Aeronautical Research Institute, University of Tokyo
Report No. 330, June 1958

First-Order Approach to a Strong Interaction Problem in Hypersonic Flow Over an Insulated Flat Plate

By

Hakuro OGUCHI

Summary. The present paper concerns with the strong interaction phenomenon over an insulated semi-infinite flat plate with a sharp leading edge. In particular the main interest is in the consistent treatment in which the boundary-layer solution may be joined continuously with the inviscid solution regarding flow variables including pressure, normal velocity, temperature (or streamwise velocity) and density.

It is shown that the behavior of the inviscid solution may be consistent with that of the boundary-layer solution to at least first-order approximation that is correct to the order of $(M/\sqrt{R_{x^*}})^{1-2/3\gamma}$, where M is the Mach number of undisturbed flow, R_{x^*} the Reynolds number based on the distance from leading edge and γ the ratio of specific heats. Then the first-order boundary-layer problem is formulated under such an external circumstance and an attempt is made for arriving at the solution.

Actual calculations are carried out for both cases of air and helium. From the solution it is found that the region in which the viscous effect plays a significant role is ranged over from 0 to a certain finite value of η, say η_δ, in terms of the similarity coordinate η in the corresponding incompressible boundary layer. The numerical results moreover indicate that the induced pressure is considerably smaller than the estimate of Lees [*7*] obtained by his approximate method in which the effect of the first-order induced pressure on the boundary layer is ignored and no survey of the first-order boundary-layer equation is made. The present results are also found to be in excellent agreement with experimental data recently obtained in helium flow by Erickson [*15*].

1. INTRODUCTION

Consider the flow over a two-dimensional semi-infinite flat plate placed parallel to the undisturbed oncoming flow of hypersonic speed. Then the strong shock waves emanate from the leading edge of the plate because of the much more remarkable viscous layer growth than at low speeds. Near the leading edge this shock wave is so strong and highly curved that a large pressure change is induced; the growth of the viscous layer itself is simultaneously influenced by this pressure change. Such an interaction phenomenon between shock wave and viscous layer is called "strong" in case when the shock wave exerts a remarkable effect on the viscous layer. If the oncoming flow has a sufficiently high Mach number and a sufficiently low Reynolds number, a strong interaction phenomenon is expected to occur over a region not only near the leading edge but also far from it so that the boundary layer approximation is valid for the viscous layer. Recently such a phenomenon has been investigated in

detail by many authors [1]–[6]. In most of these investigations the matching of the solution for the boundary layer to that for the inviscid region has been carried out on the basis of the flow model as follows: the flow field downstream of the shock wave may be separated distinctly into viscous and inviscid regions and the inviscid flow may be identified with a flow over a fictitious body whose surface is the edge of the boundary layer. Since, in fact, the amount of mass flow in the boundary layer is sufficiently small because of the anomalously high temperature near the wall, the analysis based on this flow model may be expected to provide the zeroth-order approximation to the problem. In these analyses, however, there inevitably exists a discontinuity in stream function across the junction of the inviscid and boundary-layer regions. Since, according to the analysis by Stewartson [5], the inviscid solution represents a singular behavior such that the density decreases to zero and the temperature increases to infinity as the stream function tends to zero, the simple flow model for the inviscid region is not appropriate in order to make the matching consistent regarding the temperature and density. For a finite Mach number an anomalously high temperature near the wall may cause a considerable deviation of the streamwise velocity from the undisturbed velocity even at the outer edge of the boundary layer. In view of the above fact, Lees [7] put forward the physical picture that the streamlines across the shock wave near the leading edge penetrate into the boudary layer and obtained the expressions for the streamwise velocity and vorticity in the outer region of the boundary layer, which region he called the vorticity layer. He then showed that such a change of the ambient circumference of the boundary layer leads to the correction for the zeroth-order boundary-layer solution to the order of $(M/\sqrt{R_{x*}})^{1-2/3\gamma}$, much greater than the usual "errors" made in the boundary-layer approximation, where M is the Mach number of the undisturbed flow, R_{x*} the Reynolds number based on the distance from the leading edge and γ the ratio of specific heats for gas. In his paper, however, the effect of the pressuse induced by the first-order change in boundary-layer thickness is ignored so that the result provides an estimate of the first-order change in boundary-layer thickness and induced pressure. It is also evident that the joining of the boundary-layer solution with the inviscid solution concerning the first-order pressure is not effected with success.

The present paper concerns with the strong interaction phenomenon over an insulated semi-infinite flat plate with a sharp leading edge. The viscous region is treated as usual within the framework of the boundary-layer theory, and the Prandtl number of unity and the linear viscosity-temperature relation are assumed. The main interest is in the consistent treatment in which the boundary-layer solution may be joined continuously with the inviscid solution regarding flow variables including pressure, normal velocity, temperature (or streamwise velocity) and density. First, the behavior of the inviscid solution in the vicinity of the boundary layer is derived from the inviscid solution in the form of series on the basis of the hypersonic small-disturbance theory. Second, it is shown that the behavior of the inviscid solution may in the first-order approximation be identified with that of the boundary-layer solution in the region far from the wall so far as the most predominant terms are concerned. Finally, the first-order boundary-layer problem is formulated under

the above-imposed external circumstance. Then the solution is actually effected. From the solution it is found that the region in which the viscous effect plays a significant role is ranged, in terms of the similarity coordinate η in the corresponding incompressible boundary layer, from 0 to a certain finite value of η, say η_δ. Then beyond $\eta=\eta_\delta$, there exists the overlapping region both for the inviscid and boundary-layer solutions. Thus, over this overlapping region, the smooth joining between both solutions is ensured regarding flow variables including the pressure, normal velocity, temperature (or streamwise velocity) and density. The value of η_δ is determined simultaneously with the first-order solution. According to the numerical results for both cases of air ($\gamma=7/5$) and helium ($\gamma=5/3$), the magnitude of the correction needed for the zeroth-order theory is found to be considerably less than the maximum estimate obtained by Lees [*7*].

2. The Mathematical Formulation for the Inviscid Region Behind the Shock Wave

The shock wave induced by the boundary layer alone will in general be of non-analytic form at the leading edge. Its aparture from the leading edge may be assumed to be vanishingly small because of the high Mach number of the undisturbed flow, although the shock wave must actually be slightly detached from the leading edge. According to the above assumption, we may apply the hypersonic small-disturbance theory to the present problem in a like fashion as that developed by Van Dyke [*8*] for the case with the shock wave attached to the leading edge. In this section, the ordinary differential equation governing the inviscid region behind the power-shaped shock wave is derived on the basis of the hypersonic small-disturbance theory.

In a Cartesian coordinates, let the velocity components be denoted by u^* and v^* paralle to the axes of x^* and y^*, respectively, and let the plate be defined by $y^*=0$ and $x^*\geq 0$. Let the pressure and density be denoted by p^* and ρ^*, respectively. Then the full equations of motion governing the inviscid flow are

$$\frac{\partial \rho^* u^*}{\partial x^*}+\frac{\partial \rho^* v^*}{\partial y^*}=0\,, \tag{2.1}$$

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{1}{\rho^*}\frac{\partial p^*}{\partial x^*}\,, \tag{2.2}$$

$$u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}=-\frac{1}{\rho^*}\frac{\partial p^*}{\partial y^*}\,, \tag{2.3}$$

$$u^*\frac{\partial (p^*/\rho^{*\gamma})}{\partial x^*}+v^*\frac{\partial (p^*/\rho^{*\gamma})}{\partial y^*}=0\,. \tag{2.4}$$

If the velocity, density and pressure at infinity are denoted by U, ρ^*_∞ and p^*_∞, respectively, then the following barred independent and depedent variables are conveniently introduced on the basis of the hypersonic similitude:

$$\left.\begin{aligned} &x^*=\bar{x}, \qquad y^*=\tau\bar{y}, \\ &u^*=U[1+\tau^2\bar{u}(\bar{x},\bar{y})], \quad p^*=p_\infty^*\gamma M^2\tau^2\bar{p}(\bar{x},\bar{y}), \\ &v^*=U\tau\bar{v}(\bar{x},\bar{y}), \quad \rho^*=\rho_\infty^*\bar{\rho}(\bar{x},\bar{y}), \end{aligned}\right\} \tag{2.5}$$

where $M=U/(\gamma p_\infty^*/\rho_\infty^*)^{1/2}$ is the Mach number of the undisturbed flow and τ a small parameter measuring the disturbed range. In the present paper τ is taken equal to the shock-wave height at the point $x^*=1$. It is now assumed that the terms which explicitly contain τ^2 alone may be discarded when the transformation (2.5) is substituted into the full equations of motion. Its justification will be confirmed by examining the consistency of the results. Then the full equations of motion can be simplified to the form

$$\bar{\rho}_{\bar{x}}+(\bar{\rho}\bar{v})_{\bar{y}}=0, \tag{2.6}$$

$$\bar{v}_{\bar{x}}+\bar{v}\bar{v}_{\bar{y}}+\bar{p}_{\bar{y}}/\bar{\rho}=0, \tag{2.7}$$

$$(\bar{p}/\bar{\rho}^\gamma)_{\bar{x}}+\bar{v}(\bar{p}/\bar{\rho}^\gamma)_{\bar{y}}=0, \tag{2.8}$$

where the subscripts $\bar{x}$ and $\bar{y}$ denote the differentiation with respect to $\bar{x}$ and $\bar{y}$, respectively. From these equations, $\bar{p}$, $\bar{\rho}$ and $\bar{v}$ can be determined independently of the streamwise velocity $\bar{u}$ which, if it is desired, can be determined from the energy equation of the form

$$\bar{u}+\frac{1}{2}\bar{v}^2+\frac{\gamma\bar{p}}{(\gamma-1)\bar{\rho}}=\frac{1}{(\gamma-1)M^2\tau^2}=\text{const.} \tag{2.9}$$

Let us introduce the reduced stream function $\bar{\Psi}$ defined by

$$\bar{\Psi}_{\bar{y}}=\bar{\rho}, \quad \bar{\Psi}_{\bar{x}}=-\bar{\rho}\bar{v}, \tag{2.10}$$

in order to satisfy Eq. (2.6). We introduce moreover the vorticity function $\bar{\omega}$ defined by

$$\bar{\omega}=\bar{p}/\bar{\rho}^\gamma. \tag{2.11}$$

Then it is easily seen from Eq. (2.8) that $\bar{\omega}$ is dependent only upon $\bar{\Psi}$, since, from the definition (2.10) of $\bar{\Psi}$,

$$\bar{\Psi}_{\bar{x}}+\bar{v}\bar{\Psi}_{\bar{y}}=0.$$

Eq. (2.7) can be written in terms of $\bar{\Psi}$

$$\bar{\Psi}_{\bar{y}}^2\bar{\Psi}_{\bar{x}\bar{x}}-2\bar{\Psi}_{\bar{x}}\bar{\Psi}_{\bar{y}}\bar{\Psi}_{\bar{x}\bar{y}}+\bar{\Psi}_{\bar{x}}^2\bar{\Psi}_{\bar{y}\bar{y}}=\bar{\Psi}_{\bar{y}}^{\gamma+1}\left(\gamma\omega\bar{\Psi}_{\bar{y}\bar{y}}+\frac{d\omega}{d\bar{\Psi}}\bar{\Psi}_{\bar{y}}^2\right). \tag{2.12}$$

The shock-wave conditions are also simplified by the same assumption as in the reduction of the equations of motion. If the shock-wave shape is expressed in the form

$$y^*=\tau s(x^*) \quad \text{or} \quad \bar{y}=s(\bar{x}),$$

then the simplified shock-wave conditions are

$$\left.\begin{aligned} &\bar{p}=\frac{2\gamma\kappa^2-(\gamma-1)}{\gamma(\gamma+1)\kappa^2}s'^2(\bar{x}), \quad \bar{\rho}=\frac{(\gamma+1)\kappa^2}{2+(\gamma-1)\kappa^2}, \\ &\bar{v}=\frac{2(\kappa^2-1)}{(\gamma+1)\kappa^2}s'^2(\bar{x}), \end{aligned}\right\} \tag{2.13}$$

where the prime attached to s denotes the differentiation with respect to $\bar{x}$, and κ is a hypersonic similarity parameter based on the local shock-wave slope, namely,

$$\kappa = M_T s'(\bar{x}) . \tag{2.14}$$

Eqs. (2.12) and (2.13) constitute the basis of the first-order hypersonic small-disturbance theory developed by Van Dyke [8].

