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Random Processes with Dead Time and Buffer Memories

By

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Abstract : Extending our previous theory of random processes with dead time [1], we discuss the case where events during the dead time are recorded with the aid of buffer memories. Expressions are given for the probability distribution in the finite time interval in the case of one buffer memory the detection probability for the infinite time interval and the arbitrary number of buffer memories, and the detection probability for the definite resolving time and one buffer memory. The results will be applied to the analysis of rocket data transmitted through a telemeter channel of a finite frequency response, as shown by numerical examples.

1. INTRODUCTION

In the observation of random events of a high rate, the time required for the transmission of information is not negligible. In such a case, buffer memories are emplyed to reduce missing events due to dead time at the sacrifice of information on the arrival time. The missing probability has to be known for the design of a recording system. For the precise measurement the live time has to be measured. Sometimes, the observation is carried out without live time measurement because of a tightly limited bandwidth as in the case of space observations.

The probability distribution and variance have been given for the system without buffer memories [*1*], which is referred to as A hereafter. However, general treatment of the system with buffer memories looks much complicated. In this paper the solutions of three practical problems are given along with their examples. A general method for obtaining the probability distribution in the system with one buffer memory and in a finite time interval is described in 2. This is the case of observing a certain portion of the sky with photon counters by a spinning rocket. In 3 and 4, the detection probabilities for the system with the arbitrary number of buffers in an infinite observation time and for the system with one buffer and a difinite resolving time are calculated, respectively. The latter case corresponds to a detection system with one buffer and one analog to digital converter whose conversion time is not negligible.

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2. Probability distribution of the system with one buffer memory

We consider random events of the average rate f and one buffer memory. The first event arriving during a time length τ after the arrival of the preceding event is stored in the memory and is read out just after the end of this period.

(i) Open-open case

We define $P(n, T)$ as the probability that n events are observed in the time interval T, and $Q(n, T)$ as the probability that n events are observed provided that the first event arrives at the same time of the start of time interval. $P(n, T)$ and $Q(n, T)$ are expressed as follows

$$\left.\begin{aligned}
&Q(n, T)=pP(n-1, T-\tau)+(1-p)Q(n-1, T-\tau),\\
&P(n, T)=\int_0^{T-n\tau} e^{-ft}f dt Q(n, T-t),\\
&\qquad \text{for } 1\leq n<m,
\end{aligned}\right\} \quad (1)$$

with

$$P(0, T)=e^{-fT}, Q(0, T)=0,$$

where

$$(m-1)\tau<T\leq m\tau \quad \text{and} \quad p=e^{-f\tau}.$$

Putting $P(n, T)$ and $Q(n, T)$ as

$$\begin{aligned}
&P(n, T)=\sum_{i=0}^{n-1} F_{n,i}(T-n\tau)e^{-f(T-i\tau)},\\
&Q(n, T)=\sum_{i=0}^{n-1} G_{n,i}(T-n\tau)e^{-f(T-i\tau)},
\end{aligned} \quad (2)$$

and substituting (2) in (1), we obtain

$$\left.\begin{aligned}
&F_{n+1,i}(x)=\int_0^x F_{n,i}(x)dx-F_{n,i}(x)+F_{n,i-1}(x)\\
&\qquad \text{for } 1\leq i\leq n-1,\\
&F_{n+1,0}(x)=\int_0^x F_{n,0}(x)dx-F_{n,0}(x)\\
&F_{n+1,n}(x)=F_{n,n-1}(x),\\
&G_{n,i}(x)=\frac{d}{dx}F_{n,i}(x),\\
&\text{with}\\
&F_{1,0}(x)=x.
\end{aligned}\right\} \quad (3)$$

To solve the integral equation (3), we put $F_{n,i}(x)$ as

$$F_{n,i}(x)=B_{n-1,i}(I-1)^{n-i-1}\cdot x \tag{4}$$

where I is an integral operator and $B_{n-1,i}$ are constants:

$$I\equiv\int_0^x dx.$$

Substituting (4) in (3), relations between $B_{n,i}$ are obtained as

$$\left.\begin{aligned} &B_{n+1,0}=B_{n,0}=B_{n+1,n+1}=B_{n,n}=1\,, \\ &B_{n+1,i}=B_{n,i}+B_{n,i-1}\,. \end{aligned}\right\} \tag{5}$$

