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Simultaneous Effects of Pressure Gradient and Transverse Curvature on the Boundary Layer along Slender Bodies of Revolution

By

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Summary. Calculations of the effects of pressure gradient and transverse curvature on the boundary layer along slender bodies of revolution are made for allowable body shapes by the series expansion method. Two methods of calculation are presented: the first is the perturbation expansion of flow quantities from the uniform state, and the second is that from the stagnation state. In both cases, allowable body shapes are obtained for the given outer pressure gradient. As an example of the first method, the boundary layer flow along a slender paraboloid of revolution is calculated. The results show that, as already found, the transverse curvature acts on the flow somewhat similarly as a favorable pressure gradient. Further the displacement and the longitudinal curvature effects on the viscous layer is estimated. These effects are found to appear in the second-order terms of the series expansion.

INTRODUCTION

A number of studies have been made on the solution of the two-dimensional laminar compressible boundary layer, but the extension of the method to the three-dimensional problem have been made with partial success. Especially for the axially symmetric flow, it is well known that if the ratio of the thickness of the boundary layer to body radius is very small and negligible compared with unity, Mangler's transformation can be applied so that the problem is reduced formally to that of the two-dimensional flow. Concerning the boundary layer flow along more slender bodies, Mark [*1*], and Probstein and Elliott [*2*] have given a transformation of the equations of motion to that of almot two-dimensional form, and calculated several compressible flows without pressure gradient.

The object of the present study is to extend the calculation to the more general flows containing pressure gradient. However, it is difficult to solve the general problem for all kinds of body shape accompanied by pressure gradient. Therefore, the series expansion method is used for allowable body shapes. The result obtained by this method can be referred to as an accurate one by which the accuracy

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of the approximate calculations such as momentum-integral method should be checked.

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1. Equations of Motion and Transformations

It is assumed that the specific heat at constant pressure c_p and the Prandtl number P_r are constant, and that the gas obeys the perfect gas law:

$$p=R\rho T, \tag{1}$$

where p is the static pressure, ρ the density, T the absolute temperature of the gas, and R the gas constant per unit mass. Then, the equations of motion for the axially symmetric boundary layer flow become

$$(\rho ur)_x+(\rho vr)_y=0, \tag{2}$$

$$\rho uu_x+\rho vu_y=-p_x+\frac{1}{r}(\mu ru_y)_y, \tag{3}$$

$$c_p(\rho uT_x+\rho vT_y)=up_x+\frac{c_p}{P_r\cdot r}(\mu rT_y)_y+\mu u_y^2, \tag{4}$$

where x is the distance measured along the body surface from its nose, y is the normal distance from the surface, r is the radial distance from the axis of symmetry, u and v are the velocity components in the x and y directions respectively, as shown in Fig. 1, and μ is the coefficient of viscosity. The subscripts x and y denote partial differentiations with respect to each.

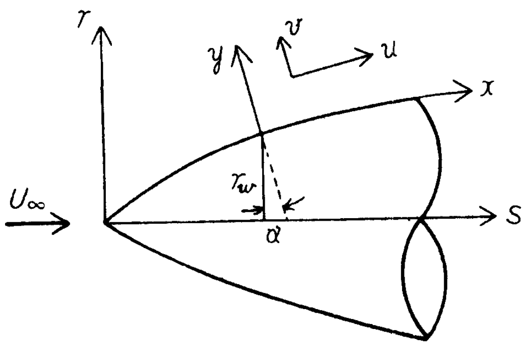

FIGURE 1

The boundary conditions at the wall, $y=0$, are

$$u=v=0,$$

and $\quad T=T_w \quad$ (non-insulated case),

or $\quad T_y=0 \quad$ (insulated case),

where the subscript w is used to denote quantities at the wall. At the outer edge of the boundary layer where $y=\delta$, the values of u and T are specified as

$$u=u_e \quad \text{and} \quad T=T_e,$$

where the subscript e is used to denote quantities outside the boundary layer.

It is well known that the above equations are reduced to almost two-dimensional form by Mark's transformation:

$$d\bar{x}=\frac{r_w^2}{L^2}dx, \quad d\bar{y}=\frac{r}{L}dy, \tag{5}$$

where L is the reference length. Then, r is expressed in terms of the new coordinates by

$$\frac{r^2}{r_w^2}=1+\frac{2L\cos\alpha}{r_w^2}\bar{y}, \tag{6}$$

where α is the angle the tangent to the meridian profile makes with the axis of symmetry.

Mark [*1*], and Probstein and Elliott [*2*] have given several solutions of the boundary layer equations for the compressible flow without pressure gradient and for the incompressible flow with pressure gradient. Probstein and Elliott have also attempted to extend their analysis to compressible flows with pressure gradient, but it has been found that series expansions of the physical quantities in integral powers of a suitably chosen parameter ξ (which characterizes the ratio of body radius to boundary layer thickness, and whose coefficients are functions of the similarity variable, η) are only possible when the Mach number of the external flow is zero and u_e=const.

Now, it is assumed that the viscosity varies linearly with the temperature,

$$\frac{\mu}{\mu_e}=C\frac{T}{T_e}, \tag{7}$$

where C denotes the Chapman-Rubesin's parameter [*3*]. By the introduction of a stream function $\bar{\Psi}$ and the variable η defined by

$$\left.\begin{aligned} \rho ru/\rho_e &= \partial\bar{\Psi}/\partial(y/L), \\ \rho rv/\rho_e &= -\partial\bar{\Psi}/\partial(x/L), \end{aligned}\right\} \tag{8}$$

$$\eta=\left(\frac{u_e}{C\nu_e\bar{x}}\right)^{1/2}\cdot\int\frac{\rho}{\rho_e}d\bar{y}, \tag{9}$$

and further by the transformation of $\bar{\Psi}$ and T/T_e given by

$$\left.\begin{aligned} \bar{\Psi} &= (C\nu_e u_e\bar{x})^{1/2}f(\bar{x},\eta), \\ T/T_e &= \lambda(\bar{x},\eta), \end{aligned}\right\} \tag{10}$$