Now let us apply this theory to the flow in the neighborhood of the leading edge when the shock wave is assumed to the expressed in the form

$$\bar{y}_s = \bar{x}^n \quad \text{or} \quad y_s^* = {}_T x^{*n} , \tag{2.15}$$

where n is assumed to be a positive number less than nuity. Then the present problem of interest is included as a special case $n=3/4$, as will be seen later. Throughout this paper the subscript s denotes the quantities at or just behind the shock wave. Since $\bar{v}=0$, $\bar{\rho}=1$ in the undisturbed flow, Eq. (2.10) yields there

$$(\bar{\Psi}_{\bar{y}})_s = 1 , \quad (\bar{\Psi}_{\bar{x}})_s = 0 ,$$

and then the integration along the shock wave yields

$$\bar{\Psi}_s = \bar{y}_s = \bar{x}^n . \tag{2.16}$$

Let us introduce a modified conical coordinate defined by

$$\theta = \bar{y}/\bar{x}^n ,$$

and let us assume the form of

$$\bar{\Psi} = \bar{x}^n f(\theta) , \tag{2.17}$$

then it is compatible with the condition at the shock, Eq. (2.16). If n is assumed to be a positive number less than unity, the parameter κ defined by Eq. (2.14) becomes sufficiently large compared with unity in the neighborhood of the leading edge. Thus, substituting $\bar{y}_s$ from Eq. (2.16) into Eq. (2.13) and retaining only the leading terms, we obtain

$$\left.\begin{aligned} \bar{p}_s &= \frac{2n^2}{\gamma+1} \bar{x}^{2(n-1)} , \\ \bar{\rho}_s &= \frac{\gamma+1}{\gamma-1} . \end{aligned}\right\} \tag{2.18}$$

With these expressions Eq. (2.11) yields the vorticity function just behind the shock

$$\bar{\omega}_s = \frac{2n^2}{\gamma+1} \left(\frac{\gamma-1}{\gamma+1} \right)^{\gamma} \bar{x}^{2(n-1)} ,$$

or in terms of $\bar{\Psi}_s$,

$$\bar{\omega}_s = \frac{2n^2}{\gamma+1} \left(\frac{\gamma-1}{\gamma+1} \right)^{\gamma} \bar{\Psi}_s^{\frac{2(n-1)}{n}} .$$

Since the vorticity function depends only on the stream function, its value downstream of the shock wave can be obtained by simply replacing $\bar{\Psi}_s$ by $\bar{\Psi}$ in the above equation, namely,

$$\bar{\omega} = \frac{2n^2}{\gamma+1} \left(\frac{\gamma-1}{\gamma+1} \right)^{\gamma} \bar{\Psi}^{\frac{2(n-1)}{n}} . \tag{2.19}$$

or using Eq. (2.17)

$$\bar{\omega}=\frac{2n^2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}\bar{x}^{2(n-1)}f^{\frac{2(n-1)}{n}}. \tag{2.20}$$

Substituting $\bar{\Psi}$ from Eq. (2.17) and $\bar{\omega}$ from Eq. (2.19) into the fundamental equation (2.12), we obtain the following ordinary differential equation for f;

$$n^2f^2f''+n(n-1)(f-f'\theta)f'^2=\frac{2n}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}f^{\frac{2(n-1)}{n}}f'^{\gamma+1}\cdot$$
$$[n\gamma f''+2(n-1)f'^2/f]. \tag{2.21}$$

Remembering $\theta=1$ at the shock wave and combining Eq. (2.16) with Eq. (2.17), we have the condition for f at the shock wave:

$$f=1 \quad \text{at} \quad \theta=1. \tag{2.22}$$

Comparing $\bar{\rho}$ from Eqs. (2.10) and (2.17) with $\bar{\rho}_s$ from Eq. (2.18), we have the condition for f':

$$f'=(\gamma+1)/(\gamma-1) \quad \text{at} \quad \theta=1. \tag{2.23}$$

Consequently the function f can be determined by solving Eq. (2.21) under the conditions (2.22) and (2.23). Here it should be noted that a singurality develops at the point $\theta=\theta_0$ where f vanishes, as we can immediately see from the construction of Eq. (2.21). Hence, in order to obtain more complete knowledge of the solution, its behavior at the limit $\theta\to\theta_0$ must be examined.

3. The Behavior of the Solution for the Inviscid Region near the Boundary Layer

In this section the solution of Eq. (2.21) is analytically found in the series form near the point $\theta=\theta_0$ so that the behaviors of flow variables there are examined. To do this, let us assume from the physical point of view that the pressure takes a non-zero bounded value even at the limit $\theta\to\theta_0$. Substituting $\bar{\Psi}$ from Eq. (2.17) into the expression for $\bar{p}$ obtained by using Eqs. (2.10), (2.11) and (2.19), we have

$$\bar{p}\simeq\frac{2n^2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}f'^{\gamma}f^{2(n-1)/n}\bar{x}^{2(n-1)}. \tag{3.1}$$

Thus the above assumption requires that $f'^{\gamma}f^{2(n-1)/n}$ tends to a non-zero finite value as $\theta\to\theta_0$. Then f must be expressed in the form

$$f\simeq A(\theta-\theta_0)^N, \tag{3.2}$$

and hence

$$f'\simeq AN(\theta\to\theta_0)^{N-1}, \tag{3.3}$$

where

$$N=n\gamma/[n\gamma+2(n-1)] \tag{3.4}$$

and A and θ_0 are constants to be determined. This expression is adequate only when N is non-negative, because the vanishing of f is required at the limit $\theta\to\theta_0$, or only when

$$n\geq 2/(\gamma+2), \tag{3.5}$$

where the lower bound may be replaced by 2/3 due to the fact that $5/3 \geq \gamma \geq 1$ for all real gases. Thus

$$1 > n \geq 2/3 . \tag{3.6}$$

It can easily be confirmed by inserting Eq. (3.2) into Eq. (2.21) that, when n fulfills the inequality (3.6), f given by Eq. (3.2) satisfies Eq. (2.21) asymptotically of $\theta \to \theta_0$. Since, for such cases, the radius of curvature of the shock wave at the leading edge becomes vanishingly small and its value provides a measure of the shock-wave aparture, the previous assumption that the shock-wave aparture may be ignored at the leading edge is valid unless the flow field under consideration is very near the leading edge. Therefore it is concluded that if $1 > n \geq 2/3$, the main part of the solution f near the point $\theta = \theta_0$ is given by Eq. (3.2). Thus the constant A and θ_0 can be determined from the asymytotic values of $(f'/N)^N f^{1-N}$ and $\theta - Nf/f'$, respectively, as the value of f tends to zero. These are readily evaluated from the solution of Eq. (2.21) obtained by step-by-step integration starting from the shock wave.† The actual calculation has been carried out for $n = 3/4$ and both for air ($\gamma = 7/5$) and helium ($\gamma = 5/3$). The results are shown in the following table.

TABLE 1.

	A	θ_0
air	4.40	0.591
helium	2.44	0.479

On remembering that the series expression of f has the form of Eq. (3.2) as the leading term, we find by the insertion into Eq. (2.21) that it has the following form

$$f = A(\theta - \theta_0)^N \{1 + a(\theta - \theta_0)^N + b(\theta - \theta_0)^{N+1} + 0[(\theta - \theta_0)^{2N}]\}, \tag{3.7}$$

where

$$a = \frac{N(1-n)(1+\gamma)}{2} \left[\frac{\gamma+1}{N(\gamma-1)}\right]^{\gamma} \frac{\theta_0 A^{1-\gamma/N}}{2n\gamma(2N-1)+6N(N--1)},$$

$$b = \frac{(N-1)(\gamma+1)}{4N} \left[\frac{\gamma+1}{N(\gamma-1)}\right]^{\gamma} \frac{[n+(1-n)N]A^{1-\gamma/N}}{n\gamma(2N+1)+(n-1)(3N+2)}.$$

It has been numerically confirmed for both cases of air and helium when $n = 3/4$ that the values of f and f' from the above expression with the values of A and θ_0 shown in Table 1 are identified with those obtained by step-by-step integration over a considerably wide range with ample accuracy. These values of f and f' are plotted against the value of θ in Figs. 1 and 2, respectively, in which the step-by-step solutions are shown by full lines while the series solutions broken lines.

Using the behavior (3.7) of f near the point $\theta = \theta_0$, $\bar{v}$ there is found from Eq. (2.10) in the form

† A similar approach can also be applicable to the case of axially symmetric flow (see Appendix A).

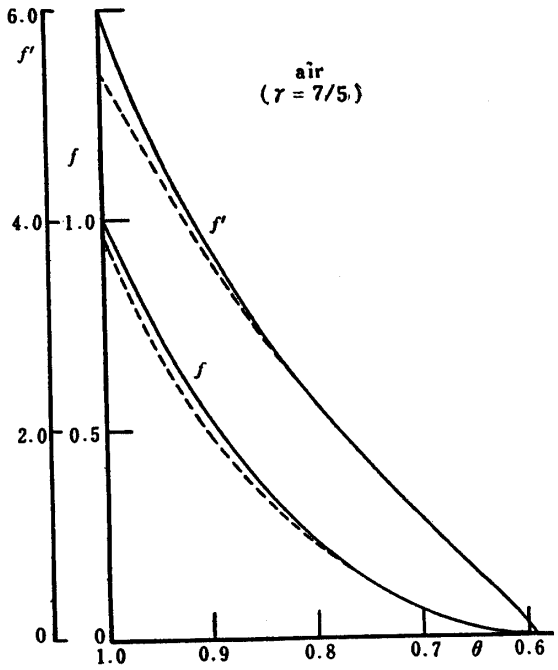

FIGURE 1.

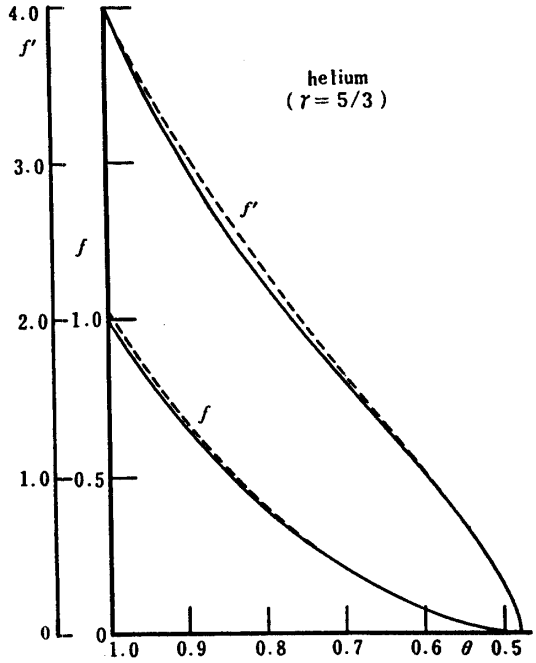

FIGURE 2.

$$\bar{v}=n\theta_0\bar{x}^{n-1}[1+0(\theta-\theta_0)]$$

and hence at the limit $\theta\to\theta_0$

$$\bar{v}=n\theta_0\bar{x}^{n-1}$$

or, from Eq. (2.5),

$$v^*=n\tau\theta_0 x^{*n-1}U. \tag{3.8}$$

On the other hand, the location where $\bar{\Psi}=0$ (body surface) is given from the definition of θ_0 by

$$y^*=\tau\theta_0 x^{*n}. \tag{3.9}$$

The normal velocity v^*, as naturally expected, is identical with the slop of the body surface given by Eq. (3.9). From Eqs. (3.1) and (3.7), the pressure $\bar{p}$ becomes near the point $\theta=\theta_0$

$$\bar{p}=\frac{2n^2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}(NA^{1/N})^{\gamma}\bar{x}^{2(n-1)}[1+0((\theta-\theta_0)^N)]$$

and hence we obtain

$$\bar{p}=\frac{2n^2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}(NA^{1/N})^{\gamma}\bar{x}^{2(n-1)},$$

$$\frac{\partial\bar{p}}{\partial\theta}=0 \quad \text{or} \quad \frac{\partial p^*}{\partial y^*}=0, \tag{3.11}$$

at the limit $\theta\to\theta_0$.