Eq. (5) shows that $B_{n,i}$ are binomial coefficients. $P(n, T)$ is given by

$$P(n, T)=\sum_{i=0}^{n-1} B_{n-1,i}e^{-f(T-i\tau)}\sum_{j=0}^{n-i-1}(-1)^j B_{n-i-1,j}\frac{(T-n\tau)^{n-i-j}}{(n-i-j)\,!}\,, \tag{6}$$

where $\quad B_{k,l}=\dfrac{k\,!}{l\,!\,(k-l)\,!}$

(ii) Effect of events in the preceding time interval

We take into account only the adjacent time interval. This is permitted in the case of $T>\tau$; if $T<\tau$, the event rate in all time intervals within τ would have to be taken into consideration.

As in *A*, we introduce G_0 and $G(s)ds$, probabilities that the final event in the preceding time interval arrives earlier than τ before the initial epoch of the observation period concerned and that the final event in the preceding time interval arrives between $\tau-s$ and $\tau-s-ds$, respectively. These satisfy the normalization condition

$$G_0+\int_0^\tau G(s)ds=1. \tag{7}$$

If the preceding time interval is infinite G_0 and $G(s)$ are independent of epoch. If we shift the time coordinate by z ($z<\tau$), the following relations hold for arbitrary z:

$$G(s)=G(z+s), \tag{8}$$

$$G_0=G_0e^{-gz}+\int_0^z G(s)e^{-g(\tau+z-s)}ds, \tag{9}$$

where g is is the average event rate in the preceding time interval. Eq. (8) implies that $G(s)$ is independent of s. Hence Eq. (9) yields

$$(1-e^{-gz})(gG_0-G_se^{-g\tau})=0; \tag{10}$$

here we denote $G(s)$ as G_s.

Combining Eq. (10) with Eq. (8), we obtain

$$\left.\begin{aligned} G_0&=\frac{e^{-g\tau}}{g\tau+e^{-g\tau}} \\ G_s&=\frac{g}{g\tau+e^{-g\tau}} \end{aligned}\right\} \tag{11}$$

This is to be compared with $1/(1+g\tau)$ and $g/(1+g\tau)$, corresponding expressions without buffer memories given in A.

(iii) Probability distribution for the general case

The probability $S(n, T)$ that the number of events observed in an arbitrary time interval T is n is expressed by

$$\begin{aligned} &S(0, T)=G_0R_1(0, T)+G_s\int_0^\tau p_sR_1(0, T-s)ds, \\ &S(n, T)=G_0R_1(n, T)+G_s\int_0^\tau [p_sR_1(n, T-s)+(1-p_s)R_1'(n, T-s)]ds \\ &S(m-1, T) \\ &=G_0R_1(m-1, T)+G_s\int_0^{T-(m-1)\tau}[p_sR_1(m-1, T-s)+(1-p_s)R_1'(m-1, T-s)]ds \\ &\quad+G_s\int_{T-(m-1)\tau}^{\tau}[p_sR_2(m-1, T-s)+(1-p_s)R_2'(m-1, T-s)]ds, \\ &S(m, T)=G_0R_2(m, T)+G_s\int_0^{T-(m-1)\tau}[p_sR_2(m, T-s)+(1-p_s)R_2'(m, T-s)]ds \end{aligned} \tag{12}$$

