

A Contribution to the Energy Decay Law of Isotropic Turbulence in the Initial Period

By

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Summary. Of the energy decay laws of isotropic turbulence in the initial period, the so-called linear decay law and Lin's decay law are well known. The linear decay law was first derived theoretically by Batchelor on the assumption of partial self-preservation of velocity correlation functions, but there is involved a self-contradiction that it eventually necessitates the premise of complete self-preservation of correlation functions. On the other hand, Lin's decay law, which was derived by assuming the similarity of spectrum for wave numbers except at small end, holds only after neglecting the self-preservation of correlation functions near $r=0$, very important characteristics confirmed by experiments. Therefore, at first in this paper, we criticize these decay laws and find out the points of self-contradiction inherent in these two theories.

Then eliminating the above mentioned points of self-contradictions and making a simple assumption that the partial self-preservation of correlation functions holds and S and G stay constant during decay, as confirmed by experiments, we try to calculate the decay curves by solving the energy decay equation (5.7),

$$\zeta \frac{d^2 \zeta}{d\xi^2} + C \left(\frac{d\zeta}{d\xi} \right)^3 - E \left(\frac{d\zeta}{d\xi} \right)^{3/2} = 0,$$

which is derived from the energy equation (2.10) and the vorticity equation (3.1), in which $\zeta = U^2/\bar{u}^2$, $\xi = x/M$, $C = (7/15)G - 2$, $E = (7/3)S\sqrt{R_M}/10$, $R_M = UM/\nu$, $S = -(\partial u/\partial x)^2/[(\partial u/\partial x)^2]^{3/2}$ and $G = \overline{u^2(\partial^2 u/\partial x^2)^2}/[(\partial u/\partial x)^2]^2$. Since this equation contains statistical quantities of turbulence, S , G and R_M , as parameters, the decay curves obtained by solving this equation vary with the initial conditions of turbulence generation and also with the values of these parameters. Thus, we can, through such parameters, examine quantitatively the effect of the initial conditions upon the energy decay of turbulence. This equation is actually solved by numerical integration for cases corresponding to the experiments done by the present author and by Batchelor and Townsend. The calculated results are in good agreement with these experimental values.

1. INTRODUCTION

It is well known that the problem of isotropic turbulence can be reduced to that of solving the Kármán-Howarth equation for the propagation of the double velocity correlation [1],

$$\frac{\partial}{\partial t}(\overline{u^2 f}) - (\overline{u^2})^{3/2} \left\{ \frac{\partial k}{\partial r} + 4 \frac{k}{r} \right\} = 2\nu \overline{u^2} \left\{ \frac{\partial^2 f}{\partial r^2} + \frac{4}{r} \frac{\partial f}{\partial r} \right\}, \quad (1.1)$$

or its Fourier transformation, i.e., the spectrum equation,

$$\frac{\partial F}{\partial t} + W = -2\nu \tilde{k}^2 F. \quad (1.2)$$

These equations govern the motion of turbulence during decay, in which t denotes the time of decay, r the distance between the two points at which the velocities are taken, ν the kinematic viscosity, $\overline{u^2}$ the mean-square of any turbulent velocity component, $f(r, t)$ the longitudinal double velocity correlation coefficient [1], $k(r, t)$ the longitudinal triple velocity correlation coefficient [1], $F(\tilde{k}, t)$ the three-dimensional energy spectrum function, $W(\tilde{k}, t)$ the three-dimensional energy transfer function, and \tilde{k} the wave-number magnitude. However, these equations can not be solved, in the general case, because of its non-linear nature. Only when the Reynolds number of turbulence is small, i.e., the decay is in the final period, the non-linear term, which expresses the transfer of energy between frequency components, can be neglected so that the solution can be obtained in an explicit form as confirmed by experiments [2] [3] [4]. On the other hand, for the case where the Reynolds number of turbulence is large and the energy transfer between frequency components plays an important role, the problem has been treated by many authors by making plausible assumptions, such as self-preservation of correlation functions or the similarity of spectrum during the process of decay. Von Kármán and Howarth themselves in 1938 made a pioneering work along these lines [1], and the prominent works by Robertson, Kolmogoroff, Heisenberg, Batchelor, and others, opened an era of systematic studies of these equations.

About ten years ago Batchelor, assuming partially self-preserving solutions for the correlation functions near $r=0$ at decay times which are not large, derived theoretically the so-called linear decay law [4],

$$\left. \begin{aligned} \overline{u^2} &\sim t^{-1}, \\ \lambda^2 &= 10\nu t, \\ R_\lambda &\equiv \sqrt{\overline{u^2}} \lambda / \nu = \text{const.}, \end{aligned} \right\} \quad (1.3)$$

in which λ is the microscale of turbulence introduced by Taylor [5], and R_λ the Reynolds number of turbulence. Batchelor and Townsend, in fact, verified by experiments that for the isotropic turbulence produced by a grid of regular mesh and in the initial period of decay, the linear decay law, (1.3), was valid for distances up to about 150 mesh-lengths from the grid [6] [7].