In the application of the above solution to the strong interaction problem, the body surface should be considered as a fictitious one, for the analysis in this section cannot be valid right up to the limit $\theta\to\theta_0$, where the viscous effect actually plays a significant role. However the analysis may be applicable to the inviscid region

near the edge of the boundary layer, in which a stream function is sufficiently small but still remains finite. It follows then from Eqs. (3.8) and (3.10) that both $\bar{v}$ and $\bar{p}$ take non-zero bounded values independent of θ or $\bar{\Psi}$ at a fixed value of $\bar{x}$ in the inviscid region near the edge of the boundary layer, so far as the most predominant terms are concerned. The value of density there is given from Eqs. (2.11) and (2.19) by

$$\bar{\rho} \simeq \left(\frac{\gamma+1}{2n^2}\right)^{1/\tau}\left(\frac{\gamma+1}{\gamma-1}\right)\bar{\Psi}^{2(1-n)/n\tau}\bar{p}^{1/\tau}, \tag{3.12}$$

where $\bar{p}$ should be taken from Eq. (3.10).

Here let us introduce the dimensionless quantities defined by

$$\left.\begin{aligned} u&=u^*/U, \quad v=v^*/U, \\ p&=p^*/p^*_\infty, \quad \rho=\rho^*/\rho^*_\infty, \end{aligned}\right\} \tag{3.13}$$

and further the usual stream function Ψ defined by

$$\frac{\partial \Psi}{\partial x^*}=-\rho v U, \qquad \frac{\partial \Psi}{\partial y^*}=\rho u U, \tag{3.14}$$

Ψ being related to $\bar{\Psi}$ defined by Eq. (2.10) by

$$\Psi \simeq \tau U \bar{\Psi}. \tag{3.15}$$

With these new variables, the behaviors of normal velocity v, pressure p and density ρ in the inviscid region near the edge of the boundary layer are obtained from Eqs. (3.8), (3.10) and (3.12) as follows:

$$v \simeq n\tau\theta_0 x^{*n-1}, \tag{3.16}$$

$$p \simeq \frac{2n^2\gamma}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{\tau}\left[\frac{n\gamma}{n\gamma+2(n-1)}\right]A^{\frac{n\tau+2(n-1)}{n}}M^2\tau^2 x^{*2(n-1)}, \tag{3.17}$$

$$\rho \simeq \left(\frac{\gamma+1}{2n^2}\right)^{1/\tau}\left(\frac{\gamma+1}{\gamma-1}\right)(p/\gamma M^2\tau^2)^{1/\tau}(\Psi/\tau U)^{2(1-n)/n\tau}, \tag{3.18}$$

so far as the most predominant terms are concerned.

4. The Zeroth-Order Approximation for the Boundary Layer

The governing equations in the viscous region near the wall are expressed within the framework of the boundary-layer theory in the form

$$\frac{\partial(\rho^* u^*)}{\partial x^*}+\frac{\partial(\rho^* v^*)}{\partial y^*}=0, \tag{4.1}$$

$$\rho^*\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)=-\frac{\partial p^*}{\partial x^*}+\frac{\partial}{\partial y^*}\left(\mu^*\frac{\partial u^*}{\partial y^*}\right), \tag{4.2}$$

$$\frac{\partial p^*}{\partial y^*}=0, \tag{4.3}$$

$$\rho^* c_p\left(u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}\right)=u^*\frac{\partial p^*}{\partial x^*}+\mu^*\left(\frac{\partial u^*}{\partial y^*}\right)^2+\frac{\partial}{\partial y^*}\left(\lambda^*\frac{\partial T^*}{\partial y^*}\right), \tag{4.4}$$

where T^* is the temperature, c_p the specific heat at constant pressure, μ^* the vis-

cosity coeficient, λ^* the heat-conductivity coeficient and other symbols have the meaning already given in the preceding section. Let us assume a gas to obey the perfect gas law:

$$p^*/\rho^* = RT^*, \tag{4.5}$$

where R is the gas constant per unit mass.

For the thermally insulated plate of main interest in this paper, the energy equation (4.4) can be replaced by a simple integral under the assumption of Prandtl number of unity, namely,

$$\frac{T^*}{T^*_\infty} = 1 + \frac{\gamma-1}{2(\gamma p^*_\infty/\rho^*_\infty)}(U^2 - u^{*2}),$$

or, in terms of dimensionless quantities refered to the values at infinity (for example, $T = T^*/T^*_\infty$)

$$\frac{T}{M^2} = \frac{\gamma-1}{2}(1-u^2) + \frac{1}{M^2} \simeq \frac{\gamma-1}{2}(1-u^2). \tag{4.6}$$

Thus, for this case we can determine the temperature from a knowledge of the velocity. Moreover let us assume the linear viscosity-temperature relation in the form

$$\mu^* = \mu^*_\infty \frac{T^*}{T^*_\infty}. \tag{4.7}$$

With these assumptions we introduce according to Stewartson [9] the following transformations into Eq. (4.2):

$$x^*_i = \int_0^{x^*} p^{\frac{3\gamma-1}{2\gamma}} dx^*, \tag{4.8}$$

$$y^*_i = p^{\frac{\gamma-1}{2\gamma}} \int_0^{y^*} \rho dy^*. \tag{4.9}$$

Then using Ψ defined by Eq. (3.14) as the dependent variable and replacing approximately $(\gamma-1)M^2/2+1$ by $(\gamma-1)M^2/2$ in view of very large values of M, we have (see Appendix B)

$$\frac{\partial\Psi}{\partial y^*_i}\frac{\partial^2\Psi}{\partial x^*_i \partial y^*_i} - \frac{\partial\Psi}{\partial x^*_i}\frac{\partial^2\Psi}{\partial y^{*2}_i} = W\frac{dW}{dx^*_i} + \nu^*_\infty \frac{\partial^3\Psi}{\partial y^{*3}_i}, \tag{4.10}$$

where ν^*_∞ is the kinetic viscosity at infinity and

$$W = Up^{(1-\gamma)/2\gamma}. \tag{4.11}$$

The equation is quite the same as that for an incompressible fluid. Then the streamwise velocity u^*_i for the corresponding incompressible boundary layer is written from Eqs. (3.14) and (4.9) as

$$u^*_i \equiv \frac{\partial\Psi}{\partial y^*_i} = uUp^{(1-\gamma)/2\gamma} = uW. \tag{4.12}$$

In this section let us consider the zeroth-order approximation for the boundary-layer solution. It will be seen from the result that the zeroth-order shock wave is expressed in the form of Eq. (2.15) and $n=3/4$ is the only adequate selection that

makes the matching between the inviscid and boundary-layer solution consistent. So let us tentatively assume that the shock wave is given by Eq. (2.15) with $n=3/4$. Then the analysis in the preceding section is applicable because its validity was ensured when $1>n\geq 2/3$. Thus, in the zeroth-order approximation, the pressure along the edge of the boundary layer is given from Eq. (3.17) by

$$p^0=Px^{*-1/2}, \tag{4.13}$$

where

$$P=\frac{9\gamma}{8(\gamma+1)}\left[\frac{\gamma-1}{(1-2/3\gamma)(\gamma+1)}\right]^{\tau}A^{\tau-2/3}M^2\tau^2. \tag{4.14}$$

The superscript o denotes the value for the zeroth-order approximation. By the use of Eq. (4.13), Eq. (4.11) becomes

$$W^0=UP^{(1-\tau)/2\tau}x^{*(\tau-1)/4\tau}. \tag{4.15}$$

Thus, since the relation between x^* and x_i^* in the zeroth-order approximation is to the zeroth order given from Eqs. (4.8) and (4.13) by

$$x^*=\left(\frac{\gamma+1}{4\gamma}\right)^{\frac{4\tau}{\tau+1}}P^{\frac{2(1-3\tau)}{\tau+1}}x_i^{*\frac{4\tau}{\tau+1}}, \tag{4.16}$$

we have

$$W^0=Qx_i^{*\frac{\tau-1}{\tau+1}}, \tag{4.17}$$

where

$$Q=U\left(\frac{\gamma+1}{4\gamma}\right)^{\frac{\tau-1}{\tau+1}}P^{-\frac{2(\tau-1)}{\tau+1}}. \tag{4.18}$$

It has already been well known that there exists a similar solution for Eq. (4.10) when W is given by the power of x_i^* as in Eq. (4.17). In fact, introducing the transformation

$$\eta=y_i^*\sqrt{\frac{\gamma}{\gamma+1}\frac{Q}{\nu_\infty^*}}x_i^{*-1/(\tau+1)}, \tag{4.19}$$

$$\Psi=\sqrt{\frac{\gamma+1}{\gamma}\nu_\infty^*Q}\,x_i^{*\tau/(\tau+1)}f_0(\eta), \tag{4.20}$$

where $f_0(\eta)$ is a function only of η, we have the differential equation

$$f_0'''+f_0f_0''+\frac{\gamma-1}{\gamma}(1-f_0'^2)=0, \tag{4.21}$$

which is one of the family discussed by Falknar and Skan [*10*], [*11*].

To solve the above equation, we must settle the boundary conditions for f_0. Substituting Ψ from Eq. (4.20) into Eq. (4.12) and then using Eqs. (4.19) and (4.17), we obtain

$$u^0=\frac{u_i^{*0}}{W^0}=f_0'(\eta). \tag{4.22}$$

From Eqs. (4.20) and (4.22) we immediately find the condition at the wall,

$$f_0=f_0'=0 \quad \text{at} \quad \eta=0. \tag{4.23}$$

If the term T/M^2 may be assumed to negligibly small in Eq. (4.6) when applied

at the edge of the boundary layer, then we have there condition

$$u^0=1\,.$$

The justification for the above assumption will be provided in the subsequent section. Thus we obtain the external condition from Eq. (4.22) in the form

$$f_0'\to 1 \quad \text{as} \quad \eta\to\infty\,. \tag{4.24}$$

Since τ or P given by Eq. (4.14) is as yet retained as unknown, we must now proceed into the determination of these quantities. This is done by requiring for the normal velocity a smooth joining between the inviscid and boundary-layer solutions at the edge of the boundary layer. As shown in Appendix C, the relation of y^* to η in the boundary layer at a fixed value of x^* is given by

$$y^*=\frac{\gamma-1}{2}\sqrt{\frac{\gamma+1}{2}\frac{\nu_\infty^*}{Q}}p^{(1-3\tau)/2\tau}x_i^{*1/(\tau+1)}\int_0^\eta\left(1-\frac{u_i^{*2}}{W^2}\right)d\eta\,. \tag{4.25}$$

By the use of the zeroth-order values of p and u_i^*/W, this equation becomes

$$y^*=\frac{\gamma-1}{2}\sqrt{\frac{\gamma+1}{2}\frac{\nu_\infty^*}{Q}}p^{0(1-3\tau)/2\tau}x_i^{*1/(\tau+1)}\int_0^\eta(1-f_0'^2)d\eta\,. \tag{4.26}$$

When η is sufficiently large, the upper bound of the integral in the right-hand side of Eq. (4.26) may be extended to infinity without any significant error, for f_0' tends exponentially to unity with increasing η. This implies that, although the boundary layer spreads over the wide range in η-coordinate, it is confined in the actual plane to the region from the wall to the edge of the boundary layer whose zerothorder y-coordinate, δ^0, is given by

$$\delta^0=\frac{\gamma-1}{2}M^2p^{0(1-3\tau)/2\tau}\sqrt{\frac{\gamma+1}{\gamma}\frac{\nu_\infty^*}{Q}}x_i^{*1/(\tau+1)}\int_0^\eta(1-f_0'^2)d\eta\,.$$

Substituting p^0 and Q from Eqs. (4.13) and (4.18), respectively, and using Eq. (4.16), the above equation is rewritten as

$$\delta^0=(\gamma-1)M^2\left(\frac{\nu_\infty^*}{U}\right)^{1/2}P^{-1/2}I_0x^{*3/4}\,, \tag{4.27}$$

with the abbreviation of

$$I_0=\int_0^\infty(1-f_0'^2)d\eta\,. \tag{4.28}$$

Therefore, in the actual plane, we may consider the boundary layer in the zeroth-order approximation as a fictitious body given by Eq. (4.27). Since, according to the results in the previous section, v at a point on the body surface is identical with the body slope there, we immediately obtain from Eq. (4.27) the zeroth-order normal velocity on the body surface or edge of the boundary layer as follows:

$$\begin{aligned} v_\delta^0&=\frac{v_\delta^{*0}}{U}=\frac{d\delta^0}{dx^*}\\ &=\frac{3}{4}(\gamma-1)M^2\left(\frac{\nu_\infty^*}{U}\right)^{1/2}P^{-1/2}I_0x^{*-1/4}\,, \end{aligned} \tag{4.29}$$

where the subscript δ denotes the value taken at the edge of the boundary layer.

