z} \tag{3.17}$$

and equations (3.14) are reduced to

$$(\nu,\kappa,\varepsilon)\frac{\partial}{\partial z}\left\{\sqrt{2U_{y0}z}\frac{\partial}{\partial z}\begin{pmatrix}U\\ \Theta\\ W\end{pmatrix}\right\}-\frac{\partial}{\partial x}\begin{pmatrix}U\\ \Theta\\ W\end{pmatrix}=-\frac{\partial\bar{\varphi}_0}{\partial x}\frac{\partial}{\partial z}\begin{pmatrix}U\\ \Theta\\ W\end{pmatrix}. \tag{3.18}$$

Firstly, let us consider the velocity field. Using the transformation of

$$\xi=\int_0^x \nu\sqrt{2U_{y0}}dx$$

and the Heaviside operator s corresponding to $\partial/\partial\xi$, we can transform equation $(3.18)_u$ to

$$\frac{\partial}{\partial z}\left(\sqrt{z}\,\frac{\partial[U-U_\infty]}{\partial z}\right)-s[U-U_\infty]=-\left[\frac{1}{\nu\sqrt{2U_{y0}}}\frac{\partial\bar{\varphi}_0}{\partial x}\frac{\partial(U-U_\infty)}{\partial z}\right],$$

where the Heaviside operated value is denoted by [ ]. By setting

$$\zeta=\sqrt{z},$$

the above equation can be transformed to a more amenable form

$$\frac{\partial^2[U-U_\infty]}{\partial\zeta^2}-4\zeta s[U-U_\infty]=-[Q_u], \tag{3.19}$$

where

$$Q_u=\frac{2}{\nu\sqrt{2U_{y0}}}\frac{\partial\bar{\varphi}_0}{\partial x}\frac{\partial(U-U_\infty)}{\partial\zeta}\approx\frac{2}{\nu}\frac{\partial\bar{\varphi}_0}{\partial x}.$$

The solution of equation (3.19) is given by

$$[U-U_\infty]=\sqrt{\zeta}\,\{I_{1/3}(Z)[A_1(\zeta)+C_1]+I_{-1/3}(Z)[A_2(\zeta)+C_2]\}, \tag{3.20}$$

where I_n is the modified Bessel function of the first kind of order n and

$$Z=\frac{4}{3}\sqrt{s}\,\zeta^{3/2}$$

$$A_1(\zeta)=-\frac{2\pi}{3\sqrt{3}}\int_0^\zeta\sqrt{\zeta}\,I_{-1/3}(Z)\cdot[Q_u]d\zeta$$

$$A_2(\zeta)=\frac{2\pi}{3\sqrt{3}}\int_0^\zeta\sqrt{\zeta}\,I_{1/3}(Z)\cdot[Q_u]d\zeta.$$

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Constants C_1 and C_2 are to be determined by the boundary conditions at $z=0$. Close to the interface,

$$[U-U_\infty]_{\zeta\to 0}=[\sqrt{2U_{y0}}\zeta-U_\infty]_{\zeta\to 0}$$

which gives

$$C_1=\Gamma\left(\frac{4}{3}\right)\cdot\left(\frac{2}{3}\sqrt{s}\right)^{-1/3}[\sqrt{2U_{y0}}] \qquad C_2=\Gamma\left(\frac{2}{3}\right)\cdot\left(\frac{2}{3}\sqrt{s}\right)^{1/3}[-U_\infty],$$

where Γ is the gamma function. The condition $[U-U_\infty]\to 0$ as $z\to\infty$ leads to

$$\Gamma\left(\frac{4}{3}\right)\cdot\left(\frac{2}{3}\sqrt{s}\right)^{-1/3}[\sqrt{2U_{y0}}]+\Gamma\left(\frac{2}{3}\right)\cdot\left(\frac{2}{3}\sqrt{s}\right)^{1/3}[-U_\infty]$$
$$=\frac{2\pi}{3\sqrt{3}}[Q_u]\int_0^\infty\sqrt{\zeta}\,\{I_{-1/3}(Z)-I_{1/3}(Z)\}d\zeta$$

of which the integral in the right hand side is $(2\sqrt{s})^{-1}$. The above relation gives

$$\Gamma\left(\frac{4}{3}\right)\cdot\left(\frac{2}{3}\right)^{-1/3}[\sqrt{2U_{y0}}]=\Gamma\left(\frac{2}{3}\right)\cdot\left(\frac{2}{3}\right)^{1/3}s^{1/3}[U_\infty]+\frac{\pi}{3\sqrt{3}}s^{-1/3}[Q_u]. \quad (3.21)$$

The inversion transform of this equation yields

$$\Gamma\left(\frac{1}{3}\right)\cdot\left(\frac{2}{3}\right)^{-1/3}\sqrt{2U_{y0}}=\left(\frac{2}{3}\right)^{1/3}U_\infty\left\{\int_0^x\nu\sqrt{2U_{y0}}dx\right\}^{-1/3}$$
$$+\frac{\pi}{3\sqrt{3}}\frac{1}{\Gamma(1/3)}\int_0^x\left\{\int_{x'}^x\nu\sqrt{2U_{y0}}dx''\right\}^{-2/3}\nu\sqrt{2U_{y0}}Q_u dx'. \quad (3.22)$$

Using the expression (F,η) for (U,y), we obtain

$$U_0=F_0' \qquad V_0=-\frac{1}{2}\sqrt{\frac{U_\infty\nu}{x}}F_0 \qquad U_{y0}=\sqrt{\frac{U_\infty}{\nu x}}F_0'' \qquad Q_u=\sqrt{\frac{U_\infty}{\nu x}}\Delta,$$

where Δ is defined as

$$\Delta=F_0-\frac{(F_0')^2}{2F_0''} \quad (3.23)$$

which represents the effect of the phase-change and the fluid motion at the interface. Setting these relations into equation (3.22) yields

$$F_0''=A(F_0')^{1/3}\{1+B(F_0'')^{1/3}\Delta\}, \quad (3.24)$$

where

$$A=\left(\frac{3}{2}\right)^{2/3}\Big/\Gamma\left(\frac{1}{3}\right)=0.489 \qquad B=\frac{\pi}{\sqrt{3}}\left(\frac{2}{3}\right)^{4/3}\Gamma\left(\frac{1}{3}\right)\Big/\Gamma\left(\frac{2}{3}\right)=2.089.$$

Solving equation (3.24) for F_0'' gives

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$$F_0''=c(1+a\varDelta)^3, \tag{3.24$'$}$$

where

$$c=A^{3/2}=0.342 \qquad a=\sqrt{A}\cdot B/2=0.730. \tag{3.25}$$

If we adjust the value of c so as to be 0.332 for the case of a flat plate flow ($\varDelta=0$), values of c and a are then

$$c=0.332 \qquad a=0.723. \tag{3.25$'$}$$

As for the temperature field, the Heaviside operator transformation corresponding to

$$\xi=\int_0^x \kappa\sqrt{2U_{y0}}dx$$

reduces equation (3.18) to

$$\frac{\partial^2[\Theta]}{\partial\zeta^2}-4\zeta s[\Theta]=-[Q_t], \tag{3.26}$$

where

$$Q_t=\frac{2}{\kappa}\frac{\Theta_{y0}}{U_{y0}}\frac{\partial\bar{\psi}_0}{\partial x} \quad . \quad \Theta_{y0}=\left(\frac{\partial\Theta}{\partial y}\right)_0.$$

Subject to the condition that at $y=0$

$$(\Theta)_{\zeta\to0}=\Theta_0+\frac{\Theta_{y0}}{U_{y0}}\{\sqrt{2U_{y0}}\zeta-U_0\}_{\zeta\to0}$$

and that $\Theta\to0$ as $\zeta\to\infty$, the solution of equation (3.26) gives the temperature gradient Θ_0' in a similar form as for F_0'';

$$\Theta_0'=A\left(\frac{\nu}{\kappa}F_0''\right)^{1/3}\left\{\left(-\Theta_0+F_0'\frac{\Theta_0'}{F_0''}\right)+B\left(\frac{\nu}{\kappa}F_0''\right)^{1/3}\frac{\Theta_0'}{F_0''}\varDelta\right\}. \tag{3.27}$$

Substituting equation (3.24)$'$ into (3.27) leads to

$$\Theta'=c\left(\frac{\nu}{\kappa}\right)^{1/3}\left\{\left(-\Theta_0+F_0'\frac{\Theta_0'}{F_0''}\right)+B\left(c\frac{\nu}{\kappa}\right)^{1/3}\frac{\Theta_0'}{F_0''}\varDelta\right\}(1+a\varDelta)$$

which can be further approximated as

$$\Theta_0'=-c\left(\frac{\nu}{\kappa}\right)^{1/3}\Theta_0\left\{1+\left(\frac{\nu}{\kappa}\right)^{1/3}F_0'+\left(1+2\left(\frac{\nu}{\kappa}\right)^{2/3}\right)a\varDelta\right\}. \tag{3.27$'$}$$

Thus, we find summarily for the gas side that

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$$\left.\begin{aligned}
&F_{10}''=c(1+a\varDelta_1)^3\\
&\Theta_{10}'=-c\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}\Theta_{10}\left\{1+\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}F_{10}'+\left(1+2\left(\frac{\nu_1}{\kappa_1}\right)^{2/3}\right)a\varDelta_1\right\}\\
&W_0'=-c\left(\frac{\nu_1}{\varepsilon}\right)^{1/3}(W_0-W_\infty)\left\{1+\left(\frac{\nu_1}{\varepsilon}\right)^{1/3}F_{10}'+\left(1+2\left(\frac{\nu_1}{\varepsilon}\right)^{2/3}\right)a\varDelta_1\right\}
\end{aligned}\right\}\quad(3.28)$$

where

$$\varDelta_1=F_{10}-\frac{(F_{10}')^2}{2F_{10}''}. \tag{3.29}$$

As for the liquid side, denoting

$$\hat{U}=U_0-U_2 \qquad \hat{\eta}=-\eta_2$$

and supposing that the liquid has a main velocity U_0 at infinity, we can follow the same manipulation leading to equation (3.24) and obtain

$$\hat{F}_0''=A(\hat{F}_0'')^{1/3}\{\hat{F}_0'+B(\hat{F}_0'')^{1/3}\hat{\varDelta}\}.$$

Since we have the relation that

$$\hat{F}_0=-F_{20} \quad \hat{F}_0'=0 \quad \hat{F}_0''=F_{20}'' \quad \hat{\varDelta}\approx 0,$$

the above equation becomes

$$F_{20}''=c(F_{10}')^{3/2}. \tag{3.30}$$

Substituting this into the boundary condition (3.5) gives

$$F_{10}'=\varLambda^{-2/3}(1+a\varDelta_1)^2\approx\varLambda^{-2/3}. \tag{3.31}$$

With this value of F_{10}', we obtain the velocity gradient of the liquid at the interface

$$F_{20}''=\varLambda^{-1}c(1+a\varDelta_1)^3\approx\varLambda^{-1}c \tag{3.32}$$

and similarly the temperature gradient of the liquid

$$\Theta_{20}'\approx c\left(\frac{\nu_2}{\kappa_2}\right)^{1/3}\Theta_{20}. \tag{3.33}$$

The temperatures Θ_{10} and Θ_{20} are determined by conditions $(3.5)_t$ and $(3.5)_q$;

$$\begin{aligned}
\Theta_{10}&=1-\Theta_{20}\\
&=\left\{-\frac{F_{10}}{c}+\frac{\lambda_2}{L}\left(\frac{\nu_2}{\kappa_2}\right)^{1/3}\right\}\left[\frac{\lambda_1}{L}\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}\left\{1+F_{10}'\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}\right.\right.\\
&\qquad\left.\left.+\left(1+2\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}\right)a\varDelta_1\right\}+\frac{\lambda_2}{L}\left(\frac{\nu_2}{\kappa_2}\right)^{1/3}\right]^{-1}.
\end{aligned} \tag{3.34}$$

The normal velocity V_{10} is obtainable from the boundary condition $(3.5)_w$ with equation $(3.28)_w$

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$$-2V_{10}\sqrt{\frac{x}{U_\infty \nu_1}}=F_{10}=\frac{2\varepsilon}{\nu_1}\frac{W_\infty-W_0}{1-W_0}c\left(\frac{\nu_1}{\varepsilon}\right)^{1/3}$$
$$\times\left\{1+F'_{10}\left(\frac{\nu_1}{\varepsilon}\right)^{1/3}+\left(1+2\left(\frac{\nu_1}{\varepsilon}\right)^{1/3}\right)a\Delta_1\right\}. \tag{3.35}$$

When the phase-change at the interface is not so intense that $\Delta_1 \ll 1$, we can determine Δ_1 by equation (3.29), F_{10} by (3.35), F'_{10} by (3.31) and W_0 by $(3.5)_e$ which are then

$$\Delta_1=F_{10}-\frac{\Lambda^{-4/3}}{2c}\quad F_{10}=2c\left(\frac{\nu_1}{\varepsilon}\right)^{2/3}\frac{W_\infty-W_0}{1-W_0}\quad W_0=\exp\left\{\frac{L}{R_v}\left(\frac{1}{T_b}-\frac{1}{T_{2\infty}}\right)\right\} \tag{3.36}$$

and finally predict F''_{10} and Θ'_{10} from equations $(3.38)_{u,t}$ and (3.34);

$$F''_{10}=c\left[1+a\left\{2c\left(\frac{\nu_1}{\varepsilon}\right)^{-2/3}\frac{W_\infty-W_0}{1-W_0}-\frac{\Lambda^{-4/3}}{2c}\right\}\right]^3 \tag{3.37}$$

$$\Theta'_{10}=-c\left(\frac{\nu_1}{\kappa_1}\right)^{-1/3}\Theta_0\left[1+\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}\Lambda^{-2/3}+\left(1+2\left(\frac{\nu_1}{\kappa_1}\right)^{2/3}\right)a\right.$$
$$\left.\times\left\{2c\left(\frac{\nu_1}{\varepsilon}\right)^{-2/3}\frac{W_\infty-W_0}{1-W_0}-\frac{\Lambda^{-4/3}}{2c}\right\}\right] \tag{3.38}$$

where

$$\Theta_{10}=\left\{-2\left(\frac{\nu_1}{\varepsilon}\right)^{-2/3}\frac{W_\infty-W_0}{1-W_0}+\frac{\lambda_2}{L}\left(\frac{\nu_2}{\kappa_2}\right)^{1/3}\right\}\left[\frac{\lambda_1}{L}\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}\left\{1+\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}\Lambda^{-2/3}\right.\right.$$
$$\left.\left.+\left(1+2\left(\frac{\nu_1}{\kappa_1}\right)^{2/3}\right)a\left(2c\left(\frac{\nu_1}{\varepsilon}\right)^{-2/3}\frac{W_\infty-W_0}{1-W_0}-\frac{\Lambda^{-4/3}}{2c}\right)\right\}+\frac{\lambda_2}{L}\left(\frac{\nu_2}{\kappa_2}\right)^{1/3}\right]^{-1}. \tag{3.39}$$

The analytical results of equation (3.37) and (3.38) with (3.25)′ are shown in Figs. 3.5 to 3.7 by dotted lines. It is shown that equations (3.37) and (3.38) could be valid even at larger amount of phase-change at the interface.