equations of momentum and energy are transformed, respectively, into

$$\frac{1}{2}\left\{\frac{\bar{x}(C\nu_e u_e)'}{C\nu_e u_e}+\frac{\bar{x}(u_e/C\nu_e)'}{u_e/C\nu_e}\right\}f_\eta^2+\bar{x}(f_\eta f_{\bar{x}\eta}-f_{\bar{x}}f_{\eta\eta})$$
$$-\frac{1}{2}\left\{1+\frac{\bar{x}(C\nu_e u_e)'}{C\nu_e u_e}\right\}ff_{\eta\eta}-f_{\eta\eta\eta}-\frac{\bar{x}}{u_e}u_{e\bar{x}}\cdot\lambda=2\xi\left(f_{\eta\eta}\int\lambda d\eta\right)_\eta, \tag{11}$$

$$
\begin{aligned}
&-\frac{1}{2}\left\{1+\frac{\bar{x}(C\nu_e u_e)'}{C\nu_e u_e}\right\}f\lambda_\eta+\bar{x}(f_\eta\lambda_{\bar{x}}-f_{\bar{x}}\lambda_\eta)+\frac{\bar{x}T_{e\bar{x}}}{T_e}f_\eta\lambda \\
&-\frac{1}{P_r}\lambda_{\eta\eta}-(\gamma-1)M_e^2\left\{f_{\eta\eta}^2-\frac{\bar{x}}{u_e}u_{e\bar{x}}\lambda\right\} \\
&=2\xi\left\{\frac{1}{P_r}\left(\lambda_\eta\int\lambda d\eta\right)_\eta+(\gamma-1)M_e^2f_{\eta\eta}^2\int\lambda d\eta\right\},
\end{aligned}
\tag{12}
$$

where

$$
\xi=\sqrt{\frac{C\nu_e\bar{x}}{u_e}}\cdot\frac{L\cos\alpha}{r_w^2},\qquad M_e^2=\frac{u_e^2}{a_e^2}=\frac{u_e^2}{\gamma RT_e},
\tag{13}
$$

and the prime is used to denote the differentiation with respect to $\bar{x}$.

2. Perturbation Expansion

It is well known that when ξ is kept constant, the similar solution exists if M_e is a constant, or if M_e is zero regardless of the pressure gradient. If the pressure gradient is small and the flow quantities can be expressed by the power series expansion from the uniform state, it is easily ascertained that the perturbation calculation is possible under a suitably chosen condition.

Let u_c, $C\nu_e$, and T_e be expanded as

$$
\left.
\begin{aligned}
u_c/u_\infty&=1+A_1(\bar{x}/L)^m+\cdots, \\
\nu_c/\nu_\infty&=1+B_1(\bar{x}/L)^m+\cdots, \\
T_c/T_\infty&=1+T_1(\bar{x}/L)^m+\cdots, \\
r_w/L&=R_0(\bar{x}/L)^n\{1+R_1(\bar{x}/L)^m+\cdots\},
\end{aligned}
\right\}
\tag{14}
$$

then, ξ can be expanded as

$$
\xi=\sqrt{\frac{C\nu_\infty L}{u_\infty}}L\cos\alpha_0R_0^{-2}(\bar{x}/L)^{\frac{1}{2}-2n}\{1+\xi_1(\bar{x}/L)^m+\cdots\}.
\tag{15}
$$

From Eqs. (11) and (12), referring to the power of $\bar{x}$ in the expression of ξ, it is seen that the power m must be of the form

$$
mq=\frac{1}{2}-2n,
\tag{16}
$$

to make similar expansions of f and λ possible, where q represents an integer. General discussion of the possibility of flow satisfying the condition (16) is not discussed here except for pointing out that the subsonic boundary layer flow along a slender paraboloid of revolution is a case.

Under the above conditions, f and λ can be expanded as follows:

$$
\left.
\begin{aligned}
f&=f_0(\eta)+(\bar{x}/L)^m f_1(\eta)+\cdots, \\
\lambda&=\lambda_0(\eta)+(\bar{x}/L)^m\lambda_1(\eta)+\cdots.
\end{aligned}
\right\}
\tag{17}
$$

In the zeroth order, if $n\neq 1/4$, the momentum equation (11) is given by

$$
2f_0'''+f_0f_0''=0,
\tag{18}
$$

and the energy equation (12) becomes

$$\frac{1}{P_r}\lambda_0''+\frac{1}{2}f_0\lambda_0'+(\gamma-1)M_\infty^2 f_0''^2=0\,. \tag{19}$$

In the first order, if the integer q equals to unity, the corresponding equations for f and λ are, respectively, given by

$$2f_1'''+f_0f_1''-2mf_0'f_1'+(2m+1)f_0''f_1+2m\left\{\frac{A_1+B_1}{2}f_0f_0''-A_1f_0'+\lambda_0\right\}$$
$$+4\sqrt{\frac{C\nu_\infty L}{u_\infty}}\,\frac{L\cos\alpha_0}{R_0\sqrt{2n+1}}\left(f_0''\int\lambda_0d\eta\right)'=0\,, \tag{20}$$

$$\frac{2}{P_r}\lambda_1''+f_0\lambda_1'-2mf_0'\lambda_1+f_1\lambda_0'+m(A_1+B_1)f_0\lambda_0'+2m\lambda_0'f_1-2mT_1f_0'\lambda_0$$
$$+2(\gamma-1)M_\infty^2(2f_0''f_1''-m\lambda_0)+4\sqrt{\frac{C\nu_\infty L}{u_\infty}}\,\frac{L\cos\alpha_0}{R_0\sqrt{2n+1}}\left\{\frac{1}{P_r}\left(\lambda_0'\int\lambda_0d\eta\right)'\right.$$
$$\left.+(\gamma-1)M_\infty^2f_0''^2\int\lambda_0d\eta\right\}=0\,. \tag{21}$$

Eqs. (18) and (19) are the same as those obtained by Probstein and Elliott [2]. Eqs. (20) and (21) are linear ordinary differential equations, and their orders are reduced by the transformation of f_1 and λ_1 into l_1 and k_1, related by

$$\left.\begin{aligned}f_1(\eta)&=f_0'(\eta)\int l_1(\eta)d\eta\,,\\ \lambda_1(\eta)&=\lambda_0'(\eta)\int k_1(\eta)d\eta\,.\end{aligned}\right\} \tag{22}$$

Especially the homogeneous part of Eq. (20) is the same as the one appeared in Howarth's problem of a boundary layer flow with a linear velocity distribution. All the higher order equations may be calculated in a similar way as above.