Later, however, Lin, and also Goldstein, had doubt about the validity of this linear decay law and derived independently the so-called Lin's decay law, i.e.,

$$\left. \begin{aligned} \overline{u^2} &= \alpha t^{-1} + \beta, \quad (\alpha \text{ and } \beta \text{ are constants}) \\ \lambda^2 &= 10\nu t \left(1 + \frac{\beta}{\alpha} t \right), \\ R_\lambda &= \sqrt{10\alpha/\nu} \left(1 + \frac{\beta}{\alpha} t \right), \end{aligned} \right\} \quad (1.4)$$

by assuming that the actual deviation from similarity of the energy spectrum would be limited only to small values of wave numbers and hence the effect of the deviation would enter only in the calculation of energy but be negligible in the computation of the rate of energy dissipation [8] [9] [10]. As will be discussed later, this decay law may be considered more reasonable than the linear decay law from the physical point of view, and has been shown to be in good

agreement with the experimental results for the turbulence behind two grids carried out by the present author as shown in Figs. 1, 2 and 3 [11] [12] [13].

These decay laws, however, contain the contents which are inconsistent with the first assumption made in the derivation of the decay law, or the defect that the decay law holds only by neglecting the facts confirmed by experiments.

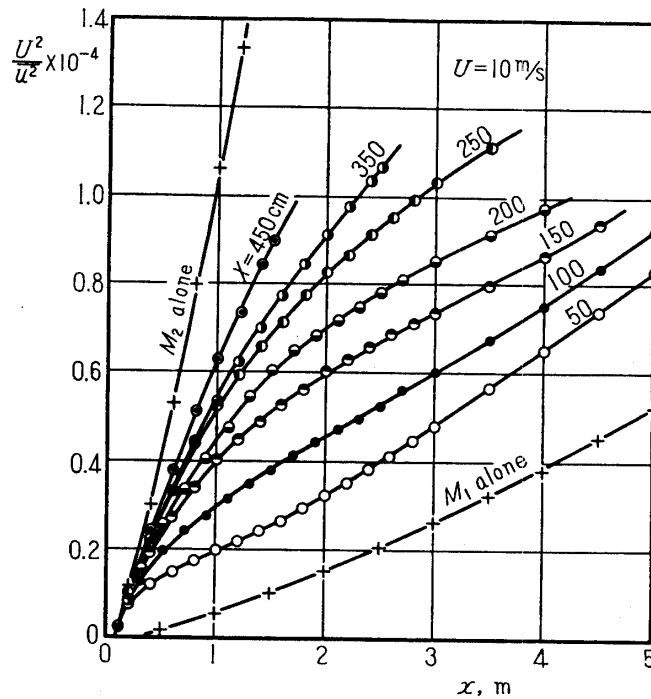


FIGURE 1. Decay of energy of turbulence behind a single grid or two grids; $M_1=5$ cm, $M_2=1$ cm, X = the distance between two grids (after Tsuji and Hama, 1953, Ref. [11]).

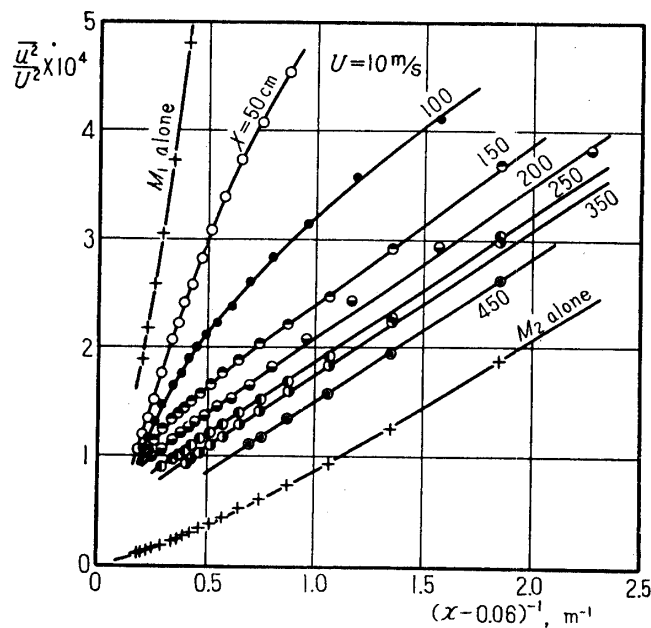


FIGURE 2. Decay of energy of turbulence behind a single grid or two grids; $M_1=5$ cm, $M_2=1$ cm, X = the distance between two grids (after Tsuji and Hama, 1953, Ref. [11]).

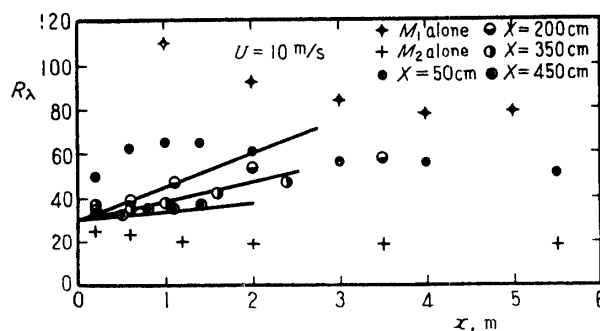


FIGURE 3. Variation of R_λ during decay of turbulence behind a single grid or two grids; $M_1=5$ cm, $M_2=1$ cm, X = the distance between two grids (after Tsuji, 1955, Ref. [12]).

Therefore, at first in this paper, we shall criticize the linear decay law and Lin's decay law, and point out the points of self-contradiction inherent in these decay laws. Then we shall try to determine the decay curves by a new method, in which only a simple assumption, confirmed by experiments, will be made and the points of self-contradiction inherent in the above decay laws will be eliminated. Consequently, we can make clear the significance of the statistical parameters of turbulence to the decay, and show how the decay curves change with the change in the relation of magnitude of these statistical parameters, and point out the effect of the initial conditions of turbulence generations upon the decay.