Certain aspects of the results shown in Figs. 3.5 to 3.7 are discussed according to these equations. Equations $(3.28)_u$ and $(3.28)_t$ or (3.37) and (3.38) show clearly the effect of the phase-change upon the coefficients, F''_{10} and Θ'_{10}. As the liquid temperature takes higher values to approach T_b (the boiling point), $|F_{10}|$ increases with the decreases of F''_{10} and $|\Theta'_{10}|$. This features are more evident at smaller values of latent heat of vaporization (Fig. 3.7). For condensation, $F_{10}>0$, the tends of F''_{10} and $|\Theta'_{10}|$ are the reverse for evaporation. As the value of T_0 approaches T_b, the values of F''_{10} and $|\Theta'_{10}|$ increase. The higher mass-fraction of vapor at infinity, W_∞ increases $F_{10}(>0)$ so as to make F''_{10} increase. Its effect upon $|\Theta'_{10}|$ has both positive term which increases with W_∞ and the negative term of Θ_{10} which decreases with W_∞. Figure 3.6 shows that the former is predominant at the smaller values of W_∞, while the latter at the larger values of W_∞. The effect of the free

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stream temperature of gas $T_{1\infty}$ is attributed mainly to the value of λ_1, which is effective upon the value of Θ_{10}. In the case of condensation, its effect is thus most remarkable.

4. Steady Gas-Liquid Flows with a Wavy Phase-Changing Interface

4.1 Disturbance flow fields

To investigate the effect of the wavy disturbances upon the velocity and temperature fields has special interest in that it might be applied to problems of wave generation by flow over a mobile boundary, for instance wind over water, as well as to heat-transfer under a wavy phase-changing interface. Such problems may be approached by considering a simple wave-train superposed on the basic state and then finding the features of disturbance fields.

Ignoring the possible instability of the flow system and accordingly taking the disturbance to be stationary relative to the wave on the interface, we can study much more clearly the interesting effect due to the presence of a wavy phase-changing interface. It is thus convenient to use a reference frame in which the wave upon the interface is stationary, moving at speed c with the wave, so that the velocity parallel to the interface is $(U-c)$ which is henceforth denoted as

$$U^* = U - c. \tag{4.1}$$

The interface elevation is taken to be

$$\delta = \delta_0 e^{i\alpha x} \tag{4.2}$$

corresponding to which the fields of velocity, temperature and concentration are to be disturbed (Fig. 4.1). The amplitude δ_0 is assumed to be small compared with the wave length $2\pi/\alpha$ so that $(\alpha\delta_0)^2$ is to be neglected.

As the characteristic length of the laminar boundary-layer flow, the measure of its thickness may be conveniently used, that is,

$$l_{r1} = \sqrt{\frac{\nu_1 x}{U_\infty}} \qquad l_{r2} = \sqrt{\frac{\nu_2 x}{U_\infty}} \tag{4.3}$$

which are assumed to be nearly constant in a reasonable distance so that the main features of the boundary-layer are largely preserved over several wave lengths, as assumed commonly in theories of boundary-layer stability. The introduction of these characteristic lengths allows similarity solutions for the base fields of velocity, temperature and concentration with a plane phase-changing interface and thus also for their perturbation fields with a wavy interface of small amplitude. With the use of these characteristic lengths, the following understanding must be taken with respect to the disturbance fields;

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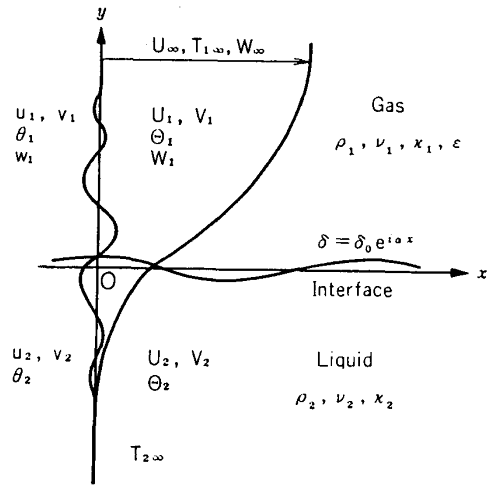

FIG. 4.1. Disturbance fields of laminar boundary-layer flows of gas and liquid having a wavy phase-changinging interface.

$$\left.\begin{aligned}
&\sqrt{\frac{\nu_1}{U_\infty x}}\to\nu_1 \qquad \sqrt{\frac{\nu_2}{U_\infty x}}\to\nu_2 \\
&\frac{\kappa_1}{\nu_1}\sqrt{\frac{\nu_1}{U_\infty x}}\to\kappa_1 \qquad \frac{\kappa_2}{\nu_2}\sqrt{\frac{\nu_2}{U_\infty x}}\to\kappa_2 \qquad \frac{\varepsilon}{\nu_1}\sqrt{\frac{\nu_1}{U_\infty x}}\to\varepsilon \\
&\frac{\lambda_1}{\rho_1 c_{p1} U_\infty}\sqrt{\frac{\nu_1 x}{U_\infty}}\to\lambda_1 \qquad \frac{\lambda_2}{\rho_1 c_{p1} U_\infty}\sqrt{\frac{\nu_2 x}{U_\infty}}\to\lambda_2
\end{aligned}\right\} \quad (4.4)$$

where ν_1^{-1} and ν_2^{-1} mean the boundary-layer Reynolds numbers with respect to the gas and liquid flows, respectively.

Taking into account the equation of continuity, we put the disturbance velocity components as

$$u = if'(y)e^{i\alpha x} \qquad v = \alpha f(y)e^{i\alpha x}. \quad (4.5)$$

Equation of motion (2.14) is then reduced to

$$\alpha(iU^*\tilde{F} - iU''f) = \nu(\tilde{F}'' - \alpha\tilde{F}) \qquad \tilde{F} \equiv f'' - \alpha^2 f. \quad (4.6)$$

Writing the corresponding disturbance temperature and concentration as

$$\left.\begin{aligned}
\theta &= g(y)e^{i\alpha x} \\
w &= h(y)e^{i\alpha x}
\end{aligned}\right\} \quad (4.7)$$

we can rewrite equations (2.14) in the form

$$\left.\begin{aligned}
\alpha(iU^*g + \Theta' f) &= \kappa(g'' - \alpha^2 g) \\
\alpha(iU^*h + W' f) &= \varepsilon(h'' - \alpha^2 h).
\end{aligned}\right\} \quad (4.8)$$

At large distances from the interface, the disturbances must vanish so that the boundary conditions at infinity are

$$f_{\pm\infty} = f'_{\pm\infty} = 0 \quad g_{\pm\infty} = 0 \quad h_\infty = 0. \quad (4.9)$$

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Since the boundary conditions at the interface must be satisfied just at the wavy interface $y=\delta$, it would seem that the boundary conditions at the mean position of the interface $y=0$ might require a further restriction on the wave amplitude so that, within the range of the distance normal to the interface less than the amplitude, the variations of U, Θ and W are negligibly small. This severe restriction can be avoided by the linearization of boundary conditions at the interface $y=\delta$ as pointed out by Landahl and Benjamin. A quantity $\Phi+\phi e^{i\alpha x}$ is linearized at $y=\delta_0 e^{i\alpha x}$ in δ_0 as

$$\Phi_0+(\Phi_0'\delta_0+\phi_0)e^{i\alpha x},$$

where the subscript 0 refers to values at the mean position of the interface, that is, at $y=0$. This means that the corresponding disturbance amplitude of the quantity at the interface $\tilde{\phi}_0 e^{i\alpha x}$ should be expressed as

$$\tilde{\phi}_0=\Phi_0'\delta+\phi_0.$$

In this way, the disturbance boundary conditions at the interface can be written in (f, g, h) expression as follows:

$$(f_1'-iU_1'\delta_1)_0=(f_2'-iU_2'\delta_2)_0 \tag{4.10$_u$}$$

$$(g_1+\Theta_1'\delta_1)_0+(g_2+\Theta_2'\delta_2)_0=0 \tag{4.10$_t$}$$

$$\rho_1\alpha_1(f_1-iU^*\delta_1)_0=\rho_2\alpha_2(f_2-iU^*\delta_2)_0 \tag{4.10$_c$}$$

$$\alpha_1(W_0-1)(f_1-iU_1^*\delta_1)_0+V_{10}(h+W'\delta_1)_0-\varepsilon(h'+W''\delta_1)_0=0 \tag{4.10$_w$}$$

$$\rho_1\nu_1(f_1''-iU_1''\delta_1+\alpha_1^2f_1)_0=\rho_2\nu_2(f_2''-iU_2''\delta_2+\alpha_2^2f_2)_0 \tag{4.10$_s$}$$

$$\begin{aligned}\rho_1\{k_1+g_1\delta_1+2\alpha_1(V_1f_1-\nu_1f_1')\}_0=&-\sigma\alpha_1^2\delta_1\\&+\rho_2\{k_2+g_2\delta_2+2\alpha_2(V_2f_2-\nu_2f_2')_0\}_0\end{aligned} \tag{4.10$_p$}$$

$$L\alpha_1(f_1-iU^*\delta_1)_0=\lambda_1(g_1'+\Theta_1''\delta_1)_0+\lambda_2(g_2'+\Theta_2''\delta_2)_0 \tag{4.10$_q$}$$

$$(h+W'\delta_1)_0=H_t(g_1+\Theta'\delta_1)_0-H_p(k_1-g_1\delta_1)_0 \tag{4.10$_e$}$$

where $(\ )_1'$ and $(\ )_2'$ denote the first derivatives with respect to y, being non-dimensionalized by l_{r1} and l_{r2}, respectively. The wave numbr α and the amplitude of the wavy interface δ have the following relationships,

$$\alpha_1=\alpha l_{r1} \qquad \alpha_2=\alpha l_{r2}$$
$$\delta=\delta_{10}l_{r1}e^{i\alpha_1x_1}=\delta_{20}l_{r2}e^{i\alpha_2x_2}$$

that is,

$$\alpha_1\delta_{10}=\alpha_2\delta_{20} \qquad \alpha_1x_1=\alpha_2x_2=\alpha x. \tag{4.11}$$

The disturbance pressure $p=ke^{i\alpha x}$ is given by equation of motion (2.5) as

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$$\alpha k=\nu\{f'''-iU'''\delta-\alpha^2(f'-iU'\delta)\} \\ -i\alpha U^*(f'-iU'\delta)-V(f''-iU''\delta)+i\alpha U'f. \tag{4.12}$$

As mentioned before, we consider only the disturbances stationary relative to the waves on the interface so that there exists a reaction of the disturbance flow to maintain the prescribed wavy disturbances. Without the external forces corresponding to such reaction, the wave number α should be determined by an eigen-value problem of the whole system as in the successive chapter. If we take the normal stress condition $(4.10)_p$ as the external necessity of maintaining the disturbance of wave number α, the disturbance pressure amplitude k is then determined as a subsequent quantity of the resulted disturbance fields.

We have thus two fourth-order ordinary differential equations for f and two second-order equations for g and one second-order for h subject to seven boundary conditions at infinity and seven at the interface. The boundary conditions at infinity restrict the number of the independent solution of the ordinary differential equations for f, g and h to be written with constants $A_1, A_2, B_1, B_2, C_1, C_2$ and D as

$$\left.\begin{aligned} f_n&=A_nf_{an}+B_nf_{bn} \\ g_n&=A_ng_{an}+B_ng_{bn}+C_ng_{cn} \qquad (n=1,2) \\ h&=A_1h_a+B_1h_b+Dh_d \end{aligned}\right\} \tag{4.13}$$

where f_a and f_b are the independent solutions of equation (4.6), g_a, g_b, h_a and h_b the solutions of equation (4.8) corresponding to f_a and f_b, respectively, and g_c and h_d those of equation (4.8) with $f\equiv 0$, all those satisfying the boundary conditions at infinity. The boundary conditions at the interface can be expressed in the form

$$a_{n1}A_1+b_{n1}B_1+a_{n2}A_2+b_{n2}B_2+c_{n1}C_1+c_{n2}C_2+d_nD+e_n\delta_0=0 \\ (n=u,s,c,t,q,w,e) \tag{4.14}$$

or in a matrix form

$$\begin{bmatrix} a_{u1} & b_{u1} & a_{u2} & b_{u2} & 0 & 0 & 0 & e_u \\ a_{s1} & b_{s1} & a_{s2} & b_{s2} & 0 & 0 & 0 & e_s \\ a_{c1} & b_{c1} & a_{c2} & b_{c2} & 0 & 0 & 0 & e_c \\ a_{t1} & b_{t1} & a_{t2} & b_{t2} & c_{t1} & c_{t2} & 0 & e_t \\ a_{q1} & b_{q1} & a_{q2} & b_{q2} & c_{q1} & c_{q2} & 0 & e_q \\ a_{w1} & b_{w1} & 0 & 0 & 0 & 0 & d_w & e_w \\ a_{e1} & b_{e1} & 0 & 0 & c_e & 0 & d_e & e_e \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ C_1 \\ C_2 \\ D \\ \delta_0 \end{bmatrix} = 0 \tag{4.14$'$}$$

where a, b, c, d and e are given by equations (4.10) as follows;

$$a_{u1}=(f'_{a1})_0 \quad b_{u1}=(f'_{b1})_0 \quad a_{u2}=-(f'_{a2})_0 \quad b_{u2}=-(f'_{b2})_0$$
$$e_u=-i(U'_{10}-U'_{20}\nu_1/\nu_2),$$
$$a_{s1}=\rho_1\nu_1(f''_{a1}+\alpha_1^2f_{a1})_0 \qquad b_{s1}=\rho_1\nu_1(f''_{b1}+\alpha_1^2f_{b1})_0$$