If the integration of above equations is made, the local skin friction coefficient C_f and the local heat-transfer rate Q at the wall are obtained as follows:

$$C_f=\frac{2(\mu u_y)_{y=0}}{\rho_\infty u_\infty^2}=2\frac{\rho_e}{\rho_\infty}\left(\frac{u_e}{u_\infty}\right)^2\frac{r_w/L}{(\bar{x}/L)^{1/2}}\left(\frac{C\nu_e}{u_eL}\right)^{1/2}\left[\sum_{n=0}^{\infty}(\bar{x}/L)^{nm}f_n''(0)\right], \tag{23}$$

$$Q=-(kT_y)_{y=0}=-\frac{c_pT_e\rho_e}{P_r}\,\frac{r_w/L}{(\bar{x}/L)^{1/2}}\left(\frac{C\nu_eu_e}{L}\right)^{1/2}\left[\sum_{n=0}^{\infty}(\bar{x}/L)^{nm}\lambda_n'(0)\right]. \tag{24}$$

3. Stagnation Flow Expansion

The perturbation method presented in the previous section is practically applicable for obtaining the down-stream solution. In the present section, an attempt will be made to obtain the solution in the case where u_e assumes the form

$$u_e/u_\infty=A_0(\bar{x}/L)^m+A_1(\bar{x}/L)^{2m}+\cdots\,. \tag{25}$$

It is then plausible to expand $C\nu_e$ and T_e as

$$\left.\begin{aligned}\nu_e/\nu_\infty&=B_0+B_1(\bar{x}/L)^{2m}+\cdots\,,\\ T_e/T_\infty&=T_0+T_1(\bar{x}/L)^{2m}+\cdots\,.\end{aligned}\right\} \tag{26}$$

Now, at the glance of the leading term of ξ, if $n \neq 1/4$, the power m must be of the form

$$q' \cdot 2m = \left(\frac{1}{2} - 2n - \frac{m}{2}\right), \tag{27}$$

where q' is an integer. Under such conditions, f and λ can be expanded, and the equations for f_n and λ_n ($n=0, 1, \cdots$) can be obtained. In the zeroth order, the momentum equation is given by

$$2f_0''' + (2m+1) f_0 f_0'' + 2m(\lambda_0 - f_0'^2) = 0\,, \tag{28}$$

and the energy equation is given by

$$\frac{1}{P_r}\lambda_0'' + \frac{2m+1}{2} f_0 \lambda_0' = 0\,. \tag{29}$$

In the first order, the corresponding equations are

$$\begin{aligned} &2f_1''' + (m+1) f_0 f_1'' - 8m f_0' f_1' + (5m+1) f_0'' f_1 + 2l\lambda_1 \\ &= -2m(A_1+B_1) f_0 f_0'' - 4mA_1(\lambda_0 - f_0'^2) - 2\xi_0\left(f_0'' \int \lambda_0 d\eta\right)', \end{aligned} \tag{30}$$

$$\begin{aligned} &\frac{1}{P_r}\lambda_1'' + \frac{1}{2}(m+1) f_0 \lambda_1' - 2m f_0' \lambda_1 + \frac{1}{2}(5m+1)\lambda_0' f_1 \\ &= -m(A_1+B_1) f_0 \lambda_0' + 2mT_1 f_0' \lambda_0 - (\gamma-1) M_\infty^2 \left(\frac{A_0^2}{T_0} f_0''^2 - m\lambda_0\right) \\ &\quad - \frac{2}{P_r}\xi_0\left(\lambda_0' \int \lambda_0 d\eta\right), \end{aligned} \tag{31}$$

respectively, where it is assumed that

$$\xi = \xi_0(\bar{x}/L)^m + \xi_1(\bar{x}/L)^{2m} + \cdots.$$

Eq. (28) coincides with the result obtained by Probstein and Elliott in the incompressible flow if $\lambda_0 = 1$. Eqs. (30) and (31) are linear differential equations, and their orders are reduced by the similar transformations as given by Eq. (22).

The skin friction coefficient C_f and the local heat-transfer rate Q at the wall are given by Eqs. (23) and (24).

4. Boundary Layer on a Slender Paraboloid of Revolution

4·1 *Inviscid Flow outside the Viscous Layer*

As a special example of the perturbation expansion, the solution of the boundary layer flow on a slender paraboloid of revolution is obtained in this section. As well known, the zeroth order solution f_0 of the boundary layer is obtained by Mark [1] assuming the axial pressure gradient zero. This assumption corresponds to limit the consideration to the flow far behind from the nose of the body. The first order solution appreciating the effect of perturbation pressure can be calculated by applying the analysis discussed in Section 2.

To find the inviscid solution in the subsonic case, the Prandtl-Glauert's approximation can be applied to the incompressible solution. The incompressible flow past a paraboloid of revolution is well known. If the radial coordinate r_w of a paraboloid of revolution is expressed by

$$r_w^2 = 4Ls\,, \tag{32}$$

where s is the axial distances from the nose of the body and L the focal length of the paraboloid as shown in Fig. 2, the inviscid solution along the surface is given by [4]

$$\frac{\Delta u^*}{u_\infty} \equiv \frac{u^* - u_\infty}{u_\infty} = -\frac{L/s}{1+(L/s)}, \qquad \frac{v^*}{u_\infty} = \frac{(L/s)^{1/2}}{1+(L/s)}, \tag{33}$$

where u^* and v^* denote the velocity components in the s and r directions respectively.

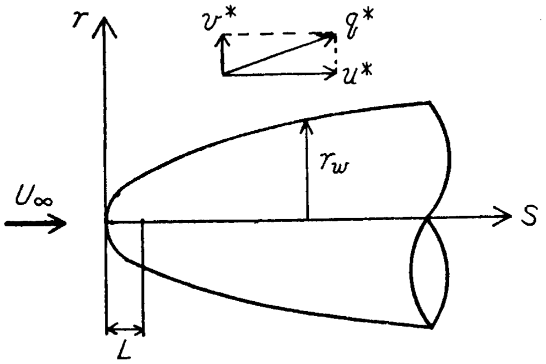

FIGURE 2

For the subsonic flow, the application of the Prandtl-Glauert's approximation gives

$$\left.\begin{aligned} \frac{\Delta u^*}{u_\infty} &= -\frac{1}{\beta^2}\frac{L\beta^2/s}{1+(L\beta^2/s)} = -\frac{L/s}{1+(L\beta^2/s)}, \\ \frac{v^*}{u_\infty} &= \frac{1}{\beta}\frac{(L\beta^2/s)^{1/2}}{1+(L\beta^2/s)} = \frac{(L/s)^{1/2}}{1+(L\beta^2/s)}, \end{aligned}\right\} \tag{34}$$

where

$$\beta = \sqrt{1 - M_\infty^2}\,.$$

For the flow far downstream from the nose where L/s is small compared with unity, the right-hand sides of the above equations can be expanded as follows:

$$\left.\begin{aligned} \frac{u^*}{u_\infty} &= 1 - \frac{L}{s} + \beta^2\left(\frac{L}{s}\right)^2 + \cdots, \\ \frac{v^*}{u_\infty} &= \left(\frac{L}{s}\right)^{1/2}\left(1 - \beta^2\frac{L}{s} + \cdots\right). \end{aligned}\right\} \tag{35}$$