2. LINEAR DECAY LAW

Let us first examine the linear decay law, $\bar{u}^2 \sim t^{-1}$. If we expand the double and triple velocity correlation coefficients $f(r, t)$ and $k(r, t)$ in powers of r , we have

$$\begin{aligned} f(r, t) &= \frac{\overline{uu'}}{\bar{u}^2} = \frac{1}{\bar{u}^2} \left\{ \bar{u}^2 + \frac{1}{2!} \overline{u \frac{\partial^2 u}{\partial x^2}} r^2 + \frac{1}{4!} \overline{u \frac{\partial^4 u}{\partial x^4}} r^4 + \frac{1}{6!} \overline{u \frac{\partial^6 u}{\partial x^6}} r^6 + \dots \right\} \\ &= 1 - \frac{\left(\frac{\partial u}{\partial x} \right)^2}{\bar{u}^2} \frac{r^2}{2!} + \frac{\left(\frac{\partial^2 u}{\partial x^2} \right)^2}{\bar{u}^2} \frac{r^4}{4!} - \frac{\left(\frac{\partial^3 u}{\partial x^3} \right)^2}{\bar{u}^2} \frac{r^6}{6!} + \dots \\ &= 1 + f_0'' \frac{r^2}{2!} + f_0^{iv} \frac{r^4}{4!} + f_0^{vi} \frac{r^6}{6!} + \dots, \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} k(r, t) &= \frac{\overline{u^2 u'}}{(\bar{u}^2)^{3/2}} = \frac{1}{(\bar{u}^2)^{3/2}} \left\{ \overline{u^2 \frac{\partial^3 u}{\partial x^3}} \frac{r^3}{3!} + \dots \right\} \\ &= \frac{\left(\frac{\partial u}{\partial x} \right)^3}{(\bar{u}^2)^{3/2}} \frac{r^3}{3!} + \dots \\ &= k_0''' \frac{r^3}{3!} + \dots, \end{aligned} \quad (2.2)$$

respectively, in which f_0'' , f_0^{iv} , f_0^{vi} and k_0''' indicate $(\partial^2 f / \partial r^2)_{r=0}$, $(\partial^4 f / \partial r^4)_{r=0}$, $(\partial^6 f / \partial r^6)_{r=0}$ and $(\partial^3 k / \partial r^3)_{r=0}$, respectively [1] [5]. Now we introduce three sta-

tistical quantities defined by the following equations;

$$\frac{1}{\lambda^2} = -f_0'' = \frac{1}{\bar{u}^2} \overline{\left(\frac{\partial u}{\partial x}\right)^2}, \quad (2.3)$$

$$G = \lambda^4 f_0^{iv} = \bar{u}^2 \overline{\left(\frac{\partial^2 u}{\partial x^2}\right)^2} / \left[\overline{\left(\frac{\partial u}{\partial x}\right)^2} \right]^2, \quad (2.4)$$

$$S = -\lambda^3 k_0''' = -\overline{\left(\frac{\partial u}{\partial x}\right)^3} / \left[\overline{\left(\frac{\partial u}{\partial x}\right)^2} \right]^{3/2}, \quad (2.5)$$

in which S is minus the skewness factor of the probability distribution of $\partial u / \partial x$ (the minus sign is introduced here, because the skewness has been found to be negative), and u is the turbulent velocity component in the direction of x -axis. By using these quantities, the above expansion equations may be rewritten as

$$f(r/\lambda, t) = 1 - \frac{1}{2!} \left(\frac{r}{\lambda}\right)^2 + \frac{G}{4!} \left(\frac{r}{\lambda}\right)^4 + \frac{\lambda^6 f_0^{vi}}{6!} \left(\frac{r}{\lambda}\right)^6 + \dots, \quad (2.6)$$

$$k(r/\lambda, t) = -\frac{S}{3!} \left(\frac{r}{\lambda}\right)^3 + \dots. \quad (2.7)$$

The linear decay law was first derived theoretically by Batchelor on an assumption of partial self-preservation of velocity correlation functions [4]. It has been known that, in the initial period of decay, the correlation functions do not obey complete self-preservation, but to some extent self-preservation exists when the decay time is not large. The function $f(r, t)$ is always parabolic in form near $r=0$. Detailed experiments, carried out by Batchelor and Townsend, of the isotropic turbulence produced by a grid of regular mesh, have indicated that the expansion of $f(r, t)$ in powers of r/λ as far as the term of fourth degree, and of $k(r, t)$ as far as the term of third degree, are independent of t at decay times which are not large, i.e., S and G are approximately constants during decay, as shown in Fig. 4 (from Figs. 7, 8 and 10 of Reference [6])* . It has also been confirmed by another experiment by Batchelor and Townsend that accurate self-preservation of the lateral double velocity correlation coefficient $g(r, t)$ holds for values of r/λ between 0 and about 1, the range becoming larger at high Reynolds numbers [7]. Considering these experimental facts, it seems reasonable to introduce the assumption of partial self-preservation for the correlation functions when the theory on isotropic turbulence in the initial period is attempted.

Now we assume that l is an unknown length and the self-preserving solutions for the correlation functions expressed as

$$f(r, t) \equiv f(\eta), \quad k(r, t) \equiv k(\eta), \quad \eta \equiv r/\lambda, \quad (2.8)$$

are valid for a range of r , $0 \leq r < l$. From the above evidence l must be at least as large as the maximum value of r , for which a fourth degree polynomial

* It has also been confirmed by measurements by H. W. Liepmann, J. Laufer and K. Liepmann that G is constant during decay [14]. Moreover, it has been revealed by later experiments by Batchelor and Townsend that the expansion of $f(r, t)$ in powers of r/λ as far as the term of sixth degree are independent of t , i.e., $\lambda^6 f_0^{vi}$ is also constant during decay [15] (See Fig. 2 of Reference [15]).

