Aeronautical Research Institute, University of Tokyo
Report No. 343, May 1959

Meta-theory of Mechanics of Continua Subject to Deformations of Arbitrary Magnitudes

(Duality of Definitions of Strain, Strain Increment and Stress for Elastic and Plastic Finite Deformations)

By

Yoshimaru YOSHIMURA

Summary. The present investigation provides a basic theory concerning the concepts of strain, strain increment and stress, which underlie mechanics of continua for small and finite deformations.

All the deformations of all the matters from fluids to solids, viewed from their mechanism, are classified into the two types, one is the elastic deformation due to the change in the distances among constituent particles and the other the plastic deformation due to the change in the mode of interconnection among particles. In order to construct a self-consistent theory for the two kinds of deformation over the whole range of small and finite deformations, the notion of strain and stress is needed to be introduced as being specific to each of the two types, and the conclusions are as follows:

1. The elastic strain is specified by the change in the geometrical configuration from the uniquely determinable undeformed state, and is defined by

$$ {}^{e}\boldsymbol{E}=\frac{1}{2}(g_{ij}-\mathring{g}_{ij})\mathring{e}^{i}\mathring{e}^{j}, $$

and the elastic strain increment by

$$ D\,{}^{e}\boldsymbol{E}=\frac{1}{2}Dg_{ij}\mathring{e}^{i}\mathring{e}^{j}. $$

Where $\mathring{g}_{ij}$ and g_{ij} represent the fundamental metric tensors before and after deformation respectively, and $\mathring{e}^{i}$ the vectors reciprocal to the basis $\mathring{e}_{i}$ in the undeformed state, referring to the Lagrangian coordinate system.

The strain ${}^{e}\boldsymbol{E}$ of a elastically deformed state does not depend on the deformation path up to the state, i.e.

$$ \oint D\,{}^{e}\boldsymbol{E}=0. $$

2. The quantity introduced primarily concerning the plastic deformation is the strain increment, which is specified by the change in the metric during the current infinitesimal deformation, as

$$ D\,{}^{p}\boldsymbol{E}=(D\,{}^{p}\varepsilon)_{ij}e^{i}e^{j}=\frac{1}{2}Dg_{ij}e^{i}e^{j}, $$

where e^{i} are the vectors reciprocal to the basis e_{i} in the deformed current state.

The plastic strain ${}^{p}\boldsymbol{E}={}^{p}\varepsilon^{ij}e^{i}e^{j}$ is obtainable by integrating the plastic strain increment $D\,{}^{p}\boldsymbol{E}$, hence the simultaneous differential equations

$$ D\,{}^{p}\varepsilon_{ij}-g^{rs}[{}^{p}\varepsilon_{rj}\nabla_{i}(Du)_{s}+{}^{p}\varepsilon_{ir}\nabla_{j}(Du)_{s}]=(D\,{}^{p}\varepsilon)_{ij}, $$

along a given path of deformation, where $(Du)_i$ are the components of the incremental displacement with regard to e^i. The plastic strain depends on the deformation path, but not on the change in the geometrical configuration, and therefore, it is regarded as a mechanical quantity nominated "strain history". This means that

$$\oint D^pE \neq 0.$$

3. Between D^eE and D^pE it holds the relation
$$D^eE = J \cdot D^pE \cdot \bar{J}, \quad J = \mathring{e}^i e_i.$$
4. The stress tensor $T = \sigma^{ij} e_i e_j$ which describes equilibrium condition is common to the both deformations, and is so defined as to give the actual force per unit of sectional area in the deformed state. This stress T is also the stress ^pT for describing, together with ^pE and D^pE, the plastic deformation.
5. The stress ^eT for describing the elastic deformation is defined by
$$^eT = \bar{J}^{-1} \cdot {}^pT \cdot J^{-1}.$$

By means of these dualistic definitions of strain and stress for the two types of deformation, the theories of elasticity and plasticity are emancipated from essential self-inconsistencies ever lurked, and reorganized from beginning under harmonious contrast, over the whole range of small and finite deformations.

1. Introduction

It will be needless to mention that various mechanics of continuum are essentially distinguished from each other by the mechanical equations of state, that is, hydrodynamics, theories of elasticity and plasticity, for example, by the compressibility and viscosity laws, the Hooke's law and the Lévy-Mises law. And these laws are described generally in terms of the strain, strain increment and stress, which are usually defined in common to all of the different kinds of deformation. That is, the strain has been defined by the difference of the geometrical configurations before and after deformation, and the strain increment as the change in such strain, whether the deformation may, for example, be elastic or plastic. Although this can be said rather natural in view of both the logical process of deriving the strain as the only geometry of deformation and the historical process of the development of the plasticity theory, which has succeeded to the theory of elasticity and inherited its methods and concepts without any modification, I can give no consent to this idea of strain and strain increment from the basic considerations on deformation as described below.

Deformations of real bodies can be classified into two groups in view of their microscopic mechanisms; one involves all the elastic deformations, such as the volume change of fluids, recoverable deformation of solids etc., due to the change in the interatomic distances, the other the plastic deformations in a broader meaning, such as the flow of fluids and metals etc., caused by the change in the mode of interatomic connection. Thus we are led to make all the deformations of all materials belong to either of the elastic and plastic deformations according as the interatomic connection is maintained or not.

The essential differences between these two types of deformation are that the former is the deformation with potential change specified by the change in the geometrical configuration from the undeformed state uniquely defined, while the latter, being an irrecoverable deformation accompanied by hysteresis phenomenon, is not specified by the change in the geometrical shape, but by the deformation path, and any deformed state can be regarded at the same time as an undeformed state.

It must be concluded from the above reasoning that *the strain for describing plastic deformation should be something dependent on the previous strain history, so that different from those for elastic deformation which are to be defined only from the change in the geometrical configuration and are independent of the path of deformation.* Though the differences between the values of these strain and strain increment for two kinds of deformation geometrically identical are second order small quantities for small deformations, they become finite for finite deformations. This implies that any erroneous idea, which has been allowed for practical purposes for small deformations, can no more be permitted for finite ones. Both the chaotic state of the coexistence of various theories of elasticity [*1*] and the failure of all attempts of general theory of plasticity [*2*] for finite deformations can be considered to be attributable to the indeterminateness of and confusion between the definitions of strain and stress for the two types of deformation. The logarithmic strain for the extensional deformation will be the only case hitherto introduced and correct as the strain for finite plastic deformation.

Another cause of the coexistence of the mutually inconsistent various theories of elasticity for finite deformation seems to be attributable to the routine use of the matrix method, which considers only the components of a vector or tensor and is not adequate for considering the change in the basic vectors or tensors due to deformation. What is correct in the basic idea among the legion of these elasticity theories can be said to be that of F.D. Murnaghan [*1*], but it also, having recourse to the matrix method, seems to involve some obscurities and mistakes not to be admitted.

The above statements will be enough to show the necessity of developing the mechanics of continuum, in particular elasticity and plasticity theories of solids, valid for the whole range of small and finite deformations, innovating the existing concepts of strain and stress and basing on their dualistic definitions proper to each kind of deformation. The theory of plasticity [*3*] based on the concept of such strain, i.e. the strain history, and valid for the finite deformation, has already been introduced by the present author, and lately the theory of elasticity [*4*] corresponding to such theory of plasticity was known to be formulated in a quite reasonable manner. In the present paper the author intends to focus his considerations to the main differences of the definitions of strain, strain increment and stress, on which the both theories are based. The systematic descriptions for the theory of elasticity are expected to be published in the near future. Thus the both theories will be seen to be reorganized from beginning under the charming contrast and harmony.

2. Essential Characters of Each of the Elastic and Plastic Deformations and the Differences between Them

It was stated in the introduction that *all the deformations can be classified into two groups of elastic and plastic deformations, according to the microscopic mechanism of deformation.* That is, *elastic deformation is that due to the change in the mutual distances among particles constituting the material,* the mode of their interconnection being unaltered. In such deformation the undeformed state with no external load is uniquely determined. Consequently *the deformed state, or the strain representing it, can be specified by the change in the geometrical configuration from the uniquely determined undeformed state, independently of the deformation path.* The strain increment can be given as the increment of such strain.