It is interesting to note that the first order perturbation term is independent of β so that the effect of compressibility is seen only in the higher order terms. The resultant velocity q^* on the surface becomes

$$\frac{q^*}{u_\infty} = \left(\frac{u^{*2}+v^{*2}}{u_\infty^2}\right)^{1/2} = 1 - \frac{1}{2}\frac{L}{s} + \cdots\,. \tag{36}$$

Then, with the aid of the energy integral in the inviscid region, the flow quantities on the body surface are determined as follows:

$$\left.\begin{aligned}
&\frac{u_e}{u_\infty}=\frac{q^*}{u_\infty}=1-\frac{1}{2}\frac{L}{s}+\cdots\quad=1-\frac{1}{2}t+\cdots,\\
&\frac{\rho_e}{\rho_\infty}=1+\frac{1}{2}M_\infty^2\frac{L}{s}+\cdots\quad=1+\frac{1}{2}M_\infty^2 t+\cdots,\\
&\frac{p_e}{p_\infty}=1+\frac{\gamma}{2}M_\infty^2\frac{L}{s}+\cdots\quad=1+\frac{\gamma}{2}M_\infty^2 t+\cdots,\\
&\frac{T_e}{T_\infty}=1+\frac{\gamma-1}{2}M_\infty^2\frac{L}{s}+\cdots=1+\frac{\gamma-1}{2}M_\infty^2 t+\cdots,\\
&\frac{\nu_e}{\nu_\infty}=1-\frac{2-\gamma}{2}M_\infty^2\frac{L}{s}+\cdots=1-\frac{2-\gamma}{2}M_\infty^2 t+\cdots,\\
&\cos\alpha=1-\frac{1}{2}\frac{L}{s}+\cdots\quad=1-\frac{1}{2}t+\cdots,\\
&\frac{s}{L}=t\left(1-\frac{1}{2}t+\cdots\right),\\
&\frac{r_w^2}{L^2}=4\frac{s}{L}=4t\left(1-\frac{1}{2}t+\cdots\right),\\
&\xi=\frac{1}{2}\xi_0\left[1+\frac{1}{4}\{1-(2-\gamma)M_\infty^2\}t+\cdots\right],
\end{aligned}\right\}\tag{37}$$

where

$$t=\left(\frac{2L}{\bar{x}}\right)^{1/2},\qquad \xi_0=\left(\frac{C\nu_\infty}{2u_\infty L}\right)^{1/2}.$$

and suffix e is used to denote quantities at the end of the inviscid layer, or those on the body surface in the zeroth approximation.

4·2 *Viscous Layer*

Before going into the calculation, the range of validity of Eqs. (2), (3) and (4), denoted in the (x, y) coordinate system, must be checked. If the longitudinal curvature $\kappa(x)$ of the body surface is taken into account, it is known that Eqs. (2), (3) and (4) are valid when the terms of order $\kappa\delta$ are neglected, where δ is the thickness of the boundary layer, and the equation of y-momentum becomes, after higher order terms are neglected,

$$\kappa\rho u^2=\partial p/\partial y\,,$$

which can be replaced by $\partial p/\partial(y/\delta)=0$, if the term of order $\kappa\delta$ is neglected. Therefore, in the present calculation, the solution is obtained for the case where $\kappa\delta$ is negligible.

Now, it is known that the outer inviscid flow solution obtained in the preceding section just corresponds to the case $n=1/4$, and $m=-1/2$ in Section 2. Therefore, if f and λ are expanded in power series of $(2L/\bar{x})^{1/2}$ as

$$\left.\begin{aligned}
&f=f_0(\eta)+(2L/\bar{x})^{1/2}f_1(\eta)+\cdots,\\
&\lambda=\lambda_0(\eta)+(2L/\bar{x})^{1/2}\lambda_1(\eta)+\cdots,
\end{aligned}\right\}\tag{38}$$

the equations which determine f_n and λ_n are, in the zeroth order,

$$\left.\begin{aligned}
&2f_0'''+f_0f_0''=-2\xi_0\Big(f_0''\int\lambda_0 d\eta\Big)',\\
&\frac{2}{P_r}\lambda_0''+f_0\lambda_0'+2(\gamma-1)M_\infty^2 f_0''^2\\
&=-2\xi_0\Big\{\frac{1}{P_r}\Big(\lambda_0'\int\lambda_0 d\eta\Big)'+(\gamma-1)M_\infty^2 f_0''^2\int\lambda_0 d\eta\Big\},
\end{aligned}\right\}\tag{39}$$

and in the first order,

$$\left.\begin{aligned}
&2f_1'''+f_0f_1''+f_0'f_1'-\frac{1}{2}f_0'^2+\frac{1+(2-\gamma)M_\infty^2}{4}f_0f_0''+\frac{1}{2}\lambda_0\\
&=-2\xi_0\Big\{\Big(f_0''\int\lambda_1 d\eta\Big)'+\Big(f_1''\int\lambda_0 d\eta\Big)'+\frac{1-(2-\gamma)M_\infty^2}{4}\Big(f_0''\int\lambda_0 d\eta\Big)'\Big\},\\
&\frac{2}{P_r}\lambda_1''+f_0\lambda_1'+f_0'\lambda_1+\frac{\gamma-1}{4}M_\infty^2 f_0''f_1''+\frac{3}{2}\lambda_0'f_1+\frac{1+(2-\gamma)M_\infty^2}{4}f_0\lambda_0'\\
&+\frac{\gamma-1}{2}M_\infty^2 f_0'\lambda_0+2(\gamma-1)M_\infty^2\Big\{\Big(1+\frac{\gamma-1}{2}M_\infty^2\Big)f_0''^2-\frac{\lambda_0}{4}\Big\}\\
&=-2\xi_0\Big[\frac{1}{P_r}\Big(\lambda_0'\int\lambda_1 d\eta+\lambda_1'\int\lambda_0 d\eta\Big)'+(\gamma-1)M_\infty^2\Big\{f_0''^2\int\lambda_1 d\eta\\
&+2f_0''f_1''\int\lambda_0 d\eta-\Big(1+\frac{\gamma-1}{2}M_\infty^2\Big)f_0''^2\int\lambda_0 d\eta\Big\}\\
&+\frac{1-(2-\gamma)M_\infty^2}{4}\Big\{\frac{1}{P_r}\Big(\lambda_0'\int\lambda_0 d\eta\Big)'+(\gamma-1)M_\infty^2 f_0''^2\int\lambda_0 d\eta\Big\}\Big].
\end{aligned}\right\}\tag{40}$$