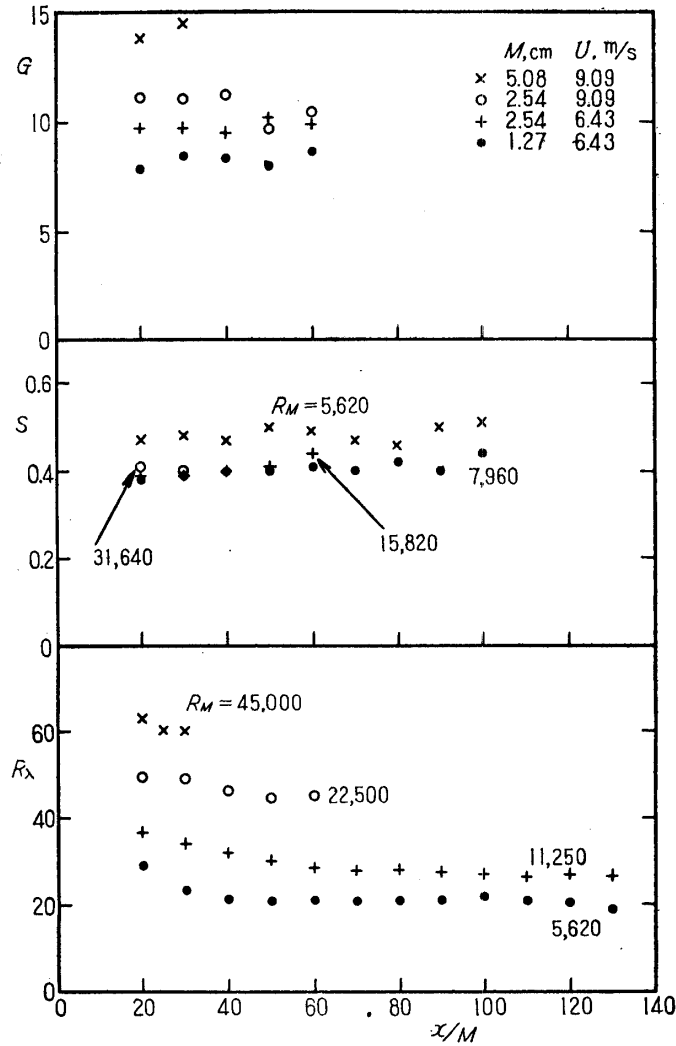


FIGURE 4. Variation of G , S and R_λ during the initial period (after Batchelor and Townsend, 1947, Ref. [6]).

gives a good representation of $f(r, t)$. The Kármán-Howarth equation (1.1) is then reduced to

$$\left(\frac{d^2 f}{d\eta^2} + \frac{4}{\eta} \frac{df}{d\eta} + \frac{5\eta}{2} \frac{df}{d\eta} + 5f \right) + \frac{\lambda^2}{2\nu R_\lambda} \frac{dR_\lambda}{dt} \left(\eta \frac{df}{d\eta} \right) + \frac{1}{2} R_\lambda \left(\frac{dk}{d\eta} + \frac{4k}{\eta} \right) = 0 \quad (2.9)$$

for the restricted range of r . If we assume that $\overline{u^2}$ decays as some power of t , the energy equation [1] [5],

$$\frac{d\overline{u^2}}{dt} = -10\nu \frac{\overline{u^2}}{\lambda^2}, \quad (2.10)$$

shows that the decay will be

$$\overline{u^2} \sim t^{-n}, \quad \lambda^2 = 10\nu t/n, \quad R_\lambda^2 \sim t^{1-n}, \quad (2.11)$$

and hence

$$\frac{\lambda^2}{2\nu R_\lambda} \frac{dR_\lambda}{dt} = \frac{5(1-n)}{2n}. \quad (2.12)$$

According to equation (2.12), the coefficient of the second term in equation (2.9)

is a constant, so that the first two groups of terms in equation (2.9) are function of η only and the equation can only be satisfied by

$$R_\lambda = \text{constant} . \quad (2.13)$$

This leads to $n=1$ provided that the constant value of R_λ is not zero (in which case we should have $n > 1$, $t = \infty$), i.e., it may be concluded that the so-called linear decay law, equation (1.3), holds and R_λ is constant during decay in the initial period. This theoretical result has often been compared with experimental results. The measured values of $\overline{u^2}$ are plotted usually in a form $U^2/\overline{u^2}$ against x/M , where x is the distance of the point of measurement from the grid of mesh size M , and U is the velocity of stream flowing past the grid. As is usual, the time of decay in the idealized theoretical problem of homogeneous turbulence is identified with the quantity x/M occurring in the experimental turbulence which is slightly nonhomogeneous (Taylor's hypothesis [5]). The abscissa x/M , is therefore, the decay time made dimensionless by the use of the factors M and U . The experimental points of $U^2/\overline{u^2}$ given by many workers may be considered to lie approximately on a straight line for the decay range $x/M=20$ to about 150 [6] [7] [14]. R_λ is also approximately constant during comparatively short decay times, so that this linear decay law has been supported generally after the study of Batchelor and Townsend [4] [6] [7].

On the other hand, as pointed out and emphasized by Lin [10], it is necessary by all means that the energy spectrum should hold the complete similarity over the whole wave-number range, i.e., the correlation should hold the complete self-preservation over the whole range of r , in order that the linear decay law should hold!!