On the contrary, *plastic deformation is that due to the change in the mode of interconnection of particles*—for example, slip in metals—, the interatomic distances being maintained almost constant. Consequently, the microscopic structural change such as the change in the group pattern of dislocations in metallic crystals, being produced, even the deformed state which have the same geometrical configuration are generally in different conditions, according to the path of deformation up to the state. This means that *the strain in plastic deformation, which specifies the deformed state, cannot be the strain determined by the geometrical configuration so that valid for elastic deformation, but other one dependent on the deformation history, i.e. a strain, which we may legitimately designate as "the strain history".* That the strain for plastic deformation should be none other than the strain history is considered as the consequence of the fact that the strain can be obtained as the result of integration of the strain increment along some prescribed deformation path, and therefore that the quantity which can be defined *imprimis* concerning the deformation is the strain increment, but not the strain.

Another important character of plastic deformation is that any deformed state can be regarded as an equilibrium state with null stress as well as it is with the non-vanishing applied stress necessary for such deformation to be continued. In fact, we cannont find in any way, from the mechanical considerations alone, the strain of a given state of a given material or its annealed virgin state, so that cannot but treat, in the mechanical theory, the given state as the origin of strain, without having the complete knowledge about its previous history of deformation. Thus we are led to consider that *any plastically deformed state is regarded at the same time as an undeformed state, i.e. as the origin of strain,* and this may be said to stand rather to reason in view of the microscopic mechanism of plastic deformation mentioned above. In consequence, *plastic deformation is essentially a sequence of successive infinitesimal deformations, each of their incipient points being assumed as an undeformed state.* Hence the strain increments corresponding to the successive deformations should also be measured on the same principle. The strain for plastic deformation dependent on its path will be obtained by intergrating such strain increment along some deformation path.

The above mentioned characteristics of elastic and plastic deformations can be reduced to the following table of contrasted form.

Elastic deformation	Plastic deformation
Change in the distances among particles	Change in the mode of interconnection among particles
Change in the geometrical configuration	Microspcopic structural change (in metals, change in the group pattern of dislocations)
Independent of deformation path	Hysteresis phenomenon dependent on deformation path
Strain increment is derived from the strain defined primarily	Strain is obtained by integration from the strain increment defined primarily
Uniquely defined undeformed state	Multiplicity of undeformed state (current state of deformation is also an undeformed state)

3. Basic Conditions to be Satisfied by the Strain, Strain Increment and Stress for Elastic and Plastic Deformations Respectively

It is believed to have become clear from the statements in the preceding sections that *the definitions of strain, strain increment and stress should be quite different from each other according as the deformation is either elastic or plastic.* Our subject is now to clarify how to define these elementary quantities, on which the mechanics of continua are based, and for this purpose it is needed to give some considerations on the basic conditions, from which the definitions of strain, strain increment and stress for each of the elastic and plastic deformations are to be deduced.

Some of these conditions are common, and others are specific to each of the elastic and plastic deformations. The former are as follows:

(I) That the strain, strain increment and stress as well as other quantities are all tensors.

Both elasticity and plasticity laws as well as all other physical laws having their meanings independent of the coordinate system to which their formulations are referred, should reasonably be described by tensor equations, which have invariant forms. This requires such quantities as strain, strain increment and stress involved in these laws to be tensors.

(II) That the principle of virtual work holds in the usual form

$$\delta(\text{work})=\int_V \text{stress} \cdot\cdot\, \delta(\text{strain})dV, \tag{3.1}$$

where δ represents variation, dots .. double scalar product, or the trace (Spur) of the tensor product, of the two tensors and V the volume of the body in the initial or deformed state as the occasion may demand.

The validity of the virtual work principle in the form of (3.1) is necessary for the existence of elastic and plastic potentials, which play an important role for the legitimate construction of mathematical theory.

On the other hand, the conditions specific to each deformation are summerized as follows from the results given in the preceding section.

(III) Conditions for elastic strain* and strain increment: In elastic deformation, the strain, not the strain increment, is introduced straightway, and is specified by the change in the geometrical configuration from the undeformed state, which can be defined uniquely. The strain increment is derived as the increment of such strain.

(IV) Conditions for plastic strain* and strain increment: In plastic deformation, the strain increment, not the strain, is introduced primarily, and is specified by the change in the geometrical shape from a deformed current state assumed as an undeformed state to a succeeding one. The strain dependent on the deformation path can be obtained by integrating the strain increment along the path.

Specific conditions for the definition of stress are not explicitly given, but it will be derived from the conditions (II), (III) and (IV) as some objects proper to each of the elastic and plastic deformations.

Even the above conditions are not sufficient for defining strain and others, since there still remains the freedom of selecting the quantity, which specifies the geometrical configuration of the body. But this problem will be answered quite unartificially by employing as such quantity the metric of the space which deforms in conformity with the body.

4. Definition of Elastic Strain

As was stated in the preceding section, elastic strain is a tensor derived from the change in the metric of the Euclidian space corresponding to the material body before and after deformation.

Representing the position vector of a material point in the initial and deformed states by $\mathring{\boldsymbol{r}}$ and $\boldsymbol{r}$ respectively, and the displacement vector by $\boldsymbol{u}$,

$$\boldsymbol{r}=\mathring{\boldsymbol{r}}+\boldsymbol{u}, \tag{4.1}$$

so that, adverting to two adjacent material points,

$$d\boldsymbol{r}=d\mathring{\boldsymbol{r}}+d\boldsymbol{u}, \tag{4.2}$$

the operator "d", indicating the differential with respect to the space fixed to the material body, is expressed as

$$d\equiv dx^i\partial_i \tag{4.3}$$

for the Lagrangian coordinates x^i, where

$$\partial_i\equiv\frac{\partial}{\partial x^i}. \tag{4.4}$$

Introducing the gradient operator

$$\mathring{\nabla}\equiv\mathring{\boldsymbol{e}}^i\partial_i, \tag{4.5}$$

* The terminology "elastic (plastic) strain" is used hereafter to mean the strain valid for describing elastic (plastic) deformation, in addition to the existing implication of being the strain produced by elastic (plastic) deformation.

where

$$\overset{\circ}{\boldsymbol{e}}_i = \partial_i \overset{\circ}{\boldsymbol{r}}, \tag{4.6}$$

and

$$\overset{\circ}{\boldsymbol{e}}_i \cdot \overset{\circ}{\boldsymbol{e}}^j = \delta_i^j, \tag{4.7}$$

d can be expressed by

$$d = d\overset{\circ}{\boldsymbol{r}} \cdot \overset{\circ}{\nabla}. \tag{4.8}$$

Then (4.2) is written as

$$d\boldsymbol{r} = d\overset{\circ}{\boldsymbol{r}} \cdot \boldsymbol{J}, \tag{4.9}$$

where

$$\boldsymbol{J} = \boldsymbol{I} + \overset{\circ}{\nabla} \boldsymbol{u}, \tag{4.10}$$

$\boldsymbol{I}$ representing the unit tensor

$$\boldsymbol{I} = \overset{\circ}{\boldsymbol{e}}^i \overset{\circ}{\boldsymbol{e}}_i. \tag{4.11}$$

The metric of the Euclidian space corresponding to the deformed state of the body is seen, from (4.9), to be specified by

$$(d\boldsymbol{r})^2 = d\overset{\circ}{\boldsymbol{r}} \cdot \boldsymbol{J} \cdot \bar{\boldsymbol{J}} \cdot d\overset{\circ}{\boldsymbol{r}}, \tag{4.12}$$

where $\bar{\boldsymbol{J}}$ represents the tensor conjugate to $\boldsymbol{J}$. Hence

$$(d\boldsymbol{r})^2 - (d\overset{\circ}{\boldsymbol{r}})^2 = d\overset{\circ}{\boldsymbol{r}} \cdot (\boldsymbol{J} \cdot \bar{\boldsymbol{J}} - \boldsymbol{I}) \cdot d\overset{\circ}{\boldsymbol{r}}, \tag{4.13}$$

and the elastic strain tensor is found reasonable to be defined by

$$^e\boldsymbol{E} = \frac{1}{2}(\boldsymbol{J} \cdot \bar{\boldsymbol{J}} - \boldsymbol{I}). \tag{4.14*}$$

It is clear, as mentioned above, that this definition of the elastic strain satisfies the conditions (I) and (III). Whether the condition (II) is fulfilled or not depends on the definition of stress and it will be so introduced later as to satisfy the condition (II).