The first order equations are the linear ordinary differential equations, whose solutions are not difficult but laborious. Further, these solutions are partly invalid as the longitudinal curvature effect is not taken into account, the effect of which is discussed in Section 4·3.

In a special case where the Prandtl number equals to unity and the surface is thermally insulated, the energy equation can be solved as a function of f as follows:

$$\lambda=\frac{T}{T_e}=1+\frac{(\gamma-1)}{2}M_e^2(1-f_\eta^2)=\lambda_0+\lambda_1\Big(\frac{2L}{\bar{x}}\Big)^{1/2}+\cdots,\tag{41}$$

where

$$\lambda_0=1+\frac{\gamma-1}{2}M_\infty^2(1-f_0'^2),\quad \lambda_1=-\frac{\gamma-1}{2}M_\infty^2\Big\{\Big(1+\frac{\gamma-1}{2}M_\infty^2\Big)(1-f_0'^2)+2f_0'f_1'\Big\}.$$

The numerical calculations of f_0 and f_1 are given in Appendices A and B respectively. Especially mathematical details of calculating f_0 are presented by Mark in his paper [1]. If ξ_0 has a small value compared with unity, the following results are obtained:

$$\left.\begin{aligned}
&f_0''(0)=0.332+\{0.288+0.117(\gamma-1)M_\infty^2\}2\xi_0+\cdots,\\
&f_1''(0)=0.679+\{0.083+0.214(\gamma-1)\}M_\infty^2+O(\xi_0).
\end{aligned}\right\}\tag{42}$$

4·3 *Effects of Displacement Thickness and Longitudinal Curvature*

In the above calculations, the effective deformation of the body shape due to the boundary layer and the longitudinal curvature of the body surface are not considered. In this section, their effects and orders of magnitude are evaluated.

The displacement thickness δ^* is calculated by

$$\int_0^{\delta}\left(1-\frac{\rho u}{\rho_e u_e}\right)\frac{r}{r_w}dy=\delta^*+\delta^{*2}\frac{\cos\alpha}{2r_w}. \tag{43}$$

Hence the following relation is obtained:

$$\frac{\delta^*}{L}=\frac{\delta_0}{L}\left(\frac{\bar{x}}{2L}\right)^{1/4}\left\{1+\left(\delta_1-\frac{1}{4}\right)\left(\frac{2L}{\bar{x}}\right)^{1/2}+\cdots\right\}, \tag{44}$$

where

$$\left.\begin{aligned}
&\delta_0=2\{(1+N_0\xi_0)^{1/2}-1\}, \qquad \delta_1=\frac{1}{\delta_0}\frac{\left\{N_0\dfrac{3-(2-\gamma)M_\infty^2}{4}+N_1\right\}2\xi_0+\delta_0^2}{2+\delta_0^2},\\
&N=\int_0^\infty(\lambda-f')d\eta=N_0+\left(\frac{2L}{\bar{x}}\right)^{1/2}N_1+\cdots,\\
&N_0=\int_0^\infty(\lambda_0-f_0')d\eta, \quad N_1=\int_0^\infty(\lambda_1-f_1')d\eta.
\end{aligned}\right\} \tag{45}$$

Therefore, the effective body shape is represented by

$$r_E=r_w+(\delta^*/\cos\alpha), \tag{46}$$

or

$$r_E^2=4L(1+N_0\xi_0)s(1+2dL/s+\cdots).$$

Then, the inviscid flow quantities for the above effective body are obtained by replacing t by εt in Eqs. (37), that is,

$$\left.\begin{aligned}
&\left(\frac{u_c}{u_\infty}\right)_1=1-\frac{\varepsilon}{2}t+\cdots,\\
&\left(\frac{\rho_c}{\rho_\infty}\right)_1=1+\frac{\varepsilon}{2}M_\infty^2t+\cdots,\\
&\left(\frac{p_c}{p_\infty}\right)_1=1+\frac{\varepsilon}{2}\gamma M_\infty^2t+\cdots,\\
&\left(\frac{T_c}{T_\infty}\right)_1=1+\frac{\varepsilon}{2}(\gamma-1)M_\infty^2t+\cdots,\\
&\left(\frac{\nu_c}{\nu_\infty}\right)_1=1+\frac{\varepsilon}{2}(\gamma-2)M_\infty^2t+\cdots,\\
&(\xi_1)=\frac{1}{2}\xi_0\left\{1+\frac{1-(2-\gamma)M_\infty^2}{4}\varepsilon t+\cdots\right\},
\end{aligned}\right\} \tag{47}$$

where

$$\varepsilon=\sqrt{1+N_0\xi_0}.$$

On the other hand, the geometrical quantities are kept unchanged, that is, expressions of $\cos\alpha$, s/L and r_w^2/L^2 keep the same form as in Eq. (37). Then, f and λ are to be expanded as follows:

$$\left.\begin{aligned} f&=f_0(\eta)+\varepsilon t f_1(\eta)+\cdots,\\ \lambda&=\lambda_0(\eta)+\varepsilon t\lambda_1(\eta)+\cdots.\end{aligned}\right\} \tag{48}$$

If it is assumed that the surface is thermally insulated, and that the value of the Prandtl number equals to unity, then, N_0 is given by

$$N_0=\int_0^\infty(1-f_0')d\eta+\frac{\gamma-1}{2}M_\infty^2\int_0^\infty(1-f_0'^2)d\eta\,. \tag{49}$$

