

On 'Master' Boltzmann Equation

By

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Summary: It is shown that Boltzmann's equation written in terms of microscopic density (viz., unaveraged Boltzmann function) has wider range of validity as well as finer resolvability for fluctuations than the conventional Boltzmann equation governing Boltzmann's function. In fact the new Boltzmann equation for ideal gases has implication as microscopically exact continuity equation like Klimontovich's equation for plasmas, and can be derived without invoking any statistical concepts, e.g., distribution functions, or molecular chaos. The Boltzmann equation in older formalism is obtained by averaging this equation only under a restricted condition of the molecular chaos. The new Boltzmann equation is seen to contain informations comparable with Liouville's equation, and serves as a master kinetic equation. A new hierarchy system is formulated in a certain parallelism to the BBGKY hierarchy. They are shown to yield an identical one-particle equation. Difference, however, between the two hierarchy systems makes first appearance in the two-particle equation. The difference is two-fold. First, the present formalism includes thermal fluctuations which are missing in the BBGKY formalism. Second, the former allows to formulate multi-time correlations as well, whereas the latter is restricted to simultaneous correlation. These two features are favorably utilized in deriving Landau-Lifshitz fluctuation law in a most straightforward manner. Also equations describing nonequilibrium interaction between thermal and fluid-dynamical fluctuations are derived.

I. INTRODUCTION

A. BBGKY and Klimontovich Formalisms

It is widely known that one of the most systematic methods in describing statistical behavior of gases in non-equilibrium state is provided by so-called BBGKY hierarchy method [1]. This method has its basis on Liouville's equation, which determines evolution of Liouville's density function, a complete specification of the state of a N -particle Hamiltonian system of a gas in terms of $6N$ variables of Γ -space.

Another density function which can be compared with the Liouville density function in the level of description has been proposed by Klimontovich in connection with plasma kinetic theory [2]. The function which is called microscopic density has explicit expression of the form

$$\hat{f}(z, t) = \sum_{s=1}^N \delta[z - z^{(s)}(t)], \quad (1)$$

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where $z = (\mathbf{x}, \mathbf{v})$ denotes a phase-space point, $z^{(s)}(t)$ gives locus of s -th particle in the phase space, δ is Dirac's delta function, and the summation is over all particles N under consideration. The microscopic density has a favorable feature in comparison with the Liouville density function: The microscopic density is defined in the (six-dimensional) phase space, in contrast to $6N$ -space of the Liouville density, and has a definite physical meaning such that the following expression

$$\frac{1}{\Delta\mathbf{x}\Delta\mathbf{v}} \int_{\Delta\mathbf{x}\Delta\mathbf{v}} \dot{f} d\mathbf{x}d\mathbf{v}$$

gives the 'exact' number density at an instant t in the phase space z . This is an immediate consequence of the fact that each integrated delta-function (1) is unity/zero depending on whether the pertaining particle is located inside/outside the volume $\Delta\mathbf{x}\Delta\mathbf{v}$, respectively.

Governing equation of the microscopic density \dot{f} has also been derived by Klimontovich [2]; this is simply a microscopic equation of continuity in the phase space

$$K\dot{f} \equiv \frac{\partial \dot{f}}{\partial t} + \mathbf{v} \cdot \frac{\partial \dot{f}}{\partial \mathbf{x}} + \frac{\dot{\mathbf{F}}}{m} \cdot \frac{\partial \dot{f}}{\partial \mathbf{v}} = 0, \quad (2)$$

where $\dot{\mathbf{F}}$ is the force exerted on a particle at (\mathbf{x}, \mathbf{v}) of the phase-space and m is the mass of the particle. We note here that no statistical processing has been involved in deriving Eq. (2): This is an exact kinematical relation to be held at any instant, expressing no generation nor disappearance of particles in the phase space.

B. Objective and Justification

Since the microscopic density \dot{f} contains physical information equivalent to the Liouville density function, and Eq. (2) to the Liouville equation, respectively, it should be possible to construct a hierarchy formalism on the basis of Eq. (2) and \dot{f} in a certain parallelism to the BBGKY formalism. Objective of the present paper is to carry out this procedure for classical ideal gases with binary molecular encounters, in a form which allows direct comparison with the BBGKY counterpart.

Efforts along this line, however, would not be very worthwhile if it simply resulted in verifying correctness of the BBGKY formalism from the different side of approach. Regarding this point we can show that the formalism proposed here unveils further details the BBGKY formalism cannot cover, reflecting a fact that the present method has finer resolvability for fluctuations than the BBGKY formalism. The subtle point out of which this difference emerges lies in a simple mathematical identity;

$$\begin{aligned} \sum_{s=1}^N \delta[\mathbf{z} - \mathbf{z}^{(s)}(t)] \sum_{\beta=1}^N \delta[\hat{\mathbf{z}} - \mathbf{z}^{(\beta)}(t)] &= \sum_{\beta \neq s}^N \delta[\mathbf{z} - \mathbf{z}^{(s)}(t)] \delta[\hat{\mathbf{z}} - \mathbf{z}^{(\beta)}(t)] \\ &+ \delta(\mathbf{z} - \hat{\mathbf{z}}) \sum_{s=1}^N \delta[\mathbf{z} - \mathbf{z}^{(s)}(t)]. \end{aligned} \quad (3)$$

Let an average, defined properly, be taken over the identity, then along with definition (1) the following equation results ;

$$\overline{\dot{f}(z)\dot{f}(\hat{z})} = f_{II}(z, \hat{z}) + \delta(z - \hat{z})f(z), \quad (4)$$

with f and f_{II} defined by

$$f(z) \equiv \overline{\dot{f}}, \quad (5)$$

$$f_{II}(z, \hat{z}) \equiv \overline{\sum_{\hat{s} \neq s}^N \delta[z - z^{(s)}(t)] \delta[\hat{z} - z^{(\hat{s})}(t)]}. \quad (6)$$

If the average is identified with what is employed in the BBGKY formalism, viz., with multiplying by the Liouville density function $f_N(z^{(1)}, \dots, z^{(N)})$ and performing integration with respect to variables $z^{(1)}, \dots, z^{(N)}$, the lefthand side of (6) gives the two-particle distribution function defined there, and in the same way, the function f of (5) coincides with the one-particle distribution, viz., the Boltzmann function. As is easily seen (for example, see (26) of Sec. III) the second term on the righthand side of (4) represents a term yielding the thermal agitation. Then expression (4) shows a clearcut difference in resolvability that the fluctuation correlation in the present formalism (the lefthand side of (4)) includes the thermal agitation which has been missing in the BBGKY formalism. This fact implies that, in the new formalism, heuristic reinterpretations of the existing kinetic theory, such as invoking Boltzmann-Langevin equation [3] to retrieve thermal agitation are no longer necessary and are replaced with a direct manipulation. (See Sec. III.)

C. A Key Version

In dealing with plasma, viz, phenomena associated with Coulomb interaction (soft collision) the Klimontovich equation (2) is successfully utilized as the master kinetic equation out of which the equation for each level of hierarchy is derived in a tractable form. This equation, however, is not convenient in dealing with an ideal gas in which molecular encounter obeys a hard collision. In fact, the force \dot{F} exerted on the molecule would be of the form of a delta-like function having small width of collision time t^* , and our interest lies in the limiting case of vanishing t^* . Therefore our task is to reformulate the microscopic continuity equation in a form free from the delta-function like force without deteriorating the level of description, viz., without recourse to any statistical concepts.