By means of (4.10), the elastic strain (4.14) can be written

$$^e\boldsymbol{E} = \frac{1}{2}[\overset{\circ}{\nabla}\boldsymbol{u} + \boldsymbol{u}\overset{\circ}{\nabla} + (\overset{\circ}{\nabla}\boldsymbol{u}) \cdot (\boldsymbol{u}\overset{\circ}{\nabla})], \tag{4.15}$$

so that setting

$$\boldsymbol{u} = \underset{\circ}{u}_i \overset{\circ}{\boldsymbol{e}}^i, \tag{4.16}$$

the analytical expression of (4.14) or (4.15) is given by

$$\left.\begin{aligned} ^e\boldsymbol{E} &= {}^e\underset{\circ}{\varepsilon}_{ij} \overset{\circ}{\boldsymbol{e}}^i \overset{\circ}{\boldsymbol{e}}^j, \\ ^e\underset{\circ}{\varepsilon}_{ij} &= \frac{1}{2}(\overset{\circ}{\nabla}_i \underset{\circ}{u}_j + \overset{\circ}{\nabla}_j \underset{\circ}{u}_i + \overset{\circ}{\nabla}_i \underset{\circ}{u}_r \overset{\circ}{\nabla}_j \underset{\circ}{u}^r), \end{aligned}\right\} \tag{4.17}$$

where

$$\overset{\circ}{\nabla}_i \underset{\circ}{u}_j = \partial_i \underset{\circ}{u}_j - \underset{\circ}{u}_r \overset{\circ}{\Gamma}{}^r_{ij}, \tag{4.18}$$

$$\left.\begin{aligned} \overset{\circ}{\Gamma}{}^k_{ij} &= \overset{\circ}{g}{}^{kr} \overset{\circ}{\Gamma}_{ij,r}, \\ \overset{\circ}{\Gamma}_{ij,k} &= \frac{1}{2}(\partial_i \overset{\circ}{g}_{jk} + \partial_j \overset{\circ}{g}_{ki} - \partial_k \overset{\circ}{g}_{ij}), \end{aligned}\right\} \tag{4.19}$$

$\overset{\circ}{g}_{ij}$ and $\overset{\circ}{g}{}^{ij}$ representing the fundamental metric tensors

* This definition is identical to that of Murnaghan (see [1]).

$$\mathring{g}_{ij}=\mathring{e}_i\cdot\mathring{e}_j\,,\quad \mathring{g}^{ij}=\mathring{e}^i\cdot\mathring{e}^j\,, \tag{4.20}$$

for the space corresponding to the undeformed state of the body.

By means of

$$d\mathring{r}=\mathring{e}_i\cdot dx^i\,,\quad dr=e_i\cdot dx^i\,, \tag{4.21}$$

$$e_i=\partial_i r\,, \tag{4.22}$$

(4.9) can be written as

$$e_i=\mathring{e}_i\cdot J\,, \tag{4.23}$$

so that it follows that

$$J=\mathring{e}^i e_i\,,\quad \bar{J}=e_i\mathring{e}^i\,. \tag{4.24}$$

By virtue of (4.24), (4.14) is expressed in the form

$${}^e E=\frac{1}{2}(g_{ij}-\mathring{g}_{ij})\mathring{e}^i\mathring{e}^j\,, \tag{4.25}$$

hence the components of ${}^e E$ as

$${}^e\underset{\circ}{\varepsilon}_{ij}=\frac{1}{2}(g_{ij}-\mathring{g}_{ij})\,. \tag{4.26}$$

Where g_{ij} is the fundamental metric tensor

$$g_{ij}=e_i\cdot e_j\,, \tag{4.27}$$

for the deformed state. The components ${}^e\underset{\circ}{\varepsilon}_{ij}$ of the elastic strain tensor are none other than those obtained so far by many authors, but it must be noticed that the basic tensors, to which they are referred, are $\mathring{e}^i\mathring{e}^j$, but not e^ie^j.

5. Elastic Strain Increment

Since the elastic strain velocity is essentially the same with the increment, we will now consider the latter on behalf of both of them. Denoting the parameter representing the time or the extent of deformation by t, and setting

$$D\equiv dt\frac{\partial}{\partial t}\,, \tag{5.1}$$

the elastic strain increment can be given as the increment of the elastic strain

$$D\,{}^e E=\frac{1}{2}(DJ\cdot\bar{J}+J\cdot D\bar{J})\,. \tag{5.2}$$

The operator $\mathring{\nabla}$ being time independent, $D\mathring{\nabla}=0$, so that we have

$$DJ=\mathring{\nabla}Du\,,\quad D\bar{J}=Du\mathring{\nabla}\,. \tag{5.3}$$

Let be introduced the operator of gradient

$$\nabla\equiv e^i\partial_i\,, \tag{5.4}$$

for the deformed state, then by virtue of

$$\mathring{\nabla}=J\cdot\nabla\,, \tag{5.5}$$

(5.3) can be written as

$$DJ=J\cdot\nabla Du\,,\quad D\bar{J}=Du\nabla\cdot\bar{J}\,. \tag{5.6}$$

Consequently, from (5.2), we obtain the result

$$D^{e}\boldsymbol{E}=\boldsymbol{J}\cdot\frac{1}{2}(\nabla D\boldsymbol{u}+D\boldsymbol{u}\nabla)\cdot\bar{\boldsymbol{J}}. \tag{5.7}$$

By putting

$$D\boldsymbol{u}=(Du)_i\boldsymbol{e}^i, \tag{5.8}$$

the analytical expression of $D^e\boldsymbol{E}$ is obtained as

$$D^{e}\boldsymbol{E}=\frac{1}{2}\left[\nabla_i(Du)_j+\nabla_j(Du)_i\right]\boldsymbol{J}\cdot\boldsymbol{e}^i\boldsymbol{e}^j\cdot\bar{\boldsymbol{J}}, \tag{5.9}$$

so that as

$$D^{e}\boldsymbol{E}=\frac{1}{2}\left[\nabla_i(Du)_j+\nabla_j(Du)_i\right]\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j. \tag{5.10}$$

Where $\nabla_i(Du)_j$ indicating the covariant derivative of $(Du)_i$,

$$\nabla_i(Du)_j=\partial_i(Du)_j-(Du)_r\Gamma^r_{ij}, \tag{5.11}$$

$$\left.\begin{aligned}\Gamma^k_{ij}&=g^{kr}\Gamma_{ij,r},\\ \Gamma_{ij,k}&=\frac{1}{2}(\partial_i g_{jk}+\partial_j g_{ki}-\partial_k g_{ij}).\end{aligned}\right\} \tag{5.12}$$

From (4.25), on the other hand, we obtain

$$D^{e}\boldsymbol{E}=\frac{1}{2}Dg_{ij}\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j. \tag{5.13}$$

These results (5.10) and (5.13) can be rewritten as

$$\left.\begin{aligned}D^{e}\boldsymbol{E}&=(D^e_{\circ}\varepsilon)_{ij}\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j,\\ (D^e_{\circ}\varepsilon)_{ij}&=\frac{1}{2}\left[\nabla_i(Du)_j+\nabla_j(Du)_i\right]=\frac{1}{2}Dg_{ij}.\end{aligned}\right\} \tag{5.14}$$

As we have, from (4.17) on the other hand, the relation

$$D^{e}\boldsymbol{E}=D^e\underset{\circ}{\varepsilon}_{ij}\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j, \tag{5.15}$$

it follows that

$$(D^e_{\circ}\varepsilon)_{ij}=D^e\underset{\circ}{\varepsilon}_{ij}. \tag{5.16}$$

That is, the components of the elastic strain increment are equal to the increment of the elastic strain components, when referred to the basic tensors $\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j$.

6. Definition of the Plastic Strain Increment

It was mentioned already that in plastic deformation the strain increment, not the strain, should be defined *imprimis* by assuming the current deformed state from which the increment are measured as an undeformed state. The theory of plasticity [3] base on such definition of strain increment and consequently valid for finite deformation and strain history phenomena has already been developed by the present author in a completely reasonable form. In the present paper, we will now give an outline of its definition with the implication of setting it against and making clear its distinction with that of the elastic strain increment.

Representing the position vector of a material point for the states t and $t+dt$ by $\boldsymbol{r}$ and $\boldsymbol{r}'$ respectively, and the displacement vector from the state t to the state $t+dt$ by $D\boldsymbol{u}$, we have

$$r' = r + Dr = r + Du, \tag{6.1}$$

so that

$$dr' = dr + dDr = dr + dDu, \tag{6.2}$$

where d and D are given by (4.3) and (5.1) respectively. Since the deformed state t is regarded, by the condition (IV), as an undeformed state, the d should be expressed by

$$d = dr \cdot \nabla, \tag{6.3}$$

as against (4.8), ∇ being the operator defined by (5.4). By means of (6.3), (6.2) can be written

$$dr' = dr \cdot (I + \nabla Du), \tag{6.4}$$

where I indicates the unit tensor

$$I = e_i e^i = g_{ij} e^i e^j = g^{ij} e_i e_j. \tag{6.5}$$

Since the displacement Du is infinitesimal, the plastic strain increment can be defined by

$$D^pE = \frac{1}{2}(\nabla Du + Du\nabla), \tag{6.6}$$

as the symmetric part of the tensor ∇Du in (6.4) or as the first order infinitesimals of the tensor D^pE specified by $(dr')^2 - (dr)^2 = dr \cdot 2D^pE \cdot dr$. It is obvious that the definition (6.6) satisfies the conditions (I) and (II), and will be proved that it fulfills the condition (III), together with some reasonable definition of stress, which will be introduced later on.