Thus, equating this value of v_δ^0 to that of v from Eq. (3.16), we find the unknown τ in the form

$$\tau=\tau_0\left(\frac{M}{\sqrt{U/\nu_\infty^*}}\right)^{1/2}, \tag{4.30}$$

where

$$\tau_0=(\gamma-1)^{1/2}\left[\frac{8(\gamma-1)}{9\gamma}\right]^{1/4}\left[\frac{\gamma-1}{(1-2/3\gamma)(\gamma+1)}\right]^{-\tau/4}A^{-\frac{1}{4}(\tau-\frac{2}{3})}I_0\,. \tag{4.31}$$

By the use of the above value of τ, Eq. (4.14) yields

$$P=p_0\frac{M^3}{\sqrt{U/\nu_\infty^*}}, \tag{4.32}$$

where

$$p_0=\frac{9\gamma}{8(\gamma+1)}\left[\frac{\gamma-1}{(1-2/3\gamma)(\gamma+1)}\right]^{\tau}A^{\tau-2/3}\tau_0^2\,. \tag{4.33}$$

Consequently we have from Eqs. (4.13), (4.27) and (4.29) with P given by Eq. (4.32)

$$\left.\begin{aligned} &p^0=p_0\frac{M^3}{\sqrt{R_{x^*}}}, \quad v_\delta^0=v_0\left(\frac{M}{\sqrt{R_{x^*}}}\right)^{1/2} \\ &\delta^0=\delta_0\left(\frac{M}{\sqrt{R_{x^*}}}\right)^{1/2}x^*, \end{aligned}\right\} \tag{4.34}$$

where

$$\left.\begin{aligned} &v_0=\frac{3}{4}\frac{(\gamma-1)}{\sqrt{p_0}}I_0, \\ &\delta_0=\frac{\gamma-1}{\sqrt{p_0}}I_0, \end{aligned}\right\} \tag{4.35}$$

Also we have from Eqs. (4.15) and (4.18)

$$W^0=\left(\frac{\gamma+1}{4\gamma}\right)^{\frac{\tau-1}{\tau+1}}U(p_0M^3/\sqrt{U/\nu_\infty^*})^{-\frac{2(\tau-1)}{\tau+1}}x_i^{*\frac{\tau-1}{\tau+1}}, \tag{4.36}$$

$$Q=\left(\frac{\gamma+1}{4\gamma}\right)^{\frac{\tau-1}{\tau+1}}U(p_0M^3/\sqrt{U/\nu_\infty^*})^{-\frac{2(\tau-1)}{\tau+1}}, \tag{4.37}$$

respectively.

From the above results it is evident that the initially assumed selection $n=3/4$ is the only appropriate one for the present problem. Thus the zeroth-order matching procedure concerning the pressure and normal velocity between the inviscid and boundary-layer solutions has completely been carried out. Since A and θ_0 are given as shown in Table 1 and the values of I_0 are

air: $I_0=1.310$, helium: $I_0=1.186$,

respectively, we can evaluate the values of τ_0, p_0, v_0 and δ_0 from Eq. (4.31), (4.33) and (4.35), respectively. These values are summarized in the following table:

TABLE 2.

	τ_0	p_0	v_0	δ_0
air	1.19	0.555	0.528	0.704
helium	1.61	1.03	0.578	0.770

The above result are, of course, in good agreement with these in reference [5], obtained by somewhat different but essentially same method.

5. THE FIRST-ORDER BEHAVIORS OF FLOW VARIABLES IN THE OUTER REGION OF THE BOUNDARY LAYER

Here it should be noted that the zeroth-order analysis in the preceding section has been settled so as to ensure the smooth joining at the edge of the boundary layer concerning only the pressure and normal velocity and, hence, regardless of the joining concerning the temperature and density. This is due to the assumption that in the application of the energy equation (4.6) to the edge of the boundary layer the term T/M^2 is ignored compared with unity. To be sure this is the case for the hypersonic inviscid region not very near the boundary layer. For the vicinity of the edge of the boundary layer, however, this point must be examined more carefully because the temperature develops a singurality there, as seen in section 3.

Representing Ψ^0 from Eq. (4.20) in terms of x^* by the use of Eq. (4.16) with the form of P given by Eq. (4.32), we have

$$\Psi^0 = 2\sqrt{\nu_\infty^* U p^0 x^*}\, f_0(\eta)\,; \tag{5.1}$$

and substituting p^0 from Eqs. (4.33) we have

$$\Psi^0 = 2\, U p_0^{1/2} x^* (M/\sqrt{R_{x^*}})^{3/2} f_0(\eta)\,. \tag{5.2}$$

In view of the meaning of a stream function Ψ, $f_0(\eta)$ must be bounded everywhere in a boundary layer up to the edge. Therefore the value of Ψ^0 is of the order of $(M/\sqrt{R_{x^*}})^{3/2}$ which is very small. Applying this form of Ψ^0 to Eq. (3.18), we obtain the density at the outer region of the boundary layer in the form

$$\rho = (p_0/D) f_0^{2/3\tau} (M/\sqrt{R_{x^*}})^{2/3\tau}\,, \tag{5.3}$$

where

$$D = 2^{-2/3\tau}(9\gamma/8)^{1/\tau}(\gamma-1)(\gamma+1)^{-(\tau+1)/\tau}\tau_0^{8/3\tau} p_0^{1-4/3\tau}\,. \tag{5.4}$$

Hence the corresponding temperature is

$$T = D f_0^{-2/3\tau} M^2 (M/\sqrt{R_{x^*}})^{1-2/3\tau}\,. \tag{5.5}$$

It is seen from Eq. (5.5) that T/M^2 is of the order of $(M/\sqrt{R_{x^*}})^{1-2/3\tau}$ at the outer region of the boundary layer. This order is much larger than that of v_δ^2 or, from Eqs. (4.34), of $M/\sqrt{R_{x^*}}$, because $5/3 \geq \gamma \geq 1$ for all real gases, but evidently much smaller when compared with unity. Substituting further the above form of T into Eq. (4.6) and discarding the term $1/M^2$, we find the streamwise velocity, u_δ, at the outer region of the boundary layer in the form

$$u_\delta = 1 - \frac{D}{\gamma-1} f_0^{-2/3\Upsilon} \left(\frac{M}{\sqrt{R_{x^*}}} \right)^{1-2/3\Upsilon} . \tag{5.6}$$

Now let us assume the expansion of the form

$$\Psi = \sqrt{\frac{\gamma+1}{\gamma}} \nu_\infty^* Q \, x_i^{*\Upsilon/(\Upsilon+1)} [f_0(\eta) + \epsilon f_1(\eta) + \cdots] , \tag{5.7}$$

$$p = p^0 [1 + \epsilon p_1 + \cdots] , \tag{5.8}$$

where p^0 and Q are given by Eqs. (4.34) and (4.37), respectively, p_1 is the unknown constant to be determined and ϵ is a small parameter depending only upon x^*. Then it is suggested from the form of Eq. (5.6) that the appropriate selection of ϵ is

$$\epsilon = \left(\frac{M}{\sqrt{R_{x^*}}} \right)^{1-2/3\Upsilon} . \tag{5.9}$$

In fact it will be seen later that this form of ϵ is the only possible selection consistent with both boundary-layer equation and boundary conditions. If ϵ is assumed to be given by Eq. (5.9), the substitution of p from Eq. (5.8) into Eq. (4.8) leads after some calculations to

$$x^* = \left(\frac{\gamma+1}{4\gamma} \right)^{\frac{4\Upsilon}{\Upsilon+1}} \left(\frac{p_0 M^3}{\sqrt{U/\nu_\infty^*}} \right)^{\frac{2(1-3\Upsilon)}{\Upsilon-1}} x_i^{*\frac{4\Upsilon}{\Upsilon+1}} \left[1 + \epsilon p_1 \frac{6(1-3\gamma)}{7-3\gamma} + \cdots \right] . \tag{5.10}$$

With this result further substitution of p into Eq. (4.11) yields

$$W = U \left(\frac{\gamma-1}{4\gamma} \right)^{\frac{\Upsilon-1}{\Upsilon+1}} \left(\frac{p_0 M^3}{\sqrt{U/\nu_\infty^*}} \right)^{-\frac{2(\Upsilon-1)}{\Upsilon+1}} x_i^{*\frac{\Upsilon-1}{\Upsilon+1}} \left[1 - \epsilon p_1 \frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)} + \cdots \right] , \tag{5.11}$$

or in accordance with Eq. (4.36)

$$W = W^0 \left[1 - \epsilon p_1 \frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)} + \cdots \right] . \tag{5.12}$$

Thus substituting Ψ, p and W from Eqs. (5.7), (5.8) and (5.12), respectively, into Eq. (4.10) and equating coefficients of ϵ on the both sides of the equation, we obtain after some calculations the differential equation

$$f_1''' + f_0 f_1'' + \frac{2}{3\gamma} f_0' f_1' - \left(1 - \frac{4}{3\gamma} \right) f_0'' f_1 = - \frac{2(\gamma-1)(3\gamma+2)}{3\gamma^2(7-3\gamma)} p_1 . \tag{5.13}$$

This is the first-order boundary-layer equation. Here it should be remembered that p_1 is as yet unknown.

Next let us consider the boundary conditions to be imposed on f_1. We have from Eq. (4.12)

$$u = \frac{u_i^*}{W} = \frac{1}{W} \frac{\partial \Psi}{\partial y_i^*} ,$$

and from Eqs. (4.19) and (5.7)

$$\partial \Psi / \partial y_i^* = W^0 [f_0' + \epsilon f_1' + \cdots] . \tag{5.14}$$

Therefore we find

$$u = \frac{u_i^*}{W} = f_0' + \epsilon \left[f_1' + p_1 \frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)} \right] + \cdots . \tag{5.15}$$

Applying this equation to the outer region of the boundary layer, we have

$$u \simeq 1+\epsilon\left[f_1'+p_1\frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)}\right]+\cdots, \tag{5.16}$$

because $f_0' \simeq 1$ there. Comparing this equation to Eq. (5.6), we can see that the initially assumed selection (5.9) for ϵ is the only appropriate one for the present problem. We obtain moreover

$$f_1'+p_1\frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)}=-\frac{D}{\gamma-1}f_0^{-2/3\gamma}. \tag{5.17}$$

As shown in Appendix D, the selection (5.9) for ϵ is found to be consistent with the condition concerning the vorticity occurring in the outer region of the boundary layer. Then we have

$$f_1'' \simeq \frac{2Df_0^{-1-2/3\gamma}}{3\gamma(\gamma-1)}. \tag{5.18}$$

As naturally expected, f_1'' given by Eq. (5.18) is directly obtained by differentiating f_1' given by Eq. (5.17) with respect to η, because $f_0(\eta)$ increases linearly with increasing η when it is large. Thus the conditions to be imposed on the function f_1 at the outer region of the boundary layer have been found to be given by Eqs. (5.17) and (5.18) in which the latter is a supplementary one. On the other hand the conditions to be imposed on f_1 at the wall are readily found from Eqs. (5.7) and (5.15), respectively, as

$$f_1=0, \quad f_1'=-\frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)}p_1 \quad \text{at} \quad \eta=0. \tag{5.19}$$