Next, the order of magnitude of the term $\kappa\delta$ or $\kappa\delta^*$ is estimated. It is known from the simple calculation that the longitudinal curvature κ of the body surface is given by

$$\kappa=-\frac{1}{2L}\left(\frac{L}{s}\right)^{3/2}\left\{1-\frac{3}{2}\frac{L}{s}+\cdots\right\}.$$

Therefore, with the aid of Eqs. (44) and (45), $\kappa\delta^*$ is calculated as

$$\begin{aligned}\kappa\delta^*&=-\frac{1}{2}\delta_0\frac{L}{s}\left\{1+O\left(\frac{L}{s}\right)\right\}\\ &=-(\varepsilon-1)t\{1+O(t)\}\,.\end{aligned}$$

Further, if ξ_0 is small compared with unity, ε_0 is expanded as

$$\varepsilon_0=1+\frac{N_{00}}{4}2\xi_0+\cdots, \tag{50}$$

where

$$\begin{aligned}N_{00}&=\int_0^\infty(1-f_{00}')d\eta+\frac{\gamma-1}{2}M_\infty^2\int_0^\infty(1-f_{00}'^2)d\eta\\ &=1.721+1.192(\gamma-1)M_\infty^2\,.\end{aligned}$$

Hence

$$\kappa\delta^*=-\frac{N_{00}}{2}\xi_0 t\{1+O(\xi_0,t)\}=O(\xi_0 t)$$

As seen from the above relation, the effects of the longitudinal curvature should be taken into account, if the term of order $\xi_0 t$ is considered. Similarly, from Eqs. (48) and (50), the displacement effects are seen in term of order $\xi_0 t$.

Therefore, if terms of order $\xi_0 t$ are neglected, the solutions are given by

$$\left.\begin{aligned} f&=f_{00}+f_{01}\cdot 2\xi_0+f_{10}\frac{L}{s}+O\left(\xi_0\frac{L}{s},\,\xi_0^2,\,\frac{L^2}{s^2}\right),\\ \lambda&=\lambda_{00}+\lambda_{01}\cdot 2\xi_0+\lambda_{10}\frac{L}{s}+O\left(\xi_0\frac{L}{s},\,\xi_0^2,\,\frac{L^2}{s^2}\right).\end{aligned}\right\} \tag{51}$$

Numerical calculations of $f_{00}\cdots f_{10}$ and $\lambda_{00}\cdots\lambda_{10}$ can be found in Appendices A and B. The resultant coefficient of skin friction can be calculated from Eq. (23).

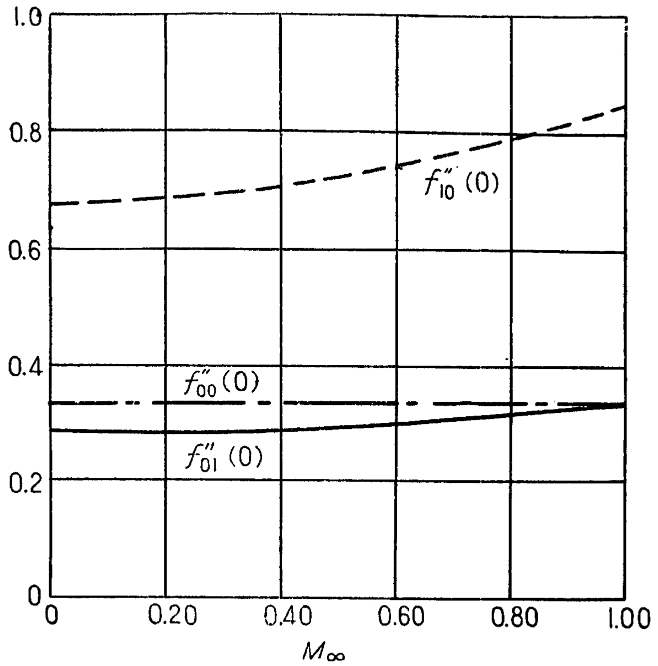

FIGURE 3

Fig. 3 show values of f''_{00}, f''_{01}, and f''_{10} at $\eta=0$ plotted against M_∞. As easily known, effects of the transverse curvature and the pressure gradient are specified by terms multiplied by ξ_0 and L/s in Eq. (51). Since all quantities $f''_{00}(0), \cdots f''_{10}(0)$ are positive, the transverse curvature and the favorable pressure gradient play a similar role in the first approximation at least in the wall skin friction.

CONCLUSION

Simultaneous effects of the pressure gradient and the transverse curvature are investigated by the perturbation expansion method. The present calculation cannot be applicable to the general body shapes accompanied by pressure gradient. Further, the perturbation equations become more and more complicated, if the higher order solutions are required. As an example of the method, the boundary layer flow on a paraboloid of revolution is calculated. From the numerical results, it is ascertained that the transverse curvature and the favorable pressure gradient play a similar role in the first approximation.

The result obtained by the present method can be referred to as an accurate one by which the accuracy of the approximate calculations such as momentum integral method should be checked.

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APPENDIX A

Numerical Integration of the Zeroth Order Momentum Equation for a Paraboloid

The zeroth order equation, as given by Eq. (39), is

$$2f_0''' + f_0 f_0'' = -2\xi_0\left(f_0'' \int \lambda_0 d\eta\right)',$$

where

$$\lambda_0 = 1 + \frac{\gamma-1}{2} M_\infty^2 (1 - f_0'^2).$$

If ξ_0 is small compared with unity, f_0 can be expanded as

$$f_0 = f_{00} + f_{01} \cdot 2\xi_0 + \cdots.$$

Then, the equations which determine f_{00} and f_{01} are

$$2f_{00}''' + f_{00} f_{00}'' = 0,$$

$$2f_{01}''' + f_{00} f_{01}'' + f_{00}'' f_{01} = -\left[f_{00}'' \int \left\{1 + \frac{\gamma-1}{2} M_\infty^2 (1 - f_{00}'^2)\right\} d\eta\right]' \equiv F_1'.$$

The upper one is the Blasius' equation and its solution is well known. The lower one can be reduced its order by the following transformation:

$$f_{01} = f_{00}' \int_0^\eta h_{01} d\eta.$$

The resultant equation becomes

$$2f_{00}' h_{01}'' + (6f_{00}'' + f_{00} f_{00}') h_{01}' + 2f_0''' h_{01} = F_1'.$$

The solution of the above equation can be obtained in the same way as that of Eq. (A-1) in "Appendix A" of reference [*2*]. The general solution is given by $h_{01} = A_1 h_{011} + h_{011'}$, where h_{011} is the complementary function, $h_{011'}$ is a particular integral, and A_1 is a constant to be determined from the boundary condition. For small η, the expansion of the homogeneous equation was obtained as

$$h_{011} = 1 + \frac{\alpha}{60}\eta^3 - \frac{\alpha^2}{2880}\eta^6 + \cdots,$$

where

$$\alpha = 0.3320573.$$

From this, h_{011} and h'_{011} can be evaluated at, say, $\eta=0$, 0.2, 0.4, and 0.6. The integration can be carried out at intervals of 0.2 in the independent variable η up to $\eta=10.0$, using numerical values of f_{00} obtained in ref. [5]. The results are presented in Fig. 4.