In plastic deformation, the current state t being assumed as an undeformed state, so that as a standard state, to which the displacement Du and others are referred, we can put

$$Du = (Du)_i e^i, \tag{6.7}$$

and therefore, obtain the analytical expression of the strain increment

$$\left.\begin{aligned} D^pE &= (D^p\varepsilon)_{ij} e^i e^j, \\ (D^p\varepsilon)_{ij} &= \frac{1}{2}\left[\nabla_i (Du)_j + \nabla_j (Du)_i\right], \end{aligned}\right\} \tag{6.8}$$

where $\nabla_i (Du)_j$ are given by (5.11) and (5.12).

The definition (6.6) is easily proved to be expressed also by

$$\left.\begin{aligned} D^pE &= (D^p\varepsilon)_{ij} e^i e^j, \\ (D^p\varepsilon)_{ij} &= \frac{1}{2} Dg_{ij}. \end{aligned}\right\} \tag{6.9}$$

7. Relation between the Elastic Strain Increment and the Plastic Strain Increment

It can easily be found by comparing (5.7) and (6.6) that it holds the relation between the elastic and plastic strain increments

$$D^eE = J \cdot D^pE \cdot \bar{J} \tag{7.1}$$

or

$$D^pE = J^{-1} \cdot D^eE \cdot \bar{J}^{-1} \tag{7.2}$$

for the same change in the geometrical configuration, where $\boldsymbol{J}$ and $\bar{\boldsymbol{J}}$ are given by (4.24), and therefor $\bar{\boldsymbol{J}}$ and $\bar{\boldsymbol{J}}^{-1}$ by

$$\bar{\boldsymbol{J}}^{-1}=\boldsymbol{e}^i\mathring{\boldsymbol{e}}_i\,,\quad \bar{\boldsymbol{J}}^{-1}=\mathring{\boldsymbol{e}}_i\boldsymbol{e}^i\,. \tag{7.3}$$

If $D\,{}^e\boldsymbol{E}$ and $D\,{}^p\boldsymbol{E}$ are expressed, referring to the basic tensors $\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j$ and $\boldsymbol{e}^i\boldsymbol{e}^j$ respectively as shown in the first equations (5.14), and (6.8) and (6.9), the relation (7.1) or (7.2) is reduced to the equality of their components

$$(D^e_{\circ}\varepsilon)_{ij}=D^e\underset{\circ}{\varepsilon}_{ij}=(D^p\varepsilon)_{ij}\,, \tag{7.4}$$

which is really seen in the second equations (5.14), and (6.8) and (6.9). That is, the elastic and plastic strain increments for an infinitesimal deformation geometrically identical are distinguished from each other by the fact that the same components are referred to the different basic tensors $\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j$ and $\boldsymbol{e}^i\boldsymbol{e}^j$ respectively. If the basic tensors are equalized, the components become different.

8. Plastic Strain, i.e. Strain History

The plastic strain, i.e. the strain history, ${}^p\boldsymbol{E}$ is obtained by integrating the plastic strain increment $D\,{}^p\boldsymbol{E}$ along a given deformation path, that is, by

$${}^p\boldsymbol{E}=\int_0^t D^p\boldsymbol{E}\,. \tag{8.1}$$

If ${}^p\boldsymbol{E}$ is assumed to have been obtained in any way as

$${}^p\boldsymbol{E}={}^p\varepsilon_{ij}\boldsymbol{e}^i\boldsymbol{e}^j\,, \tag{8.2}$$

it follows that

$$D({}^p\varepsilon_{ij}\boldsymbol{e}^i\boldsymbol{e}^j)=(D^p\varepsilon)_{ij}\boldsymbol{e}^i\boldsymbol{e}^j \tag{8.3}$$

so that, by means of

$$D\boldsymbol{e}^i=-g^{ir}\dot{\nabla}_s(Du)_r\boldsymbol{e}^s\,, \tag{8.4}$$

the simultaneous differential equations for the components

$$D^p\varepsilon_{ij}-g^{rs}[{}^p\varepsilon_{rj}\nabla_i(Du)_s+{}^p\varepsilon_{ir}\nabla_j(Du)_s]=(D^p\varepsilon)_{ij}\,. \tag{8.5}$$

The presence of the coefficients $\nabla_i(Du)_s$, $\nabla_j(Du)_s$ and $(D^p\varepsilon)_{ij}$ in the equations represents the dependence of their solution ${}^p\varepsilon_{ij}$ on the deformation path, i.e. generally

$$\oint D\,{}^p\boldsymbol{E}\neq 0\,. \tag{8.6}$$

It will be needless to mention, on the contrary, that it holds

$$\oint D\,{}^e\boldsymbol{E}=0, \tag{8.7}$$

for the elastic deformation.

While the elastic strain components ${}^e\underset{\circ}{\varepsilon}_{ij}$ obtained in section 4 are referred to the basic tensors $\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j$, the plastic strain components ${}^p\varepsilon_{ij}$, which are to be obtained from (8.5), are referred to $\boldsymbol{e}^i\boldsymbol{e}^j$. The components of ${}^p\boldsymbol{E}$ referred to $\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j$ are shown to be obtained by the method of local coordinate system [3], by which the frame in the neighbourhood of each material point, so that the base vectors, are convected with the point without deformation. Now we will indicate such coordinates, which are non-holonomic in the deformed state by dx^i and such basic vectors equal to $\mathring{e}_i$ by e_i with Gothic index, then the components ${}^p\varepsilon_{ij}$ referred to the basic tensors $e^i e^j (=\mathring{e}^i \mathring{e}^j)$ are obtained by integrating the differential equations

$$(D^p\varepsilon)_{ij} = D^p\varepsilon_{ij}. \tag{8.8}$$

The implication of (8.8) is the same with that of (8.5), only their analytical expressions being different.

9. Comparison between the Elastic and the Plastic Strains for Some Particular Deformations

We will now exemplify, for some particular deformations, how the elastic and the plastic strains and their increments so far introduced are different from each other for the geometrically identical deformation.

(1) *Extensional Deformation*

As shown in Fig. 1, let the Lagrangian coordinate system for $t=0$ be coincident with the rectangular Cartesian coordinate system, and the cube with the edges of the length l_0 parallel to the axes be converted into a parallelepiped with the edges l_1, l_2 and l_3, being elongated by

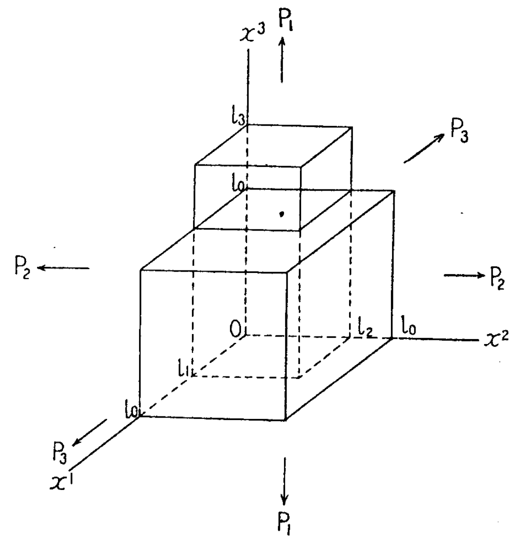

Figure 1.