Here p_s is the probability that no event arrives during τ after the last event in the preceding time interval,

$$p_s=e^{-g\tau-(f-g)s} \tag{13}$$

$R(n, T)$ and $R'(n, T)$ are the distributions when the initial epoch is open and the fiinal epoch is not always open, the definition being the same as $P(n, T)$ and $Q(n, T)$ respectively in 2 and expressed

$$R(n, T)=\begin{cases} R_1(n, T) \\ R_2(n, \tau) \end{cases}, \qquad R'(n, T)=\begin{cases} R_1'(n, T) & T\geq n\tau \\ R_2'(n, T) & (n-1)\tau<T\leq n\tau, \end{cases}$$

$$\left.\begin{aligned} &R_1'(n, T)=pR_1(n-1, T-\tau)+(1-p)R_1'(n-1, T-\tau), \\ &R_2'(n, T)=pR_2(n-1, T-\tau)+(1-p)R_2'(n-1, T-\tau), \\ &R_1(n, T)=\int_0^{T-n\tau} e^{-ft}fdtR_1'(n, T-t) \\ &\qquad+\int_{T-n\tau}^{T-(n-1)\tau} e^{-ft}fdtR_2'(n, T-t), \end{aligned}\right\} \tag{14}$$

$$R_2(n,T)=\int_0^{T-(n-1)\tau} e^{-ft} f dt R_2'(n,T-t),$$

with

$$R_1(0,T)=e^{-fT},$$
$$R_1'(0,T)=R_2'(0,T)=0,$$
$$R_2'(1,T)=1.$$

We do not give general explicit expressions, but show some examples for $m=1$, 2 and 3. Firstly, we give examples of $R(n,T)$ and $R'(n,T)$.

$$\left.\begin{aligned} &R_1'(0,T)=R_2'(0,T)=0,\\ &R_1(0,T)=e^{-fT},\\ &R_2(0,T)=\text{indefinite}.\end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} &R_1'(1,T)=e^{-fT},\\ &R_2'(1,T)=1,\\ &R_1(1,T)=\{f(T-\tau)-1\}e^{-fT}+e^{-f(T-\tau)},\\ &R_2(1,T)=1-e^{-fT}.\end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} &R_1'(2,T)=\{f(T-2\tau)-2e^{-fT}+2\}e^{-f(T-\tau)},\\ &R_2'(2,T)=1-e^{-fT},\\ &R_1(2,T)=\left\{\frac{f^2(T-2\tau)^2}{2}-2f(T-2\tau)-f\tau\right\}e^{-fT}\\ &\qquad +\{2f(T-2\tau)-1\}e^{-f(T-\tau)}+e^{-f(T-2\tau)},\\ &R_2(2,T)=-f(T-\tau)e^{-fT}-e^{-f(T-\tau)}+1.\end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} &R_1'(3,T)=\left\{\frac{f^2(T-3\tau)^2}{2}-3f(T-3\tau)-f\tau+2\right\}e^{-fT}\\ &\qquad +\{3f(T-3\tau)-5\}e^{-f(T-\tau)}+3e^{-f(T-2\tau)},\\ &R_2'(3,T)=\{-f(T-2\tau)+1\}e^{-fT}-2e^{-f(T-\tau)}+1,\\ &R_1(3,T)=\left[\frac{f^3(T-3\tau)^3}{6}-\frac{3f^2(T-3\tau)^2}{2}+2f(T-3\tau)-f\tau\left(fT-\frac{5}{2}f\tau-1\right)\right]\\ &\qquad \times e^{-fT}+\left\{\frac{3f^2(T-3\tau)^2}{2}+5f(T-3\tau)-2f\tau\right\}e^{-f(T-\tau)}\\ &\qquad +\{3f(T-3\tau)-1\}e^{-f(T-2\tau)}+e^{-f(T-3\tau)},\\ &R_2(3,T)=\left\{-\frac{f^2(T-2\tau)^2}{2}+f(T-2\tau)\right\}e^{-fT}-2f(T-2\tau)e^{-f(T-\tau)}\\ &\qquad -e^{-f(T-2\tau)}+1.\end{aligned}\right\} \tag{18}$$

$S(n, T)$ for various values of m are expressed as follows.