D. Brief Account of Approach

Derivation of the master kinetic equation along the line meeting the above requirement is attempted in Sec. II. It turns out that the prospective master equation has the form of the Boltzmann equation. It cannot be overemphasized that *this* Boltzmann equation differs from the conventional Boltzmann equation in that it governs the microscopic density \dot{f} (not the Boltzmann function f) and that it involves no statistical processing, such as molecular chaos. The conventional Boltzmann equation is obtained as its moment equation of the lowest order by averaging the

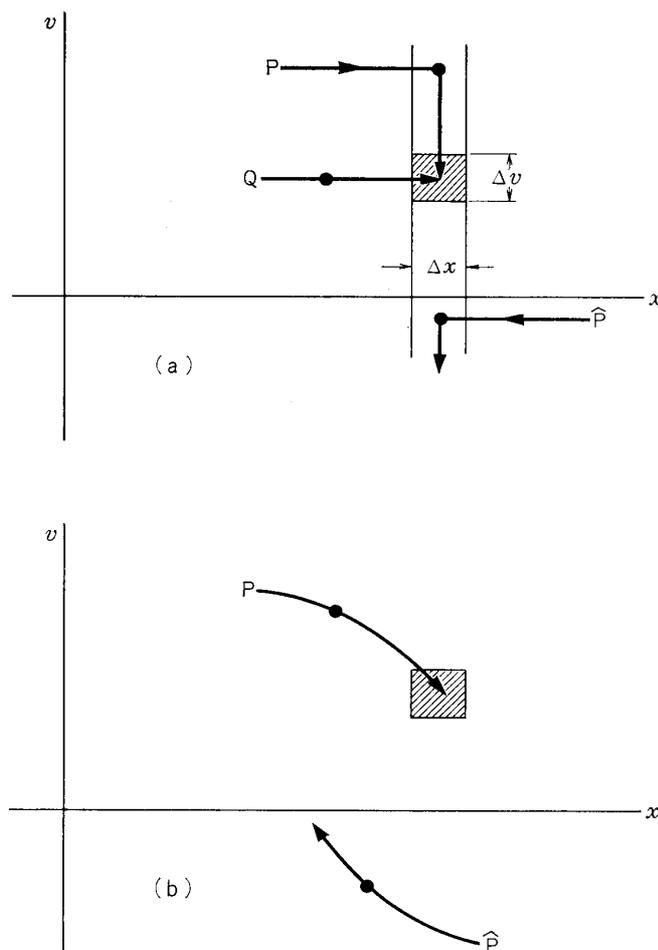


FIG. 1. Schematic representation of six-dimensional (x, v) space and of molecular encounters for a hard (a) and a soft (b) collisions

equation and by postulating (binary) molecular chaos. In Sec. III, equations of the higher stages of hierarchy are derived and are compared with their BBGKY counterparts. They are shown to be identical in the homogeneous part of equations but differ in that the present hierarchy equations comprise inhomogeneous terms which attribute to the presence of thermal agitations. In order to check if the thermodynamic fluctuation is included correctly in the present formalism an attempt is made to derive Landau-Lifshitz's formula for spontaneous stress and heat flow (Sec. IV). In Sec. V moment equations describing evolution, in the physical space, of turbulent fluctuation in the course of nonequilibrium interaction with the thermal agitation are derived.

II. 'MASTER' BOLTZMANN EQUATION

A. Limiting Form of the Klimontovich Continuity Equation

A guideline in reformulating the Klimontovich continuity equation to apply to a gas with hard molecules is the following: As we have seen previously the vanish-

ing collision time ($t^*=0$) characterizing the hard collision, on one hand, makes it difficult to employ Eq. (2). On the other hand, however, it enables to single out effects due to collision as instantaneous, far-reaching effects in the velocity space, thereby to formulate molecular interaction in an integral form instead of a divergence form: Fig. 1 illustrates loci of particles in a binary encounter for hard (a) and soft (b) collisions, respectively. A salient feature of the collision of type (a) is that a particle (say P) which lies on a three-dimensional hyperplane ($x=\text{const.}$, v ; arbitrary) at time t can reach a marked volume Δz located at (\mathbf{x}, \mathbf{v}) *instantly* upon collision provided that velocity $\hat{\mathbf{v}}'$ of the collision partner \hat{P} satisfies the conditions,

$$\left. \begin{aligned} \hat{\mathbf{v}}' + \mathbf{v}' &= \hat{\mathbf{v}} + \mathbf{v}, \\ \hat{v}'^2 + v'^2 &= \hat{v}^2 + v^2, \end{aligned} \right\} \quad (7)$$

where \mathbf{v}' is the velocity of the particle P before collision and where the velocity \mathbf{v} of the particle P after collision has been prescribed. Then, change in the number of particles in the volume Δz during time interval Δt , viz.,

$$\frac{\partial}{\partial t} \int_{\Delta z} f dz \cdot \Delta t, \quad (8)$$

depends on two fluxes, viz, convective flux Q across surfaces $\mathbf{x}=\text{const.}$ and $\mathbf{x} + \Delta \mathbf{x} = \text{const.}$ and collisional flux P reaching instantly from any part of the hyperplane in a manner analogous to radiative flux in the physical space.

Peculiarity of the present problem, viz., the vanishing collision time, lies in the fact that convective flux is allowed only across the surfaces $\mathbf{x}=\text{const.}$, $\mathbf{x} + \Delta \mathbf{x}=\text{const.}$, of the six dimensional volume Δz whereas the 'radiative' flux is permeable only across the surfaces $\mathbf{v}=\text{const.}$, $\mathbf{v} + \Delta \mathbf{v}=\text{const.}$; the two fluxes are apparently discernible because they are perpendicular to each other. No such distinction is possible in the case of a soft collision, e.g., Coulomb interaction (Fig. 1b). The first contribution amounts to

$$-\frac{\partial}{\partial \mathbf{x}} \cdot \int_{\Delta z} \mathbf{v} f dz \Delta t, \quad (9)$$

and the second one is expressed by a difference of integrals spanning over the whole hyperplane, counting number of particles shot into and out of the marked volume;

$$\int_{\Omega'} f(z') f(\hat{z}') dz' d\hat{z}' - \int_{\Omega} f(z) f(\hat{z}) dz d\hat{z}. \quad (10)$$

In the above expression Ω' and Ω , respectively, are integral regions of the collision partners (z', \hat{z}') and (z, \hat{z}) which are located between the hyperplanes $\mathbf{x}=\text{const.}$, and $\mathbf{x} + \Delta \mathbf{x}=\text{const.}$ and subject to additional conditions (7) and the condition that the collision partners are to collide within time interval Δt . In expression (10), we transform variables as

$$\begin{aligned} dz' d\hat{z}' &= d\mathbf{x}' d\hat{\mathbf{x}}' d\mathbf{v}' d\hat{\mathbf{v}}' \\ &= d\mathbf{x}'_G d\mathbf{r}' d\mathbf{v}' d\hat{\mathbf{v}}', \end{aligned} \quad (11)$$

which is easily confirmed in view of the relations

$$\begin{aligned}\partial(\mathbf{x}', \hat{\mathbf{x}}')/\partial(\mathbf{x}'_G, \mathbf{r}') &= 1, \\ \partial(\hat{\mathbf{v}}', \mathbf{v}')/\partial(\hat{\mathbf{v}}, \mathbf{v}) &= 1,\end{aligned}$$

where $\mathbf{x}'_G(=\frac{1}{2}(\mathbf{x}' + \hat{\mathbf{x}}'))$ and $\mathbf{r}'(=\mathbf{x}' - \hat{\mathbf{x}}')$ denote vectors specifying center of gravity and the relative location of two particles, respectively. In the similar way we have