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$$a_{s2}=-\rho_2\nu_2(f''_{a2}+\alpha_2^2 f_{a2})_0 \qquad b_{s2}=-\rho_2\nu_2(f''_{b2}+\alpha_2^2 f_{b2})_0$$
$$e_s=-i(\rho_1\nu_1 U''_{10}-\rho_2\nu_2 U''_{20}\nu_1/\nu_2),$$
$$a_{c1}=\rho_1(f_{a1})_0 \quad b_{c1}=\rho_1(f_{b1})_0 \quad a_{c2}=-\rho_2(f_{a2})_0\nu_2/\nu_1$$
$$b_{c2}=-\rho_2(f_{b2})_0\nu_2/\nu_1 \qquad e_c=-i(\rho_1-\rho_2)U_0,$$
$$a_{t1}=(g_{a1})_0 \quad b_{t1}=(g_{b1})_0 \quad a_{t2}=(g_{a2})_0 \quad b_{t2}=(g_{b2})_0$$
$$c_{t1}=(g_{c1})_0 \quad c_{t2}=(g_{c2})_0 \quad e_t=\Theta'_{10}+\Theta'_{20}\nu_1/\nu_2,$$
$$a_{q1}=\alpha_1(f_{a1})_0-\lambda_1/L(g'_{a1})_0 \qquad b_{q1}=\alpha_1(f_{b1})_0-\lambda_1/L(g'_{b1})_0$$
$$a_{q2}=-\lambda_2/L(g'_{a2})_0 \qquad b_{q2}=-\lambda_2/L(g'_{b2})_0$$
$$c_{q1}=-\lambda_1/L(g'_{c1})_0 \qquad c_{q2}=-\lambda_2/L(g'_{c2})_0$$
$$e_q=-i\alpha_1 U_0^*-(\lambda_1\Theta''_{10}+\nu_1/\nu_2\cdot\lambda_2\Theta''_{20})/L,$$
$$a_{w1}=\{\alpha_1(W-1)f_{a1}+V_1 h_{a1}-\varepsilon h'_{a1}\}_0$$
$$b_{w1}=\{\alpha_1(W-1)f_{b1}+V_1 h_{b1}-\varepsilon h'_{b1}\}_0$$
$$d_w=(V_1 h_d-\varepsilon h'_d)_0 \qquad e_w=\{-i\alpha_1(W-1)U^*+V_1 W'-\varepsilon W''\}_0,$$
$$a_{e1}=(h_a-H_t g_{a1}+H_p k_{a1})_0 \qquad b_{e1}=(h_b-H_t g_{b1}+H_p k_{b1})_0$$
$$c_{a1}=-H_t(g_{c1})_0 \quad d_e=(h_d)_0 \quad e_e=-H_p g_1+W'_0-H_t\Theta'_{10}.$$

Eliminating C_1, C_2 and D in equations (4.14) gives

$$\begin{bmatrix} a_{u1} & b_{u1} & a_{u2} & b_{u2} & e_u \\ a_{s1} & b_{s1} & a_{s2} & b_{s2} & e_s \\ a_{c1} & b_{c1} & a_{c2} & b_{c2} & e_c \\ a_{q1}^* & b_{q1}^* & a_{q2}^* & b_{q2}^* & e_q^* \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ \delta_0 \end{bmatrix} = 0 \tag{4.15}$$

where

$$a_{q1}^*=\alpha_1(f_{a1})_0+a_{q1}^{**}$$
$$a_{q1}^{**}\equiv\frac{\lambda_1}{L}\left(\frac{g_{a1}}{g_{c1}}g'_{c1}-g'_{a1}\right)_0+\frac{1}{H_t}\left(\frac{\lambda_2}{L}\frac{g'_{c2}}{g_{c2}}-\frac{\lambda_1}{L}\frac{g'_{c1}}{g_{c1}}\right)_0$$
$$\times\left\{H_p k_{a1}+\left[(1-W)\alpha_1 f_{a1}+\varepsilon h_{a1}\left(\frac{h'_{a1}}{h_{a1}}-\frac{h'_d}{h_d}\right)\right]_0\left(V_1-\varepsilon\frac{h'_d}{h_d}\right)_0^{-1}\right\}$$
$$b_{q1}^*=\alpha_1(f_{b1})_0+b_{q1}^{**} \qquad b_{q1}^{**}\equiv(a_{q1}^{**})_{a\to b}$$
$$a_{q2}^*=\frac{\lambda_2}{L}\left(\frac{g_{a2}}{g_{c2}}g'_{c2}-g'_{a2}\right)_0 \qquad b_{q2}^*\equiv(a_{q2}^*)_{a\to b}$$
$$e_q^*=-i\alpha_1 U_0^* e_q^{**}$$
$$e_q^{**}\equiv\frac{\lambda_1}{L}\left(\frac{g'_{c1}}{g_{c1}}\Theta'_{10}-\Theta''_{10}\right)+\frac{\nu_1}{\nu_2}\frac{\lambda_2}{L}\left(\frac{g'_{c2}}{g_{c2}}\Theta'_{20}-\Theta''_{20}\right)$$
$$+\frac{1}{H_t}\left(\frac{\lambda_2}{L}\frac{g'_{c2}}{g_{c2}}-\frac{\lambda_1}{L}\frac{g'_{c1}}{g_{c1}}\right)_0\left\{-H_p g_1+\left(V_1-\varepsilon\frac{h'_d}{h_d}\right)_0\right.$$
$$\left.\times\left[i(W-1)\alpha_1 U^*+\varepsilon W'\left(\frac{W''}{W'}-\frac{h'_d}{h_d}\right)\right]_0\right\}.$$

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The solution of equations (4.14) and (4.15) gives constants A_n, B_n, C_n and D corresponding to δ_0. Since equations (4.14) are linear, we can set $\delta_0=1$ for simplicity, thus, values of A_n, B_n, C_n and D are to be interpreted finally as a multiple of δ_0. Substituting these constants in equations (4.13), we obtain the disturbance fields of velocitty, emperature and concentration.

4.2 *Effect of the wavy interface*

To examine quantitatively the effect of the wavy interface, we treat the problem analytically with an approximation of linear profiles for the mean field. Since the disturbances f, g and h diminish very rapidly with increasing y, the region where the magnitude of disturbances f, g and h is significant in comparison with their values at the interface can be assumed to be largely covered with the linear profile region of mean fields, U, Θ and W, over which we can assume approximately

$$U=U_0+U_0'y \quad \Theta=\Theta_0+\Theta_0'y \quad W=W_0+W_0'y. \tag{4.16}$$

With the linear distribution of mean velocity, equation (4.6) can be reduced to

$$F''+\frac{\alpha}{i\nu}U_0'\left(y+\frac{U_0^*-i\alpha\nu}{U_0'}\right)F=0. \tag{4.17}$$

Defining a coordinate z as

$$z=y+z_0 \qquad z_0\equiv(U_0^*-i\alpha\nu)/U_0', \tag{4.18}$$

we can rewrite the above equation for the upper fluid,

$$F_1''+\beta_1 z_1 F_1=0 \qquad \beta_1\equiv -i\alpha_1 U_{10}'/\nu_1, \tag{4.17$'$}$$

of which the solution is a linear combination of functions $F^{(1)}(z)$ and $F^{(2)}(z)$;

$$F^{(1,2)}(z)=\sqrt{z}\,H_{1/3}^{(1,2)}\left(\frac{2}{3}\sqrt{\beta}\,z^{3/2}\right),$$

where $H_{1/3}^{(1)}$ and $H_{1/3}^{(2)}$ are the Hankel functions of order one-third of the first and the second kind, respectively. Accordingly, equation (4.6)

$$f_1''-\alpha_1^2 f_1=F_1$$

subject to the boundary condition $f_1=f_1'=0$ at $y_1=\infty$ has the following solution.

$$f_1=A_1e^{-\alpha_1 y_1}+B_1\left[e^{\alpha_1 y_1}-\frac{2\int_{z_{10}}^{z_1}F_1^{(2)}(\xi)\sinh\{\alpha_1(z_1-\xi)\}d\xi}{e^{\alpha_1 z_{10}}\int_{z_{10}}^{\infty}F_1^{(2)}(\xi)e^{-\alpha_1\xi}d\xi}\right].$$

For the lower liquid flow, denoting

$$\hat{z}_2=\hat{y}_2+\hat{z}_{20} \quad \hat{y}_2=-y_2 \quad \hat{z}_{20}=-z_{20} \quad \beta_2\equiv i\alpha_2 U_{20}'/\nu_2 \tag{4.21}$$

we can take the same manipulation as the above to obtain

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$$f_2 = A_2 e^{-\alpha_2 \hat{y}_2} + B_2 \left[e^{\alpha_2 \hat{y}_2} - \frac{2\int_{\hat{z}_{20}}^{\hat{z}_2} F_2^{(1)}(\xi) \sinh\{\alpha_2(\hat{z}_2 - \xi)\} d\xi}{e^{\alpha_2 \hat{z}_{20}} \int_{\hat{z}_{20}}^{\infty} F_2^{(1)}(\xi) e^{-\alpha_2 \xi} d\xi} \right]. \tag{4.22}$$

Equations (4.20) and (4.22) give the values of f, f' and f'' at the interface as

$$\left.\begin{aligned} f_{n0} &= A_n + B_n \\ f'_{n0} &= (-1)^n \alpha_n (A_n - B_n) \\ f''_{n0} &= \alpha_n^2 \{A_n + (1-\tau_n) B_n\}, \end{aligned}\right\} \quad (n=1,2) \tag{4.23}$$

where

$$\tau_1 = \frac{2F_1^{(2)}(z_{10})}{\alpha_1 e^{\alpha_1 z_{10}} \int_{z_{10}}^{\infty} F_1^{(2)}(\xi) e^{-\alpha_1 \xi} d\xi} \qquad \tau_2 = \frac{2F_2^{(1)}(\hat{z}_{20})}{\alpha_2 e^{\alpha_2 \hat{z}_{20}} \int_{\hat{z}_{20}}^{\infty} F_2^{(1)}(\xi) e^{-\alpha_2 \xi} d\xi}. \tag{4.24}$$

Next, we consider the temperature field. Equation (4.8) with (4.16) becomes

$$g'' + \beta_\kappa z_\kappa g = \frac{\alpha}{\kappa} \Theta'_0 f, \tag{4.25}$$

where

$$z_\kappa = y + z_{\kappa 0} \quad z_{\kappa 0} = (U_0^* - i\alpha\kappa)/U'_0 \quad \beta_\kappa = -i\alpha U'_0/\kappa. \tag{4.26}$$

Defining functions $G^{(1)}$ and $G^{(2)}$ as

$$G^{(1,2)}(z_\kappa) = \sqrt{z_\kappa}\, H_{1/3}^{(1,2)}\left(\frac{2}{3}\sqrt{\beta_\kappa}\, z_\kappa^{3/2}\right),$$

we obtain the following solution of equation (4.25) for the upper and lower fluids, respectively,

$$\left.\begin{aligned} g_1 &= C_1 G_1^{(2)} + \frac{i\pi}{6} \frac{\alpha_1}{\kappa_1} \Theta'_{10} \left\{ G_1^{(2)} \int_{z_{\kappa 10}}^{z_{\kappa 1}} G_1^{(1)} f_1 d\xi + G_1^{(1)} \int_{z_{\kappa 1}}^{\infty} G_1^{(2)} f_1 d\xi \right\} \\ g_2 &= C_2 G_2^{(1)} + \frac{i\pi}{6} \frac{\alpha_2}{\kappa_2} \Theta'_{20} \left\{ G_2^{(2)} \int_{\hat{z}_{\kappa 2}}^{\infty} G_2^{(1)} f_2 d\xi + G_2^{(1)} \int_{\hat{z}_{\kappa 20}}^{\hat{z}_{\kappa 2}} G_2^{(2)} f_2 d\xi \right\}. \end{aligned}\right\} \tag{4.27}$$

With equations (4.20) and (4.22), equations (4.27) give values of g and g' at the interface

$$\left.\begin{aligned} g_{n0} &= A_n G_{an0} + B_n G_{bn0} + C_n G_{cn0} \\ g'_{n0} &= A_n G'_{an0} + B_n G'_{bn0} + C_n G'_{cn0} \end{aligned}\right\} \quad (n=1,2) \tag{4.28}$$

where

$$\left.\begin{aligned} & G_{mn0} = G_{mn} G_n^{(n)}(z_{\kappa n0}) \quad G'_{mn0} = G_{mn} G_n^{(n)\prime}(z_{\kappa n0}) \quad (m=a,\, b) \\ & G_{m1} = \frac{i\pi}{6} \frac{\alpha_1}{\kappa_1} \Theta'_{10} \int_{z_{\kappa 10}}^{\infty} G_1^{(1)} F_{m1} d\xi \qquad G_{m2} = \frac{i\pi}{6} \frac{\alpha_2}{\kappa_2} \Theta'_{20} \int_{\hat{z}_{\kappa 20}}^{\infty} G_2^{(2)} F_{m2} d\xi \\ & G_{c10} = G_1^{(2)}(z_{\kappa 10}) \quad G'_{c10} = G_1^{(2)\prime}(z_{\kappa 10}) \quad G_{c20} = G_2^{(1)}(\hat{z}_{\kappa 20}) \quad G'_{c20} = G_2^{(1)\prime}(\hat{z}_{\kappa 20}) \end{aligned}\right\} \tag{4.29}$$

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where F_{an} and F_{bn} are the coefficients of A_n and B_n on the right hand side of equations (4.20) and (4.22), respectively.