$$n_1 = \frac{l_1}{l_0}, \quad n_2 = \frac{l_2}{l_0} \quad \text{and} \quad n_3 = \frac{l_3}{l_0}, \tag{9.1}$$

times in the axial directions respectively, then, from (4.26), the components of the elastic strain for such deformation are

$$\left.\begin{aligned} {}^e\varepsilon_{11} &= \frac{1}{2}\frac{l_1^2 - l_0^2}{l_0^2} = \frac{1}{2}(n_1^2 - 1), \\ {}^e\varepsilon_{22} &= \frac{1}{2}\frac{l_2^2 - l_0^2}{l_0^2} = \frac{1}{2}(n_2^2 - 1), \end{aligned}\right\} \tag{9.2}$$

$$\left.\begin{aligned}{}^{e}\underset{\circ}{\varepsilon}_{33}&=\frac{1}{2}\frac{l_3^2-l_0^2}{l_0^2}=\frac{1}{2}(n_3^2-1)\,,\\ {}^{e}\underset{\circ}{\varepsilon}_{23}&={}^{e}\underset{\circ}{\varepsilon}_{31}={}^{e}\underset{\circ}{\varepsilon}_{12}=0\,,\end{aligned}\right\}$$

so that the components of the elastic strain increment are

$$\left.\begin{aligned}D^{e}\underset{\circ}{\varepsilon}_{11}&=\frac{l_1Dl_1}{l_0^2}=n_1Dn_1\,,\\ D^{e}\underset{\circ}{\varepsilon}_{22}&=\frac{l_2Dl_2}{l_0^2}=n_2Dn_2\,,\\ D^{e}\underset{\circ}{\varepsilon}_{33}&=\frac{l_3Dl_3}{l_0^2}=n_3Dn_3\,,\\ D^{e}\underset{\circ}{\varepsilon}_{23}&=D^{e}\underset{\circ}{\varepsilon}_{31}=D^{e}\underset{\circ}{\varepsilon}_{12}=0\,,\end{aligned}\right\}\tag{9.3}$$

referring to the basic tensors $\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j$ composed of the unit base vectors $\mathring{\boldsymbol{e}}^i$ for the rectangular Cartesian coordinate system.

On the other hand, the components of the plastic strain increment for the same infinitesimal extension Dl_1, Dl_2, Dl_3 in the axial directions are given by

$$\left.\begin{aligned}(D^{p}\varepsilon)_{11}&=\frac{Dl_1}{l_1}=\frac{Dn_1}{n_1}\,,\\ (D^{p}\varepsilon)_{22}&=\frac{Dl_2}{l_2}=\frac{Dn_2}{n_2}\,,\\ (D^{p}\varepsilon)_{33}&=\frac{Dl_3}{l_3}=\frac{Dn_3}{n_3}\,,\\ (D^{p}\varepsilon)_{23}&=(D^{p}\varepsilon)_{31}=(D^{p}\varepsilon)_{12}=0\,,\end{aligned}\right\}\tag{9.4}$$

referring to the basic tensors $\boldsymbol{e}^i\boldsymbol{e}^j$ for the rectangular Cartesian local coordinate system, equivalent to $\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j$. The components of the plastic strain for the same extensional deformation, as in the case of elasic deformation, are obtained as

$$\left.\begin{aligned}{}^{p}\varepsilon_{11}&=\log n_1\,,\\ {}^{p}\varepsilon_{22}&=\log n_2\,,\\ {}^{p}\varepsilon_{33}&=\log n_3\,,\\ {}^{p}\varepsilon_{23}&={}^{p}\varepsilon_{31}={}^{p}\varepsilon_{12}=0\,,\end{aligned}\right\}\tag{9.5}$$

by integrating (9.4) along the prescribed deformation path. In the case of plastic deformation, in particular, it holds the incompressibility relation

$$n_1n_2n_3=1\,.\tag{9.6}$$

The comparison between (9.2) and (9.5), and (9.3) and (9.4) gives us clear information how the elastic and the plastic strains and strain increments differ from each other for the triaxial extensional deformation geometrically identical.

(2) *Simple Shear*

As shown in Fig. 2, let the cube OABCLMNP with the edges of unit length in the axial directions be converted into a parallelepiped OABC′L′M′NP′ by the simple shear specified by $CC'=\gamma$, then the elastic strain components referred to the basic tensors $\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j$ for such simple shear are obtained, by means of (4.26), as

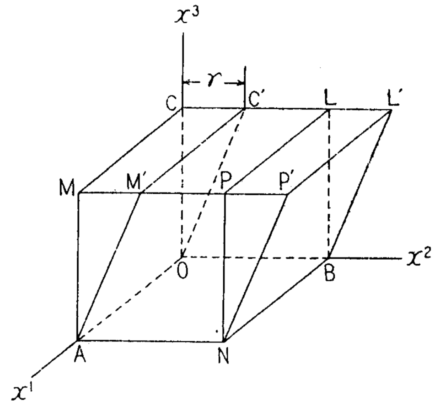

FIGURE 2.

$$\left.\begin{array}{l} {}^{e}\underset{\circ}{\varepsilon}_{23}=\dfrac{1}{2}\gamma\,, \qquad {}^{e}\underset{\circ}{\varepsilon}_{33}=\dfrac{1}{2}\gamma^{2}\,, \\ \text{the other} \quad {}^{e}\underset{\circ}{\varepsilon}_{ij}{}'s=0\,, \end{array}\right\} \tag{9.7}$$

so that the components of the elastic strain increment as

$$\left.\begin{array}{l} D^{e}\underset{\circ}{\varepsilon}_{23}=\dfrac{1}{2}D\gamma\,, \quad D^{e}\underset{\circ}{\varepsilon}_{33}=\gamma D\gamma\,, \\ \text{the other} \quad D^{e}\underset{\circ}{\varepsilon}_{ij}{}'s=0\,. \end{array}\right\} \tag{9.8}$$

The components of the plastic strain increment for the same simple shear as above are obtained, from (6.9), as

$$\left.\begin{array}{l} D^{p}\varepsilon_{23}=\dfrac{1}{2}D\gamma\,, \\ \text{the other} \quad D^{p}\varepsilon_{ij}{}'s=0\,, \end{array}\right\} \tag{9.9}$$

referring to $\boldsymbol{e}^i\boldsymbol{e}^j$ equivalent to $\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j$, so that the components of the plastic strain also for the same simple shear as

$$\left.\begin{array}{l} {}^{p}\varepsilon_{23}=\dfrac{1}{2}\gamma\,, \\ \text{the other} \quad {}^{p}\varepsilon_{ij}{}'s=0\,. \end{array}\right\} \tag{9.10}$$

It is found from (9.7) and (9.10) that for elastic simple shear the normal strain component in the x^3 direction has some value but zero, and therefore the principal direction makes an angle larger than 45° with the x^2 axis, but for plastic simple shear the nomal component vanishing, the angle of the principal direction is equal to 45°, no matter how large the deformation may be.

(3) *Combination of Triaxial Extension and Simple Shear*

We suppose that, as shown in Fig. 3, the unit cube is converted into a parallelepiped by the combination of axial extensions by n_1, n_2, n_3 times and simple shear specified by the value γ' of the parameter γ. When this deformation is carried out plastically, its path comes into question in general, but when it is elastic, the deformation path does not matter in determining the strain.

The elastic strain is given by the components referred to the basic tensors $\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j$

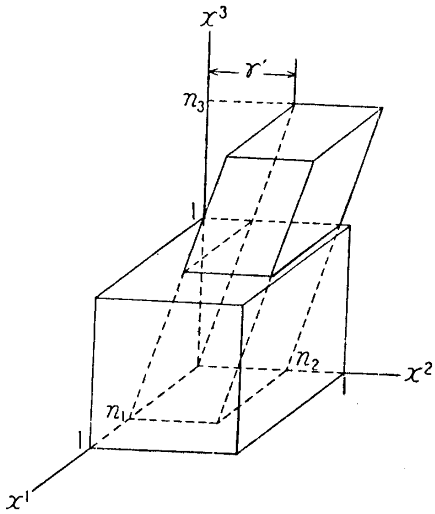

FIGURE 3.

$$\left.\begin{array}{l} {}^e\varepsilon_{11}=\frac{1}{2}(n_1^2-1), \quad {}^e\varepsilon_{22}=\frac{1}{2}(n_2^2-1), \quad {}^e\varepsilon_{33}=\frac{1}{2}(n_3^2+\gamma'^2-1), \\ {}^e\varepsilon_{23}=\frac{1}{2}n_2\gamma', \quad {}^e\varepsilon_{31}={}^e\varepsilon_{12}=0, \end{array}\right\} \quad (9.11)$$

and the elastic strain increment by the components

$$\left.\begin{array}{l} D^e\varepsilon_{11}=n_1Dn_1, \quad D^e\varepsilon_{22}=n_2Dn_2, \quad D^e\varepsilon_{33}=n_3Dn_3+\gamma D\gamma, \\ D^e\varepsilon_{23}=\frac{1}{2}(n_2D\gamma+\gamma Dn_2), \quad D^e\varepsilon_{31}=D^e\varepsilon_{12}=0. \end{array}\right\} \quad (9.12)$$

The plastic strain increment is given by

$$\left.\begin{array}{l} D^p\varepsilon_{11}=\frac{Dn_1}{n_1}, \quad D^p\varepsilon_{22}=\frac{Dn_2}{n_2}, \quad D^p\varepsilon_{33}=\frac{Dn_3}{n_3}, \\ D^p\varepsilon_{23}=-\frac{1}{2}\frac{\gamma}{n_3}\frac{Dn_2}{n_2}+\frac{1}{2}\frac{D\gamma}{n_3}, \quad D^p\varepsilon_{31}=D^p\varepsilon_{12}=0, \end{array}\right\} \quad (9.13)$$

referring to the basic tensors $\boldsymbol{e}^i\boldsymbol{e}^j$ equivalent to $\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j$.