Before we proceed into the solution of Eq. (5.13), let us examine the asymptotic behavior of f_1 when η is large. In the region where η is large, Eq. (5.13) takes the asymptotic form

$$f_1'''+(\eta-\beta)f_1''+\frac{2}{3\gamma}f_1'=-\frac{2(\gamma-1)(3\gamma+2)}{3\gamma^2(7-3\gamma)}p_1, \tag{5.20}$$

because $f_0'' \simeq 0$, $f_0' \simeq 1$ and $f_0 \simeq \eta-\beta$ where β is a constant known from the asymptotic behavior of f_0. Then the general solution for the above equation is expressed in the form

$$\begin{aligned} f_1'=(\eta-\beta)e^{-(\eta-\beta)^2/2}\Big\{&a_1\,{}_1F_1\left[1-\frac{1}{3\gamma},\frac{3}{2};\frac{1}{2}(\eta-\beta)^2\right] \\ &+a_2(\eta-\beta)^{-1}\,{}_1F_1\left[\frac{1}{2}-\frac{1}{3\gamma},\frac{1}{2};\frac{1}{2}(\eta-\beta)^2\right]\Big\}-\frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)}p_1, \end{aligned} \tag{5.21}$$

where ${}_1F_1$ represents a confluent hypergeometric function, and a_1 and a_2 are certain constants. By making use of the asymptotic expression of the confluent hypergeometric function

$${}_1F_1(\alpha,\beta;\sigma) \simeq \Gamma(\beta)/\Gamma(\alpha)\cdot e^{\sigma}\sigma^{\alpha-\beta},$$

Eq. (5.21) can be reduced for large values of η to the form

$$f_1' \simeq a(\eta-\beta)^{-2/3\gamma}-\frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)}p_1, \tag{5.22}$$

where a is a certain constant, or, since $f_0 \simeq \eta-\beta$,

$$f_1' \simeq a f_0^{-2/3\gamma} - \frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)} p_1 . \tag{5.23}$$

On comparing this asymptotic solution with Eq. (5.17), we can certainly expect the joining of the boundary-layer solution to inviscid solution provided the constant a is selected as

$$a = -\frac{D}{\gamma-1} . \tag{5.24}$$

Then, since f_1'' from Eq. (5.23) is identical with f_1'' given by Eq. (5.18), the respective values of f_1' and f_1'' given by the inviscid and boundary-layer solutions are found to be overlapped each other concerning their most predominant terms. Therefore the condition to be imposed on f_1 at the large values of η may be specified by Eqs. (5.22) or (5.23) with a given by Eq. (5.24). Evidently the fulfillment of this condition ensures the smooth joining between the inviscid and boundary-layer solutions concerning the temperature (or streamwise velocity) as well as vorticity.

6. The Approximate Approach to the First-Order Boundary-Laler Solution

In the preceding section we have derived the first-order boundary-layer equation (5.13) with the boundary conditions (5.19) and (5.23). However the constant p_1 involved in the equation is as yet retained as unknown. This constant is to be determined so as to fulfill the requirement of the smooth joining between the inviscid and boundary-layer solutions concerning the first-order induced pressure.

Before making the determination of p_1 the approximate approach to the first-order boundary-layer solution is first considered. The substitution of f_1' from Eq. (5.23) into Eq. (5.20) leads to the argument that $f_1''' \simeq 0$ at the overlapping region. With the fact in mind that f_1''' represents the viscous effect, this means that the flow in the overlapping region is substantially inviscid. Therefore the boundary layer may be considered to be ranged from 0 to a certain large value of η, say η_δ, in η-coordinate and joined thereafter to the overlapping region. Then we may adopt the following boundary conditions as the approximate conditions to be imposed on f_1:

$$f_1' = -\frac{D}{\gamma-1} f_0^{-2/3\gamma} - \frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)} p_1 ,$$
$$f_1''' = 0 \quad \text{at} \quad \eta = \eta_\delta , \tag{6.1}$$

where η_δ is as yet unknown. Suppose that we can determine the solution of Eq. (5.13) satisfying the boundary conditions (5.19) and (6.1) in such a way that the value of η_δ is large enough to be regarded as that of the edge of the boundary layer.

It is now convenient to introduce the function $F(\eta)$ defined by

$$F(\eta) = \frac{[f_0(\eta_\delta)]^{2/3\gamma}}{D} \left[f_1(\eta) + \frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)} p_1 \eta \right] . \tag{6.2}$$

Then Eq. (5.13) and the boundary conditions (5.19) and (6.1) are expressed in terms

of F as

$$F'''+f_0F''+\frac{2}{3\gamma}f_0'F'-\left(1-\frac{4}{3\gamma}\right)f_0''F=-\frac{2}{3\gamma}L\left[1-f_0'-2\left(1-\frac{3\gamma}{4}\right)f_0''\eta\right], \tag{6.3}$$

$$F=F'=0 \quad \text{at} \quad \eta=0, \tag{6.4}$$

$$F'=-\frac{1}{\gamma-1}, \quad F'''=0 \quad \text{at } \eta=\eta_\delta, \tag{6.5}$$

with the abbreviation of

$$L=\frac{[f_0(\eta_\delta)]^{2/3\gamma}}{D}\frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)}p_1. \tag{6.6}$$

Thus, if we know the relation of p_1 to η_δ, then η_δ will be the single unknown quantity involved in the present problem. Then, since the four boundary conditions are imposed on the solution of the third-order differential equation (6.3), it becomes possible to determine the unknown η_δ and the solution simultaneously.

As shown in Appendix C, the relation of y^* to η in the boundary layer at a fixed value of x^* is given by Eq. (4.25). Substituting p and u_i^*/W from Eqs. (5.8) and (5.15), respectively, into Eq. (4.25), we obtain in the first-order approximation

$$y^*=\frac{(\gamma-1)\sqrt{M}x^*}{\sqrt{p_0}(R_{x^*})^{1/4}}\left[1+\epsilon p_1\frac{1-3\gamma}{2\gamma}\right]\cdot$$
$$\int_0^\eta\left[(1-f_0'^2)-2\epsilon f_0'\left\{f_1'+\frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)}p_1\right\}\right]d\eta. \tag{6.7}$$

For sufficiently large values of η, since

$$\int_0^\eta(1-f_0'^2)d\eta\simeq\int_0^\infty(1-f_0'^2)d\eta\equiv I_0,$$

this equation becomes

$$y^*\simeq\frac{(\gamma-1)\sqrt{M}x^*}{\sqrt{p_0}(R_{x^*})^{1/4}}I_0\left[1+\epsilon\left(I_1+\frac{1-3\gamma}{2\gamma}p_1\right)\right], \tag{6.8}$$

where

$$I_1=-\frac{2}{I_0}\int_0^\eta f_0'\left[f_1'+\frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)}p_1\right]d\eta.$$

Thus we obtain the value of y^* corresponding to η_δ

$$\delta^1=\delta^0\left[1+\epsilon\left(I_1|_{\eta=\eta_\delta}+\frac{1-3\gamma}{2\gamma}p_1\right)\right],$$

where $I_1|_{\eta=\eta_\delta}$ denotes the value of I_1 at $\eta=\eta_\delta$, namely,

$$I_1|_{\eta=\eta_\delta}=-\frac{2}{I_0}\int_0^{\eta_\delta}f_0'\left[f_1'+\frac{(\gamma-1)(3\gamma+2)}{\gamma(7-3\gamma)}p_1\right]d\eta$$

or, by the use of Eq. (6.2),

$$I_1|_{\eta=\eta_\delta}=-\frac{2}{I_0}\frac{D}{[f_0(\eta_\delta)]^{2/3\gamma}}\int_0^{\eta_\delta}f_0'F'd\eta. \tag{6.9}$$

Also, with the abbreviation of

$$\delta_1 = I_1|_{\eta=\eta_\delta} + \frac{1-3\gamma}{2\gamma} p_1 , \tag{6.10}$$

δ^1 is given by

$$\delta^1 = \delta^0[1+\epsilon\delta_1] . \tag{6.11}$$

As seen from the meaning of η_δ previously mentioned, δ^1 may be regarded as the boundary-layer thickness in the first-order approximation. Hence, remembering $\delta^0 \sim x^{*3/4}$ and $\epsilon \sim x^{*-\frac{1}{2}(1-\frac{2}{3\gamma})}$, we obtain from Eq. (6.11) the slope of the edge of the boundary layer

$$\frac{d\delta^1}{dx^*} = \frac{d\delta^0}{dx^*}\left[1+\epsilon\left(\frac{1}{3}+\frac{4}{9\gamma}\right)\delta_1\right] . \tag{6.12}$$

Now, in accordance with the usual hypersonic boundary-layer consideration, the slope of the edge of the boundary layer may be assumed to be identical with the flow inclination. Then the first-order change in pressure induced by the first-order change in slope of the edge of the boundary layer can be found by means of the tangent-wedge approximation [*12*], namely,

$$p_1 = 2\left(\frac{1}{3}+\frac{4}{9\gamma}\right)\delta_1$$

or

$$p_1 = 2\left(\frac{1}{3}+\frac{4}{9\gamma}\right)\left(I_1|_{\eta=\eta_\delta} + \frac{1-3\gamma}{2\gamma}p_1\right) . \tag{6.13}$$

Hence, by the use of Eq. (6.10) and (6.9), we obtain

$$p_1 = -\frac{4\gamma(3\gamma+4)}{18\gamma^2+9\gamma-4}\frac{D}{[f_0(\eta_\delta)]^{2/3\gamma}I_0}\int_0^{\eta_\delta} f_0'F'd\eta . \tag{6.14}$$

The smooth joining of the first-order pressure at $\eta=\eta_\delta$ is to be fulfilled only when the above relation is valid and thus the condition must be imposed on the function F in order to settle the first-order problem completely. Then L defined by Eq. (6.6) becomes

$$L = -\frac{4(\gamma-1)(3\gamma+2)(3\gamma+4)}{(7-3\gamma)(18\gamma^2+9\gamma-4)}\frac{1}{I_0}\int_0^{\eta_\delta} f_0'F'd\eta . \tag{6.15}$$

Now η_δ is the single unknown constant involved in the first-order boundary-layer problem. Therefore it becomes possible to determine the solution and the unknown constant η_δ on the basis of Eqs. (6.3), (6.4), (6.5) and (6.15).

7. The Numerical Results for the First-Order Boundary-Layer Problem

Before proceeding to the actual determination of the boundary-layer problem described in the preceding section, we consider first the simplification made by ignoring the effect of the first-order change in induced pressure on the boundary layer. Then, since $p_1=0$ and, from Eq. (6.6), $L=0$, Eq. (6.3) reduces simply to

$$F'''+f_0F''+\frac{2}{3\gamma}f_0'F'-\left(1-\frac{4}{3\gamma}\right)f_0''F=0. \tag{7.1}$$

If the value of F'' is prescribed at the wall, the solution of Eq. (7.1) can uniquely be evaluated with the condition at the wall, Eq. (6.4). Thus we obtain a family of solutions satisfying only the conditions (6.4), with $F''(0)$ as a parameter. In this way the equation (7.1) has numerically been integrated for both cases of air and helium. The values of F', F'' and F''' are shown in Figs. 3 and 4 for air and

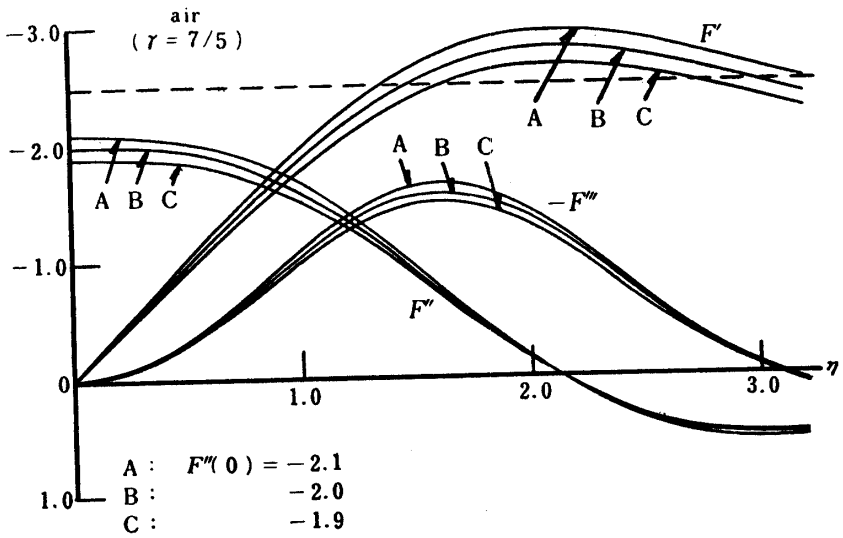

FIGURE 3.