Replacing κ, Θ_0' by ε and W_0' in equation (4.27), we obtain in the same way values of h and h' at the interface

$$\left.\begin{aligned} h_0&=A_1H_{a0}+B_1H_{b0}+DH_{d0}\\ h_0'&=A_1H_{a0}'+B_1H_{b0}'+DH_{d0}' \end{aligned}\right\}(4.30)$$

where

$$\left.\begin{aligned} &H_{m0}=H_mH_\varepsilon^{(1)}(z_{\varepsilon 0}) \quad H_{m0}'=H_mH_\varepsilon^{(1)\prime}(z_{\varepsilon 0}) \quad (m=a,b)\\ &H_m=\frac{i\pi}{6}\frac{\alpha_1}{\varepsilon}W_0'\int_{z_{\varepsilon 0}}^{\infty}H_\varepsilon^{(1)}F_{m1}d\xi \qquad H_\varepsilon^{(1)}=\sqrt{z}\,H_{1/3}^{(1)}\Big(\frac{2}{3}\sqrt{\beta_\varepsilon}\,z^{3/2}\Big). \end{aligned}\right\}(4.31)$$

Since we may consider the length $|\beta|^{-1/3}$ as a measure of the effective thickness of the disturbance fields, the assumption that this thickness is small compared with the thickness of linear profile of the mean fields can therefore be expressed as $|\beta|^{-1/3}\alpha\ll 1$. Thus, we may take $|\beta^{1/2}z_0^{3/2}|\ll 1$ and approximate τ_n as

$$\tau_n=\frac{2\cdot 3^{1/3}}{\Gamma\Big(\frac{2}{3}\Big)}\frac{\beta_n^{1/3}}{\alpha_n}e\Big(n,-\frac{i\pi}{3}\Big) \qquad e(n,x)\equiv\exp\{-(-1)^nx\} \quad (4.32)$$

which means $|\tau_n|\gg 1$, and G_{mn}, $G_n^{(m)}$ and $G_n^{(m)\prime}$ as

$$\left.\begin{aligned} &G_{an}=G_{bn}=\frac{i\pi}{6}\frac{2}{\sqrt{3}}\frac{\alpha_n}{\kappa_n}\Theta_{n0}'\beta_{\kappa n}^{-1/2}e\Big(n,\frac{i\pi}{6}\Big)\\ &G_n^{(m)}(z_{\kappa n0})=\frac{2}{\sqrt{3}}\frac{3^{1/3}}{\Gamma\Big(\frac{2}{3}\Big)}\beta_{\kappa n}^{-1/6}e\Big(m,-\frac{i\pi}{2}\Big)\\ &G_n^{(m)\prime}(z_{\kappa n0})=\frac{2}{\sqrt{3}}\frac{3^{2/3}}{\Gamma\Big(\frac{1}{3}\Big)}\beta_{\kappa n}^{1/6}e\Big(m,\frac{i\pi}{6}\Big) \end{aligned}\right\}(4.33)$$

where the following approximate relations are used

$$F_1^{(2)}(z_{10})\approx F_1^{(2)}(0) \qquad \int_0^\infty F_1^{(2)}(\xi)d\xi=\frac{1}{\sqrt{\beta_1}}\frac{2i}{\sqrt{3}}e^{-(\pi/3)i}$$

$$F_2^{(1)}(z_{20})\approx F_2^{(1)}(0) \qquad \int_0^\infty F_2^{(1)}(\xi)d\xi=-\frac{1}{\sqrt{\beta_2}}\frac{2i}{\sqrt{3}}e^{(\pi/3)i}$$

$$\int_{z_{10}}^\infty f_{a1}G_1^{(2)}d\xi\approx\int_0^\infty G_1^{(2)}d\xi \qquad \int_{z_{20}}^\infty f_{a2}G_2^{(1)}d\xi\approx\int_0^\infty G_2^{(1)}d\xi.$$

Equations (4.33) hold for H and H' by replacing κ and Θ_0' by ε and W_0', respectively.

Substituting the obtained values of f, g and h at the interface into the boundary conditions (4.15) yields

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$$
\begin{bmatrix}
-\alpha_1 & \alpha_1 & -\alpha_2 & \alpha_2 & -i\left(U'_{10}-U'_{20}\dfrac{\nu_1}{\nu_2}\right) \\
\dfrac{\rho_1}{\rho_2}\left(\dfrac{\nu_1}{\nu_2}\right)^3 & \dfrac{\rho_1}{\rho_2}\left(\dfrac{\nu_1}{\nu_2}\right)^3\left(1-\dfrac{\tau_2}{2}\right) & -1 & -\left(1-\dfrac{\tau_2}{2}\right) & 0 \\
\dfrac{\rho_1}{\rho_2} & \dfrac{\rho_1}{\rho_2} & -\dfrac{\nu_2}{\nu_1} & -\dfrac{\nu_2}{\nu_1} & -i\left(\dfrac{\rho_1}{\rho_2}-1\right)U_0^* \\
\alpha_1+a_1^* & \alpha_2+b_1^* & a_2^* & b_2^* & -i\alpha_1 U_0^*+e^*
\end{bmatrix}
\begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ 1 \end{bmatrix}=0
\tag{4.34}
$$

where

$$
a_1^*=b_1^*=e^{-(\pi/3)i}3^{1/3}\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\frac{\lambda_1}{L}\beta_{\kappa 1}^{1/3}\left[i3^{-2/3}\Gamma\left(\frac{1}{3}\right)\beta_{\kappa 1}^{1/3}\frac{\Theta'_{10}}{U'_{10}}\right.
$$

$$
\left.+e^{(2\pi/3)i}\frac{\alpha_1}{H_t}\left\{1+e^{-(2\pi/3)i}\left(\frac{\beta_{\kappa 2}}{\beta_{\kappa 1}}\right)^{1/3}\frac{\lambda_2}{\lambda_1}\right\}\frac{1-W_0-3^{-1/3}\Gamma\left(\frac{2}{3}\right)\beta_\varepsilon^{-1/3}W'_0}{V_{10}+e^{(\pi/3)i}3^{1/3}\Gamma\left(\frac{2}{3}\right)\Big/\Gamma\left(\frac{1}{3}\right)\varepsilon\beta_\varepsilon^{1/3}}\right]
$$

$$
a_2^*=b_2^*=-ie^{(\pi/3)i}3^{-1/3}\Gamma\left(\frac{2}{3}\right)\frac{\lambda_1}{L}\beta_{\kappa 2}^{2/3}\frac{\Theta'_{20}}{U'_{20}}
$$

$$
e^*=-e^{(\pi/3)i}3^{1/3}\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\frac{\lambda_1}{L}\beta_{\kappa 1}^{1/3}\left[\left\{-\Theta'_{10}+e^{-(2\pi/3)i}\left(\frac{\beta_{\kappa 2}}{\beta_{\kappa 1}}\right)^{1/3}\frac{\nu_1}{\nu_2}\frac{\lambda_2}{\lambda_1}\Theta'_{20}\right\}+\frac{1}{H_t}\right.
$$

$$
\left.\times\left\{1+e^{-(2\pi/3)}\left(\frac{\beta_{\kappa 2}}{\beta_{\kappa 1}}\right)^{1/3}\frac{\lambda_2}{\lambda_1}\right\}\frac{i\alpha_1(W_0-1)U_0^*+e^{(\pi/3)i}3^{1/3}\Gamma\left(\frac{2}{3}\right)\Big/\Gamma\left(\frac{1}{3}\right)\varepsilon\beta_\varepsilon^{1/3}W'_0}{V_{10}+e^{(\pi/3)i}3^{1/3}\Gamma\left(\frac{2}{3}\right)\Big/\Gamma\left(\frac{1}{3}\right)\varepsilon\beta_\varepsilon^{1/3}}\right].
$$

Equations (4.34) give the constants A_n and B_n as

$$
\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}=\frac{1}{2}\left\{\mp\frac{i}{\alpha_1}\left(U'_{10}-\frac{\nu_1}{\nu_2}U'_{20}\right)\pm i\,\frac{4-\tau_2}{\tau_2}U_0^*\right.
$$

$$
\left.-i\,\frac{\nu_1}{\nu_2}\,\frac{a_2^*}{\alpha_1+a_1^*}\,U_0^*-\frac{-i\alpha_1 U_0^*+e^*}{\alpha_1+a_1^*}\right\}
$$

$$
\begin{pmatrix} A_2 \\ B_2 \end{pmatrix}=i\left(\frac{1}{2}\mp\frac{4-\tau_2}{2\tau_2}\right)\frac{\nu_1}{\nu_2}U_0^*
$$

where $\rho_1/\rho_2 \ll 1$ is assumed. Thus, we obtain

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$$\left.\begin{aligned} f_{10}&=A_1+B_1=\frac{1}{\alpha_1+a_1^*}\left\{i\alpha_1\left(1-\frac{a_2^*}{\alpha_2}\right)U_0^*-e^*\right\} \\ f_{20}&=A_2+B_2=i\frac{\nu_1}{\nu_2}U_0^* \\ f_{10}'&=\alpha_1(-A_1+B_1)=i\left(U_{10}'-\frac{\nu_1}{\nu_2}U_{20}'\right)-i\frac{4-\tau_2}{\tau_2}U_0^* \\ f_{10}''&=\alpha_1^2(A_1+B_1)-\alpha_1^2\tau_1B_1=\alpha_1^2\left\{f_{10}-\frac{\tau_1}{2}\left(f_{10}+\frac{f_{10}'}{2}\right)\right\}. \end{aligned}\right\} \quad (4.35)$$

By eliminating C_2 with equations $(4.14)_t$ and $(4.14)_q$ to obtain C_1, g_{10} and g_{10}' can be written as

$$\begin{aligned} g_{10}=&\left(\frac{\lambda_2}{\lambda_1}\frac{g_{c1}}{g_{c1}'}\frac{g_{c2}'}{g_{c2}}-1\right)_0^{-1}\left[-\frac{\lambda_2}{\lambda_1}\frac{g_{c2}'}{g_{c2}}\left\{\frac{\lambda_1}{\lambda_2}\frac{g_{c2}}{g_{c2}'}g_{a1}\left(1-\frac{g_{c1}}{g_{c1}'}\frac{g_{a1}'}{g_{a1}}\right)(A_1+B_1)\right.\right. \\ &\left.\left.+\frac{g_{c1}}{g_{c1}'}g_{a2}\left(1-\frac{g_{c2}}{g_{c2}'}\frac{g_{a2}'}{g_{a2}}\right)(A_2+B_2)+\frac{g_{c1}}{g_{c1}'}e_t\right\}-\frac{g_{c1}}{g_{c1}'}\frac{L}{\lambda_1}\left\{e_q+\alpha_1(A_1+B_1)\right\}\right]_0 \\ g_{10}'=&\left(\frac{\lambda_2}{\lambda_1}\frac{g_{c1}}{g_{c1}'}\frac{g_{c2}'}{g_{c2}}-1\right)_0^{-1}\left[-\frac{\lambda_2}{\lambda_1}\frac{g_{c2}'}{g_{c2}}\left\{g_{a1}\left(1-\frac{g_{c1}}{g_{c1}'}\frac{g_{a1}'}{g_{a1}}\right)(A_1+B_1)\right.\right. \\ &\left.\left.+g_{a2}\left(1-\frac{g_{c2}}{g_{c2}'}\frac{g_{a2}'}{g_{a2}}\right)(A_2+B_2)+e_t\right\}-\frac{L}{\lambda_1}\{e_q+\alpha_1(A_1+B_1)\}\right]_0 \end{aligned} \quad (4.36)$$

where, with equation (4.33), we can use the following approximation

$$\frac{g_{an0}'}{g_{an0}}=i3^{1/3}\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\beta_{\kappa n}^{1/3}e\left(n,\frac{i\pi}{6}\right)$$

$$\frac{g_{cn0}'}{g_{cn0}}=-i3^{1/3}\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\beta_{\kappa n}^{1/3}e\left(n,-\frac{i\pi}{6}\right)$$

$$g_{an0}=i\frac{2\pi}{9}\frac{3^{1/3}}{\Gamma\left(\frac{2}{3}\right)}\frac{\alpha_n}{\kappa_n}\beta_{\kappa n}^{-2/3}\Theta_{n0}'e\left(n,-\frac{i\pi}{3}\right)$$

where $n=1,2$.

The disturbance normal velocity at the interface $\alpha_1 f_{10}$ is then explicitly expressed as

$$\begin{aligned} f_{10}&\left[\alpha_1+e^{-(\pi/3)i}\Gamma_0\beta_{\kappa1}^{1/3}\frac{\lambda_1}{L}\left[i3^{-2/3}\Gamma\left(\frac{1}{3}\right)\beta_{\kappa1}^{1/3}\frac{\Theta_{10}'}{U_{10}'}+e^{(2\pi/3)i}\frac{\alpha_1}{H_t}\right.\right. \\ &\left.\left.\times\left\{1+e^{-(2\pi/3)}\frac{\lambda_2}{\lambda_1}\left(\frac{\beta_{\kappa2}}{\beta_{\kappa1}}\right)^{1/3}\right\}\left\{H_pk_{a1}+\frac{1-W_0-3^{-1/3}\Gamma\left(\frac{2}{3}\right)\beta_\epsilon^{-1/3}W_0'}{V_{10}+e^{(\pi/3)i}\Gamma_0\epsilon\beta_\epsilon^{1/3}}\right\}\right]\right] \end{aligned}$$

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$$
\begin{aligned}
&= iU_0^*\Big\{\alpha_1 + ie^{(\pi/3)i}3^{-1/3}\Gamma\Big(\frac{2}{3}\Big)\beta_{\kappa 2}^{2/3}\frac{\lambda_2}{L}\frac{\nu_1}{\nu_2}\frac{\Theta'_{20}}{U'_{20}} \\
&\quad + e^{(\pi/3)i}\Gamma_0\beta_{\kappa 1}^{1/3}\frac{\lambda_1}{L}\Big[\Big\{-\Theta'_{10} + e^{-(2\pi/3)i}\frac{\lambda_2}{\lambda_1}\frac{\nu_1}{\nu_2}\Big(\frac{\beta_{\kappa 2}}{\beta_{\kappa 1}}\Big)^{1/3}\Theta'_{20}\Big\} \\
&\quad + \frac{1}{H_t}\Big\{1 + e^{-(2\pi/3)i}\frac{\lambda_2}{\lambda_1}\Big(\frac{\beta_{\kappa 2}}{\beta_{\kappa 1}}\Big)^{1/3}\Big\}\Big\{-H_p g_1 + \frac{i\alpha_1(W_0-1)U_0^* + e^{(\pi/3)i}\Gamma_0\epsilon\beta_\epsilon^{1/3}W'_0}{V_{10} + e^{(\pi/3)i}\Gamma_0\epsilon\beta_\epsilon^{1/3}}\Big\}\Big].
\end{aligned} \tag{4.37}
$$

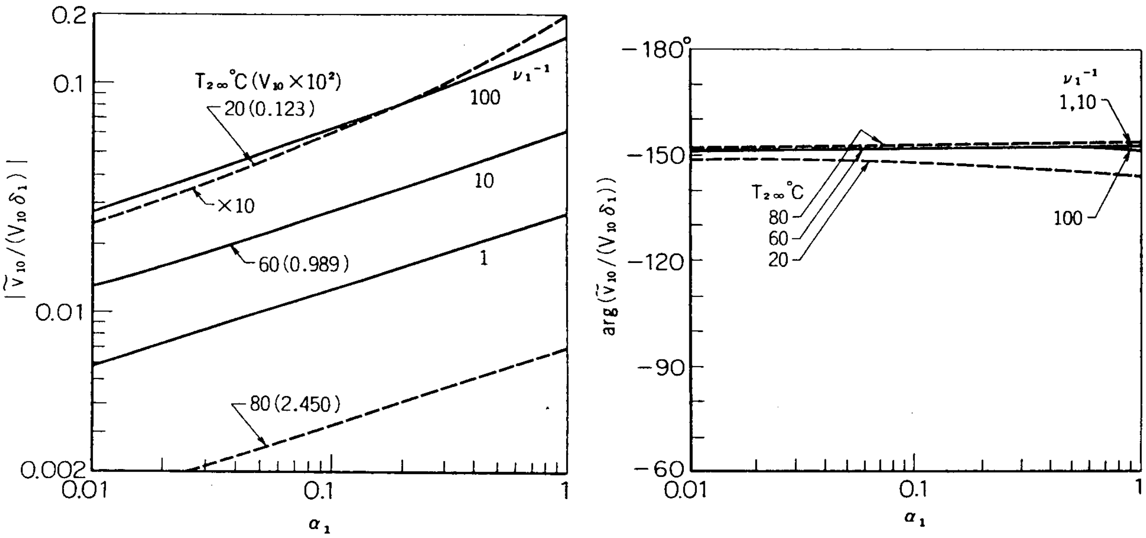

FIG. 4.2.1. Disturbance rate of phase-change (Water-air, $T_{1\infty}=100$°C, $c=0$; ——, $T_{2\infty}=60$°C, $W_\infty=0$; - - - -, $\nu_1^{-1}=10$, $W_\infty=0$).