$$dzd\hat{z} = dx_G dr dv d\hat{v},$$

where the quantities \mathbf{x}_G and \mathbf{r} are defined similarly with respect to the unprimed collision partners (z, \hat{z}). For sufficiently small Δt , we can put $\mathbf{x}'_G = \mathbf{x}_G = \mathbf{x}$. Integral region of \mathbf{r}' which forms a subspace of Ω' constitutes a cylindrical shell with height $|\hat{\mathbf{v}}' - \mathbf{v}'| \Delta t$ and with radius b (the impact parameter), standing on a target particle z' at the center;

$$d\mathbf{r}' = 2\pi |\hat{\mathbf{v}}' - \mathbf{v}'| \Delta t b db = d\mathbf{r},$$

where invariance of the magnitude of relative velocities

$$V = |\hat{\mathbf{v}}' - \mathbf{v}'| = |\hat{\mathbf{v}} - \mathbf{v}|, \quad (12)$$

an immediate consequence of Eqs. (7), has been utilized. Then the integral (10) is rewritten in the form

$$\Delta t \int_{\Delta z} dx dv \int [f(\hat{z}')f(z') - f(\hat{z})f(z)] dK d\hat{v}, \quad (13)$$

with

$$dK \equiv 2\pi V b db.$$

It should be stressed here that integral expression (10) counts exact number of pertinent collisions; in other words, no statistical processing, viz., no concept of expectation values, nor hypothesis of the molecular chaos, is involved there. In fact, the above manipulation is based only on two factors; the exact number density f and deterministic Newton mechanics.

Equation of continuity, which expresses no generation nor disappearance of particles in the phase space, requires that the total change in the number of molecules in the volume Δz during time interval Δt attributes entirely to the two origins; gain/loss due to convection (9) and collision (13). We have, then,

$$\frac{\partial}{\partial t} \int_{\Delta z} f dz = - \frac{\partial}{\partial \mathbf{x}} \cdot \int_{\Delta z} \mathbf{v} f dz + \int_{\Delta z} dz \int [f(\hat{z}')f(z') - f(\hat{z})f(z)] dK d\hat{v}. \quad (14)$$

Thus we have obtained an ideal gas version of the plasma kinetic equation (2). If, in the above equation, we introduce a formal procedure of dividing Eq. (14) by Δz and taking a limit of $\Delta z \rightarrow 0$, then we have

$$B\dot{f} \equiv \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \dot{f} - J(\hat{z}|z)[\dot{f}(z)\dot{f}(\hat{z})] = 0, \quad (15)$$

with

$$J(\hat{z}|z)[g(z, \hat{z})] \equiv \int [g(z', \hat{z}') - g(z, \hat{z})] d\mathbf{K} d\hat{\mathbf{v}}. \quad (16)$$

This is simply Boltzmann's equation, written in terms of the microscopic density in place of the Boltzmann function.

In what follows this equation will play the role of a master equation from which a series of hierarchy equations is generated, just in a similar manner that the BBGKY hierarchy is derived from the Liouville equation. For this reason Eq. (15) is referred to as the Master Boltzmann equation (MBE) in the following.

The above formulation is exact only for gases with elastic spherical molecules. Classification into two orthogonal fluxes, the one written in a divergence form and the other in an integral form of (15), is obscured if the above reasoning is to be extended to include power force law

$$|\mathbf{F}| \sim r^{-s}$$

of the molecular interaction. In fact, then, both loci P and Q of Fig. 1a have finite curvatures in entering the volume Δz and are not perpendicular to each other. If, however, we can choose the size Δx_i of the volume such that it is larger than the radius of curvature in *almost* all collisions, but is smaller than a characteristic length of long-range effects, the above method is seen still valid. Thus the incorporation of the softer collision into the framework of the present formalism is made possible at the risk of deteriorating temporal resolution of the MBE: It is not capable of resolving events with time constant smaller than $t^* = r^*/(RT)^{1/2}$, where r^* denotes the characteristic radius of curvature of a particle upon collision. This does not affect, however, on resolution capability of the present formalism of the thermal agitation.

The extended validity of the MBE for softer molecular encounter is only conditional: Limitation arises from the fact that the collision integral does not converge [4] for

$$s < 3. \quad (17)$$

This implies that, under this condition, we cannot find the proper volume size Δx_i meeting the requirement stated above, thereby making the two fluxes discriminate. Thus the condition $s \leq 3$ provides a criterion as to which of the master kinetic equation (2) or (15) is to be employed in the problem.

B. Two Methods of Solving MBE

Because of "rugged" or discrete structure of the function \dot{f} , mathematical tools in solving MBE (15) are limited to either of the following schemes: The one is a direct method in which distribution of the particles is simulated by Monte Carlo

method, and evolution at subsequent time is determined according to Eq. (15) for a given initial condition specified properly. A most successful application of this method is seen in the study of rarefied gasdynamics. For example, in the shock structure problem the direct Monte Carlo method yields solution which agrees with experiments better than those by analytical methods. This is because well-designed Monte Carlo method utilizes no averaging procedure, nor concept of distribution function [5], thereby simulates direct solution of the MBE (15), whereas the analytical methods [6, 7] solve the conventional Boltzmann equation (Eq. (21) below) where non-equilibrium correction (ϕ -term) is lacking.

The second method of solving MBE is to replace the function f with a continuous function by means of a certain smoothing procedure, e.g., averaging. Standard analytical methods are made available only via this procedure, and the following part of the paper is devoted to the analytical approach by means of average taken at various statistical levels of description.

C. One-particle Equation. Comparison with Traditional Boltzmann Equation

In order to compare Eq. (15) with the Boltzmann equation in older formalism, we take average of Eq. (15), and utilize expression (4) and (5), then we have

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}}\right) f(z) = J(\hat{z}|z)[f_{\text{II}}(z, \hat{z})]. \quad (18)$$

In deriving the equation use is made of the fact that the term $\delta(z - \hat{z})f$, representing thermodynamic fluctuation, and supposed to appear in the collision integral is seen to have no contribution: Actually we have, for an arbitrary moment function $\beta(\mathbf{v})$,

$$\begin{aligned} & \int \beta(\mathbf{v}) \mathcal{T}(\hat{z}|z) [\delta(z - \hat{z})f] d\mathbf{v} \\ &= \delta(\mathbf{x} - \hat{\mathbf{x}}) \int [\beta(\mathbf{v}') - \beta(\mathbf{v})] \delta[\mathbf{v} - \hat{\mathbf{v}}] f(\mathbf{v}) d\mathbf{K} d\mathbf{v} d\hat{\mathbf{v}} = 0, \end{aligned}$$

because the factor $(\beta' - \beta)\delta(\mathbf{v} - \hat{\mathbf{v}})$ vanishes in view of the relationship [4]

$$\left. \begin{aligned} \mathbf{v}' &= \mathbf{v} + \boldsymbol{\alpha}(\boldsymbol{\alpha} \cdot \mathbf{V}), \\ \hat{\mathbf{v}}' &= \hat{\mathbf{v}} - \boldsymbol{\alpha}(\boldsymbol{\alpha} \cdot \mathbf{V}), \end{aligned} \right\} \quad (19)$$

where $\boldsymbol{\alpha}$ is a constant vector defined by the first of Eqs. (19) and \mathbf{V} has been defined by (12). Then we readily see that Eq. (18) is identical with the one-particle equation in the BBGKY formalism [1], [4], [8], [9] provided that the average taken in (18) to have f and f_{II} is the same as described in Sec. IB.