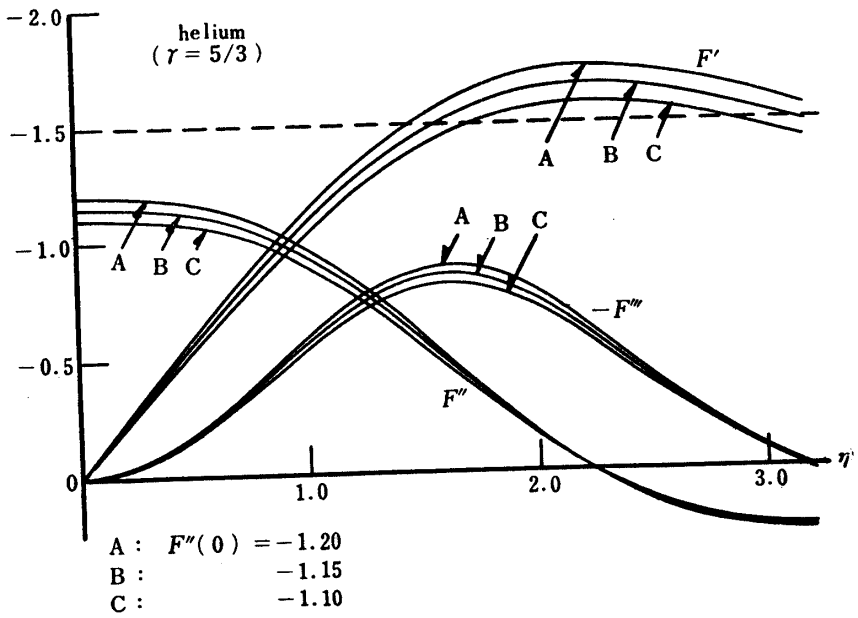

FIGURE 4.

helium, respectively. As seen from these results, the location in which F''' vanishes is comparatively insensitive to the variation of $F''(0)$ although the corresponding value of F' is rather sensitive to it. These circumstances make the determination

of η_δ easy, thus considerably reducing the amount of labor in numerical calculation. As a matter of fact, the final solution satisfying the boundary conditions (6.4) and (6.5) can be determined by interpolating between the two integral curves, for the one of which the value of F' is slightly smaller, and for the other of which the value of F' is slightly larger than $-1/(\gamma-1)$ at the location where F''' vanishes. The values of η_δ and $F''(0)$ thus determined are

$$\begin{array}{lll} \text{air:} & \eta_\delta=3.09, & F''(0)=-2.04, \\ \text{helium:} & \eta_\delta=3.20, & F''(0)=-1.15. \end{array}$$

Using the solution of Eq. (4.21) integrated by Hartree [*11*], the values of f_0 at $\eta=\eta_\delta$ are

$$\text{air:}\quad f_0=2.17, \qquad \text{helium:}\quad f_0=2.36.$$

Also using the zeroth-order values of τ_0 and p_0 shown in Table 2, the values of D defined by Eq. (5.4) are

$$\text{air:}\quad D=0.120, \qquad \text{helium:}\quad D=0.330.$$

Thus all the numerical constants needed for the first-order approximation have been evaluated for both cases of air and helium.

By the use of Eq. (6.9) with the above obtained values of F' and numerical constants, we obtain

$$\text{air:}\quad I_1|_{\eta=\eta_\delta}=0.673, \qquad \text{helium:}\quad I_1|_{\eta=\eta_\delta}=1.32.$$

Under the present simplification, Eq. (6.13) becomes

$$p_1=2\left(\frac{1}{3}+\frac{4}{9\gamma}\right)I_1|_{\eta=\eta_\delta}.$$

By the use of this equation we obtain

$$\text{air:}\quad p_1=0.876, \qquad \text{helium:}\quad p_1=1.58.$$

Consequently the estimate for the pressure along the wall surface is given by

$$\left.\begin{array}{l} \text{air:}\quad p=0.555\dfrac{M^3}{\sqrt{R_{x^*}}}\left[1+0.876\left(\dfrac{M}{\sqrt{R_{x^*}}}\right)^{0.824}\right], \\ \text{helium:}\ p=1.03\dfrac{M^3}{\sqrt{R_{x^*}}}\left[1+1.58\left(\dfrac{M}{\sqrt{R_{x^*}}}\right)^{0.60}\right]. \end{array}\right\} \tag{7.2}$$

Now we proceed to the straightforward solution of the present boundary-layer problem. The calculation may conveniently be carried out by the iterative scheme and by starting with a suitably assumed value for the constants L on the right-hand side of Eq. (6.3). The solution of Eq. (6.3) satisfying both conditions (6.4) and (6.5) can be determined in quite the same fashion as for the previous simplified case. It has been found also in the present case that the location $F'''=0$ is almost insensitive to the variation of the initial value $F''(0)$. After the solution and the value of η_δ have heen determined, we can evaluate by Eq. (6.15) the value of L, which, however, will in general be different from the initially assumed one. The solution can be adopted as consistent only when the initially assumed and finally calculated values of L are found in agreement.

Following the above mentioned scheme, the actual calculations were carried out

for both cases of air ($\gamma=7/5$) and helium ($\gamma=5/3$). Starting with a value of L guessed from the results of the above mentioned simplified case, we arrived at the consistent value of L within the accuracy of 1% after three or four iterations. The final values of numerical constants are summarized in the following table.

TABLE 3.

	L	η_δ	$F''(0)$
air	2.8	3.30	−1.14
helium	4.0	3.64	0.149

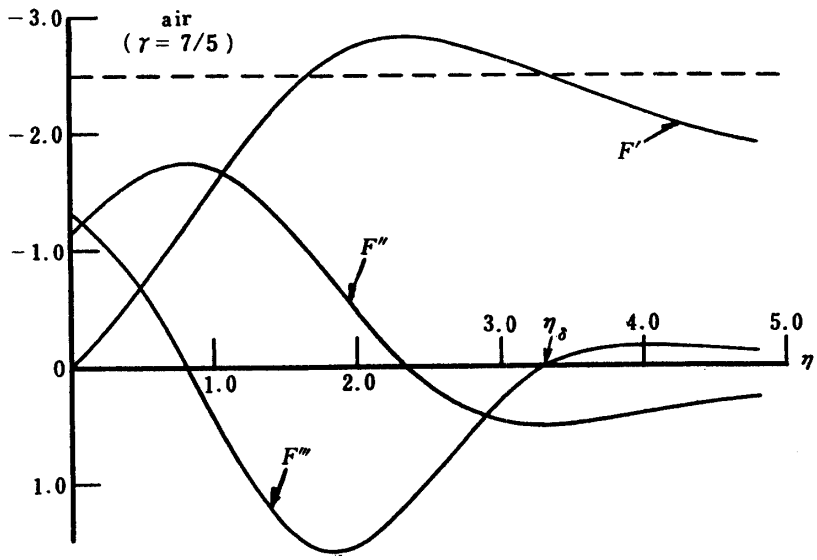

FIGURE 5.

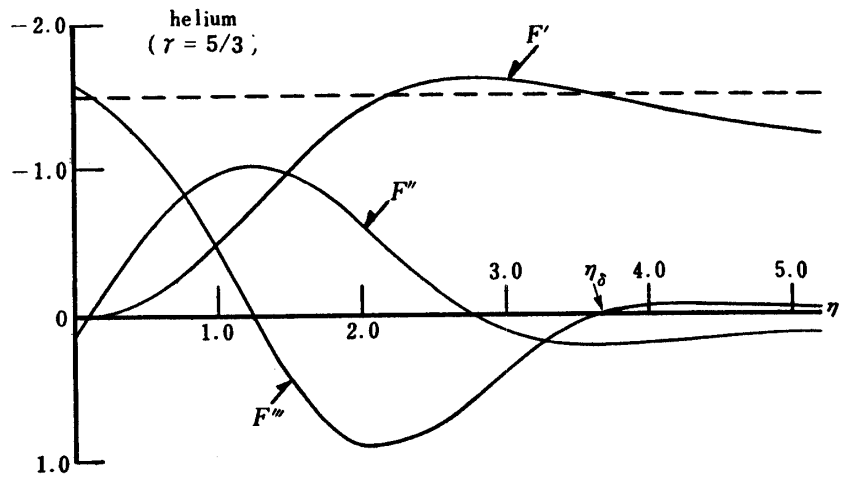

FIGURE 6.

The solution F' and its derivatives are shown for the cases of air and helium in Figs. 5 and 6, respectively. By the use of Eq. (6.6) with the values of L and η_δ above evaluated, we obtain

$$\text{air:}\quad p_1=0.35, \qquad \text{helium:}\quad p_1=0.63.$$

Thus the pressure on the surface, given by Eq. (5.8), is

$$\left.\begin{aligned} &\text{air:} \quad p=0.555\frac{M^3}{\sqrt{R_{x^*}}}\left[1+0.35\left(\frac{M}{\sqrt{R_{x^*}}}\right)^{0.824}\right], \\ &\text{helium:} \quad p=1.03\frac{M^3}{\sqrt{R_{x^*}}}\left[1+0.63\left(\frac{M}{\sqrt{R_{x^*}}}\right)^{0.60}\right], \end{aligned}\right\} \tag{7.3}$$

correct to the first order.

Following Eqs. D-1 and D-2 in Appendix D, we obtain the shear stress

$$\mu^*\frac{\partial u^*}{\partial y^*}=(f_1''+\epsilon f_0'')\frac{\mu_\infty^*}{2}\frac{\mu U}{T}\sqrt{\frac{Up}{\nu_\infty^* x^*}}\,.$$

By the use of the linear viscosity-temperature relation, the above equation becomes

$$\mu^*\frac{\partial u^*}{\partial y^*}=(f_1''+\epsilon f_0'')\frac{\mu_\infty^* U}{2}\sqrt{\frac{Up}{\nu_\infty^* x^*}}\,.$$

Hence we have the following expression for the local skin-friction coeficient

$$\begin{aligned} C_f &\equiv \mu^*\frac{\partial u^*}{\partial y^*}\Big/\frac{1}{2}\rho_\infty^* U^2 \\ &=\sqrt{\frac{p}{R_{x^*}}}(f_0''+\epsilon f_1'')\,. \end{aligned}$$

The substitution of p from Eq. (5.8) into the above expression yields

$$C_f=p_0^{1/2}\left(\frac{M}{\sqrt{R_{x^*}}}\right)^{3/2}\left[f_0''+\left(f_1''+\frac{1}{2}p_1 f_0''\right)\left(\frac{M}{\sqrt{R_{x^*}}}\right)^{1-2/3\gamma}\right]$$

which is correct to the first order. The local skin-friction coeficient at the wall, $(C_f)_w$, has been evaluated as follows:

$$\begin{aligned} &\text{air:} \quad (C_f)_w=\left(\frac{M}{\sqrt{R_{x^*}}}\right)^{3/2}\left[0.568+0.0323\left(\frac{M}{\sqrt{R_{x^*}}}\right)^{0.824}\right], \\ &\text{helium:} \quad (C_f)_w=\left(\frac{M}{\sqrt{R_{x^*}}}\right)^{3/2}\left[0.867+0.306\left(\frac{M}{\sqrt{R_{x^*}}}\right)^{0.60}\right]. \end{aligned}$$

As seen from Figs. 5 and 6, the magnitude of F''' is much smaller compared with those of F'' and F' in the region where $\eta\geq\eta_\delta$. This implies that in this region the effect of deviation of streamwise velocity from the uniform value and the effect of vorticity (which are due to the magnitudes of F' and F'', respectively) play much more significant role compared with the viscous effect (which is due to the magnitude of F'''). As a matter of fact, in the region $\eta\geq\eta_\delta$, F' can be confirmed to be numerically in good agreement with its asymptotic solution derived from Eqs. (5.23) and (6.2), namely,

$$F'\simeq-\frac{1}{\gamma-1}\left[\frac{f_0(\eta_\delta)}{f_0(\eta)}\right]^{2/3\gamma}.$$

Moreover, the evaluated value of η_δ is found large enough to be regarded as the edge of the boundary layer. This can also be confirmed from the comparison with the value of $\eta_{\delta_{K-P}}$, the zeroth-order boundary-layer thickness in Kármán-Pohlhausen sense, which is easily obtained from the well known method as follows:

$$\text{air:} \quad \eta_{\delta_{K-P}}=3.38, \qquad \text{helium:} \quad \eta_{\delta_{K-P}}=3.20.$$

This argument seems to provide a justification for the procedure applied in the present analysis.