(a) Amplitude (b) Phase angle

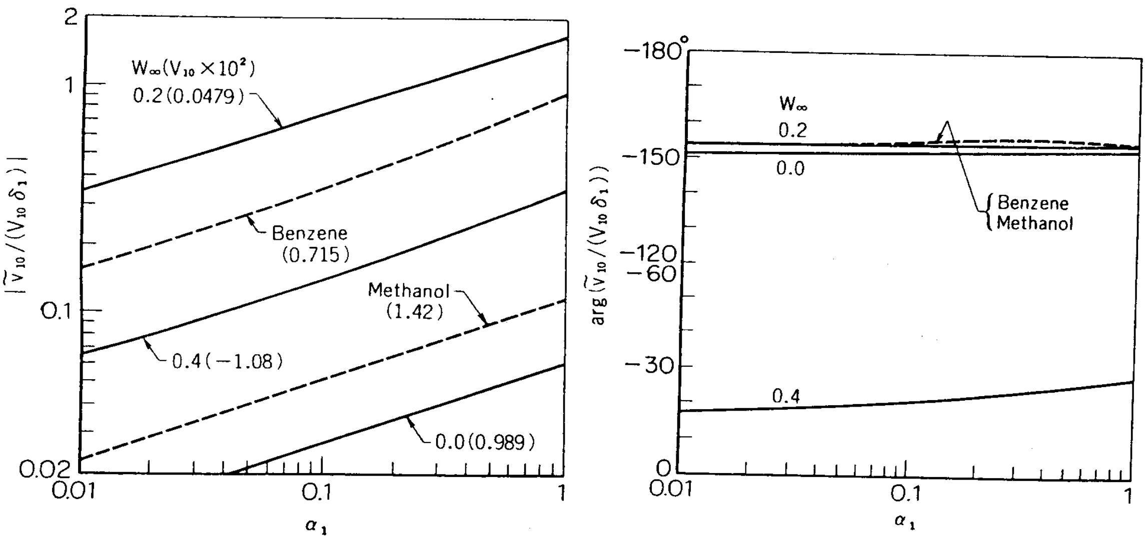

FIG. 4.2.2. Disturbance rate of phase-change (Air, $T_{1\infty}=100$°C, $c=0$, $\nu_1^{-1}=10$; ——, water, $T_{2\infty}=60$°C; - - - -, $T_{2\infty}=40$°C, $W_\infty=0$).

(a) Amplitude (b) Phase angle

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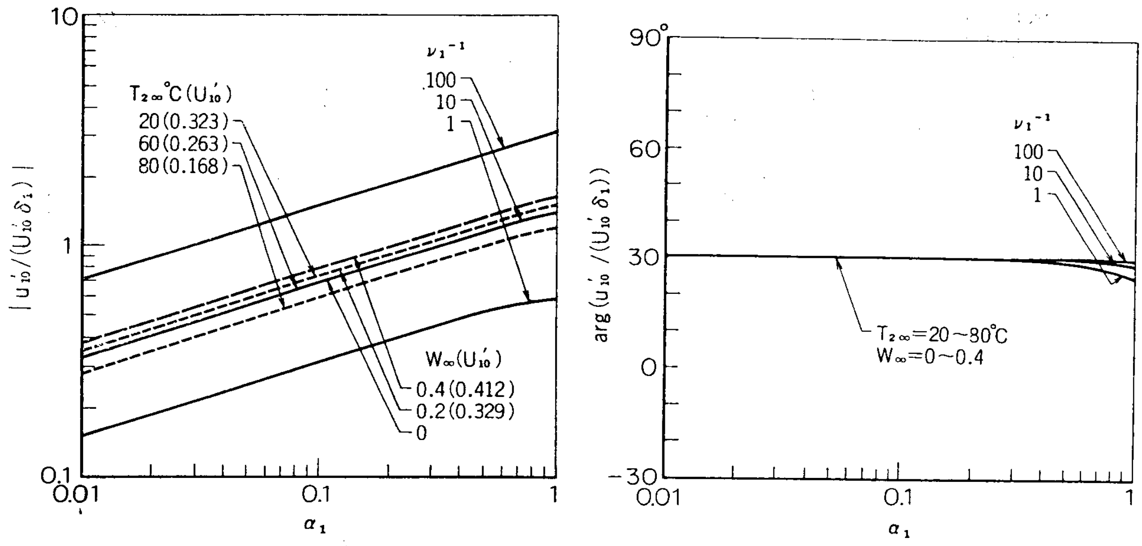

FIG. 4.3. Disturbance velocity gradient at the interface (Water-air, $T_{1\infty}=100°C$, $c=0$; ——, $T_{2\infty}=60°C$, $W_\infty=0$; ----, $\nu_1^{-1}=10$, $W_\infty=0$; – – –, $T_{2\infty}=60°C$, $\nu_1^{-1}=10$).
(a) Amplitude (b) Phase angle

where $\Gamma_0=3^{1/3}\Gamma\left(\frac{2}{3}\right)\Big/\Gamma\left(\frac{1}{3}\right)$. The right side of the above equation means the disturbance heat flux into the interface and the left is the disturbance amount of phase-change. Both sides phisically consist of three terms, attributed to the disturbed convection field, the disturbed temperature field and the change in the interface temperature, respectively. The disturbance rate of phase-change $\tilde{v}_{10}/(V_{10}\delta_1)=\alpha_1(f_{10}-iU_0^*)/V_{10}$ is graphically shown in Fig. 4.2, where V_{10} is $(-\nu_1 F_{10}/2)$ in the term of the preceding chapter. The order estimation of the terms of equation (4.37) shows that the most predominant on the left side is the one attributed to the disturbance convection field and on the right those due to the disturbed temperature field and due to the disturbed convection field, so that the disturbance normal velocity or the disturbance rate of phase-change can be approximated as

$$\alpha_1 f_{10}=e^{(\pi/3)i}\Gamma_0\beta_{\kappa 1}^{1/3}\frac{\lambda_1}{L}(-\Theta'_{10})^*+i\alpha_1 U_0^* \tag{4.38}$$

$$(-\Theta'_{10})^*\equiv-\Theta'_{10}+e^{-(2\pi/3)}\frac{\lambda_2}{\lambda_1}\frac{\nu_1}{\nu_2}\left(\frac{\beta_{\kappa 2}}{\beta_{\kappa 1}}\right)^{1/3}$$

or with equation (4.26)

$$\tilde{v}_{10}=e^{(7\pi/6)i}\frac{\Gamma_0}{-L}\left(\frac{\alpha_1 U'_{10}}{\kappa_1}\right)^{1/3}\kappa_1(-\Theta'_{10})^*\delta_1 \tag{4.38$'$}$$

$$(-\Theta'_{10})^*\equiv-\Theta'_{10}+e^{-(\pi/3)i}\left(\frac{\rho_1\kappa_1}{\rho_2\kappa_2}\right)^{1/3}\frac{\lambda_2}{\lambda_1}\Theta'_{20}$$

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where $U'_{20}/U'_{10}=\rho_1\nu_1/(\rho_2\nu_2)$ is used. $\tilde{v}_{10}$ is thus proportional to the one-third power of the wave number with the phase lag of 150° relative to the interface, as shown in the figure.

In Fig. 4.3, the disturbance velocity gradient at the interface $\tilde{u}'_{10}/(U'_{10}\delta_1)=if''_{10}/U'_{10}$ given by equation (4.35) is shown. With equation (4.38) and the relation that $|\tau_1|, |\tau_2|\gg 1$, we obtain the following approximate expression for f''_{10},

$$f''_{10}=e^{(5\pi/3)i}\Gamma_1\alpha_1\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{1/3}U_0^*+e^{(5\pi/3)i}\Gamma_1\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{1/3}\left(U'_{10}-\frac{\nu_1}{\nu_2}U'_{20}\right)$$
$$+e^{(\pi/3)i}\frac{\Gamma_2}{-L}\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{2/3}(\nu_1\kappa_1^2)^{1/3}(-\Theta_{10})^* \tag{4.39}$$
$$+e^{(5\pi/6)i}2U_0^*\alpha_1^2\frac{\nu_2}{\nu_1}\left(\frac{U'_{10}}{U'_{20}}\right)^{1/3}$$

where

$$\Gamma_0=\frac{3^{1/3}\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\quad \Gamma_1=\frac{3^{1/3}}{\Gamma\left(\frac{2}{3}\right)}\quad \Gamma_2=\frac{3^{2/3}}{\Gamma\left(\frac{1}{3}\right)}.$$

The right side of the above equation comprises two contributions from the disturbance fields of the modified mean field of velocity (the first and second terms) and of the disturbed velocity field due to the phase-change (the third and fourth terms). Since the former, especially the second term, is the most predominant as shown in the figure, f''_{10} or $\tilde{u}'_{10}(=if''_{10})$ is then roughly approximated as

$$\frac{u'_{10}}{U'_{10}\delta_1}=e^{(\pi/6)i}\frac{3^{1/3}}{\Gamma\left(\frac{2}{3}\right)}\left(1-\frac{\rho_1\nu_1^2}{\rho_2\nu_2^2}\right)U'_{10}\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{1/3} \tag{4.40}$$

which is proportional to $(\alpha_1/\nu_1)^{1/3}$ and 30° in advance of the interface. The left hand side of the above equation implies the disturbance friction coefficient c_f defined by equation (3.11).

The disturbance pressure $p_{10}/\delta_1=k_{10}$ given by equation (4.12) and shown in Fig. 4.4, is proportional to $(\alpha_1/\nu_1)^{-1/3}$ and $(\alpha_1/\nu_1)^{2/3}$ at smaller and larger wave numbers, respectively. In the analytical expression of k by equation (4.12), the most predominant is $\nu f'''$, that is

$$\alpha_1 k_{10}\approx\nu_1 f'''_{10}.$$

Since equation (4.20) gives f'''_{10} approximately as

$$f'''_{10}=\alpha_1^2\left\{f'_1-\frac{1}{2}\left(f_1+\frac{f'_1}{\alpha_1}\right)\frac{F_1^{(2)\prime}}{F_1^{(2)}}\tau_1\right\}_0\approx e^{(\pi/3)i}\alpha_1\left(f_{10}+\frac{f'_{10}}{\alpha_1}\right)\frac{3^{2/3}}{\Gamma\left(\frac{1}{3}\right)}\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{2/3}.$$

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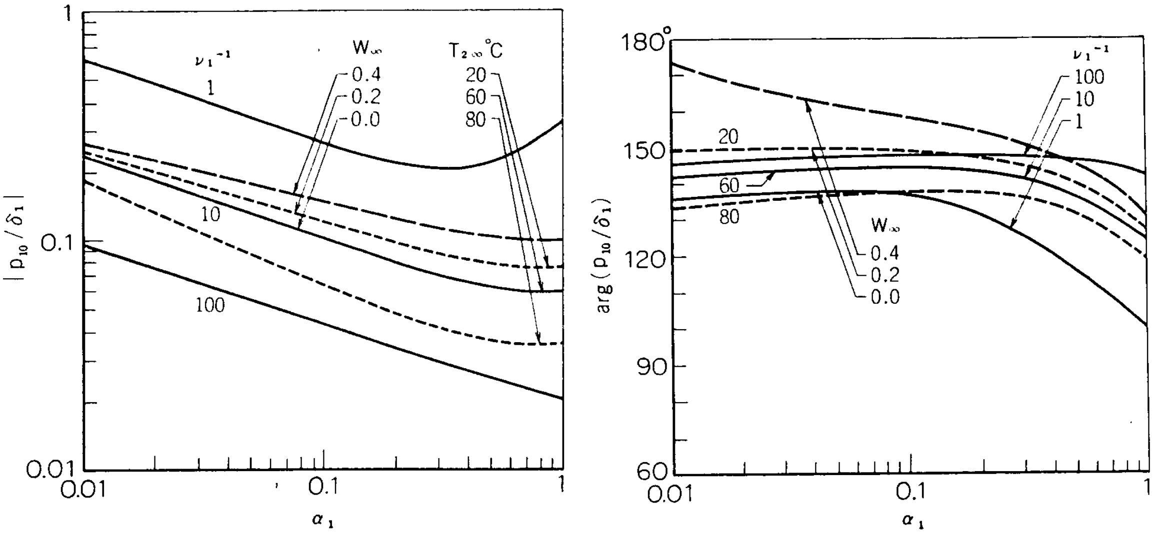

FIG. 4.4. Disturbance pressure at the interface (Water-air, $T_{1\infty}=100°C$, $c=0$; ——, $T_{2\infty}=60°C$, $W_\infty=0$; ----, $\nu_1^{-1}=10$, $W_\infty=0$; - - -, $T_{2\infty}=60°C$, $\nu_1^{-1}=10$).
(a) Amplitude (b) Phase angle

with equation (4.38) and (4.35), k_{10} is then expressed by

$$k_{10}=e^{(\pi/3)i}\Gamma_2\left\{e^{(7\pi/6)i}\frac{\Gamma_0}{-L}(U'_{10})^{1/3}\left(\frac{\kappa_1}{\alpha_1}\right)^{2/3}(-\Theta)^*+iU_0+\frac{iU'_{10}}{\alpha_1}\right\}\nu_1\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{2/3} \tag{4.41}$$

of which the third and second terms become more effective at smaller and larger wave numbers, being proportional to $\alpha_1^{-1/3}$ and $\alpha_1^{2/3}$ with the phase advance of 150°, respectively.

An interesting feature of equations (4.40) and (4.41) is the phase relation between the stresses and the wave of the interface. The shearing stress ($\sim u'_{10}$) is approximately 30° in advance of the wave. The phase of the normal stress is about 150° in advance. These phase relations are accordant with the Benjamin's result of linear or boundary-layer profiles models [6], which may be interpreted as a kind of Jeffereys' 'sheltering' effect that the stresses are distributed as if the leeward slopes of the wavy interface were sheltered and a wake were formed behind each wave crest. The effect of the phase-change upon the stresses at the interface is expressed in the terms of $(-\Theta'_{10})^*$ which become relatively predominant at larger wave numbers or for the case of higher vapor concentrations at infinity (condensation taking place at the interface), having the phase relation of 120° for u'_{10} and $-90°$ for k_{10} in advance, respectively, which act to weaken the 'sheltering' effect in the phase relation. In the evaporation case, the 'sheltering' effect may be also reduced in amplitude by the decrease of U'_{10} and $|\Theta'_{10}|$.