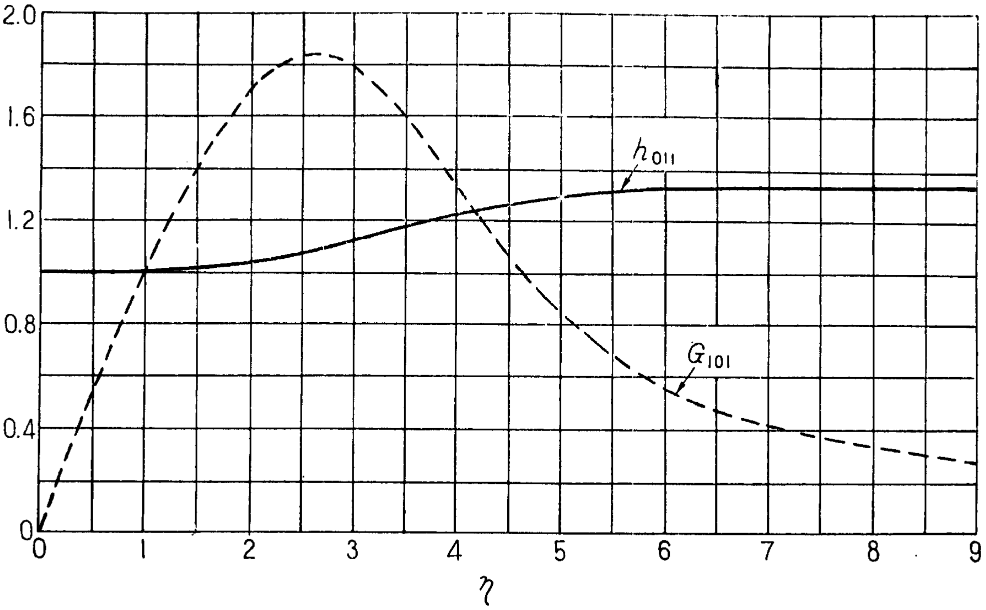

FIGURE 4

Once the homogeneous solution is known, the particular integral is given by

$$h_{01P}=-\frac{h_{011}}{2}\int_0^\eta \frac{f''_{00}}{h_{011}^2 f'^3_{00}}\left\{\left(\lambda_{00}-\frac{f_{00}}{2}\int_0^\eta \lambda_{00}d\eta\right)h_{011}f'^2_{00}d\eta\right\}d\eta\,.$$

The constant of integration A_1 can be determined from the boundary condition that h_{01} must vanish at the edge of the boundary layer, say, at $\eta=10.0$. Thus

$$A_1=-h_{01P}(10)/h_{011}(10)\,,$$

and therefore

$$h_{01}=\frac{J_1(10)}{h_{011}(10)}h_{011}(\eta)-J_1(\eta)+\frac{\gamma-1}{2}M_\infty^2\left\{\frac{J_1(10)}{h_{011}(10)}h_{011}(\eta)-J_2(\eta)\right\}$$

$$\equiv S_1(\eta)+\frac{\gamma-1}{2}M_\infty^2 S_2(\eta)\,,$$

where

$$J_1(\eta)=h_{011}\int_0^\eta \frac{f''_{00}}{h_{011}^2 f'^3_{00}}\left\{\int_0^\eta\left(1-\frac{f_{00}}{2}\eta\right)h_{011}f'^2_{00}d\eta\right\}d\eta\,,$$

$$J_2(\eta)=h_{011}\int_0^\eta \frac{f''_{00}}{h_{011}^2 f'^3_{00}}\left[\int_0^\eta\left\{1-f'^2_{00}-\frac{f_{00}}{2}\int_0^\eta(1-f'^2_{00})d\eta\right\}h_{011}f'^2_{00}d\eta\right]d\eta\,.$$

Thus the zeroth order solution can be obtained in the following form:

$$f_0=f_{00}+f_{01}\cdot 2\xi_0+\cdots\,,$$

where

$$f_{01}=f'_{00}\int_0^{\eta}S_1(\eta)d\eta+\frac{\gamma-1}{2}M_{\infty}^2f'_{00}\int_0^{\eta}S_2(\eta)d\eta\,.$$

In particular,

$$f''_{01}(0)=2f''_{00}A_1=\frac{2\alpha}{h_{011}(10)}\left\{J_1(10)+\frac{\gamma-1}{2}M_{\infty}^2J_2(10)\right\}$$
$$=0.288+0.117(\gamma-1)M_{\infty}^2\,.$$

In "Appendix A" of reference [2], the constant of integration A (A_1 in the present paper) are calculated for both a cone and a cylinder. It is connected with the perturbation friction from Mangler's value by the relation