$$
\left.
\begin{aligned}
m=1:\quad & S(0, T)=\frac{e^{-fT+g(T-\tau)}}{g\tau+e^{-g\tau}}, \\
& S(1, T)=\frac{1}{g\tau+e^{-g\tau}}\{gT+e^{-g\tau}-e^{-fT+g(T-\tau)}\}.
\end{aligned}
\right\} \tag{19}
$$

$g(\tau-T)/(g\tau+e^{-g\tau})$ has to be added to $S(0, T)$ which is the probability that the time interval is entirely blocked by the last event in the preceding time interval.

$$
\left.
\begin{aligned}
m=2:\quad & S(0, T)=\frac{e^{-fT}}{g\tau+e^{-g\tau}}, \\
& S(1, T)=\frac{1}{g\tau+e^{-g\tau}}\Big[g(2\tau-T)+\alpha e^{-f\tau}-(1+\alpha)e^{-fT} \\
& \qquad +\left(1-\frac{1}{\alpha}\right)e^{-fT-g\tau}-\left(1-\frac{1}{\alpha}-e^{-f\tau}\right)e^{-fT+g(T-2\tau)}\Big], \\
& S(2, T)=\frac{1}{g\tau+e^{-g\tau}}\Big[g(T-\tau)+e^{-g\tau}-\alpha e^{-f\tau}+\alpha e^{-fT} \\
& \qquad -\left(1-\frac{1}{\alpha}\right)e^{-fT-g\tau}+\left(1-\frac{1}{\alpha}-e^{-f\tau}\right)e^{-fT+g(T-2\tau)}\Big], \\
& \text{where}\quad \alpha\equiv g/f.
\end{aligned}
\right\} \tag{20}
$$

$$
\left.
\begin{aligned}
& m=3: \\
& S(0, T)=\frac{e^{-fT}}{g\tau+e^{-g\tau}}, \\
& S(1, T)=\frac{e^{-fT}}{g\tau+e^{-g\tau}}\left\{f(T-2\tau)-2-\alpha+\frac{1}{\alpha}+\left(1+\frac{1}{\alpha}\right)e^{-g\tau}+(1+\alpha)e^{f\tau}\right\}, \\
& S(2, T)=\frac{1}{g\tau+e^{-g\tau}}\Big[g(3\tau-T)+2\alpha e^{-f\tau} \\
& \qquad +\left\{1-(f+g)(T-2\tau)+3\alpha-\frac{1}{\alpha}\right\}e^{-fT}-(1+3\alpha)e^{-f(T-\tau)} \\
& \qquad -\left\{2-f(T-2\tau)+\frac{3}{\alpha}-\frac{f}{\alpha}(T-2\tau)+\frac{1}{2\alpha^2}\right\}e^{-fT-g\tau} \\
& \qquad +2\left(1+\frac{1}{\alpha}\right)e^{-f(T-\tau)-g\tau}+\left(1-\frac{1}{\alpha}\right)^2 e^{-fT+g(T-3\tau)} \\
& \qquad -2\left(1-\frac{1}{\alpha}\right)e^{-f(T-\tau)+g(T-3\tau)}+e^{-f(T-2\tau)+g(T-3\tau)}\Big],
\end{aligned}
\right\} \tag{21}
$$

$$S(3,\,T)=\frac{1}{g\tau+e^{-g\tau}}\Big[g(T-2\tau)+e^{-f\tau}-2\alpha e^{-f\tau}+\{g(T-2\tau)-2\alpha\}\,e^{-fT}$$
$$+\left\{1-f(T-2\tau)+\frac{2}{\alpha}-\frac{f}{\alpha}(T-2\tau)+\frac{1}{2\alpha^2}\right\}e^{-fT-g\tau}$$
$$-2\left(1+\frac{1}{\alpha}\right)e^{-f(T-\tau)-g\tau}-\left(1-\frac{1}{\alpha}\right)^2 e^{-fT+g(T-3\tau)}$$
$$+2\left(1-\frac{1}{\alpha}\right)e^{-f(T-\tau)+g(T-3\tau)}-e^{-f(T-2\tau)+g(T-3\tau)}\Big].$$