Concluding Remarks and Discussion

To summarize, it is concluded that the boundary-layer solution may be joined up to the inviscid solution, at least to the order of $(M/\sqrt{R_{x^*}})^{1-2/3\tau}$, with respect to flow variables including temperature (or streamwise velocity) and vorticity. Then the region in which the viscous effect plays a significant role has been found to be ranged over from 0 to η_δ in terms of the non-dimensional variable η. In the region for which $\eta \geq \eta_\delta$, the main part of the boundary-layer solution has been shown to overlap on that of the inviscid solution, the solution being essentially subjected to the inviscid one.

Here a note should be added concerning the viscosity-temperature relation. If we assume, instead of the form of Eq. (4.7), the form proposed by Chapmann and Rubesin [*13*]

$$\mu^* = C\mu^*_\infty \frac{T^*}{T^*_\infty},$$

where $C=(\mu^*_w/\mu^*_\infty)/(T^*_w/T^*_\infty)$ and μ^*_w and T^*_w denote the values of the viscosity coeficient and temperature at the wall, respectively, we should replace μ^* by μ^*/C throughout the present analysis.

Numerical calculations have been carried out for both cases of air and helium, first, under the simplification of ignoring the effect of the first-order pressure on the boundary layer. Such a simplification results in the maximum estimate concerning the first-order change in pressure exerting a favorable effect on the boundary-layer growth.

An estimate for the induced pressure on the wall obtained by Lees [*7*] under the same simplification is

$$p=0.92\frac{\sqrt{C}M^3}{\sqrt{R_{x^*}}}\left[1+1.7\left(\frac{\sqrt{C}M}{\sqrt{R_{x^*}}}\right)^{0.6}\right]$$

for the case of helium. In spite of his rough treatment without surveying for the first-order boundary-layer equation, the value of $p_1=1.7$ is surprisingly good when compared to the present one, $p_1=1.58$.

The calculation has been effected without any simplification for both cases of air and helium. As seen from Eq. (7.3), the first-order change in induced pressure along the wall is considerably smaller compared with the corresponding maximum estimate. As regards the skin friction on the wall, the first-order change is found to increase its magnitude as in the induced pressure. These first-order changes of the order of $(M/\sqrt{R_{x^*}})^{1-2/3\tau}$ are appreciable at large value of $M^3/\sqrt{R_{x^*}}$.

At present there are only a few experimental data available for the large value of $M^3/\sqrt{R_{x^*}}$. The pressure distribution on the flat plate placed in helium flow was measured by Hammitt and Bodgonoff [*14*] up to Mach number 14.3 and, recently,

by Erickson [*15*] up to 18. Erickson also indicated from the measurements of temperature recovery that essentially insulated conditions existed on the model during the tests in helium tunnel. Such data may be compared with the present theoretical results. The induced pressure for helium calculated from Eq. (7.3) is shown in Fig. 7 together with the experimental results obtained by Hammitt and Bogdonoff

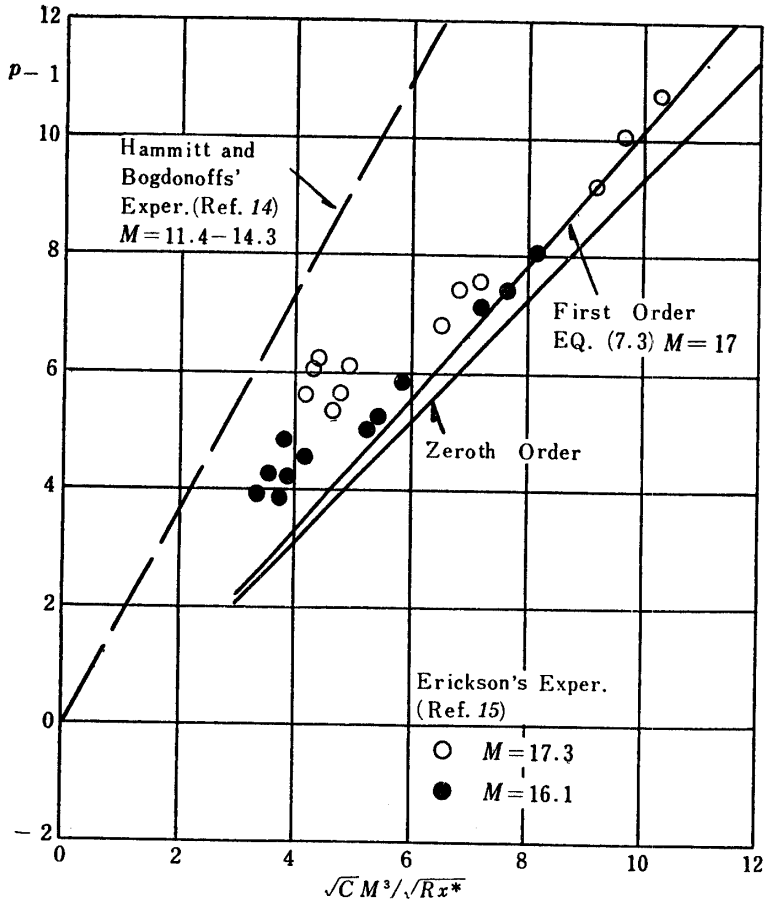

FIGURE 7.

and by Erickson. The theoretical prediction is in fairy good agreement with Erickson's result so that the pertinent flow phenomenon can be seen to be accounted for by the present analysis without necessity of any complementary hypothesis. Nevertheless there is a considerable difference between the results of the two experimants, no explanation being available at present. Further experimantal investigations are urgently needed.

ACKNOWLEDGEMENT

The author wishes to express his sincere thanks to Prof. Itiro Tani for his valuable discussions and kind advices, and also to Prof. Ryuma Kawamura for his helpful suggestions and continual encouragement throughout this work.

Department of Aerodynamics
Aeronautical Research Institute
University of Tokyo, Tokyo, May 21, 1958

References

[1] Bertram, M. H.: An Approximate Method for Determining the Displacement Effects and Viscous Drag of Laminar Boundary Layers in Two-Dimensional Hypersonic Flow, NACA TN 2773, 1952.

[2] Shen, S. F.: An Estimate of Viscosity Effect on the Hypersonic Flow Over an Insulated Wedge. Jour. Math. Phys., Vol. 31, No. 3, p 192, 1952.

[3] Nagamatsu, H. T. and Li, T. Y.: Shock Wave Effects on the Laminar Skin Friction of an Insulated Flat Plate at Hypersonic Speeds. Jour. Aero. Sci., Vol. 20, No. 5, p 345, 1953.

[4] Lees, L.: On the Boundary-Layer Equations in Hypersonic Flow and Their Approximate Solutions. Jour. Aero. Sci., Vol. 20, No. 2, p 143, 1953.

[5] Stewartson, K.: On the Motion of a Flat Plate at High Speeds in a Viscous Compressible Fluid-II. Steady Motion. Jour. Aero. Sci., Vol. 22, No. 5, p 303, 1955.

[6] Kuo, Y. H.: Viscous Flow Along a Flat Plate Moving at High Supersonic Speeds. Jour. Aero. Sci., Vol. 23, No. 2, p 125, 1956.

[7] Lees, L.: Influence of the Leading-Edge Shock Waves on the Laminar Boundary Layer at Hypersonic Speeds. Jour. Aero. Sci., Vol. 23, No. 6, p 594, 1956.

[8] Van Dyke, M. D.: A Study of Hypersonic Small-Disturbance Theory. NACA TN 3173, 1954.

[9] Stewartson, K.: Correlated Incompressible and Compressible Boundary Layers. Proc. Roy. Soc., A, Vol. 200, No. A1060, p 84, 1949.

[10] Falknar, V. N. and Skan, S. W.: Some Approximate Solutions of the Boundary Layer Equations, Phil. Mag., Vol. 12, p 865, 1931.

[11] Hartree, D. R.: On an Equation Occuring in Falknar and Skan's Approximate Treatment of the Equations of the Boundary Layer. Proc. Cambr. Phil. Soc., Vol. 33, Part II, p 223, 1937.

[12] Lees, L.: Hypersonic Flow. Pro. the Fifth Inter. Conf. p 241, 1955.

[13] Chapman, E. R. and Rubesin, M. W.: Temperature and Velocity Profile in Compressible Laminar Boundary Layer with Arbitrary Distribution of Surface Temperature. Jour. Aero. Sci., Vol. 16, No. 9, p 547, 1949.

[14] Hammitt, A. G. and Bogdonoff, S. M.: Hypersonic Studies of the Leading Edge Effect on the Flow Over a Flat Plate. Jet Propulsion, Vol. 26, No. 4, p 241, 1956.

[15] Erickson, W. D.: Study of Pressure Distributions on Simple Sharp-Nosed Models at Mach Numbers from 16 to 18 in Helium Flow. NACA TN 4113, 1957.

[16] Lees, L. and Kubota, T.: Inviscid Hypersonic Flow Over Blunt-Nosed Slender Bodies. Jour. Aero. Sci., Vol. 24, No. 3, p 195, 1957.

Appendix A

The inviscid solution for the case of axially symmetric flow

Similar treatment can be applied to the axially symmetric flow, in which the shock wave is given by the power form, namely,

$$r^* = \tau x^{*n},$$

where the cylindrical coordinates (x^*, r^*) are used instead of the Cartesian coordinates (x^*, y^*) for the case of plane flow. The barred quantities are introduced by Eq. (2.5) in which the symbol r is used instead of the symbol y. Let us introduce the reduced stream function $\overline{\Psi}$ defined by

$$\partial\bar{\Psi}/\partial\bar{r}=\bar{r}\bar{\rho}\,,\quad \partial\bar{\Psi}/\partial\bar{x}=-\bar{r}\bar{\rho}\bar{v}\,,$$

then, according to reference [8], the basic equation is given by

$$\bar{\Psi}_{\bar{r}}^2\bar{\Psi}_{\bar{x}\bar{x}}-2\bar{\Psi}_{\bar{x}}\bar{\Psi}_{\bar{r}}\bar{\Psi}_{\bar{x}\bar{r}}+\bar{\Psi}_{\bar{x}}^2\bar{\Psi}_{\bar{r}\bar{r}}=\frac{\bar{\Psi}_{\bar{r}}^{\gamma+1}}{\bar{r}^{\gamma-1}}\left[\gamma\omega\left(\bar{\Psi}_{\bar{r}\bar{r}}-\frac{\bar{\Psi}_{\bar{r}}}{\bar{r}}\right)+\frac{d\omega}{d\bar{\Psi}}\bar{\Psi}_{\bar{r}}^2\right], \qquad \text{A–1}$$

where ω is dependent only on $\bar{\Psi}$ and given by

$$\omega(\bar{\Psi})=\frac{2n^2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}(2\bar{\Psi})^{\frac{n-1}{n}}.$$

With the nearly conical coordinate

$$\theta=\bar{r}/\bar{x}^n\,,$$

and the assumption

$$\bar{\Psi}=\bar{x}^2 f(\theta)\,, \qquad \text{A–2}$$

the form of $\bar{\Psi}$ is compatible with the shock-wave condition which is given by Eq. (2.13). In the similar manner as the case of plane flow we can derive the following form of ω

$$\omega=\frac{2^{\frac{2n-1}{n}}n^2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}f^{\frac{n-1}{n}}\bar{x}^{2(n-1)}. \qquad \text{A–3}$$

Substituting Eqs. A-2 and A-3 into Eq. A-1, we obtain the ordinary differential equation

$$\begin{aligned}&4n^2f^2f''+n(1-n)\theta f'^3-2nff'^2\\&=\frac{2^{\frac{2n-1}{n}}n}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}\frac{f'^{\gamma+1}}{\theta^{\gamma-1}}f^{1-1/n}\left[\gamma n\left(f''-\frac{f'}{\theta}\right)+(n-1)\frac{f'^2}{f}\right].\end{aligned} \qquad \text{A–4}$$

In the neighborhood of the point $\theta=\theta_0$ in which $\bar{\Psi}=0$, the predominant part of the solution f is readily found to be expressed in the form

$$f(\theta)=B(\theta-\theta_0)^N\,,$$

where

$$N=\frac{n\gamma}{n\gamma+(n-1)}.$$

The requirement of vanishing of f at the point $\theta=\theta_0$ is fulfilled only when

$$n\geq\frac{1}{1+\gamma}\geq\frac{1}{2}$$

which affords the criterion of n for the case of axially symmetric flow. For the case of plane flow the criterion is given by Eq. (3.5). These criterions of n for both cases of plane flow and axially symmetric flow have already been found by Lees and Kubota [16] with analogy to the blast wave theory.