The linear decay law, which was derived by making the assumption of the partial self-preservation of the correlation functions, necessitates, in fact, such important premise that the correlation should hold the complete self-preservation. Therefore, it is inconsistent with the assumption from which the linear decay law was derived. Why does such a self-contradiction arise? The assumption of the partial self-preservation of the correlation functions is quite correct, but in the process of deriving the decay law from this assumption, an unreasonable method, inconsistent with this assumption, was used. As will be mentioned later again, it must be noticed that using $\sqrt{\overline{u^2}}$ and λ as the similarity parameters and, moreover, assuming the power decay law to hold are nothing but assuming the complete self-preservation of the correlation functions implicitly!! The self-preservation which was assumed to hold at first for the range $0 \leq r < l$ was extended to hold over the whole range of r , i.e., $0 \leq r < \infty$, in consequence of the assumption of the power decay law. In the general case, in which the complete self-preservation is not assumed, the decay curve, $(\overline{u^2})^{-1}$ vs. t , is not expressed by a simple power law, but usually as a polynomial of t (Lin's decay law is an example of such cases).

Therefore, it may be easily concluded that the linear decay law which was derived only on the assumption of the complete self-preservation of correlation functions is not correct for the isotropic turbulence in the initial period of decay

for which only the partial self-preservation holds actually. In fact, examining in detail the results of the experiments conducted not only by the present author [11] [12] [13], but also by Batchelor and Townsend [6] [7] and other workers, it may be pointed out that the linear decay law does not hold, as shown in Fig. 1 as an example. Also R_λ is not constant during the initial period of decay as shown in Figs. 3 and 4.

3. LIN'S DECAY LAW

The so-called Lin's decay law, which has been originally put forward by Lin [8] and later endowed with firmer reasoning by Goldstein [9], may be derived on the assumption that the actual deviation from the similarity of the spectrum will be limited only to small values of wave numbers and hence the effect of deviation will enter only in the calculation of energy but be negligible in the computation of the rate of energy dissipation [10],

$$-\frac{d\bar{u}^2}{dt} = \varepsilon = 2\nu \int_0^\infty \tilde{k}^2 F(\tilde{k}, t) d\tilde{k}. \quad (3.1)$$

This decay law may be considered to be more reasonable than the linear decay law from the physical point of view, and experiments by the present author have shown it to be valid for the turbulence either behind a single grid or with a superposed disturbances of low frequencies as shown in Figs. 1, 2 and 3 [11] [12] [13].

However, reference velocity $V(t)$ and scale $L(t)$, which are employed as the similarity parameters in deriving this decay law, are not exactly equal to the turbulent intensity $\sqrt{\bar{u}^2}$ and microscale λ , respectively. Therefore, for example, if the correlation functions are expressed non-dimensionally by using V and L as the similarity parameters, the correlation curves are not expressed as a single curve near $r=0$. Hence, this theory developed by Lin and Goldstein neglects the self-preservation of the correlation functions near $r=0$ (very important characteristics confirmed by the experiments as discussed in §2 [6] [7] [12] [13])!! The fact that Lin's decay law holds after neglecting the self-preservation of the correlation function near $r=0$ will be verified theoretically by using the vorticity equation of turbulence.

If we expand both sides of equation (1.1) in powers of r , and equate coefficients, the constant term gives simply the well-known energy equation [1] [5],

$$\frac{d\bar{u}^2}{dt} = -10\nu \frac{\bar{u}^2}{\lambda^2}. \quad (2.10)$$

The coefficient of r^2 gives

$$\frac{d(\bar{u}^2 f_0'')}{dt} - \frac{7}{3} (\bar{u}^2)^{3/2} k_0''' = \frac{14}{3} \nu \bar{u}^2 f_0^{iv}. \quad (3.1)$$

This equation describes the rate of change of mean square vorticity, and may be called the vorticity equation [1]. Now the mean square of the component of vorticity in any direction, $\bar{\omega}^2$, is expressed as

$$\bar{\omega}^2 = -5\bar{u}^2 f_0'' = 5\bar{u}^2 / \lambda^2. \quad (3.2)$$

Using this $\overline{\omega^2}$, and introducing G and S defined by the equations (2.4) and (2.5) respectively, equation (3.1) may be written in a form as

$$\frac{d\overline{\omega^2}}{dt} = \frac{7(\overline{\omega^2})^{3/2}}{3\sqrt{5}} \left[S - \frac{2G}{R_\lambda} \right], \quad (3.3)$$

or

$$-\frac{d}{dt} \left(\frac{1}{\sqrt{\overline{\omega^2}}} \right) = \frac{7}{6\sqrt{5}} \left[S - \frac{2G}{R_\lambda} \right]. \quad (3.4)$$

If we assume the right-hand side of equation (3.4) is a constant independent of time, then the equation (3.4) can be integrated immediately. Now, if

$$\frac{7}{6\sqrt{5}} \left[S - \frac{2G}{R_\lambda} \right] = \text{constant} = - \left(\frac{2}{R} \right)^{1/2}, \quad \text{say,} \quad (3.5)$$

equation (3.4) becomes

$$\frac{d}{dt} \left(\frac{1}{\sqrt{\overline{\omega^2}}} \right) = \left(\frac{2}{R} \right)^{1/2}. \quad (3.6)$$

Integrating equation (3.6), we get

$$\frac{1}{\sqrt{\overline{\omega^2}}} = \left(\frac{2}{R} \right)^{1/2} t,$$

i.e.,

$$\overline{\omega^2} = \frac{R}{2t^2}. \quad (3.7)$$

Using equations (2.10) and (3.3), we obtain

$$\overline{\omega^2} = - \frac{1}{2\nu} \frac{d\overline{u^2}}{dt} = \frac{R}{2t^2}. \quad (3.8)$$

Integrating equation (3.8) once more, we finally obtain the decay law,

$$\overline{u^2} = \frac{\nu R}{t} (1 - At) = \nu R (t^{-1} - A), \quad (3.9)$$

in which A is an integrating constant. Equation (3.9) is nothing but Lin's decay law.