The mean number of events,

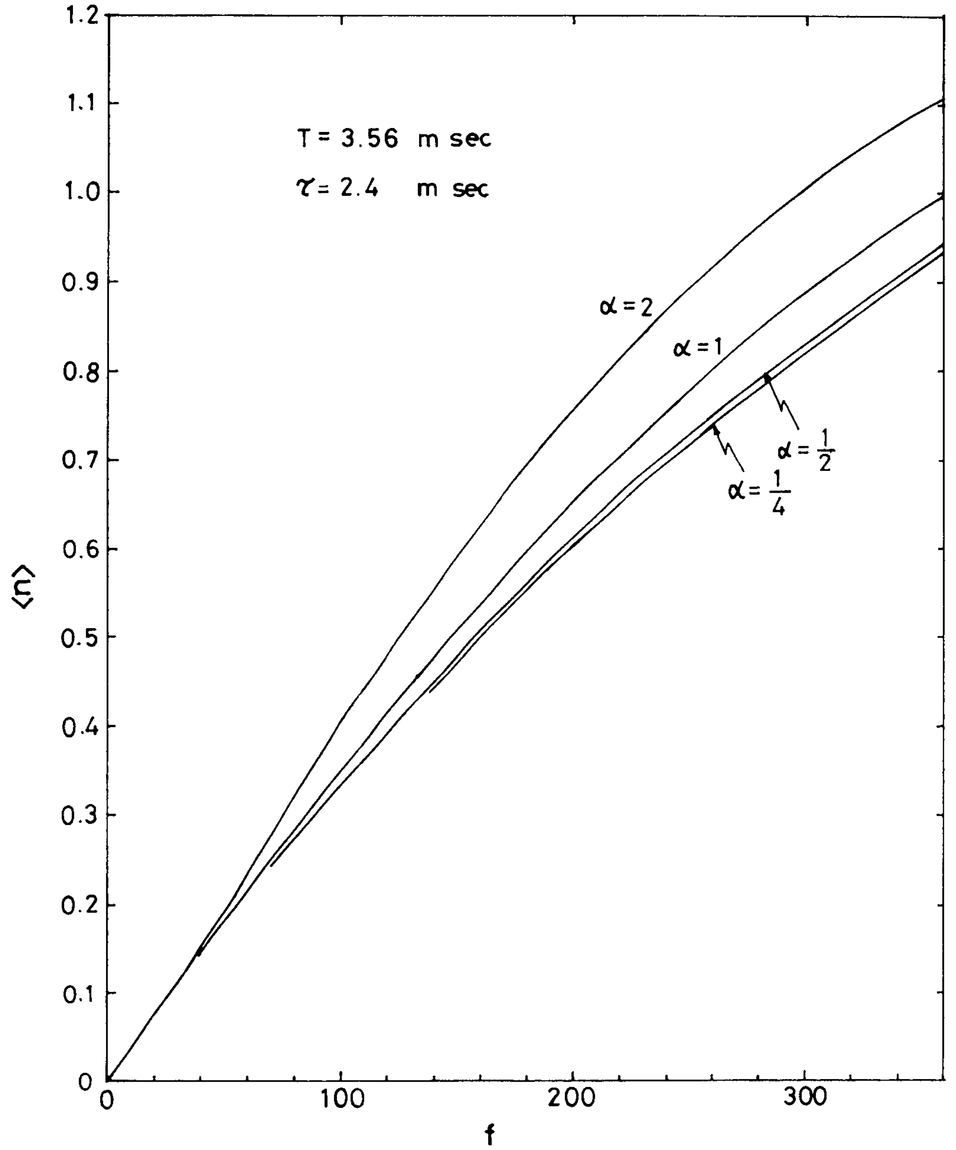

FIG. 1. The average number of counts $<n>$ versus the event rate f for $m=2$ ($T=3.56$ m sec, $\tau=2.4$ m sec). The value of $<n>$ depends on the event rate in the preceding period g through $\alpha\equiv g/f$.

$$\langle n \rangle = S(1, T) + 2S(2, T) \tag{22}$$

for $m=2$, is given as a function of f and f/g in Fig. 1. The value of τ is chosen for a telemeter channel of the highest response frequency available for the K-9M rocket observation, and the value of T corresponds to the spin angle of 3° for the flight of K-9M-44.