The plastic strain is obtained by integrating (9.13) along a prescribed path of deformation. As shown in Fig. 4, when the unit cube is first extended by n_1, n_2,

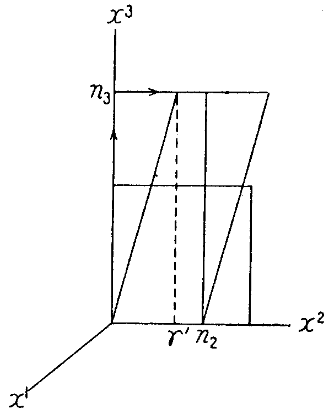

FIGURE 4.

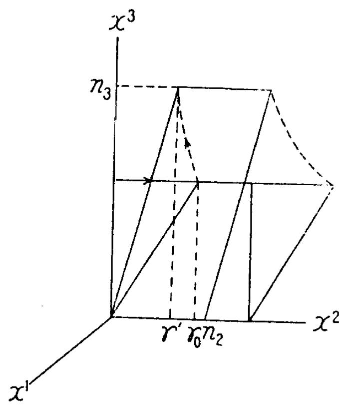

FIGURE 5.

n_3 times in the x^1, x^2, x^3 directions respectively and then subjected to simple shear up to the state $\gamma=\gamma'$, keeping n_1, n_2, n_3 constant, the strain for the final state specified by n_1, n_2, n_3 and γ' is given by

$$\left.\begin{array}{l} {}^p\varepsilon_{11}=\log n_1\,, \quad {}^p\varepsilon_{22}=\log n_2\,, \quad {}^p\varepsilon_{33}=\log n_3\,, \\ {}^p\varepsilon_{23}=\dfrac{1}{2}\dfrac{\gamma'}{n_3}\,, \quad {}^p\varepsilon_{31}=0\,, \quad {}^p\varepsilon_{12}=0\,. \end{array}\right\} \tag{9.14}$$

We suppose, on the other hand, the order of deformation being reversed, the unit cube is first subjected to simple shear γ_0 and then extended by n_1, n_2, n_3 times in the respective axial directions as shown in Fig. 5. In this case, the value of γ varies in the second stage of the deformation process, i.e. the stage of triaxial extension, according to the relation

$$\frac{\gamma}{\gamma_0}=n_2\,, \tag{9.15}$$

(here n_2 means current value) and finally assumes the value γ' for the prescribed magnifications n_1, n_2 and n_3. The strain is obtained by integrating (9.13) under the condition (9.15), as

$$\left.\begin{array}{l} {}^p\varepsilon_{11}=\log n_1\,, \quad {}^p\varepsilon_{22}=\log n_2\,, \quad {}^p\varepsilon_{33}=\log n_3\,, \\ {}^p\varepsilon_{23}=\dfrac{1}{2}\gamma_0=\dfrac{1}{2}\dfrac{\gamma'}{n_2}\,, \quad {}^p\varepsilon_{31}={}^p\varepsilon_{12}=0\,. \end{array}\right\} \tag{9.16}$$

The comparison among the results (9.11), (9.14) and (9.16) gives us the clear information that the strain of itself is not same for the geometrically same final state of deformation, according as the deformation is either elastic or plastic, and in particular, when plastic, according to the deformation path up to the state.

10. Stress

The next fundamental problem for solution is that how to difine stresses corresponding to the strains and strain increments already introduced for each of the elastic and plastic deformations. Such stresses, together with the strain increments, were seen to be so defined as to satisfy the basic conditions (I) and (II) for tensority and the virtual work principle, so that are supposed to be specific for each kind of the deformations.

On the other hand, the stress fields for the two state of deformation, one the elastic and the other the plastic, are to be identical, when they are caused by the same external forces exerted on the bodies in a state of the same geometrical configuration. This is considered to mean that the stress to describe the equilibrium equations is defined independently of the kind of deformation. If so, then arises the problem how such stress are related to those which are to describe the state equations and specific to each of the elastic and plastic deformations.

Before proceeding to such cardinal subject as above, we must now begin with the stress which is caused by the external forces so that governs the equilibrium condition. The physical state of such stress is definite, but its mathematical expression as a tensor would be given in various ways. Namely, expressed analogously to the case of simple extension, we can think of, for example, the nominal stress as well as the true stress. What stress would be used is at our disposal, only influencing the form of the equilibrium equations, but not their physical implication. But the true stress in its general meaning, that is, the stress referred to unit of area in the deformed state, is most fitted, as a matter of course, for the purpose of describing the equilibrium condition. Any other stress, when necessary to consider, can be derived from this generalized true stress.

Denoting the surface element in the deformed state by $\boldsymbol{n}dS$, $\boldsymbol{n}$ being unit normal, and the force exerted through it by $\boldsymbol{f}dS$, the generalized true stress is defined by the tensor $\boldsymbol{T}$ which satisfies

$$(\boldsymbol{n}dS)\cdot\boldsymbol{T}=\boldsymbol{f}dS,$$

so that

$$\boldsymbol{n}\cdot\boldsymbol{T}=\boldsymbol{f}, \tag{10.1}$$

whether the deformation may be elastic or plastic.

The stress tensor $\boldsymbol{T}$ which satisfies (10.1) is determined by prescribing the stress vectors $\boldsymbol{f}_{\text{I}}$, $\boldsymbol{f}_{\text{II}}$ and $\boldsymbol{f}_{\text{III}}$ corresponding to the three mutually independent directions $\boldsymbol{n}_{\text{I}}$, $\boldsymbol{n}_{\text{II}}$ and $\boldsymbol{n}_{\text{III}}$ as

$$\boldsymbol{T}=\boldsymbol{n}^{\lambda}\boldsymbol{f}_{\lambda}, \tag{10.2}$$

where the Greek index, say λ, corresponding to some directions different in general from the axial directions with the Roman index represents I, II, III.

If now $\boldsymbol{n}^{\lambda}$ and $\boldsymbol{f}_{\lambda}$ are represented as

$$\boldsymbol{n}^{\lambda}=n^{\lambda i}\boldsymbol{e}_i, \tag{10.3}$$

$$\boldsymbol{f}_{\lambda}=f_{\lambda}^{i}\boldsymbol{e}_i \tag{10.4}$$

(10.2) can be written as

$$\left.\begin{aligned}\boldsymbol{T}&=\sigma^{ij}\boldsymbol{e}_i\boldsymbol{e}_j,\\ \sigma^{ij}&=n^{\lambda i}f_{\lambda}{}^{j}.\end{aligned}\right\} \tag{10.5}$$

If, in particular, $\boldsymbol{n}_{\text{I}}$, $\boldsymbol{n}_{\text{II}}$ and $\boldsymbol{n}_{\text{III}}$ are chosen equal to $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, so that $\boldsymbol{n}^{\text{I}}$, $\boldsymbol{n}^{\text{II}}$ and $\boldsymbol{n}^{\text{III}}$ to $\boldsymbol{e}^1$, $\boldsymbol{e}^2$ and $\boldsymbol{e}^3$ respectively, $\boldsymbol{T}$ is expressed as

$$\boldsymbol{T}=\boldsymbol{e}^{l}\boldsymbol{f}_{l}, \tag{10.6}$$

instead of (10.2), bacause $\boldsymbol{f}_{\lambda}$ is written $\boldsymbol{f}_{l}$. (10.3) and (10.4) being replaced by

$$\boldsymbol{e}^{l}=g^{li}\boldsymbol{e}_i, \tag{10.7}$$

$$\boldsymbol{f}_{l}=f_{l}{}^{i}\boldsymbol{e}_i, \tag{10.8}$$

we have

$$\left.\begin{aligned}\boldsymbol{T}&=\sigma^{ij}\boldsymbol{e}_i\boldsymbol{e}_j,\\ \sigma^{ij}&=g^{li}f_{l}^{j}=f^{ij}.\end{aligned}\right\} \tag{10.9}$$

When the local coordinate system is used in particular, the stress $\boldsymbol{T}$ is expressed by

$$\boldsymbol{T}=\sigma^{ij}\boldsymbol{e}_i\boldsymbol{e}_j. \tag{10.10}$$

If the deformation, so that the relation between $\boldsymbol{e}_i$ and $\boldsymbol{e}_i$, is known, σ^{ij} are obtainable from σ^{ij}.