As will be well understood by the above discussion, we must notice that Lin's decay law holds under a major premise,

$$S - \frac{2G}{R_\lambda} = \text{const.} \quad (3.10)$$

Then, if we investigate what kind of combination exists between S , G and R_λ to satisfy the necessary condition (3.10) of Lin's decay law, the following five cases may be noticed.

	S	G/R_λ	G	R_λ
a.	const.	const.	const.	const.
b.	const.	const.	not const.	not const.
c.	not const.	not const.	not const.	const.
d.	not const.	not const.	const.	not const.
e.	not const.	not const.	not const.	not const.

In the case (a), $R_2 = \text{const.}$ Then the linear decay law holds and the correlation functions show the complete self-preservation*. This is the special case of Lin's decay law (which corresponds to the case of $\beta = 0$ in equation (1.4)). In the case (c), as $R_2 = \text{const.}$, the correlation functions must show the complete self-preservation and hence S and G must be constants. Therefore, it contradicts the condition imposed on S and G in this case and such a case does not exist actually. In other cases (b), (d) and (e), it is required that both S and G are, or at least either one of them is, necessarily not constant. Namely, it is required as the premise of validity of Lin's decay law that, in general case of this law except the special case of $\beta = 0$, both S and G are not constant or at least either one of them is necessarily not constant, and it becomes clear that this decay law holds only by neglecting the fact that S and G are constant during decay as confirmed by the experimental results. In other words, Lin's decay law may be considered to be the theory in which the self-preservation of the correlation functions near $r = 0$ as confirmed by the experiments is denied.

Since Lin's decay law was derived on the assumption of the complete similarity of the vorticity spectrum, not energy spectrum, the assumption used is reasonable from the physical point of view and, in fact, it is supported by experiments as shown in Figs. 1 and 2. Therefore, Lin's decay law may be thought to be better than the linear decay law at these points. But on the other hand, this decay law contains a defect that it holds only by neglecting the self-preservation, confirmed by experiments, of the correlation functions near $r = 0$. Hence we must say that Lin's decay law, too, contains self-contradiction.

4. AN ATTEMPT TO DERIVE A NEW DECAY LAW

In the preceding two chapters, we have discussed on the two decay laws of isotropic turbulence in the initial period of decay. However, the linear decay law holds under the premise of the complete self-preservation of the correlation functions, and, on the other hand, Lin's decay law holds only by neglecting the self-preservation of the correlation functions near $r = 0$, so that both of these decay laws involve characteristics, which are inconsistent with the fact confirmed by the experiments. Considering the incompleteness of these existing decay laws, we have reconsidered the decay law over again, and using a simple assumption which is confirmed by experiments, we have tried to derive a new decay law by a new method, in which the discrepancies that appear in the above two decay laws are excluded.

The most reliable assumption to be used seems to be the partial self-preservation of the correlation functions near $r = 0$, which was confirmed by experiment done by Batchelor and Townsend [6] [7] and employed in the theoretical study developed by Batchelor [4]. Namely, careful experiments carried out by Batchelor and Townsend on the isotropic turbulence produced by a grid of regular

* If we solve the energy equation (2.10) under the condition of $R_2 = \text{const.}$, we can get easily the solution, $\overline{u^2} \sim t^{-1}$.

mesh have indicated that the expansion of $f(r, t)$ in powers of r/λ as far as the term of fourth degree, and of $k(r, t)$ as far as the term of third degree, are independent of t at decay times which are not large, i.e., S and G are approximately constant during decay, as shown in Fig. 4 [6]. It has also been confirmed by another experiment by Batchelor and Townsend that accurate self-preservation of the lateral double velocity correlation coefficient $g(r, t)$ seems to hold for values of r/λ between 0 and some figure of the order of 1 [7]. These characteristics of $g(r, t)$ was also confirmed by the present author by experiments carried out on the isotropic turbulence behind a single grid and two grids [12]. Namely, as confirmed in Figs. 5 and 6, the precise self-preservation of the measured $g(r/\lambda, t)$

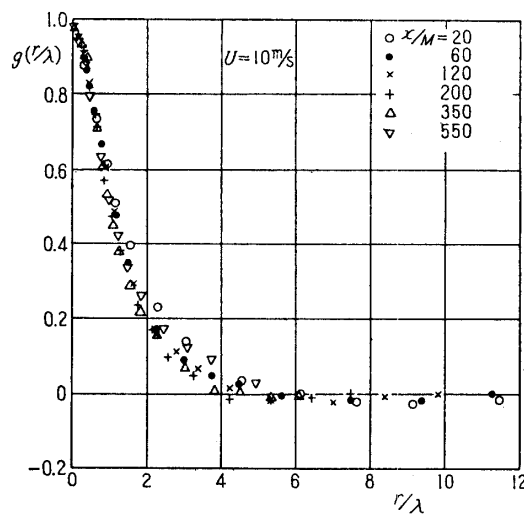


FIGURE 5. Correlation coefficient $g(r/\lambda)$ of turbulence behind a single grid; $M=1$ cm, $U=10$ m/s (after Tsuji, 1955, Ref. [12]).