The disturbance temperature gradient at the interface $\theta'_{10}/(\Theta'_{10}\delta)=g'_{10}/\Theta'_{10}$ given by equation (3.36) is shown in Fig. 4.5, being proportional to $\alpha_1^{0\sim 2/3}$, were $-\theta'_{10}$ implies the disturbance heat-transfer coefficient c_h defined by equation (3.11). The approximate expression for g'_{10} is

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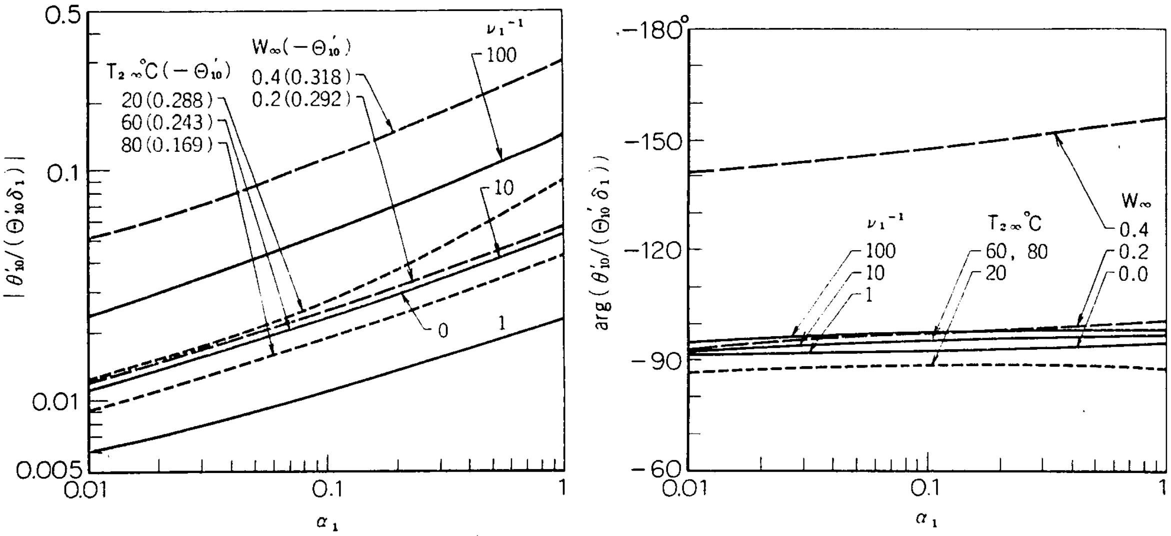

FIG. 4.5. Disturbance temperature gradient at the interface (Water-air, $T_{1\infty}=100$°C, $c=0$; ——, $T_{2\infty}=60$°C, $W_\infty=0$; ----, $\nu_1^{-1}=10$, $W_\infty=0$; - - -, $T_{2\infty}=60$°C, $\nu_1^{-1}=10$).

(a) (b)

$$
\begin{aligned}
g'_{10}=\;& ie^{(2\pi/3)i}\Gamma_0(U'_{10})^{1/3}\left(\frac{\alpha_1}{\kappa_1}\right)^{1/3} \\
&\times\Bigg[(f_{10}-iU_0^*)(-\Theta'_{10})\left\{1+e^{(2\pi/3)i}\Gamma_3(U'_{10})^{-2/3}\left(\frac{\alpha_1}{\kappa_1}\right)^{1/3}\right\} \\
&+e^{(\pi/6)i}(f_{10}-iU_0^*)\frac{-L}{\Gamma_0}\,\frac{\lambda_1}{\lambda_2}\left(\frac{\nu_2}{\nu_1}\,\frac{\kappa_1}{\kappa_2}U'_{20}\right)^{-1/3}\left(\frac{\alpha_1}{\kappa_1}\right)^{2/3} \\
&+iU_0^*\Bigg[(-\Theta'_{10})\left\{1+e^{(2\pi/3)i}\Gamma_3(U'_{10})^{-2/3}\left(\frac{\alpha_1}{\kappa_1}\right)^{1/3}\right\} \\
&\quad+\Theta'_{20}\left\{1+e^{(\pi/3)i}\Gamma_3\frac{\nu_2}{\nu_1}\left(\frac{\nu_2}{\nu_1}\,\frac{\kappa_1}{\kappa_2}\right)^{1/3}(U'_{20})^{-2/3}\left(\frac{\alpha_1}{\kappa_1}\right)^{1/3}\right\}\Bigg]\Bigg]
\end{aligned}
\tag{4.42}
$$

where $\Gamma_3=3^{-7/6}/\Gamma\left(\frac{2}{3}\right)$, which comprises the effects of the disturbed temperature field due to the disturbed flow (the first, third and fourth terms) and the disturbance amount of heat for phase-change (the second term). The predominant terms of them yield

$$
\begin{aligned}
\frac{\theta'_{10}}{\Theta'_{10}\delta_1}=\;&\frac{\Gamma\left(\frac{2}{3}\right)\pi}{\sqrt{3}\left\{\Gamma\left(\frac{1}{3}\right)\right\}^2}\,\frac{1}{-L}(-\Theta'_{10})^* \\
&+\frac{3^{1/3}\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\left\{e^{-(\pi/2)i}\frac{\lambda_1}{\lambda_2}\left(\frac{\rho_1\kappa_1}{\rho_2\kappa_2}\right)^{-1/3}\frac{(-\Theta_{10})^*}{-\Theta_{10}}\right.
\end{aligned}
\tag{4.42'}
$$

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$$+e^{(2\pi/3)i}U_0^*\left(1-\frac{\Theta_{20}'}{\Theta_{10}'}\right)\Big\}\left(\frac{\alpha_1 U_{10}'}{\kappa_1}\right)^{1/3}$$

Further, for intense evaporation at the interface which gives smaller values of $|\Theta_{10}'|$, $|\Theta_{20}'|$ and U_0, the above equation can be roughly expressed as

$$\frac{\theta_{10}'}{\Theta_{10}'\delta_1}\approx e^{-(\pi/2)i}\frac{3^{1/3}\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\frac{\lambda_1}{\lambda_2}\left(\frac{\rho_1\kappa_1}{\rho_2\kappa_2}\right)^{-1/3}\left(\frac{\alpha_1 U_{10}'}{\kappa_1}\right)^{1/3} \tag{4.42)''$$

which shows the proportionality to $(\alpha_1/\kappa_1)^{1/3}$ and the phase lag of 90°, although at further larger and smaller wave numbers terms proportional to $\alpha_1^{2/3}$ and $\alpha_1^{-1/3}$, respectively, in equation (4.36) come to be more influential.

Comparing equation (4.42)″ with (4.40) gives

$$\left|\frac{\theta_{10}'}{\Theta_{10}'}\right|\Big/\left|\frac{u_{10}'}{U_{10}'}\right|\approx\frac{\lambda_1}{\lambda_2}\left(\frac{\rho_2\kappa_2}{\rho_1\kappa_1}\right)^{1/3}\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}$$

of which the order of magnitude is about 0.1. This indicates that the waviness of the interface causes disturbance to the heat-transfer coefficient by one order of magnitude less than to the skin-friction. As for the temperature field, the wavy disturbance at the interface is almost absorbed into the liquid because of its high heat conductivity and the heat flux required for the disturbance rate of phase-change at the interface is supplied mostly by heat conduction through the liquid.

The disturbance coefficient of heat-transfer of the liquid side is then given by

$$g_{20}'\approx e^{(7\pi/6)i}\frac{3^{1/3}\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\frac{\lambda_1}{\lambda_2}\left(\frac{\alpha_1 U_{10}'}{\kappa_1}\right)^{1/3}(-\Theta_{10}') \tag{4.43}$$

of which the ratio to g_{10}' is $(\rho_1\kappa_1/\rho_2\kappa_2)^{1/3}=0.1\sim 1$. On the other hand, the ratio of f_{20}'' to f_{10}'' is $\rho_1\nu_1/\rho_2\nu_2=10^{-2}\sim 10^{-3}$.

The disturbance u-velocity and temperature at the interface given by equations (4.35) and (4.36) are illustrated in Figs. 4.6 and 4.7, respectively. They are approximated by

$$if_{10}'=-U_{10}'\left(1-\frac{\rho_1\nu_1^2}{\rho_2\nu_2^2}\right) \tag{4.44}$$

$$g_{10}=\frac{\lambda_1}{\lambda_2}\left(\frac{\rho_1\kappa_1}{\rho_2\kappa_2}\right)^{-1/3}\Bigg[e^{(7\pi/6)i}\frac{\pi}{3^{5/6}\Gamma\left(\frac{1}{3}\right)}\frac{1}{-L}\left(\frac{\alpha_1 U_{10}'}{\kappa_1}\right)^{-1/3}(-\Theta_{10}')^*(-\Theta_{10}')$$
$$+e^{-(2\pi/3)i}(-\Theta_{10}')^*+U_0^*\left\{e^{(\pi/2)i}(-\Theta_{10}')+e^{(7\pi/6)i}\frac{\lambda_2}{\lambda_1}\left(\frac{\rho_1\kappa_1}{\rho_2\kappa_2}\right)^{1/3}\Theta_{20}'\right\}\Bigg]. \tag{4.45}$$

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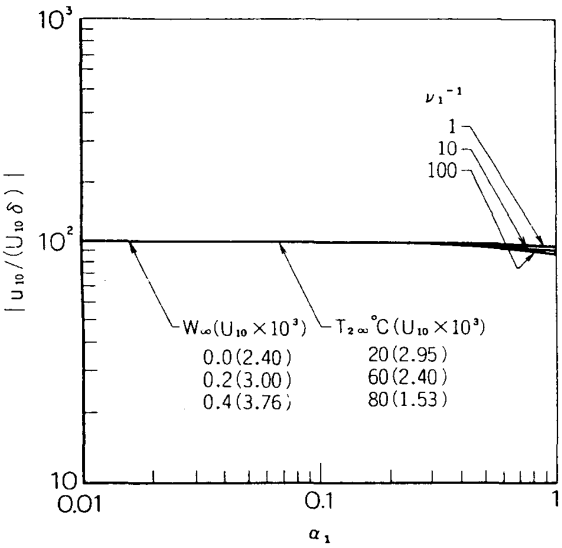

FIG. 4.6. Disturbance u-velocity at the interface (Water-air, $T_{1\infty}=100°C$, $c=0$; ——, $T_{2\infty}=60°C$, $W_{2\infty}=60°C$, $W_\infty=0$; ----, $\nu_1^{-1}=10$, $W_\infty=0$; - - -, $T_{2\infty}=60°C$, $\nu_1^{-1}=10$).

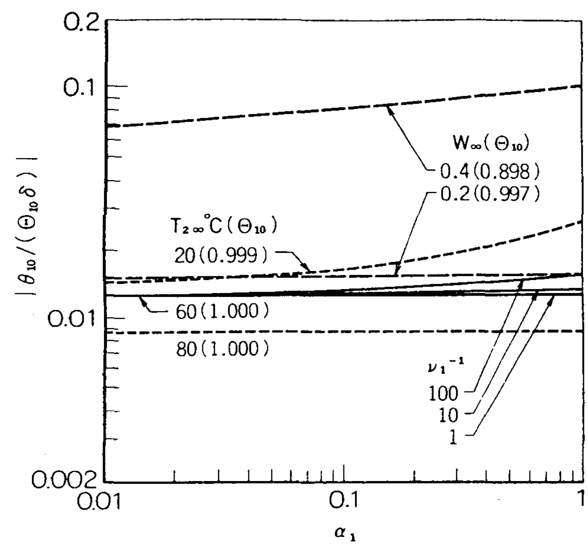

FIG. 4.7. Disturbance temperature at the interface (Water-air, $T_{1\infty}=100°C$, $c=0$; ——, $T_{2\infty}=60°C$, $W_\infty=0$; ----, $\nu_1^{-1}=10$, $W_\infty=0$; - - -, $T_{2\infty}=60°C$, $\nu_1^{-1}=10$).

Corresponding to equation (4.42), the latter may be further rewritten as

$$g_{10}\approx e^{-(2\pi/3)}\frac{\lambda_1}{\lambda_2}\left(\frac{\rho_1\kappa_1}{\rho_2\kappa_2}\right)^{-1/3}(-\Theta'_{10}) \tag{4.45'}$$

which has the phase lag of 120° relative to the interface. To first approximation, both disturbances are relevant to neither wave number nor Reynolds number. These equations show that the overall disturbance u-velocity at the interface $\tilde{u}_{10}$ which is the sum of $if'_{10}\delta_1$ and $U'_{10}\delta_1$ is nearly equal to zero, that is, the overall u-velocity at the interface remains undisturbed. Since $|g_{10}|\ll|\Theta'_{10}|$, the overall disturbance temperature at the interface ($\tilde{\theta}_{10}=g_{10}\delta_1+\Theta'_{10}\delta_1$) is then given by $\tilde{\theta}_{10}=\Theta'_{10}\delta_1$.

5. Hydrodynamic Instability of Gas-Liquid Laminar Flows with a Phase-Changing Interface

5.1 Non-stationary disturbance fields

In the preceding chapter, under phase-changing at the interface, an account was given of the effect of stationary disturbances upon the features of laminar boundary-layer flows of gas and liquid. An alternative approach to the instability problem of the motion with respect to small wavy disturbances may be to examine directly the solution of the linearized system of the whole field in the usual way for problems of boundary-layer stability, as used by Wuest [*22*], Lock [*19*], Feldman [*17*] and Miles [*20*] for the isothermal cases.