3. Detection probability of the system with the arbitrary number of buffer memories for an infinite time interval

In the random process with l buffer meories, we define A_i as the probability that the number of unoccupied memories is $i(i=1, 2, \cdots l)$ just after the initiation of the events. A_i are determined from the following linear equations.

$$\left.\begin{aligned} &\sum_{i=1}^{l} A_i = 1, \\ &A_1 = \sum_{i=1}^{l} A_i P(i \leq), \\ &A_j = \sum_{i=j-1}^{l} A_i P(i-j+1),\ j=2, 3, \cdots\cdots l-1, \\ &A_l = \sum_{i=l-1}^{l} A_i P(i-l+1) + A_l P(0), \end{aligned}\right\} \tag{24}$$

where $\quad P(i \leq) = \sum_{k=i}^{\infty} P(k) \quad$ and $\quad P(k) = \dfrac{(f\tau)^k e^{-f\tau}}{k!}.$

There are $l+1$ equations in Eq. (23), but the second equation can be derived from the third and the fourth equations. The mean dead time T_D is expressed as

$$T_D = \sum_{i=1}^{l} A_i T_{Di}.$$

where T_{Di} is the dead time when the available number of memories is i and is given by

$$\begin{aligned} T_{Di} &= \int_0^{\tau} e^{-ft_1} f dt_1 \int_0^{\tau-t_1} e^{-ft_2} f dt_2 \cdots \int_0^{\tau-t_1\cdots-t_{i-1}} e^{-ft_i} f dt_i (\tau - t_1 - t_2 \cdots - t_i) \\ &= \tau - \frac{i}{f} + \frac{e^{-f\tau}}{f} \sum_{k=0}^{i-1} \frac{i-k}{k!} (f\tau)^k. \end{aligned} \tag{25}$$

The detection probability f'/f, the ratio of the apparent event rate f' to the true rate f, is