$$C_f/(C_f)_{\text{Mangler}}=1+A_1\cdot 2\xi_0\,,$$

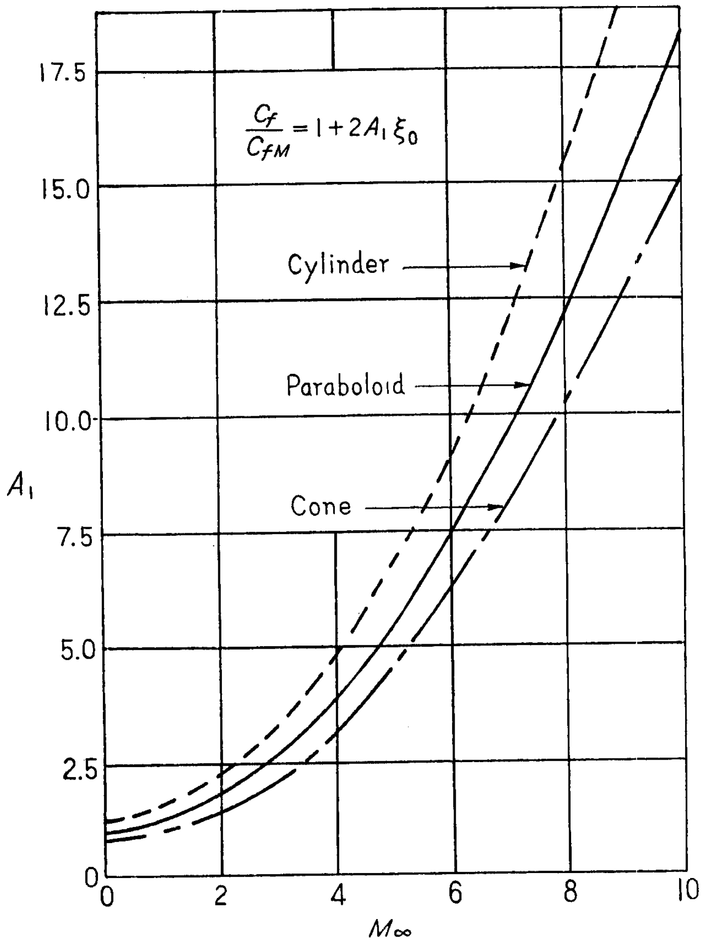

FIGURE 5

where ξ_0 denotes the slenderness parameter. This result is applicable to the case of arbitrary Mach number only when the inviscid outer flow is uniform. In Fig. 5, values of A_1 against M_∞ when $\gamma=1.4$ are given for a cone, a paraboloid of revolution and a cylinder assuming $P_r=1$ and $(\partial T/\partial y)_w=0$.

Appendix B

Numerical Integration of the First Order Momentum Equation for a Paraboloid

The first order equation for f_1 as given by Eq. (40) is

$$2f_1''' + f_0 f_1'' + f_0' f_1' + \frac{1+(2-\gamma)M_\infty^2}{4} f_0 f_0'' + \frac{1}{2}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)(1-f_0'^2)$$

$$= -2\xi_0\left[\left(f_0''\int\lambda_1 d\eta\right)' + \left(f_1''\int\lambda_0 d\eta\right)' + \frac{1}{4}\{1-(2-\gamma)M_\infty^2\}\left(f_0''\int\lambda_0 d\eta\right)'\right].$$

In the above equation, if the Prandtl number equals to unity and the surface is thermally insulated, then

$$\lambda_0 = 1+\frac{\gamma-1}{2}M_\infty^2(1-f_0'^2),$$

$$\lambda_1 = -\frac{\gamma-1}{2}M_\infty^2\left[\left(1+\frac{\gamma-1}{2}M_\infty^2\right)(1-f_0'^2)+2f_0'f_1'\right].$$

Transforming f_1 into $G_1 = f_1'$, and expanding G_1 in a power series of ξ_0 as

$$G_1 = G_{10} + O(\xi_0),$$

then, the equation to determine G_{10} is given by

$$2G_{10}'' + f_{00}G_{10}' + f_{00}'G_{10} = -\frac{1+(2-\gamma)M_\infty^2}{4} f_{00}f_{00}'' - \frac{1}{2}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)(1-f_{00}'^2) \equiv F_2.$$

Integrating once, the above one is reduced to

$$2G_{10}' + f_{00}G_{10} = 2A_2 + \int_0^\eta F_2 d\eta.$$

Integrating once more, the solution of G_{10} is obtained, giving

$$G_{10} = f_{00}''\left\{\int_0^\eta \left(2A_2 + \int_0^\eta F_2 d\eta\right)\frac{d\eta}{f_{00}''} + B_2\right\}.$$

From the boundary condition that $G_{10}(0)=0$, it is deduced that B_2 must be vanished. While from the condition $G_{10}(\infty)=0$, the value of A_2 cannot be determined as the right-hand side tends to vanish regardless of the value of A_2, due to the asymptotic nature of f_{00}. In the boundary layer theory, the method of steepest descent is usually applied, as a plausible procedure, in such cases to determine unknown constant. If such postulation is made in the present problem, A_2 can be found to be

$$A_2 = -\frac{1}{2}\int_0^\infty F_2 d\eta = \frac{1+(2-\gamma)M_\infty^2}{4}\alpha + \frac{1}{4}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\omega,$$

$$\alpha = f_{00}''(0) = 0.332, \quad \omega = \int_0^\infty (1-f_{00}'^2)d\eta = 2.385,$$

and therefore, f_{10}' and $f_{10}''(0)$ are given by

$$f'_{10}=G_{10}=\frac{1+(2-\gamma)M_\infty^2}{2}f''_{00}\eta+\frac{1}{2}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)f''_{00}\int_0^\eta \frac{\omega-\int_0^\eta(1-f'^2_{00})d\eta}{f''_{00}}d\eta\,,$$

$$f''_{10}(0)=G'_{10}(0)=A_2=0.679+\{0.083+0.215(\gamma-1)\}M_\infty^2\,.$$

Thus, G_{10} is known numerically as a function of f_{00}.

Last, it is required to see the asymptotic nature of the term

$$f''_{00}\int_0^\eta \frac{1}{f''_{00}}d\eta\equiv G_{101}\,,$$

appeared in the final expression of G_{10}, as $f''_{00}(\eta)$ tends to zero exponentially when $\eta\to\infty$, inversely the term in the integrand tends to increase infinitely. The solution of the Blasius' equation as $\eta\to\infty$ is given by [6]

$$f_{00}\approx\eta-\bar{\beta}+\bar{\gamma}\int_\infty^\eta\int_\infty^\eta e^{-\frac{1}{4}(\eta-\bar{\beta})^2}d\eta\,,$$

where

$$\bar{\gamma}=0.231\,,\qquad \bar{\beta}=1.721\,,$$

therefore

$$G_{101}(\eta\to\infty)\approx e^{-\frac{1}{4}(\eta-\bar{\beta})^2}\int_0^\eta e^{\frac{1}{4}(\eta-\bar{\beta})^2}d\eta$$

$$=\frac{2}{\eta-\bar{\beta}}+\frac{4}{(\eta-\bar{\beta})^3}+\cdots\,,$$

which diminishes by $O\,(1/\eta)$ as η increases, and the convergency is ascertained. In Fig. 4, G_{101} is also plotted against η, showing the convergency of this term.