The disturbances are considered to be of the form

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$$\phi(y)e^{i\alpha(x-ct)},$$

where c is the complex wave velocity which may be expressed as

$$c=c_r+ic_i,$$

where c_r is the wave velocity and c_i allows for amplification of disturbances if $c_i>0$, damping of disturbances if $c_i<0$, and neutral disturbances if $c_i=0$. The corresponding disturbance elevation of the interface is taken to be

$$\delta=\delta_0 e^{i\alpha(x-ct)}. \tag{5.1}$$

In the same way as in the preceding chapter, we can write the disturbances of velocity, temperature and concentration as, respectively,

$$u=if'(y)e^{i\alpha(x-ct)} \qquad v=\alpha f(y)e^{i\alpha(x-ct)}, \tag{5.2}$$

$$\theta=g(y)e^{i\alpha(x-ct)} \qquad w=h(y)e^{i\alpha(x-ct)}, \tag{5.3}$$

and obtain the following governing equations and boundary equations in the same form as equation (4.6), (4.8) and (4.10);

(a) Equations:

$$\left.\begin{aligned}&\alpha\{i(U-c)\tilde{F}-iU''f\}=\nu(\tilde{F}''-\alpha^2\tilde{F})\\&\quad\tilde{F}\equiv f''-\alpha f,\\&\alpha\{i(U-c)g+\Theta'f\}=\kappa(g''-\alpha^2 g),\\&\alpha\{i(U-c)h+W'f\}=\varepsilon(h''-\alpha^2 h),\end{aligned}\right\} \tag{5.4}$$

(b) Boundary conditions:

$$\left.\begin{aligned}&f_{1\infty}=f'_{1\infty}=0 \qquad f_{2\infty}=f'_{2\infty}=0\\&g_{1\infty}=0 \qquad\quad g_{2\infty}=0 \qquad\quad h_\infty=0,\end{aligned}\right\} \tag{5.5}$$

$$(f_1'-iU_1'\delta_1)_0=(f_2'-iU_2'\delta_2)_0 \tag{5.6$_u$}$$

$$(g_1+\Theta_1'\delta_1)_0+(g_2+\Theta_2'\delta_2)_0=0 \tag{5.6$_t$}$$

$$\rho_1\alpha_1(f_1-iU\delta_1)_0=\rho_2\alpha_2(f_2-iU\delta_2)_0 \tag{5.6$_c$}$$

$$\alpha_1(W_0-1)(f_1-iU\delta_1)_0+V_{10}(h+W'\delta_1)_0-\varepsilon(h'+W''\delta_1)_0=0 \tag{5.6$_w$}$$

$$\rho_1\nu_1(f_1''-iU_1''\delta_1+\alpha_1^2 f_1)_0=\rho_2\nu_2(f_2''-iU_2''\delta_2+\alpha^2 f_2)_0 \tag{5.6$_s$}$$

$$\rho_1\{k_1+g_1\delta_1+2\alpha_1(V_1f_1-\nu_1f_1')\}_0=-\sigma_1\alpha_1^2\delta_1+\rho_2\{k_2+g_2\delta_2+2\alpha_2(V_2f_2-\nu_2f_2')\}_0 \tag{5.6$_p$}$$

$$L\alpha_1(f_1-iU\delta_1)_0=\lambda_1(g_1'+\Theta_1''\delta_1)_0+\lambda_2(g_2'+\Theta_2''\delta_2)_0 \tag{5.6$_q$}$$

$$(h+W'\delta_1)_0=H_t(g_1+\Theta_1'\delta_1)_0-H_p(k_1-g_1\delta_1)_0 \tag{5.6$_e$}$$

$$\alpha k=\nu\{f'''-iU'''\delta-\alpha^2(f'-iU'\delta)\}-i\alpha U(f'-iU'\delta)-V(f''-iU''\delta)+i\alpha U'f,$$

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where we used the same reference lengths as equation (4.3) so that all quantities appeared in above equations should be defined by equations (4.4), and

$$g_1=\frac{g}{U_\infty^2}\sqrt{\frac{\nu_1 x}{U_\infty}}, \qquad \sigma_1=\frac{\sigma}{\rho_1 U_\infty^2}\sqrt{\frac{U_\infty}{\nu_1 x}}.$$

The gravitational field is assumed to act downwards in the flow configuration.

We have now two fourth-order ordinary differential equations for f_1 and f_2, two second-order for g_1 and g_2, and one second-order for h, snbject to seven boundary conditions at infinity and eight at the interface. The boundary conditions at infinity restrict the number of independent solutions of the fourth-order differential equation to two and of the second-order to one. The general solutions of these differential equations must be therefore of the form as before

$$\left.\begin{aligned} f_n&=A_n f_{an}+A_n f_{bn} \\ g_n&=A_n g_{an}+B_n g_{bn}+C_n g_{cn} \\ h&=A_1 h_a+B_1 h_b+D h_d \end{aligned}\right\} \qquad (n=1,2), \tag{5.7}$$

where A_n, B_n, C_n and D are arbitrary constants, f_a and f_b the solutions of equations $(5.4)_u$, g_a, g_b, h_a and h_b the solutions of equations $(5.4)_t$ and $(5.4)_c$ corresponding to f_a and f_b, respectively, and g_c and h_d those of equations $(5.4)_t$ and $(5.4)_c$ with $f\equiv 0$. The boundary conditions at the interface may be then expressed in the form

$$a_{n1}A_1+b_{n1}B_1+a_{n2}A_2+b_{n2}B_2+c_{n1}C_1+c_{n2}C_2+c_{n2}C_2+d_nD+e_n\delta_0=0 \tag{5.8}$$
$$(n=u, s, c, p, t, q, w, e),$$

or in a matrix form,

$$\begin{pmatrix} a_{u1} & b_{u1} & a_{u2} & b_{u2} & 0 & 0 & 0 & e_u \\ a_{s1} & b_{s1} & a_{s2} & b_{s2} & 0 & 0 & 0 & e_s \\ a_{c1} & b_{c1} & a_{c2} & b_{c2} & 0 & 0 & 0 & e_c \\ a_{p1} & b_{p1} & a_{p2} & b_{p2} & 0 & 0 & 0 & e_p \\ a_{t1} & b_{t1} & a_{t2} & b_{t2} & c_{t1} & c_{t2} & 0 & e_t \\ a_{q1} & b_{q1} & a_{q2} & b_{q2} & c_{q1} & c_{q2} & 0 & e_q \\ a_{w1} & b_{w1} & 0 & 0 & 0 & 0 & d_w & e_w \\ a_{e1} & b_{e1} & 0 & 0 & c_{e1} & 0 & d_e & e_e \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ C_1 \\ C_2 \\ D \\ \delta_0 \end{pmatrix} = 0 \tag{5.8$'$}$$

where a, b, c, d and e are given by equation (5.6) as follows;

$$a_{u1}=(f'_{a1})_0 \quad b_{u1}=(f'_{b1})_0 \quad a_{u2}=-(f'_{a2})_0 \quad b_{u2}=-(f'_{b2})_0$$
$$e_u=-i(U'_{10}-U'_{20}\nu_1/\nu_2),$$
$$a_{s1}=\rho_1\nu_1(f''_{a1}+\alpha_1^2 f_{a1})_0 \qquad b_{s1}=\rho_1\nu_1(f''_{b1}+\alpha_1^2 f_{b1})_0$$
$$a_{s2}=-\rho_2\nu_2(f''_{a2}+\alpha_2^2 f_{a2})_0 \qquad b_{s2}=-\rho_2\nu_2(f''_{b2}+\alpha_2^2 f_{b2})_0$$
$$e_s=-i(\rho_1\nu_1 U''_{10}-\rho_2\nu_2 U''_{20}\nu_1/\nu_2),$$
$$a_{c1}=\rho_1(f_{a1})_0 \quad b_{c1}=\rho_1(f_{b1})_0 \quad a_{c2}=-\rho_2(f_{a2})_0\nu_2/\nu_1$$

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$$b_{c2} = -\rho_2(f_{b2})_0\nu_2/\nu_1 \qquad e_c = -i(\rho_1-\rho_2)U_0,$$
$$a_{p1} = \rho_1\{k_{a1}+2\alpha_1(V_1 f_{a1}-\nu_1 f'_{a1})\}_0 \qquad b_{p1} = \rho_1\{k_{b1}+2\alpha_1(V_1 f_{b1}-\nu_1 f'_{b1})\}_0$$
$$a_{p2} = -\rho_2\{k_{a2}+2\alpha_2(V_2 f_{a2}-\nu_2 f'_{a2})\}_0 \qquad b_{p2} = -\rho_2\{k_{b2}+2\alpha_2(V_2 f_{b2}-\nu_2 f'_b)\}_0$$
$$e_p = \alpha_1^2\sigma_1 + (\rho_2-\rho_1)g_1 + \rho_1 k_{e1} - \rho_2 k_{e2}$$
$$k_{a,b} = [\nu(f'''-\alpha^2 f') - \{i\alpha(U-c)f' + Vf''\}]/\alpha + iU'f$$
$$k_e = [-i\nu(U'''-\alpha^2 U')_0 + i\{i\alpha(U-c)U' - VU''\}_0]/\alpha$$
$$a_{t1} = (g_{a1})_0 \quad b_{t1} = (g_{b1})_0 \quad a_{t2} = (g_{a2})_0 \quad b_{t2} = (g_{b2})_0$$
$$c_{t1} = (g_{c1})_0 \quad c_{t2} = (g_{c2})_0 \quad e_t = \Theta'_{10} + \Theta'_{20}\nu_1/\nu_2,$$
$$a_{q1} = \alpha_1(f_{a1})_0 - \lambda_1/L(g'_{a1})_0 \qquad b_{q1} = \alpha_1(f_{b1})_0 - \lambda_1/L(g'_{b1})_0$$
$$a_{q2} = -\lambda_2/L(g'_{a2})_0 \qquad b_{q2} = -\lambda_2/L(g'_{b2})_0$$
$$c_{q1} = -\lambda_1/L(g'_{c1})_0 \qquad c_{q2} = -\lambda_2/L(g'_{c2})_0$$
$$e_q = -i\alpha_1 U_0 - (\lambda_1/L\Theta''_{10} + \nu_1/\nu_2\cdot\lambda_2/L\Theta''_{20}),$$
$$a_{w1} = \{\alpha_1(W-1)f_{a1} + V_1 h_a - \varepsilon h'_a\}_0$$
$$b_{w1} = \{\alpha_1(W-1)f_{b1} + V_1 h_b - \varepsilon h'_b\}_0$$
$$d_w = (V_1 h_d - \varepsilon h'_d)_0 \qquad e_w = \{-i\alpha_1(W-1)U + V_1 W' - \varepsilon W''\}_0,$$
$$a_{e1} = (h_a - H_t g_{a1} + H_p k_{a1})_0 \qquad b_{e1} = (h_b - H_t g_{b1} + H_p k_{b1})_0$$
$$c_{e1} = -H_t(g_{c1})_0 \quad d_e = (h_d)_0 \quad e_e = -H_p g_1 + (W' - H_t\Theta'_1)_0.$$

In order that the set of equation (5.8)′ is to have a non-trivial solution for A, B, C and D, the following relation, the so-called secular equation of an eighth-order determinant, must hold

$$\Delta \equiv \begin{vmatrix} a_{u1} & b_{u1} & a_{u2} & b_{u2} & 0 & 0 & 0 & e_u \\ a_{s1} & b_{s1} & a_{s2} & b_{s2} & 0 & 0 & 0 & e_s \\ a_{c1} & b_{c1} & a_{c2} & b_{c2} & 0 & 0 & 0 & e_c \\ a_{p1} & b_{p1} & a_{p2} & b_{p2} & 0 & 0 & 0 & e_p \\ a_{t1} & b_{t1} & a_{t2} & b_{t2} & c_{t1} & c_{t2} & 0 & e_t \\ a_{q1} & b_{q1} & a_{q2} & b_{q2} & c_{q1} & c_{q2} & 0 & e_q \\ a_{w1} & b_{w1} & 0 & 0 & 0 & 0 & d_w & e_w \\ a_{e1} & b_{e1} & 0 & 0 & c_e & 0 & d_e & e_e \end{vmatrix} = 0 \qquad (5.9)$$

of which the solution for eigen values c, α and ν has to be obtained numerically.

To obtain numerical solution of equations (5.4) in the form of equations (5.7), we must find their solutions at infinity where $f=f'=0$ and $g=h=0$. As $|y|\to\infty$, equations (5.4) become

$$\tilde{F}'' - \left\{\alpha^2 + \frac{i\alpha}{\nu}(1-c)\right\}\tilde{F} = 0 \qquad \tilde{F} = f'' - \alpha^2 f$$
$$g'' - \left\{\alpha^2 + \frac{i\alpha}{\kappa}(1-c)\right\}g = 0 \qquad h'' - \left\{\alpha^2 + \frac{i\alpha}{\varepsilon}(1-c)\right\}h = 0$$

of which solutions that $f=f'=0$ and $g=h=0$ at infinity are

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$$\left.\begin{aligned}
&(f_a)_{\pm\infty}=\exp(\mp\alpha y) \qquad (f_b)_{\pm\infty}=\exp\left[\mp\left\{\alpha^2+\frac{i\alpha}{\nu}(1-c)\right\}^{1/2}y\right]\\
&(g_c)_{\pm\infty}=\exp\left[\mp\left\{\alpha^2+\frac{i\alpha}{\kappa}(1-c)\right\}^{1/2}y\right]\\
&(h_d)_{+\infty}=\exp\left[-\left\{\alpha^2+\frac{i\alpha}{\varepsilon}(1-c)\right\}^{1/2}y\right].
\end{aligned}\right\}\quad(5.10)$$

Starting with these solutions at infinity, we can solve numerically equation $(5.4)_{u,t,c}$ using the Runge-Kutta integration;

$$\left.\begin{aligned}
&f'''-\left\{2\alpha^2+i(U-c)\frac{\alpha}{\nu}\right\}f''+\left[\alpha^4+i\frac{\alpha}{\nu}\left\{\alpha^2(U-c)+U''\right\}\right]f=0\\
&g''-\left\{\alpha^2+i(U-c)\frac{\alpha}{\kappa}\right\}g=0\\
&h''-\left\{\alpha^2+i(U-c)\frac{\alpha}{\varepsilon}\right\}h=0
\end{aligned}\right\}\quad(5.11)$$

to obtain f_{an}, f_{bn}, g_{cn} and $h_d(n=1,2)$, respectively, and equations (5.4)

$$\left.\begin{aligned}
&g''-\left\{\alpha^2+i(U-c)\frac{\alpha}{\kappa}\right\}g=i\frac{\alpha^2}{\kappa}\Theta' f\\
&h''-\left\{\alpha^2+i(U-c)\frac{\alpha}{\varepsilon}\right\}h=i\frac{\alpha^2}{\varepsilon}W' f
\end{aligned}\right\}\quad(5.12)$$

to obtain g_{an}, g_{bn}, h_a and h_b corresponding to f_a and f_b, respectively. It is noted that the solutions of f, g and h obtained in this manner may often grow to very large magnitude so that the computer is unable to proceed. To avoid such a difficulty, the solutions at infinity $f_{\pm\infty}, g_{\pm\infty}$ and $h_{+\infty}$ must start with initial values as small as possible. The used step size Δy was 0.05. The effect of step size was checked by recomputing some of results with 0.001. To obtain proper convergence, double precision arithmetic was used throughout the calculations.