Description in terms of the smoothed distribution functions, viz., of the Boltzmann function f , the two-point function f_{II} , etc., is unavoidably connected with a difficulty of indeterminacy [10]: Eq. (18) for the one-particle equation is linked with effects due to two-particle interaction, also the same is true for two-particle equation where three-particle interaction intervenes (See Table 1, below). To solve the

equation the chain should be terminated at a certain stage by introducing a closure condition. Quality of the theory depends on two factors, viz., the quality of the closure condition adopted and the stage in the hierarchy at which the truncation is effected. A most primitive form of the closure condition was proposed by Boltzmann in 1872, and has been known as the hypothesis of (binary) molecular chaos;

$$f_{11}(z, \hat{z}) - f(z)f(\hat{z}) = 0. \quad (20)$$

With this closure condition, Eq. (18) is written as

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) f = J(\hat{z}|z)[f(z)f(\hat{z})], \quad (21)$$

which is nothing but the Boltzmann equation in its conventional form. It would be self-explanatory, from the above procedure leading to Eq. (21), that the equation is valid only under far more restrictive conditions than Eq. (15).

D. Breakdown of the Molecular Chaos

Binary molecular chaos hypothesis (20) has reigned the kinetic theory over a century, and has served as a key hypothesis on which the classical Boltzmann equation (21) rests. In ref. [9], however, region of validity of the molecular chaos has been examined critically, leading to the following findings: Let us define by

$$\phi(z, \hat{z}) \equiv f_{11}(z, \hat{z}) - (1 - N^{-1})f(z)f(\hat{z}), \quad (22) [11]$$

a measure for deviation from the molecular chaos. It is shown that leading terms of ϕ decays according to a diffusion equation for a gas in equilibrium. In nonequilibrium (shear flow) situation, the function ϕ is shown to die off eventually so far as the basic flow is stable with respect to criterion by hydrodynamical stability theory. For these cases, therefore, there will be not much danger of putting $\phi=0$ from the beginning. However, if the basic flow is predicted to be unstable according to the stability criterion, ϕ is shown to grow drastically. Thus we see that the molecular chaos is misleading in unstable flow situations. Crucial point is that the deviation from the molecular chaos now at issue is not the one aroused by the dense-gas effect which is of $O(n^{-1})$ (n ; the mean number density), or by a spontaneous correlation which survives only for a few collisions, but is a result of macroscopic correlation of $O(1)$, which persists over a hydrodynamic length, viz., over thousands mean free paths. The direct connection of nonvanishing ϕ with the unstable flow circumstances has made the macroscopic (turbulent) correlation incorporated into the framework of the kinetic theory. On the other hand, it has turned out that the traditional Boltzmann equation (21) is invalidated for these cases and is to be replaced with the one-particle equation (18). Closure at one-particle stage turns out incorrect and under these circumstances we should address to the multi-particle equations.

III. A NEW HIERARCHY SYSTEM

A. Two-point Two-time Correlation

The Liouville equation, viz., the master equation of BBGKY formalism, describes evolution of the Liouville density function f_N in the space of (z_1, \dots, z_N, t) after the standard way of description of multiparticle Hamiltonian mechanics. In statistical mechanics it is often more convenient to discuss correlation in a multi-particle, multi-time space, viz., in the space of $[\zeta(a_1), \dots, \zeta(a_N)]$, where $\zeta(a)$ denotes a seven-dimensional space $[z(a), t(a)]$. Formalism along this line is made possible by adopting MBE in the following manner: Let us denote by $\hat{f}(a)$ microscopic density in $\zeta(a)$ -space and define the Boltzmann operator $B(a)$ by

$$B(a)\hat{f}(a) \equiv \left[\frac{\partial}{\partial t(a)} + \mathbf{v}(a) \cdot \frac{\partial}{\partial \mathbf{x}(a)} \right] \hat{f}(a) - J(\hat{a}|a)[\hat{f}(a)\hat{f}(\hat{a})]. \quad (23)$$

Let us form

$$\overline{\Delta f(b)B(a)\hat{f}(a)} = 0, \quad (24)$$

where

$$\Delta f(b) = \hat{f}(b) - f(b)$$

is instantaneous fluctuation in the number density in $\zeta(b)$ space, viz., at a phase space point $z(b)$ and at time $t(b)$. Then, after taking average, we have an equation of the form

$$\begin{aligned} & \left[\frac{\partial}{\partial t(a)} + \mathbf{v}(a) \cdot \frac{\partial}{\partial \mathbf{x}(a)} \right] [\overline{\hat{f}(a)\hat{f}(b)} - f(a)f(b)] \\ & = J(\hat{a}|a)[\overline{\hat{f}(a)\hat{f}(\hat{a})\hat{f}(b)} - \overline{\hat{f}(a)\hat{f}(\hat{a})}f(b)]. \end{aligned} \quad (25)$$

In view of definition (1) for the microscopic density the quantity $\overline{\hat{f}(a)\hat{f}(b)}$ is decomposed as

$$\overline{\hat{f}(a)\hat{f}(b)} = f_{\text{II}}(a; b) + g(a; b), \quad (26)$$

where f_{II} and g are two-point, two-time distribution function and self-correlation function defined, respectively, by

$$f_{\text{II}}(a; b) \equiv \sum_{s_a \neq s_b} \overline{\delta[z(a) - z^{(s_a)}(t(a))]\delta[z(b) - z^{(s_b)}(t(b))]}, \quad (27)$$

and

$$g(a; b) \equiv \sum_{s=1}^N \overline{\delta[z(a) - z^{(s)}(t(a))]\delta[z(b) - z^{(s)}(t(b))]}. \quad (28)$$

Note that two arguments separated by semi-colon (;) mean two points with correlation taken at different times. The physical meaning of $f_{\text{II}}(a; b)$ would be straight-

foward; it denotes two-point probability density of different particles at different times. On the other hand, the self-correlation $g(a; b)$ expresses the probability density of finding the same particle at the point $z=z(a)$ and at the time $t=t(a)$ as was located initially, viz., at $t=t(b)$ at the point $z=z(b)$. As is easily confirmed the functions $f_{II}(a; b)$ and $g(a; b)$ reduce, in the limit $t(a) \rightarrow t(b)$, respectively, to

$$f_{II}(a; b) = f_{II}(a, b), \quad (29)$$

$$g(a; b) = \delta[z(a) - z(b)]f(a), \quad (30)$$

where $f_{II}(a, b)$ is the two-point, one-time distribution function defined by (6).

Extending expression (3) to triple product and taking average, we are led to the following expression for a three-point two-time distribution function as

$$\left. \begin{aligned} \overline{\dot{f}(a)\dot{f}(\hat{a})\dot{f}(b)} &= f_{III}(a, \hat{a}; b) + \delta[z(a) - z(\hat{a})]f_{II}(a; b) + g_{II}(a, \hat{a}; b) \\ &+ g_{II}(\hat{a}, a; b) + \delta[z(a) - z(\hat{a})]g(a; b), \end{aligned} \right\} \quad (31)$$

with

$$t(a) = t(\hat{a}),$$

where the following functions have been newly introduced :

$$f_{III}(a, \hat{a}; b) \equiv \frac{\sum_{s_a \neq s_{\hat{a}} \neq s_b \neq s_a} \delta[z(a) - z^{(s_a)}(t(a))] \delta[z(\hat{a}) - z^{(s_{\hat{a}})}(t(\hat{a}))]}{\times \delta[z(b) - z^{(s_b)}(t(b))]}, \quad (32)$$

$$g_{II}(a, \hat{a}; b) \equiv \frac{\sum_{s_a \neq s_b} \delta[z(a) - z^{(s_a)}(t(a))]}{\times \delta[z(\hat{a}) - z^{(s_b)}(t(a))] \delta[z(b) - z^{(s_b)}(t(b))]} \quad (33)$$