Appendix B

The derivation of the corresponding incompressible boundary-layer equation

According to the transformations (4.8) and (4.9), we have

$$\left.\begin{aligned}
u^* &\equiv \frac{1}{\rho}\frac{\partial\Psi}{\partial y^*}=p^{\frac{\Upsilon-1}{2\Upsilon}}\frac{\partial\Psi}{\partial y_i^*},\\
v^* &\equiv -\frac{1}{\rho}\frac{\partial\Psi}{\partial x^*}=-\frac{1}{\rho}\left(p^{\frac{3\Upsilon-1}{2\Upsilon}}\frac{\partial\Psi}{\partial x_i^*}+\frac{\partial\Psi}{\partial y_i^*}\frac{\partial y_i^*}{\partial x^*}\right),\\
\frac{\partial u^*}{\partial x^*} &= p^{\frac{2\Upsilon-1}{\Upsilon}}\frac{\partial^2\Psi}{\partial x_i^*\partial y_i^*}+\frac{\gamma-1}{2\gamma p}u^*\frac{dp}{dx^*}+p^{\frac{\Upsilon-1}{2\Upsilon}}\frac{\partial^2\Psi}{\partial y_i^{*2}}\frac{\partial y_i^*}{\partial x^*},\\
\frac{\partial u^*}{\partial y^*} &= \rho p^{\frac{\Upsilon-1}{\Upsilon}}\frac{\partial^2\Psi}{\partial y_i^{*2}}.
\end{aligned}\right\} \qquad \text{B–1}$$

Then, since from Eqs. (4.5) and (4.7)

$$\rho\mu^*=\rho_\infty^*\nu_\infty^*,$$

the viscosity term in Eq. (4.2) is expressed in the form

$$\frac{\partial}{\partial y^*}\left(\mu^*\frac{\partial u^*}{\partial y^*}\right)=\rho^*\nu_\infty^* p^{\frac{5\Upsilon-2}{2\Upsilon}}\frac{\partial^3\Psi}{\partial y_i^{*3}}. \qquad \text{B–2}$$

Substituting Eqs. B-1 and B-2 into Eq. (4.2), we obtain

$$\frac{\partial\Psi}{\partial y_i^*}\frac{\partial^2\Psi}{\partial x_i^*\partial y_i^*}-\frac{\partial\Psi}{\partial x_i^*}\frac{\partial^2\Psi}{\partial y_i^{*2}}=-p^{-\frac{5\Upsilon-3}{2\Upsilon}}\frac{dp}{dx^*}\left[\frac{p_\infty^*}{\rho^*}+\frac{\gamma-1}{2\gamma}\frac{u^{*2}}{p}\right]+\nu_\infty^*\frac{\partial^3\Psi}{\partial y_i^{*3}}. \qquad \text{B–3}$$

By the use of Eq. (4.6), we have

$$\frac{p_\infty^*}{\rho^*}+\frac{\gamma-1}{2\gamma}\frac{u^{*2}}{p}=\frac{a_\infty^*}{\gamma p}\left(1+\frac{\gamma-1}{2}M^2\right), \quad a_\infty^*=\sqrt{\gamma p_\infty^*/\rho_\infty^*}.$$

Since $(\gamma-1)M^2/2\gg 1$, we have

$$\frac{p_\infty^*}{\rho_\infty^*}+\frac{\gamma-1}{2\gamma}\frac{u^{*2}}{p}\simeq\frac{(\gamma-1)a_\infty^*}{2\gamma p}M^2.$$

Hence the pressure term in Eq. B-3 becomes

$$-p^{-\frac{5\Upsilon-3}{2\Upsilon}}\frac{dp}{dx^*}\left[\frac{p_\infty^*}{\rho^*}+\frac{\gamma-1}{2\gamma}\frac{u^{*2}}{p}\right]\simeq-\frac{\gamma-1}{2\gamma}U^2p^{-\frac{7\Upsilon-3}{2\Upsilon}}\frac{dp}{dx^*}.$$

Here let us introduce W defined by

$$W=Up^{(1-\Upsilon)/2\Upsilon},$$

then

$$\frac{\gamma-1}{2\gamma}U^2p^{-\frac{7\Upsilon-3}{2\Upsilon}}\frac{dp}{dx^*}=-W\frac{dW}{dx_i^*},$$

so that Eq. B-3 becomes

$$\frac{\partial\Psi}{\partial y_i^*}\frac{\partial^2\Psi}{\partial x_i^*\partial y_i^*}-\frac{\partial\Psi}{\partial x_i^*}\frac{\partial^2\Psi}{\partial y_i^{*2}}=W\frac{dW}{dx_i^*}+\nu_\infty^*\frac{\partial^3\Psi}{\partial y_i^{*3}}.$$

Appendix C

The relation between y^* and η in the boundary layer at a fixed value of x^*

When x^* is held fixed, we obtain from Eq. (4.9)

$$y^*=p^{\frac{1-\Upsilon}{2\Upsilon}}\int_0^{y_i^*}\frac{dy_i^*}{\rho}, \qquad \text{C–1}$$

where the integrand is written as

$$1/\rho=(1/p)[1+(\gamma-1)(1-u^2)M^2/2]\simeq(\gamma-1)(1-u^2)M^2/2p$$

in view of very large values of M. Hence Eq. C-1 becomes

$$y^*=\frac{\gamma-1}{2}M^2p^{\frac{1-3\Gamma}{2\Gamma}}\int_0^{y_i^*}(1-u^2)dy_i^*,$$

or, since $u=u_i^*/W$ from Eq. (4.12),

$$y^*=\frac{\gamma-1}{2}M^2p^{\frac{1-3\Gamma}{2\Gamma}}\int_0^{y_i^*}\left[1-\left(\frac{u_i^*}{W}\right)^2\right]dy_i^*. \qquad \text{C–2}$$

By the use of Eq. (4.19), Eq. C-2 becomes

$$y^*=\frac{\gamma-1}{2}\sqrt{\frac{\gamma+1}{2}\frac{\nu_\infty^*}{Q}}\,p^{(1-3\Gamma)/2\Gamma}x_i^{*1/(\Gamma+1)}\int_0^{\eta}\left[1-\left(\frac{u_i^*}{W}\right)^2\right]d\eta.$$

Appendix D

The expression for f_1'' in the outer region of the boundary layer

The differentiation of Eq. (5.15) with respect to y^* yields

$$\frac{\partial u}{\partial y^*}=(f_0''+\epsilon f_1''+\cdots)\frac{\partial\eta}{\partial y^*}, \qquad \text{D–1}$$

where for a fixed value of x^* or x_i^*

$$\partial\eta/\partial y^*=(\partial\eta/\partial y_i^*)(\partial y_i^*/\partial y^*).$$

Using $\partial\eta/\partial y_i^*$ and $\partial y_i^*/\partial y$ obtained from the differentiations of Eqs. (4.19) and (4.9), respectively, we find

$$\frac{\partial\eta}{\partial y^*}=\frac{1}{2}\sqrt{\frac{Up}{\nu_\infty^*x^*}}\frac{1}{T}. \qquad \text{D–2}$$

Thus Eq. D-1 becomes

$$\frac{\partial u}{\partial y^*}=(f_0''+\epsilon f_1''+\cdots)\frac{1}{2T}\sqrt{\frac{Up}{\nu_\infty^*x^*}}.$$

Applying this equation to the outer region of the boundary layer and remembering $f_0''\simeq 0$ there, we obtain

$$\frac{\partial u}{\partial y^*}\simeq\frac{\epsilon}{2T}\sqrt{\frac{Up}{\nu_\infty^*x^*}}f_1''.$$

Substituting the zeroth-order values of p and T given by Eqs. (4.34) and (5.5), respectively, we have

$$\frac{\partial u}{\partial y^*}\simeq\frac{\sqrt{p_0}}{2x^*}\frac{f_0^{2/3\Gamma}}{D}\left(\frac{M}{\sqrt{R_{x^*}}}\right)^{-1/2}f_1''\left(\frac{M}{\sqrt{R_{x^*}}}\right)^{-1+2/3\Gamma}\epsilon. \qquad \text{D–3}$$

On the other hand the vorticity is related to the entropy gradient by the Crocco's theorem as

$$\frac{\partial v}{\partial x^*}-\frac{\partial u}{\partial y^*}=\frac{p^*}{RU\rho_\infty^*}\frac{dS^*}{d\Psi}$$

at the outer region of the boundary layer, in which S^* is the entropy per unit mass. Since $\bar{\omega}$ defined by Eq. (2.11) is also written as

$$\bar{\omega} = \text{const.}\, e^{S^*/c_v}$$

where c_v is the specific heat at constant volume,

$$\frac{dS^*}{d\Psi} = \frac{c_v}{\bar{\omega}} \frac{d\bar{\omega}}{d\bar{\Psi}} \frac{d\bar{\Psi}}{d\Psi}.$$

By means of Eq. (3.15), the above equation becomes at the outer region of the boundary layer

$$\frac{dS^*}{d\Psi} \simeq \frac{c_v}{\tau U} \frac{1}{\bar{\omega}} \frac{d\bar{\omega}}{d\bar{\Psi}}. \qquad \text{D–4}$$

Setting $n=3/4$ is Eq. (2.19), we see that ω is proportional to $\bar{\Psi}^{-2/3}$, Eq. D-4 therefore becomes

$$\frac{dS^*}{d\Psi} \simeq -\frac{2c_v}{3\tau U\bar{\Psi}} \simeq -\frac{2}{3}\frac{c_v}{\Psi}.$$

Thus we obtain

$$\frac{\partial v}{\partial x^*} - \frac{\partial u}{\partial y^*} \simeq -\frac{2}{3(\gamma-1)} \frac{p^*}{\rho_\infty^* U\Psi} = -\frac{2a_\infty^{*2} p}{3(\gamma-1)\gamma U\Psi}.$$

Substituting for p and Ψ the zeroth-order values from Eqs. (4.34) and (5.2), respectively, we obtain

$$\frac{\partial v}{\partial x^*} - \frac{\partial u}{\partial y^*} \simeq -\frac{\sqrt{p_0}}{3\gamma(\gamma-1)f_0}\left(\frac{M}{\sqrt{R_{x^*}}}\right)^{-1/2} x^{*-1},$$

in which, however, $\partial v/\partial x^*$ may be discarded compared with $\partial u^*/\partial y^*$. We have therefore

$$\frac{\partial u}{\partial y^*} \simeq \frac{\sqrt{p_0}}{3\gamma(\gamma-1)f_0}\left(\frac{M}{\sqrt{R_{x^*}}}\right)^{-1/2} x^{*-1}. \qquad \text{D–5}$$

Comparing Eq. D-5 with Eq. D-3 we can see as before that the initially assumed selection (5.9) is also appropriate for the present problem. Then we get

$$f_1'' \simeq \frac{2Df_0^{-1-2/3\tau}}{3\gamma(\gamma-1)}.$$