5.2 *Stability curves*

The eigen values obtained in this manner are shown in Figs. 5.1 to 5.4. The results indicate that the shape of the neutral stability curves in the wave-number vs. Reynolds-number plane is similar to the case of the boundary-layer flow on a rigid wall (Wazzan [*21*]), although the latter is more stabilized than the former (Fig. 5.1). In the figure, the result calculated by Lock ($U_\infty=100$cm/s)[*19*] for the case of no phase-change at the interface is also illustrated in the present dimension of the wave-number vs. Reynolds-number plane. It shows the similar destabilization of the flow compared with that in the case of a rigid wall.

The behaviour of the amplification or damping of the disturbances in the neighbourhood of the neutral curve is shown in Fig. 5.2. The curves of constant c_i are packed close together in the neighbourhood of the upper branch of the neutral line, especially at higher liquid temperature.

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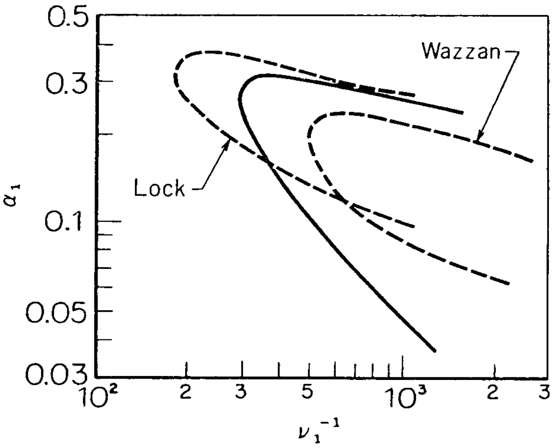

FIG. 5.1. Neutral stability curves of wave-number against Reynolds number (Water-air, $T_{1\infty}=100°C$, $T_{2\infty}=20°C$, $W_\infty=80°C$, $W_\infty=0$).

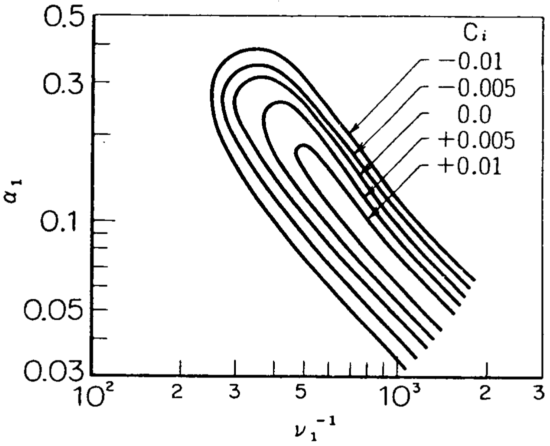

FIG. 5.2. Stability curves (Water-air, $T_{1\infty}=100°C$, $T_{2\infty}=80°C$, $W_\infty=0$).

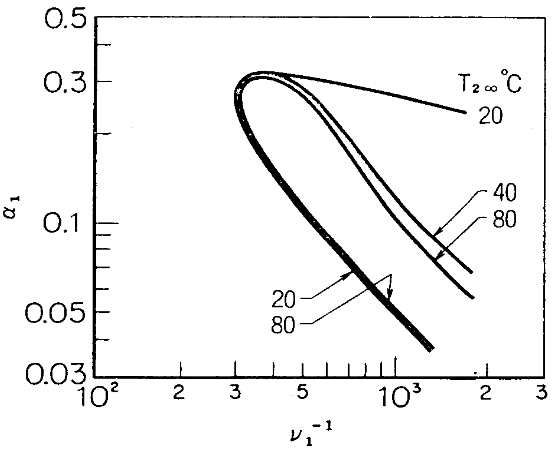

FIG. 5.3. Effect of phase-change (liquid temperature) on stability curves (Water-air, $T_{1\infty}=100°C$, $W_\infty=0$).

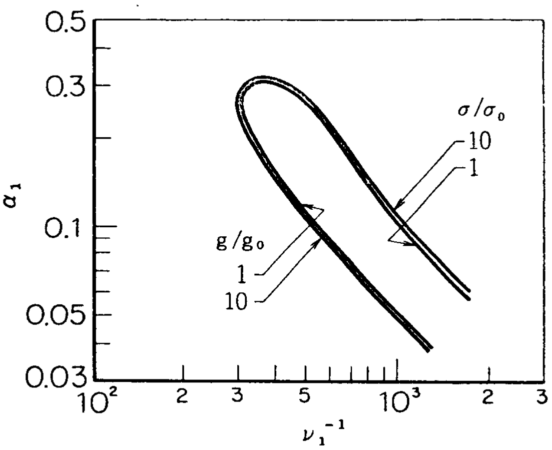

FIG. 5.4. Effect of gravity and surface tension on stability curves (Water-air, $T_{1\infty}=100°C$, $T_{2\infty}=80°C$, $W_\infty=0$).

The influence of phase-change at the interface is shown in Fig. 5.3. The lower branch of the neutral stability is slightly affected by the change in the liquid temperature, though the upper branch is considerably influenced by the phase-change so as to stabilize the flow at higher Reynolds numbers. It implies that, when intense evaporation takes place at the interface, there exist disturbances which are once amplified and then decay rapidly with the increase of the stream velocity. The phase-change at the interface thus has less influence upon the critical Reynolds-number at which the first neutral disturbance with a certain wave-number comes to appear.

Gravity and surface tension have a destabilizing effect as shown in Fig. 5.4, although even at their large values it is hardly distinguishable. Since the effect of

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surface tension is to vanish at smaller wave numbers, it becomes more evident at larger wave numbers, that is, at the upper branch of the curve. These features are qualitatively accordant with the Feldman's result of uniform shearing flows of two viscous and isothermal fluids.

The aspects discussed above are hardly changed with the physical quantities of the fluid or the kind of liquid. The combination of air-benzene or air-methylalcohol slightly stabilizes the flow in comparison with the air-water case.

6. Conclusion

Laminar boundary-layer flows of gas and liquid having a phase-changing interface at their common boundary are studied theoretically to predict their aspect of flow and thermal fields first in the steady state. Using the results of steady fields and examining their perturbed fields with small wavy disturbances, we then proceed to investigate the aspect of the distubance fields and the hydrodynamic instability of the system.

(i) Steady field

The normal velocity or the rate of phase-change at the interface takes an important role on the velocity and temperature profiles, thus on the coefficients of skin-friction and heat-transfer. As the liquid temperature becomes higher so that violent evaporation takes place at the interface, these profiles come to be of the more diffuse one, being evidently S-shaped. The analytical result with an approximation of the u-velocity of a linear profile shows that the non-dimensional coefficients of skin-friction and heat-transfer, F''_{10} and Θ'_{10} are

$$\left[1+0.723\left\{F_{10}-\frac{(F'_{10})^2}{2F''_{10}}\right\}\right]^3$$

and

$$\left[1+\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}F'_{10}+0.723\left\{1+2\left(\frac{\nu_1}{\kappa_1}\right)^{2/3}\right\}\left\{F_{10}-\frac{(F'_{10})^2}{2F''_{10}}\right\}\right]\Theta_{10}$$

times those for a solid flat-plate, that is, 0.332, respectively, where $-F_{10}$, F'_{10} and Θ_{10} are non-dimensional y and x-components of velocity and non-dimensional temperature at the interface.

The normal velocity, the u-velocity and the temperature at the interface are given by

$$-F_{10}=-0.664\left(\frac{\nu_1}{\varepsilon}\right)^{-2/3}\frac{W_\infty-W_0}{1-W_0}\left[1+\left(\frac{\nu_1}{\varepsilon}\right)^{1/3}F'_{10}\right.$$
$$\left.+0.723\left\{1+2\left(\frac{\nu_1}{\varepsilon}\right)^{1/3}\right\}\left\{F_{10}-\frac{(F'_{10})^2}{2F''_{10}}\right\}\right]$$

$$F'_{10}=\Lambda^{-2/3}\left[1+0.723\left\{F_{10}-\frac{(F'_{10})^2}{2F''_{10}}\right\}\right]$$

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$$\Theta_{10}=\left\{-\frac{F_{10}}{0.332}+\frac{\lambda_2}{L}\left(\frac{\nu_2}{\kappa_2}\right)^{1/3}\right\}\left[\frac{\lambda_1}{L}\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}F'_{10}\right.$$
$$\left.+0.723\left\{1+2\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}\right\}\left\{F_{10}-\frac{(F'_{10})^2}{2F''_{10}}\right\}\right]+\frac{\lambda_2}{L}\left(\frac{\nu_2}{\kappa_2}\right)^{1/3}\Bigg]^{-1},$$

where W_0 and W_∞ are vapor-concentration at the interface and at infinity, respectively.

For cases of weakly phase-changing, these factors of coefficient become

$$\left[1+0.723\left\{0.664\left(\frac{\nu_1}{\varepsilon}\right)^{-2/3}\frac{W_\infty-W_0}{1-W_0}-1.51\Lambda^{-4/3}\right\}\right]^3$$

and

$$\left[1+\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}\Lambda^{-2/3}+0.723\left\{1+2\left(\frac{\nu_1}{\kappa_1}\right)^{2/3}\right\}\left\{0.664\left(\frac{\nu_1}{\varepsilon}\right)^{-2/3}\frac{W_\infty-W_0}{1-W_0}\right.\right.$$
$$\left.\left.-1.51\Lambda^{-4/3}\right\}\right]\Theta_{10}$$

respectively, with the approximation that

$$-F_{10}\simeq-0.664\left(\frac{\nu_1}{\varepsilon}\right)^{-2/3}\frac{W_\infty-W_0}{1-W_0}$$
$$F'_{10}\simeq\Lambda^{-2/3}$$
$$\Theta_{10}\simeq\left\{-2\left(\frac{\nu_1}{\varepsilon}\right)^{-2/3}\frac{W_\infty-W_0}{1-W_0}+\frac{\lambda_2}{L}\left(\frac{\nu_2}{\kappa_2}\right)^{1/3}\right\}\left\{\frac{\lambda_1}{L}\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}+\frac{\lambda_2}{L}\left(\frac{\nu_2}{\kappa_2}\right)^{1/3}\right\}^{-1}.$$

In cases of intense evaporation, the liquid tends to stem the air flow and the rate of phase-change is dominated almost by the thermal field of the gas side.

(ii) Stationary disturbance field

Corresponding to the disturbance elevation of the wavy phase-changing interface $\delta_0 e^{i\alpha x}$, the disturbance u-velocity gradient at the interface, that is, the skin-friction coefficient is approximately

$$\frac{u'_{10}}{U'_{10}}=\left\{1.066\left(1-\frac{\rho_1\nu_1^2}{\rho_2\nu_2^2}\right)\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{1/3}e^{(\pi/6)i}\right.$$
$$\left.+0.776\frac{(\nu_1\kappa_1^2)^{1/3}}{-L}\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{2/3}\frac{(-\Theta'_{10})^*}{U'_{10}}e^{(5\pi/6)i}\right\}\delta_{10}e^{i\alpha x}$$

which is proportional to the 1/3th power of the wave number and the Reynolds number wth the phase advance of 30° relative to the interface. The pressure acting on the interface is roughly given by

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$$p_{10}=\left\{0.565\frac{1}{-L}\left(\frac{\kappa_1}{\nu_1}\right)^{2/3}U'_{10}(-\Theta'_{10})^* e^{-(\pi/2)i}\right.$$
$$\left.+0.776\left(U_0+\frac{U'_{10}}{\alpha_1}\right)\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{2/3}e^{(5\pi/6)i}\right\}\nu_1\delta_{10}e^{i\alpha x}.$$

The phase relation of the shearing stress (+30°) and the normal stress (+150°) relative to the interface is accordant with the Benjamin's result for the case of isothermal flows, showing the 'sheltering' effect. The phase-change at the interface of evaporation acts to weaken this effect both in magnitude and in phase relation, especially at larger wave numbers.

The disturbance rate of phase-change at the interface is approximately estimated as

$$\tilde{v}_{10}=0.728\frac{\lambda_1}{-L}\left(\frac{\alpha_1 U'_{10}}{\kappa_1}\right)^{1/3}(-\Theta'_{10})^*\delta_{10}e^{i(\alpha x-5\pi/6)}$$

which is proportional to $(\alpha_1/\kappa_1)^{1/3}$ with the phase lag of 150°. The temperature gradient at the interface, that is, the heat transfer coefficient is

$$\frac{\theta'_{10}}{\Theta'_{10}}=\left[0.342\frac{(-\Theta'_{10})^*}{-L}+0.728\left(\frac{\alpha_1 U'_{10}}{\kappa_1}\right)^{1/3}\right.$$
$$\left.\times\left\{\frac{\lambda_1}{\lambda^2}\left(\frac{\rho_1\kappa_1}{\rho_2\kappa_2}\right)^{-1/3}\frac{(-\Theta'_{10})^*}{-\Theta'_{10}}e^{-(\pi/2)i}+U_0^*\left(1-\frac{\Theta'_{20}}{\Theta'_{10}}\right)e^{(2\pi/3)i}\right\}\right]\delta_{10}e^{i\alpha x}$$

which, for intense evaporation, is proportional to $(\alpha_1/\kappa_1)^{1/3}$ with the phase lag of 90°.

Because of the high heat conductivity of the liquid, the heat flux required for the phase-change at the interface is supplied mainly by heat conduction through the liquid layer, so that the wavy disturbance has less influence on the heat-transfer than on the skin-friction. The ratio of the former to the latter is about $\lambda_1/\lambda_2(\rho_2\kappa_2/\rho_1\kappa_1)^{1/3}\simeq 0.1$.

The disturbance u-velocity and temperature at the interface are approximately given by

$$u_{10}\simeq U'_{10}\left(1-\frac{\rho_1\nu_1^2}{\rho_2\nu_2^2}\right)\delta_{10}e^{i(\alpha x+\pi)}$$
$$\theta_{10}\simeq\frac{\lambda_1}{\lambda_2}\left(\frac{\rho_1\kappa_1}{\rho_2\kappa_2}\right)^{-1/3}(-\Theta'_{10})\delta_{10}e^{i(\alpha x+2\pi/3)}$$

respectively, which imply that the overall disturbance u-velocity and temperature are roughly

$$\tilde{u}_{10}\simeq 0,\qquad \tilde{\theta}_{10}\simeq\Theta'_{10}\delta_{10}e^{i\alpha x}.$$

(iii) Instability curves

The hydrodynamic instabiity of Tollmien-Schlichting type, the neutral stability curve in the wave-number vs. the Reynolds number plane is similar to the case of the boundary-layer flow on a rigid wall, although the former is more destabilized.

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The phase-change at the interface has a considerable effect on the upper branch of the neutral stability that the flow comes to be again stabilized at higher Reynolds numbers. The lower branch, thus the critical Reynolds number is not appreciably affected by the phase-change. Surface tension and gravity have a slightly destabilizing effect of the flow.

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