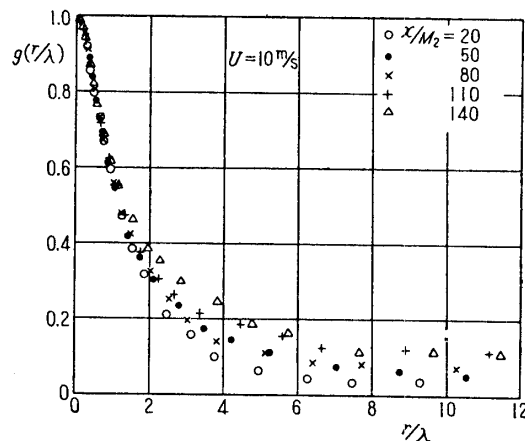


FIGURE 6. Correlation coefficient $g(r/\lambda)$ of turbulence behind two grids; $M_1=5$ cm, $M_2=1$ cm, $X=450$ cm, $U=10$ m/s (after Tsuji, 1955, Ref. [12]).

holds for values of r/λ less than 1 for the turbulence behind a single grid or the turbulence behind two grids with the exception of the case where high frequency fluctuations are superposed. Considering these experimental results, it seems to

be reasonable to assume that S and G are constant during decay for the turbulence in the initial period of decay. Therefore, we shall use this assumption as a starting point of the present study*.

Now, if we assume that S and G are constant during decay, we can expect that it is convenient to study the decay process by using the vorticity equation (3.1) which contains S and G explicitly. This equation is an ordinary differential equation with \bar{u}^2 and λ as dependent variables, so that $\bar{u}^2(t)$ and $\lambda(t)$ can not be determined only from this equation even if S and G are constant. Lin's decay law is derived from this equation by using another condition, $S - 2G/R_\lambda = \text{const.}$, which relates $\bar{u}^2(t)$ and $\lambda(t)$ through R_λ . On the other hand, linear decay law is derived from the assumption of power decay law and the condition, $R_\lambda = \text{const.}$ As discussed in the preceding two chapters, both conditions, $S - 2G/R_\lambda = \text{const.}$, and $R_\lambda = \text{const.}$, contain characteristics inconsistent with experimental results. Therefore, we reject both of these conditions and propose a new method in which $\bar{u}^2(t)$ and $\lambda(t)$ are determined by solving the vorticity equation (3.1) and the energy equation (2.10) simultaneously under the simple condition, $S = \text{const.}$ and $G = \text{const.}$

For the sake of convenience for comparison, the assumptions on the parameters, S , G and R_λ , for linear decay law, Lin's decay law and the new decay law are shown in the following table.

	S	G	R_λ	$S - 2G/R_\lambda$
Linear Decay Law	const.	const.	const.	const.
Lin's Decay Law	$\left\{ \begin{array}{l} \text{const.} \\ \text{not const.} \\ \text{not const.} \end{array} \right.$	$\left\{ \begin{array}{l} \text{not const.} \\ \text{const.} \\ \text{not const.} \end{array} \right.$	$\left\{ \begin{array}{l} \text{not const.} \\ \text{not const.} \\ \text{not const.} \end{array} \right.$	const.
New Decay Law	const.	const.	not const.	not const.

5. THE FUNDAMENTAL EQUATION GOVERNING THE ENERGY DECAY IN THE INITIAL PERIOD

By using λ , the vorticity equation (3.1) may be written as

$$-\frac{d(\bar{u}^2/\lambda^2)}{dt} + \frac{7}{3} S \frac{(\bar{u}^2)^{3/2}}{\lambda^3} = \frac{14}{3} \nu \frac{\bar{u}^2}{\lambda^4} G. \quad (5.1)$$

Hence, the problem to derive the energy decay law is reduced to solving the equation (5.1) and the energy equation,

$$\frac{d\bar{u}^2}{dt} = -10\nu \frac{\bar{u}^2}{\lambda^2}, \quad (2.10)$$

* In the experiments by the present author, S and G were not measured and the fact that S and G are constant during decay was not confirmed directly. However, since it was confirmed that the self-preservation of the correlation coefficient $g(r, t)$ holds for the range $r/\lambda < 1$, considering the experimental results by Batchelor and Townsend, there would be no objection to conclude that S and G are constant in the case of our experiments.

simultaneously under suitable initial conditions ($\overline{u^2} = \overline{u_0^2}$ and $\lambda = \lambda_0$ at $t = t_0$)* for the given constant values of S and G . If we substitute equation (2.10) into equation (5.1) so as to eliminate λ , equation (5.1) becomes

$$\frac{d^2 \overline{u^2}}{dt^2} + \frac{7}{3} S \left(\frac{1}{10\nu} \right)^{1/2} \left(-\frac{d\overline{u^2}}{dt} \right)^{3/2} = \frac{7}{15} \frac{G}{\overline{u^2}} \left(-\frac{d\overline{u^2}}{dt} \right)^2, \quad (5.2)$$

so that we obtain a second-order non-linear ordinary differential equation. Equation (5.2) is the fundamental equation which governs the energy decay of isotropic turbulence in the initial period of decay†. Therefore, if we solve this equation under the initial conditions,

$$\overline{u^2} = \overline{u_0^2} \quad \text{and} \quad \frac{d\overline{u^2}}{dt} = \left(\frac{d\overline{u^2}}{dt} \right)_0 \quad \text{at} \quad t = t_0^\dagger,$$

the decay curve may be easily obtained. Now, if we put

$$Z = \frac{1}{\overline{u^2}}, \quad A = \frac{7}{3} S \left(\frac{1}{10\nu} \right)^{1/2}, \quad B = \frac{7}{15} G, \quad (5.3)$$

and substitute equation (5.3) into equation (5.2), equation (5.2) is transformed into

$$\boxed{Z \frac{d^2 Z}{dt^2} + (B-2) \left(\frac{dZ}{dt} \right)^2 - A \left(\frac{dZ}{dt} \right)^{3/2} = 0}, \quad (5.4)$$

which is the differential equation on $Z(t)$. It is more convenient to use equation (5.4) in general, in order to discuss the decay process.