We note here that the function g_{II} has mixed character of ordinary correlation (between different particles) and of self-correlation, the latter being characterized by two arguments separated by semi-colon. Therefore, g_{II} concerns with two particles although it is specified by three arguments. Similarly to what has led to (30), definition (33) reduces, for $t(a) \rightarrow t(b)$, to

$$g_{II}(a, \hat{a}; b) \rightarrow \delta[z(\hat{a}) - z(b)]f_{II}(a, \hat{a}). \quad (34)$$

As will turn out later, decomposition formulae (26) and (31) separate out the thermodynamic part designated by g 's from the total fluctuation. The rest parts, viz., hydrodynamic fluctuation correlations are given in terms of multi-point correlations ;

$$\phi_{II}(a; b) = f_{II}(a; b) - (1 - N^{-1})f(a)f(b), \quad (35)$$

$$\begin{aligned} \phi_{III}(a; b; c) &= f_{III}(a; b; c) - (1 - N^{-1})(1 - 2N^{-1})f(a)f(b)f(c) \\ &- (1 - 2N^{-1})f(a)\phi_{II}(b; c) - (1 - 2N^{-1})f(b)\phi_{II}(c; a) \\ &- (1 - 2N^{-1})f(c)\phi_{II}(a; b). \end{aligned} \quad (36)$$

These relations are so designed that integrating condition (36) with respect to $z(c)$ yields condition (35). This requirement is necessary in order for error caused by truncation (e.g., $\phi_{\text{III}}=0$) employed at a certain stage not to affect on exactness of the equations prior to that stage [12]. The factors $1-N^{-1}$, $1-2N^{-1}$ etc. are inserted for the following conditions

$$\left. \begin{aligned} \int \phi_{\text{III}}(a; b; c) dz(c) &= 0, \\ \int \phi_{\text{II}}(a; b) dz(b) &= 0, \end{aligned} \right\} \quad (37)$$

to be fulfilled. (Note that f_{II} and f obey the conditions, respectively, $\int f_{\text{II}}(a; b) dz(b) = (N-1)f(a)$, $\int f(a) dz(a) = N$.) Condition (35) reduces, at $t(a)=t(b)$, to (22) as it should.

In order to close the system at two-particle level let us introduce an assumption of “ternary” molecular chaos,

$$\phi_{\text{III}}(a; b; c) = 0, \quad (38)$$

which makes it possible to express the three particle distribution function via effects due to binary correlation. (See Eq. (36).) Then Eq. (25), when conditions (26) and (35) on its lefthand side, also those (31), (36), (38) on its righthand side, are utilized, yields the following equation;

$$\begin{aligned} & \left[\frac{\partial}{\partial t(a)} + \mathbf{v}(a) \cdot \frac{\partial}{\partial \mathbf{x}(a)} \right] [\phi(a; b) + g(a; b)] = J(\hat{a}|a) \\ & \times [f(a)\phi(\hat{a}; b) + f(\hat{a})\phi(a; b) + g_{\text{II}}(a, \hat{a}; b) + g_{\text{II}}(\hat{a}, a; b)] + O(N^{-1}). \end{aligned} \quad (39)$$

Terms of $O(N^{-1})$ are unimportant in what follows; only the limiting case $N \rightarrow \infty$ with N/V (V ; total volume under consideration) retained finite need be considered.

B. Self-correlation in Equilibrium Gases

For a gas in equilibrium leading terms of the hydrodynamic correlation obey a diffusion equation, so are shown to die off eventually [9]. Therefore, we may put from the beginning,

$$\phi = 0 \quad (40)$$

in Eq. (39). This condition, then, allows us to put

$$g_{\text{II}}(a, \hat{a}; b) = f(a)g(\hat{a}; b), \quad (41)$$

which follows directly from initial condition (34) g_{II} should obey, along with (30) and the general rule that two particles in an equilibrium gas are uncorrelated if they are so initially. Thus Eq. (39) is written for an equilibrium gas, in the form

$$\begin{aligned} & [\partial/\partial\tau + \mathbf{v}(a) \cdot \partial/\partial\mathbf{x}(a)]g(a; b) \\ & = J(\hat{a}|a)[f(a)g(\hat{a}; b) + f(\hat{a})g(a; b)], \end{aligned} \quad (42)$$

where τ is defined by

$$\tau \equiv t(a) - t(b), \quad (43)$$

which we shall assume as positive in what follows. Eq. (42) is in agreement with the one derived on different bases by Fox and Uhlenbeck [13], Hinton [14], Chappell [15]. It, however, differs from the one due to Montgomery [16] in that the integrand function of the collision integral is symmetrical with respect to a and \hat{a} . This symmetry is a consequence of the ternary molecular chaos (38) we have employed, which requires implicitly that the particle should experience many collisions in the course of random flight from $z(b)$ to $z(a)$, viz., during elapse of time τ . If, instead, we seek self-correlation of a particle for a very short time during which few collisions are effected, we may put, in the collision term of (25),

$$\overline{f(a)f(\hat{a})f(b)} \sim \overline{f(a)f(b)f(\hat{a})}.$$

In fact, for such small τ correlation between a and b of an identical particle will not be so destroyed as to be comparable with that with a foreign particle \hat{a} . With this decomposition rule (together with (26)) a self-correlation equation [16] of the form (42), with the first term on the righthand side lacking, follows. Thus we see that Eq. (42) and that derived in ref. [16] are to be responsible for large and for small τ 's, respectively.

C. Two-Point Simultaneous Correlation

Now we turn to constructing an equation which is symmetric with respect to the two space-time points $\zeta(a)$ and $\zeta(b)$. The symmetry with respect to $z(a)$ and $z(b)$ in the distribution function is favorable in comparing the present theory with the BBGKY theory, in which such symmetry is warranted a priori. Such an equation can be obtained simply by adding to Eq. (39) a transposed equation in which the role of the variables $z(a)$ and $z(b)$ are interchanged. If, in the equation, time variables are transformed to

$$\left. \begin{aligned} t &= \frac{1}{2}[t(a) + t(b)], \\ \tau &= t(a) - t(b), \end{aligned} \right\}$$

we have the following equation

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \mathbf{v}(a) \cdot \frac{\partial}{\partial \mathbf{x}(a)} + \mathbf{v}(b) \cdot \frac{\partial}{\partial \mathbf{x}(b)} \right] [\phi(a; b) + g(a; b)] \\ & = \mathcal{F}[\hat{a}|a][f(a)\phi(\hat{a}; b) + g_{\text{II}}(a, \hat{a}; b)] \\ & \quad + \mathcal{F}[\hat{b}|b][f(b)\phi(a; b) + g_{\text{II}}(a; b, \hat{b})], \end{aligned} \quad (44)$$

with the operator \mathcal{T} defined by

$$\mathcal{T}[\hat{a}|a]h(a, \hat{a}; b) \equiv J(\hat{a}|a)[h(a, \hat{a}; b) + h(\hat{a}, a; b)]. \quad (45)$$

Special interest lies in the case of simultaneous correlation, $\tau=0$. Then the equation in which g and g_{II} have been eliminated by means of (30) and (34) reads

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \mathbf{v}(a) \cdot \frac{\partial}{\partial \mathbf{x}(a)} + \mathbf{v}(b) \cdot \frac{\partial}{\partial \mathbf{x}(b)} \right] \phi(a, b) \\ & - \mathcal{T}[c|a][f(a)\phi(c, b)] - \mathcal{T}[c|b][f(b)\phi(c, a)] \\ & = \mathcal{T}[c|a]\{\delta[z(a) - z(b)]f_{\text{II}}(a, c)\} + \mathcal{T}[c|b]\{\delta[z(a) - z(b)]f_{\text{II}}(b, c)\} \\ & - \delta[z(a) - z(b)]J(c|a)[f_{\text{II}}(a, c)]. \end{aligned} \quad (46)$$

D. Comparison with BBGKY Hierarchy

In Eq. (46) terms on the lefthand side are connected with macroscale correlation, whereas those on the righthand side (inhomogeneous terms) have a common factor of a delta function, so are seen to represent effects due to thermodynamic fluctuations which have been missing in the BBGKY formalism. If terms on the righthand side are put zero, Eq. (46) is in exact agreement with the two-point BBGKY equation [8], [9] (with the ternary molecular chaos).