11. Equilibrium Equations

The equilibrium conditions for the portion of the body which has in the deformed state the volume V and the surface S as shown in Fig. 6 are given by

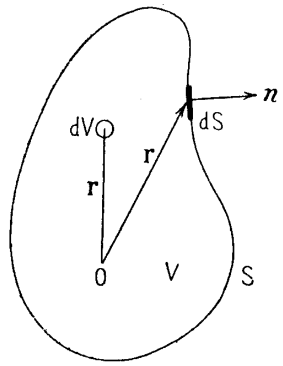

Figure 6.

$$\int \rho \boldsymbol{F} dV + \int \boldsymbol{n} \cdot \boldsymbol{T} dS = 0 \tag{11.1}$$

and

$$\int \boldsymbol{r} \times \rho \boldsymbol{F} dV + \int \boldsymbol{r} \times (\boldsymbol{n} \cdot \boldsymbol{T}) dS = 0 , \tag{11.2}$$

whether the deformation may be elastic or plastic. Where ρ indicates the density in the deformed state, $\boldsymbol{F}$ the body force per unit of mass.

Applying the Gauss' theorem

$$\int_S \boldsymbol{n} \cdot \boldsymbol{T} dS = \int_V \nabla \cdot \boldsymbol{T} dV \tag{11.3}$$

to (11.1), we obtain

$$\int_V (\rho \boldsymbol{F} + \nabla \cdot \boldsymbol{T}) dV = 0 , \tag{11.4}$$

so that

$$\nabla \cdot \boldsymbol{T} + \rho \boldsymbol{F} = 0 . \tag{11.5}$$

This is the equilibrium equation aimed at and valid in common to the both kinds of deformation.

Then applying to (11.2) the Gauss' theorem (11.3) whose $\boldsymbol{T}$ is replaced by $\boldsymbol{T} \times \boldsymbol{r}$, and considering (11.5), we obtain the relation

$$\boldsymbol{e}^r \cdot \boldsymbol{T} \times \boldsymbol{e}_r = 0 , \tag{11.6}$$

representing the tensor $\boldsymbol{T}$ to be symmetric.

12. Principle of Virtual Work

The work when the portion of the body which is in equilibrium in the deformed state as shown in Fig. 6 is displaced virtually by $\delta \boldsymbol{u}$ is given by

$$\delta W = \int_V \rho \boldsymbol{F} \cdot \delta \boldsymbol{u} dV + \int_S \boldsymbol{n} \cdot \boldsymbol{T} \cdot \delta \boldsymbol{u} dS . \tag{12.1}$$

whether the deformation may be elastic or plastic. By applying the Gauss' theorem (11.3) and then the equilibrium equation (11.5), to the second term in the right hand side of (12.1), it can be transformed to a volume integral

$$\delta W=\int \boldsymbol{T}\cdot\cdot\frac{1}{2}(\nabla\delta\boldsymbol{u}+\delta\boldsymbol{u}\nabla)dV. \tag{12.2}$$

By (6.6), (12.2) is written

$$\delta W=\int \boldsymbol{T}\cdot\cdot\delta^p\boldsymbol{E}dV. \tag{12.3}$$

This is the expression of the virtual work principle for the plastic deformation.

On account of the relation (7.1) between $D^p\boldsymbol{E}$ and $D^e\boldsymbol{E}$, (12.3) is further converted to the form

$$\delta W=\int \bar{\boldsymbol{J}}^{-1}\cdot\boldsymbol{T}\cdot\boldsymbol{J}^{-1}\cdot\cdot\delta^e\boldsymbol{E}dV. \tag{12.4}$$

Putting

$${}^e\boldsymbol{T}=\bar{\boldsymbol{J}}^{-1}\cdot\boldsymbol{T}\cdot\boldsymbol{J}^{-1}, \tag{12.5}$$

(12.4) is rewritten as

$$\delta W=\int {}^e\boldsymbol{T}\cdot\cdot\delta^e\boldsymbol{E}dV. \tag{12.6}$$

This is the virtual work principle for the elastic deformation, and means that *the elastic stress corresponding to the elastic strain increment $\delta^e\boldsymbol{E}$ is not $\boldsymbol{T}$, but ${}^e\boldsymbol{T}$ defined by* (12.5).

It is natural to express the stress tensor $\boldsymbol{T}$ by

$$\boldsymbol{T}=\sigma^{ij}\boldsymbol{e}_i\boldsymbol{e}_j, \tag{12.7}$$

as seen in (10.5), consequently the stress tensor ${}^e\boldsymbol{T}$ given by (12.5) is obtained as

$${}^e\boldsymbol{T}=\sigma^{ij}\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j, \tag{12.8}$$

by the use of (4.24) and (7.3). It is found that ${}^e\boldsymbol{T}$ *is the tensor obtained from* $\boldsymbol{T}$ *by converting its basic tensor* $\boldsymbol{e}_i\boldsymbol{e}_j$ *into* $\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j$ *keeping its components unchanged.*

By means of (6.8) and (12.7), the virtual work principle (12.3) is expressed in terms of components, as

$$\delta W=\int \sigma^{ij}(\delta^p\varepsilon)_{ij}dV. \tag{12.9}$$

Similarly by means of (5.14) and (12.8), (12.6) is represented as

$$\delta W=\int_e \sigma^{ij}\delta^e\underset{\circ}{\varepsilon}_{ij}dV. \tag{12.10}$$

As it holds $(\delta^p\varepsilon)_{ij}=\varepsilon^e\underset{\circ}{\varepsilon}_{ij}$, as shown in (7.4), the expressions (12.9) and (12.10) of the virtual work principle are not distinguished at all, whether the deformation is elastic or plastic. But this being all a mere appearance, the stress and strain increment tensors of themselves are distingished by the basic tensors, to which their components are referred.

In fact, if we use, in the case of plastic deformation in particular, the local coordinate system whose basic tensors $\boldsymbol{e}_i\boldsymbol{e}_j$ and $\boldsymbol{e}^i\boldsymbol{e}^j$ are equal to the basic tensors $\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j$ and $\mathring{\boldsymbol{e}}^i\mathring{\boldsymbol{e}}^j$ respectively natural for analysing elastic deformation, (12.9) is replaced by the expression

$$\delta W=\int \sigma^{ij}\delta^{p}\varepsilon_{ij}dV. \tag{12.11}$$

which is clearly distinguishable from (12.10).

13. The Stress, Strain and Strain Increment for the Purpose of Describing State Equations

The fact that the virtual work principle assumes as simple the forms (12.3) and (12.6) for the finite elastic and plastic deformations as for the small ones, warrants the existence of the elastic and plastic potentials, consequently the possibility of the derivation of the state equations such as the laws of elasticity and plasticity, quite similarly to the case of small deformation. Thus the respective combinations of stress, strain increment and strain (${}^{e}\boldsymbol{T}$, $D^{e}\boldsymbol{E}$, ${}^{e}\boldsymbol{E}$) and ($\boldsymbol{T}$, $D^{p}\boldsymbol{E}$, ${}^{p}\boldsymbol{E}$) are seen to be reasonable for the description of each of the state equations for the elastic and the plastic deformations.

14. An Example of the Stresses

The stress $\boldsymbol{T}$, representing the field of force of itself at the point under consideration, is not effected by deformation. But its expression as shown in (10.5) is not possible, unless the state of deformation is known. The stress ${}^{e}\boldsymbol{T}$, on the other hand, is further associated with deformation by the tensor $\boldsymbol{J}$. So that, in order to illustrate $\boldsymbol{T}$ and ${}^{e}\boldsymbol{T}$ and the distinction between them, it is necessary that the type of stress and that of the corresponding strain are known. And for this we need the elasticity and plasticity laws to be known in general. The only case that we can know the types of stress and strain without any information about those laws is that of extension (uni- to tri-axial) of isotropic bodies, in which tensile stress is clear to produce extension in its direction, whether it is elastic or plastic.