$$\frac{f'}{f} = \frac{1}{1+fT_D}. \tag{26}$$

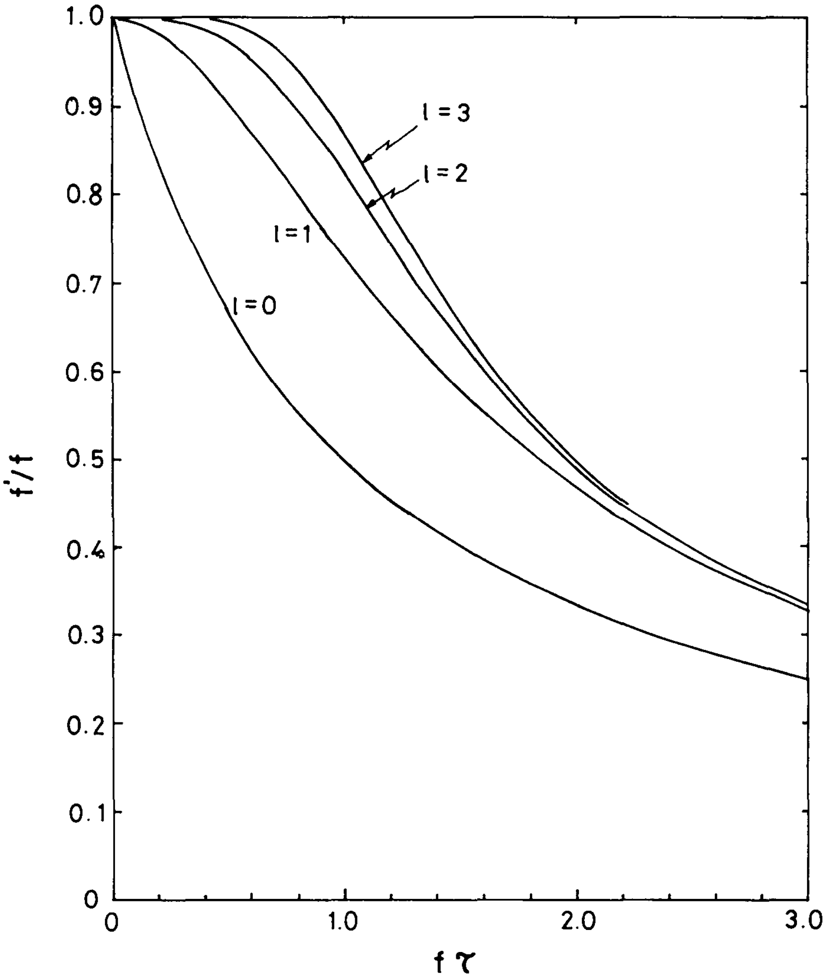

FIG. 2. The detection probability f'/f of the system with buffer memories versus $f\tau$ for various number of buffer memories.

We give explicit expressions for $l=1, 2$ and 3.

$l=1$: $A_1=1$,

$$T_D=\tau-\frac{1}{f}+\frac{e^{-f\tau}}{f},$$

$$\frac{f'}{f}=(f\tau+e^{-f\tau})^{-1} \tag{27}$$

$l=2$: $A_1=1-\dfrac{e^{-f\tau}}{1-f\tau e^{-f\tau}}$,

$$A_2=\frac{e^{-f\tau}}{1-f\tau e^{-f\tau}},$$

$$T_D=\tau-\frac{1}{f}+\frac{e^{-2f\tau}}{f(1-f\tau e^{-f\tau})},$$

$$\frac{f'}{f}=\left\{f\tau+\frac{e^{-2f\tau}}{1-f\tau e^{-f\tau}}\right\}^{-1}, \tag{28}$$

$$l=3:\quad A_1=1-\frac{2e^{-f\tau}(1-f\tau e^{-f\tau})}{2-4f\tau e^{-f\tau}+(f\tau)^2e^{-2f\tau}},$$

$$A_2=\frac{2e^{-f\tau}(1-e^{-f\tau}-f\tau e^{-f\tau})}{2-4f\tau e^{-f\tau}+(f\tau)^2e^{-2f\tau}},$$

$$A_3=\frac{2e^{-2f\tau}}{2-4f\tau e^{-f\tau}+(f\tau)^2e^{-2f\tau}},$$

$$T_D=\tau-\frac{1}{f}+\frac{2e^{-3f\tau}}{f\{2-4f\tau e^{-f\tau}+(f\tau)^2e^{-2f\tau}\}},$$

$$\frac{f'}{f}=\left\{f\tau+\frac{e^{-3f\tau}}{1-2f\tau e^{-f\tau}+\frac{(f\tau)^2}{2}e^{-2f\tau}}\right\}^{-1} \tag{29}$$

The values of f'/f against $f\tau$ are graphically shown in Fig. 2.

4. Detection probability of the system with resolving time and one buffer memory

The mean dead time of the system with resolving τ_1, during which the system is insensitive, and one buffer memory is given by

$$\left.\begin{aligned} T_D&=\tau_1+\int_0^{\tau-\tau_1}e^{-ft}(\tau-\tau_1-t)fdt && (\tau\geq\tau_1)\\ &=\tau-\frac{1}{f}+\frac{e^{-f(\tau-\tau_1)}}{f}, && \\ T_D&=\tau_1. && (\tau\leq\tau_1)\end{aligned}\right\} \tag{30}$$

The detection probability is given by

$$\left.\begin{aligned} \frac{f'}{f}&=\frac{1}{f\tau+e^{-f(\tau-\tau_1)}}, && (\tau\geq\tau_1)\\ &=\frac{1}{1+f\tau_1}. && (\tau\leq\tau_1)\end{aligned}\right\} \tag{31}$$

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Reference

[1] S. Hayakawa, F. Makino and F. Nagase, ISAS Report No. 513, **39**, 181 (1974).