TABLE 1. Comparison of the present hierarchy equations with the BBGKY counterparts.

	BBGKY hierarchy	Present hierarchy
one-particle equation	$\int \dots \int_{N-1}^L f_N = 0^*$	$\overline{B\dot{f}} = 0^*$
two-particle equation	$\int \dots \int_{N-2}^L f_N = 0^{**}$	$\overline{\Delta f(a)B\dot{f}(b) + \Delta f(b)B\dot{f}(a)} = 0^{***}$
three-particle equation	$\int \dots \int_{N-3}^L f_N = 0$	$\overline{\sum \Delta f(a)\Delta f(b)B\dot{f}(c)} = 0$ (a, b, c ; cycl. permut.)
---	---	---
master equation	$L f_N = 0$ (Liouville)	$B\dot{f} = 0^\dagger$ (Master Boltzmann)

*: Eq. (18), **: Eq. (47) ***: Eq. (48), †: Eq. (15)

Comparison of the BBGKY with the present hierarchy system is summarized in Table 1: The one-particle equations in both formalisms have turned out to be identical if the bar average is meant by

$$\bar{g} = \int \dot{g} f_N^\dagger dz^{(1)} \dots dz^{(N)}$$

as has been identified in Sec. I, where f_N^\dagger is a certain function symmetrical with respect to its N arguments $[z^{(1)}, \dots, z^{(N)}]$, but not necessarily obeying the Liouville equation. The two-particle equation in the BBGKY formalism with no closure condition is written in the form,

$$\begin{aligned} \mathcal{L}[\phi, \phi_{\text{III}}] \equiv & \left[\frac{\partial}{\partial t} + \mathbf{v}(a) \cdot \frac{\partial}{\partial \mathbf{x}(a)} + \mathbf{v}(b) \cdot \frac{\partial}{\partial \mathbf{x}(b)} \right] \phi(a, b) \\ & - \mathcal{T}[c|a][f(a)\phi(c, b)] - \mathcal{T}[c|b][f(b)\phi(c, a)] \\ & - \{\mathcal{T}[c|a] + \mathcal{T}[c|b]\}[\phi_{\text{III}}(a, b, c)] = 0. \end{aligned} \quad (47)$$

The corresponding equation in the present formalism reads

$$\begin{aligned} \mathcal{L}[\phi, \phi_{\text{III}}] = & \mathcal{T}[c|a]\{\delta[z(a) - z(b)]f_{\text{II}}(a, c)\} \\ & + \mathcal{T}[c|b]\{\delta[z(a) - z(b)]f_{\text{II}}(b, c)\} \\ & - \delta[z(a) - z(b)]J(c|a)f_{\text{II}}(a, c). \end{aligned} \quad (48)$$

If 'ternary' molecular chaos ($\phi_{\text{III}} = 0$) is imposed on the equation, it reduces to Eq. (46) as it should.

Constructing three- and more-particle equations in the form which allows direct comparison each other is rather straightforward: As to the three-particle case, the prospective equation in the present formalism is given by

$$\overline{\sum_{\substack{(a,b,c: \\ \text{cyclic permut.})}} \Delta f(a)\Delta f(b)B\dot{f}(c)} = 0, \quad (48)'$$

where the summation is over cyclic permutations of the three space-time points (a, b, c) , and where the operator B has been defined by (23). As in the two-particle case this equation contains more information than the corresponding BBGKY equation

$$\underbrace{\int \dots \int}_{N-3} Lf_N = 0,$$

where L is the Liouville operator, in the sense that Eq. (48)' gives three-time correlation in general, and that it includes thermal agitations as forcing terms. It should be noted at this point that range of applicability of the two hierarchy systems are not identical: As was discussed in Sec. IIA the present formalism applies only to hard molecules, whereas the BBGKY formalism has no such limitation in principle. The same procedure of generating a hierarchy system on the basis of Klimontovich equation (2), viz.,

$$\begin{aligned} \overline{K\dot{f}(a)} &= 0, \\ \overline{\Delta f(a)K\dot{f}(b) + \Delta f(b)K\dot{f}(a)} &= 0, \\ \overline{\sum_{\substack{(a,b,c: \\ \text{cyclic permut.})}} \Delta f(a)\Delta f(b)K\dot{f}(a)} &= 0, \\ \dots & \end{aligned}$$

supplements to cover whole range of intermolecular force law, and to form a complete set.

IV. DERIVATION OF LANDAU-LIFSHITZ FORMURAE

A. Moment Expansion of Function g

We present here an illustration that the present formalism gives correct account for phenomena ascribed to thermal agitation of the gas. We will show, under equilibrium condition ($u_i=0$, $V=0$), that Eq. (42) for self-correlation g yields a system of moment equations out of which the well-known Landau-Lifshitz fluctuation formulae for spontaneous stress and heat flow are derived [17]. For this purpose we expand the function g in a double series of Hermite's polynomials as

$$g(a; b) = \omega(a)\omega(b) \sum \frac{Q_{ij\dots,lm\dots}^{(J,K)}}{c^{J+K} J! K!} \mathcal{H}_{ij\dots}^{(J)}(a) \mathcal{H}_{lm\dots}^{(K)}(b), \quad (49)$$

where c is the isothermal speed of sound ($=\sqrt{RT}$), \mathcal{H} is the three-dimensional Hermite polynomial [4], and ω is a function defined by

$$\omega(a) \equiv \frac{1}{(2\pi c^2)^{3/2}} \exp \left[-\frac{v(a)^2}{2c^2} \right]. \quad (50)$$

Note that the Hermite polynomials have the property of ortho-normality with the weight function ω , so that the expansion coefficients Q are given as

$$Q_{ij\dots,lm\dots}^{(J,K)}(\mathbf{x}(a) - \mathbf{x}(b), \tau) = \int c^{J+K} \mathcal{H}_{ij\dots}^{(J)}(a) \mathcal{H}_{lm\dots}^{(K)}(b) g(a, b) d\mathbf{v}(a) d\mathbf{v}(b). \quad (51)$$

Each Q of (51) is shown to be of $O(n)$, which is seen from its initial condition

$$[g(a; b)]_{\tau=0} = \delta[\mathbf{z}(a) - \mathbf{z}(b)] f(a). \quad (52)$$

When this initial value of g is substituted in Eq. (51) we have the general expression for initial values of the expansion coefficients Q 's:

$$Q_{ij\dots,lm\dots}^{(J,K)}(\mathbf{x}(a) - \mathbf{x}(b), \tau) = n c^{J+K} \delta[\mathbf{x}(a) - \mathbf{x}(b)] \delta_{JK} \sum \delta_{i_1} \delta_{j_2} \dots, \quad (53)$$

where the summation is over $N!$ permutations in the subscripts of Kronecker's deltas. Especially we have

$$\left. \begin{aligned} [Q^{(0,0)}]_{\tau=0} &= n \delta[\mathbf{x}(a) - \mathbf{x}(b)], \\ [n^{-2} Q_{i_1 i_1}^{(1,1)}]_{\tau=0} &= n^{-1} c^2 \delta_{i_1} \delta[\mathbf{x}(a) - \mathbf{x}(b)], \end{aligned} \right\} \quad (54)$$

which give thermodynamic parts of the density and the velocity correlation of fluctuation, respectively. Furthermore, if we take, in Eq. (51), the moment function

$$(m/3)^2 c^4 [\mathcal{H}^{(2)}(a) + 3][\mathcal{H}^{(2)}(b) + 3] \quad (55)$$

in place of $c^{J+K} \mathcal{H}^{(J)}(a) \mathcal{H}^{(K)}(b)$, we have the pressure fluctuation formula

$$\begin{aligned}\overline{\Delta p(a)\Delta p(b)} &= (m/3)^2\{9c^4[Q^{(0,0)}]_0 + [Q^{(2,2)}]_0\} \\ &= p^2n^{-1}(5/3)\delta[\mathbf{x}(a) - \mathbf{x}(b)].\end{aligned}\quad (56)$$