Suppose that the cube with the edges of length l_0 is deformed into a rectangular parallelepiped with the edges of length l_1, l_2 and l_3 as shown in Fig. 1, under the action of some triaxial tensile loads P_1, P_2 and P_3. Then remembering (9.1), the orthogonal unit base vectors $\mathring{\boldsymbol{e}}_1$, $\mathring{\boldsymbol{e}}_2$ and $\mathring{\boldsymbol{e}}_3$ are converted into

$$\boldsymbol{e}_1=n_1\mathring{\boldsymbol{e}}_1,\quad \boldsymbol{e}_2=n_2\mathring{\boldsymbol{e}}_2,\quad \boldsymbol{e}_3=n_3\mathring{\boldsymbol{e}}_3 \tag{14.1}$$

respectively by the extensional deformation.

As the tensile loads P_1, P_2 and P_3 are exerted on the sufaces of the parallelepiped with the areas l_2l_3, l_3l_1 and l_1l_2, the normal stresses on these surfaces are

$$\sigma_1=\frac{P_1}{l_2l_3},\quad \sigma_2=\frac{P_2}{l_3l_1},\quad \sigma_3=\frac{P_3}{l_1l_2}, \tag{14.2}$$

and therefore the stress tensor is given by

$$\boldsymbol{T}=\sigma_1\mathring{\boldsymbol{e}}_1\mathring{\boldsymbol{e}}_1+\sigma_2\mathring{\boldsymbol{e}}_2\mathring{\boldsymbol{e}}_2+\sigma_3\mathring{\boldsymbol{e}}_3\mathring{\boldsymbol{e}}_3. \tag{14.3}$$

$\mathring{\boldsymbol{e}}_i$ being equal to $\boldsymbol{e}_i$, σ_1, σ_2 and σ_3 in (14.3) are the components σ^{ij} regarding to the local coordinate system [see (10.10)].

By means of (14.1), (14.3) is written as

$$T=\frac{\sigma_1}{n_1^2}e_1e_1+\frac{\sigma_2}{n_2^2}e_2e_2+\frac{\sigma_3}{n_3^2}e_3e_3 \tag{14.4}$$

and these components σ_1/n_1^2, σ_2/n_2^2 and σ_3/n_3^2 are none other than those for the Lagrangian coordinate, σ^{ij}, given in (10.5).

The elastic stress ${}^eT$ is obtained from (14.4) as

$${}^eT=\frac{\sigma_1}{n_1^2}\mathring{e}_1\mathring{e}_1+\frac{\sigma_2}{n_2^2}\mathring{e}_2\mathring{e}_2+\frac{\sigma_3}{n_3^2}\mathring{e}_3\mathring{e}_3\,. \tag{14.5}$$

The distinction between T and ${}^eT$ is clearly shown in (14.3), (14.4) and (14.5). That is, referred to the same basic tensors $\mathring{e}_i\mathring{e}_j$ for the undeformed state, the components of the plastic stress T being σ_1, σ_2 and σ_3, those of the elastic stress ${}^eT$ are σ_1/n_1^2, σ_2/n_2^2 and σ_3/n_3^2; while the components of T referred to e_ie_j and those of ${}^eT$ referred to $\mathring{e}_i\mathring{e}_j$ are equal to each other and also to σ_1/n_1^2, σ_2/n_2^2 and σ_3/n_3^2.

15. Conclusion

Deformations of continua from fluids to solids can be classified into the two groups, one is the elastic deformation due to the change in the distances among particles constituting the material and the other the plastic deformation due to the change in the mode of the interconnection of particles. On account of this distinction of the mechanism of deformation, the difinitions of the strain, strain increment and stress for descibing deformation are essentially different, according as the deformation is either elastic or plastic.

That is, the elastic strain is specified by the difference of the geometrical configuration, so that of the metric, before and after deformation, and the elastic strain increment is derived as the increment of the elastic strain thus defined. The quantity introduced *imprimis* in plastic deformation is the strain increment, and it is specified by the change in the geometrical configuration from the state t to the state $t+dt$, the current state t being assumed as an undeformed state. The plastic strain is obtained by integrating this strain increment along a given path of deformation. Consequently, while the elastic strain is independent of the deformation path, being specified only by the geometrical shape of the deformed state, the plastic strain depends on the deformation path up to the final state. There are, however, close relation between the both strain increments.

The plastic stress, that is the stress adequate for describing the plasticity law together with the plastic strain increment, is the stress resulting from the generalization of the true stress for the case of simple tension. The elastic stress reasonable for describing elasticity law together with the elastic strain is a modified one, which can be deduced from the plastic stress. Contrary to the stress valid for the description of the state equation, that for the description of the equilibrium equation is the same for the both deformations, and equal to the plastic stress. But as for this view of the common stress for describing the equilibrium condition, it might not necessarily be said that there remains no ambiguity.

These differences of strain, strain increment and stress between the two kinds of deformation geometrically identical are of finite order for finite deformation, and are negligible for small one. But essentially this will not give any justification, even for the case of small deformation, to their common definitions accepted so far in general. Thus the theories of elasticity and plasticity are regarded to stand on the dualistic foundations respecting the basic concepts of strain, strain increment and stress as well as the state equations. And without this idea, it will be impossible to construct theories of elasticity and plasticity consistent for the whole range of small and finite deformations. Such a theory of plasticity has already been proposed by the present author, and that of elasticity whose outline has also been established in his mind will be published in the near future.

Department of Aerodynamics and Structures
Aeronautical Research Institute
University of Tokyo, Tokyo
March 24, 1959

REFERENCES

[1] For instance, Biot, M. A.: Theory of Elasticity with Large Displacements and Rotations, Proc. 5th Intern. Congr. Appl. Mech., 1938, pp. 117–122.
Biot, M. A.: Non-linear Theory of Elasticity and the Linearized Case for a Body under Initial Stress, Phil, Mag., Ser. 7, **27** (1939), pp. 468–489.
Murnaghan, F. D.: Finite Deformations of an Elastic Solid, Amer. Journ. Math., **59** (1937), pp. 235–260.
Kappus, R.: Zur Elastizitätstheorie endlicher Verschiebungen, ZAMM, **19** (1939), pp. 271–285, pp. 344–361.
Green, A. E. and Zerna, W.: Theory of Elasticity in General Coordinates, Phil. Mag., **41** (1950), p. 313.
Green, A. E. and Shield, R. T.: Finite Elastic Deformation of Incompressible Isotropic Bodies, Phil. Mag., **42** (1952), pp. 407–419.

[2] For instance, Dorn, J. E. and Latter, A. J.: Stress-Strain Relations for Finite Elastoplastic Deformations, Journ. Appl. Mech., **15** (1948), pp. 234–236.
Phillips, A.: Variational Principles in the Theory of Finite Plastic Deformations, Quart. Appl. Math., **7** (1950), pp. 110–114.
Nadai, A.: Theory of Flow and Fracture of Solids, McGraw-Hill, 1950, pp. 117–150.

[3] Yoshimura, Y.: Basic Theory of Finite Plastic Deformation, Read at the 32th Annual Meeting JSME, April, 1955.
Yoshimura, Y.: Work Principles and Plasticity Law in Finite Plastic Deformation, Read at the 5th Japan National Congr. Appl. Mech., Sept., 1955.
Yoshimura, Y.: Some Applications of the General Theory of Plastic Deformation (Examples of Plastic Strain or Strain History), Read at the 33th Annual Meeting JSME, April, 1956.
Yoshimura, Y.: Theory of the Dependence of Anisotropy and the Bauschinger Effect in Plastic Deformation on Strain History, Read at the 6th Japan National Congr. Appl. Mech., Oct., 1956.
Yoshimura, Y. and Takenaka, Y.: Yield Condition and Work Hardening Rate Proper to a Metal and Their Dependence on its Strain History, First Report, Transactions JSME, Vol. 25, No. 151 (1959), pp. 133–139.
Yoshimura, Y., Takenaka, Y. and Abe, S.: ditto, Second Report, Transactions JSME, Vol. 25, No. 151 (1959), pp. 140–147.

Some of these papers, whose publication has been delayed for more than two years due to some mistake in carrying out the business, e.g. its procedure, are included for the greater part of them in my text-book "Sosei Rikigaku (Mechanics of Plastic Solids)", Kyoritsu Shuppan, Tokyo, 1957.

[4] Yoshimura, Y.: Elasticity Theory Valid for Non-linear Stress-Strain Law and Finite Deformation, Read at the 8th Japan National Congr. Appl. Mech., Sept., 1958.