It is easily examined that fluctuation formulae (54) and (56) are in agreement with those found in text books of physics. In a similar fashion we can calculate the fluctuation of viscous stress and heat flow: These two quantities are defined, in the language of the kinetic theory, by

$$\begin{aligned}q_{ij,lm}^{(2,2)} &= \int (mc^2)^2 [\mathcal{H}_{ij}^{(2)}(a) - \frac{1}{3}\delta_{ij}\mathcal{H}^{(2)}(a)][\mathcal{H}_{lm}^{(2)}(b) - \frac{1}{3}\delta_{lm}\mathcal{H}^{(2)}]g d\mathbf{v}(a)d\mathbf{v}(b), \\ q_{i,i}^{(3,3)} &= \int \left(\frac{m}{2}c^3\right)^2 \mathcal{H}_i^{(3)}(a)\mathcal{H}_i^{(3)}(b)g d\mathbf{v}(a)d\mathbf{v}(b),\end{aligned}$$

where $\mathcal{H}^{(2)}$ and $\mathcal{H}_i^{(3)}$ are Hermite polynomials contracted with respect to two indices appearing in their subscripts.

In view of (51), these relations read, respectively,

$$\begin{aligned}q_{ij,lm}^{(2,2)} &= m^2[Q_{ij,lm}^{(2,2)} - \frac{1}{3}\delta_{ij}Q_{,lm}^{(2,2)} - \frac{1}{3}\delta_{lm}Q_{ij}^{(2,2)} + \frac{1}{9}\delta_{ij}\delta_{lm}Q^{(2,2)}], \\ q_{i,i}^{(3,3)} &= \frac{m^2}{4}Q_{i,i}^{(3,3)}.\end{aligned}$$

Their initial values are obtained from (53) as

$$[q_{ij,lm}^{(2,2)}]_{\tau=0} = \frac{p^2}{n}\delta[\mathbf{x}(a) - \mathbf{x}(b)](\delta_{il}\delta_{jm} + \delta_{jl}\delta_{im} - \frac{2}{3}\delta_{ij}\delta_{lm}), \quad (57)$$

$$[q_{i,i}^{(3,3)}]_{\tau=0} = \frac{5}{2}\frac{p^2}{n}RT\delta[\mathbf{x}(a) - \mathbf{x}(b)]\delta_{ii}. \quad (58)$$

B. Moment Equations

Substituting (49) into Eq. (42), multiplying the equation by moment function $c^{2J}\mathcal{H}_{ij}^{(J)}\dots(a)\mathcal{H}_{lm}^{(J)}\dots(b)$, and performing integrals with respect to $\mathbf{v}(a)$ and $\mathbf{v}(b)$, then we have a series of moment equations for Maxwellian molecules in a manner similar to thirteen-moment method [4];

$$(\partial/\partial\tau)Q^{(0,0)} = (\partial/\partial\tau)Q_{i,i}^{(1,1)} = (\partial/\partial\tau)Q^{(2,2)} = 0, \quad (59)$$

$$(\partial/\partial\tau)q_{ij,lm}^{(2,2)} + 6nBq_{ij,lm}^{(2,2)} = 0, \quad (60)$$

$$(\partial/\partial\tau)q_{i,i}^{(3,3)} + 4nBq_{i,i}^{(3,3)} = 0, \quad (61)$$

where B is a constant related to the viscosity and the thermal conductivity coefficients as follows;

$$\left. \begin{aligned}\mu &= mRT/6B, \\ \lambda &= 5mR^2T/8B.\end{aligned}\right\} \quad (62)$$

Solving Eqs. (60) and (61), eliminating B by the use of (62), and employing the initial conditions (57) and (58), respectively, we are led to the following formulae,

$$q_{ij,lm}^{(2,2)} = n^{-1} p^2 \exp(-\tau p / \mu) \delta[\mathbf{x}(a) - \mathbf{x}(b)] (\delta_{il} \delta_{jm} + \delta_{im} \delta_{jl} - \frac{2}{3} \delta_{ij} \delta_{lm}), \quad (63)$$

$$q_{i,l}^{(3,3)} = n^{-1} p^2 c_p T \exp(-\tau p c_p / \lambda) \delta[\mathbf{x}(a) - \mathbf{x}(b)] \delta_{il}, \quad (64)$$

where $c_p = \frac{5}{2} R$ is the specific heat under constant pressure of a monatomic gas. If, in Eqs. (63) and (64), the factor $e^{-a\tau} (a \gg 1)$ is replaced by $(2/a)\delta(\tau)$, we have familiar forms for the Landau-Lifshitz formulae as it was derived first on purely macroscopic considerations [18]. Thus, use of the kinetic theory, coupled with the thirteen-moment method, has enabled us to show exponential decay of temporal correlation in agreement with general rule of the relaxation phenomena.

V. NONEQUILIBRIUM INTERACTION BETWEEN THERMAL AND MACROSCALE FLUCTUATIONS

Derivation of the Landau-Lifshitz formulae, as shown above on the basis of the present formalism, has assured accuracy of our basic equation (15), therefore, the general kinetic fluctuation equation (46) in elucidating phenomena associated with thermodynamic fluctuations. We will discuss in what follows how the thermal agitation interacts with other fluctuations in nonequilibrium situations ($\psi \neq 0$). In order to see how thermodynamic fluctuations interact with the macroscopic ones in the physical space, we multiply moment function $\alpha(a)\beta(b)$ on Eq. (46) and carry out integration with respect to $\mathbf{v}(a)$ and $\mathbf{v}(b)$, then we have the following expression for moment equations:

$$\begin{aligned} & \left\langle \left\langle \frac{\mathcal{D}\alpha\beta\psi}{\mathcal{D}t} + \frac{\partial}{\partial x_k(a)} \alpha\beta c(a) \xi_k(a) \psi + \frac{\partial}{\partial x_k(b)} \alpha\beta c(b) \psi \xi_k(b) - \left(\alpha \frac{D\beta}{Dt} + \beta \frac{D\alpha}{Dt} \right) \psi \right. \right. \\ & \quad \left. \left. - c(a) \xi_k(a) \frac{\partial \alpha}{\partial x_k(a)} \beta \psi - c(b) \xi_k(b) \alpha \frac{\partial \beta}{\partial x_k(b)} \psi \right\rangle \right\rangle_{ab} - \langle \langle [\alpha] \beta f(a) \psi(\hat{a}, b) \rangle \rangle_{a\hat{a}}^{\dagger} \rangle_b \\ & \quad - \langle \langle [\beta] \alpha f(b) \psi(a, \hat{b}) \rangle \rangle_{\hat{b}b}^{\dagger} \rangle_a = \delta[\mathbf{x}(a) - \mathbf{x}(b)] \mathfrak{S}, \end{aligned} \quad (65)$$

with

$$\mathfrak{S} \equiv \frac{1}{2} \langle \langle \{ -[\alpha(a)\beta(a)] + [\alpha(a) + \alpha(\hat{a})][\beta(a)] + [\beta(a) + \beta(\hat{a})][\alpha(a)] \} f_{\text{II}}(a, \hat{a}) \rangle \rangle_{a\hat{a}}^{\dagger} \quad (66)$$

where the following abbreviations are employed:

$$\left. \begin{aligned} \langle Z \rangle_a & \equiv \int Z d\mathbf{v}(a), \\ \langle \langle Z \rangle \rangle_{a\hat{a}}^{\dagger} & \equiv \int Z d\mathbf{K}(a, \hat{a}) d\mathbf{v}(a) d\mathbf{v}(\hat{a}), \\ [\alpha(a)] & \equiv \alpha(a') + \alpha(\hat{a}') - \alpha(a) - \alpha(\hat{a}), \\ \frac{\mathcal{D}Z}{\mathcal{D}t} & \equiv \frac{\partial Z}{\partial t} + \frac{\partial u_k(a)Z}{\partial x_k(a)} + \frac{\partial u_k(b)Z}{\partial x_k(b)}, \\ \frac{D}{Dt} & \equiv \frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k}, \end{aligned} \right\} \quad (67)$$

and

$$\xi_k \equiv c^{-1}(v_k - u_k). \quad (68)$$

In the above formulae, u_k denotes the hydrodynamic velocity.

We expand the function ϕ as

$$\phi(a, b) = \omega(a)\omega(b) \sum_{(J, K)=(0,0)}^{(\infty, \infty)} \frac{R_{ij\dots lm\dots}^{(J, K)}}{c^{J+K} J! K!} \mathcal{H}_{ij\dots}^{(J)}(a) \mathcal{H}_{lm\dots}^{(K)}(b), \quad (69)$$

where ω is given by (50) in which v_k/c is replaced by ξ_k of (68). Expansion coefficients R represent correlation functions in the physical space, and their evolution equations are obtained by Eq. (65) in which the moment functions (α, β) are chosen as combination of the following set of polynomials;

$$m, \quad mc\mathcal{H}_i^{(1)}, \quad mc^2\mathcal{H}_{ij}^{(2)} \quad \text{and} \quad mc^3\mathcal{H}_i^{(3)}$$

where m is the mass of a particle, and $\mathcal{H}_i^{(3)}$ is the third-order Hermite polynomial with two indices contracted. This exhausts whole set necessary in the thirteen-moment level of description.

Let us consider the simplest case of incompressible flows, then expansion (69) starts with $(J, K) = (1, 1)$, and the moment functions up to $\mathcal{H}_{ij}^{(2)}$ suffice to close the equations. In this case moment of the lowest order is $(\alpha, \beta) = (m, m\mathcal{H}_i^{(1)})$, and the associated moment equation gives

$$\partial R_{i,i}^{(1,1)} / \partial x_i(b) = 0, \quad (70)$$

where no contribution from the thermal agitation is observed. For moment functions of higher order, however, effects due to the thermal motion make their appearance, and the two-point Navier-Stokes equation corresponding to the choice of moment $(\alpha, \beta) = (m\mathcal{H}_i^{(1)}(a), m\mathcal{H}_i^{(1)}(b))$ reads

$$\begin{aligned} \frac{\mathcal{D}R_{i,l}^{(1,1)}}{\mathcal{D}t} + \frac{\partial u_i(a)}{\partial x_k(a)} R_{k,l}^{(1,1)} + \frac{\partial u_l(b)}{\partial x_k(b)} R_{i,k}^{(1,1)} + \frac{\partial R_{i,l}^{(2,1)}}{\partial x_i(a)} + \frac{\partial R_{i,l}^{(1,2)}}{\partial x_l(b)} \\ - \nu [\nabla^2(a) + \nabla^2(b)] R_{i,i}^{(1,1)} = \frac{pp_{il}}{\nu n} \delta[\mathbf{x}(a) - \mathbf{x}(b)], \end{aligned} \quad (71)$$

where p_{il} is the stress deviator tensor and ν is the kinematic viscosity. On the right-hand side terms including $R_{i,i}^{(1,1)}$ are neglected because it is smaller by $O(n^{-1})$ than those standing on the left-hand side. It is seen in Eq. (71) that the inhomogeneous term on the right-hand side is aroused by shearing motion of a gas. This term vanishes for a gas in equilibrium as it should.

In ordinary fluid mechanics the thermal agitation is not the controlling mechanism of triggering turbulence; it is too small ($O(n^{-1}) \sim 10^{-20}$) to affect on macroscopic phenomena under normal conditions. Turbulence initiation is affected more by free-stream turbulence of the windtunnel or noise which enter in the formalism as initial value of $R_{i,i}^{(1,1)}$ and not as inhomogeneous terms. The thermal agitation con-

tributes to generating macroscopic correlations only under hypothetical conditions of perfectly quiescent windtunnel.

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REFERENCES AND FOOTNOTES

- [1] D. Montgomery, The Foundations of Classical Kinetic Theory, in *Lectures in Theoretical Physics* (Gordon and Breach Science Publishers Inc., New York, 1967) Vol. 9C, p. 15.
- [2] Yu. L. Klimontovich, *The Statistical Theory of Nonequilibrium Processes in a Plasma* (M. I. T. Press, Cambridge, Mass., 1967).
- [3] M. Bixon and R. Zwanzig, *Phys. Rev.* **187**, 857 (1969).
- [4] H. Grad, in *Handbuch der Physik*, edited by S. Flügge (Springer, Berlin, 1958) Vol. 12, p. 205.
- [5] G. A. Bird, *Phys. Fluids*, **13**, 2676 (1970), also in *Rarefied Gasdynamics*, L. Trilling and H. Y. Wachman, Eds. (Academic Press Inc., New York, 1969) Vol. I, p. 301.
- [6] H. M. Mott-Smith, *Phys. Rev.* **82**, 885 (1951).
- [7] H. Salwen, C. Grosch and S. Ziering, *Phys. Fluids*, **7**, 180 (1964).
- [8] V. N. Zhigulev, *Dokl. Akad. Nauk, SSSR* **165**, 502 (1965), [*Sov. Phys.—Dokl.* **10**, 1003 (1966)].
- [9] S. Tsugé, *Phys. Fluids*, **17**, 22 (1974).
- [10] Indeterminacy of this kind is inherent to phenomena which are stochastic and nonlinear. For example, we encounter the similar situation in solving the Navier-Stokes equation for turbulent flows.
- [11] The factor N^{-1} is unimportant in discussing fluid-dynamical behavior of ϕ but is inserted in order to fulfil exactly the condition $\int \phi(z, \hat{z}) dz = 0$. (See Sec. IIIA below.)
- [12] C. C. A. Sastri, Long-Range Correlations in the Kinetic Theory, Air Force Office of Scientific Research Report TR-1238, 1973.
- [13] R. E. Fox and G. E. Uhlenbeck, *Phys. Fluids*, **13**, 2881 (1970).
- [14] F. L. Hinton, *Phys. Fluids*, **13**, 857 (1970).
- [15] W. R. Chappell, *J. Statist. Phys.*, **2**, 267 (1970).
- [16] D. Montgomery, *Phys. Fluids*, **12**, 804 (1969).
- [17] K. Sagara and S. Tsugé, *Phys. Letters*, **48A**, 53 (1974).
- [18] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, (Addison-Wesley, 1959) Ch. 17.