In the study of isotropic turbulence, however, we often consider the so-called frozen pattern in order to compare with the wind-tunnel experiments, and discuss the decay process in the form $U^2/\overline{u^2}$ against x/M instead of the from $1/\overline{u^2}$ against t . Hence using the relation,

$$t = \frac{x}{U} = \frac{M}{U} \cdot \frac{x}{M}, \quad (5.5)$$

which means the so-called Taylor's hypothesis [5], let us transform the independent variable t in equation (5.4) into x/M . If we put

$$\left. \begin{aligned} \zeta &= \frac{U^2}{\overline{u^2}}, & \xi &= \frac{x}{M} = \frac{U}{M} t, \\ C &= \frac{7}{15} G - 2 = B - 2, \\ E &= \frac{7}{3} S \sqrt{R_M/10}, & R_M &= \frac{UM}{\nu}, \end{aligned} \right\} \quad (5.6)$$

* As the result, the Reynolds number of turbulence $R_{\lambda_0} = \sqrt{\overline{u_0^2}} \lambda_0 / \nu$ at $t = t_0$ is given.

† If we assume the power decay law, $\overline{u^2}(t) \sim t^{-n}$, and substitute it into equation (5.2), we obtain

$$(n+1) + \frac{7}{3} S \left(\frac{1}{10\nu} \right)^{1/2} n^{1/2} t^{-(n/2)+1} = \frac{7}{15} n G.$$

Therefore, if S and G are constant, $n=1$ is the solution and in this case $R_\lambda = (10/\nu n)^{1/2} t^{-(n/2)+(1/2)} = \text{const.}$ This is nothing but the solution in the case of complete self-preservation of correlation functions. Namely, even though we assume the partial self-preservation of correlation function near $r=0$ (i.e., $S = \text{const.}$ and $G = \text{const.}$), if we assume, in addition, the power decay law, the solution reduces to that of the complete self-preservation in the end.

‡ To give the initial condition, $(d\overline{u^2}/dt)_0$ at $t = t_0$, has the same meaning as to give the initial condition, $\lambda = \lambda_0$ at $t = t_0$, on account of equation (2.10).

and substitute into equation (5.4), then equation (5.4) is reduced to

$$\boxed{\zeta \frac{d^2 \zeta}{d\xi^2} + C \left(\frac{d\zeta}{d\xi} \right)^2 - E \left(\frac{d\zeta}{d\xi} \right)^{3/2} = 0} \quad (5.7)$$

This is the fundamental energy decay equation in the non-dimensional form for the turbulence in the initial period of decay. The variables and coefficients that appear in this equation are expressed in such a form that we can measure them directly in the experiments. Consequently, if we solve this equation, we can readily compare the solution with experiments, so that we shall discuss hereafter by using this equation. If we solve this equation under the initial conditions, $\zeta = \zeta_0 = (U^2/\bar{u}^2)_0$ and $d\zeta/d\xi = (d\zeta/d\xi)_0 = [d(U^2/\bar{u}^2)/d(x/M)]_0$ at $\xi = \xi_0 = (x/M)_0$, the decay curve, U^2/\bar{u}^2 vs. x/M , can be readily obtained, but this equation involves C and E , i.e., R_M , G and S as the parameters. Therefore, it may be expected that the form of the decay curve may vary with the values of these parameters and the initial conditions. After the relation between U/\bar{u} and x/M is obtained, the relation between λ^2 and x/M can be derived from the energy equation (2.10). Namely,

$$\lambda^2 = \frac{10M^2}{R_M} \cdot \frac{\zeta}{d\zeta/d\xi} = \frac{10M^2}{R_M} \cdot \frac{(U^2/\bar{u}^2)}{d(U^2/\bar{u}^2)/d(x/M)} \quad (5.8)$$

The Reynolds number of turbulence can be calculated by the equation,

$$R_\lambda^2 = \frac{\bar{u}^2 \lambda^2}{\nu^2} = \frac{10R_M}{d\zeta/d\xi} = \frac{10R_M}{d(U^2/\bar{u}^2)/d(x/M)} \quad (5.9)$$

6. THE GENERAL CHARACTERISTICS OF THE SOLUTION OF THE ENERGY DECAY EQUATION

Before we try to solve the equation (5.7) for the examples, let us examine the general characteristics of the solution of this equation. Equation (5.7) may be written in the form,

$$\begin{aligned} \zeta'' &= \frac{1}{\zeta} [E(\zeta')^{3/2} - C(\zeta')^2] = \frac{(\zeta')^2}{\zeta} \left[\frac{E}{\sqrt{\zeta'}} - C \right] \\ &= \frac{(\zeta')^2}{\zeta} [D - C], \end{aligned} \quad (6.1)$$

in which

$$D \equiv E/\sqrt{\zeta'} \quad (6.2)$$

and the dashes denote differentiation with respect to ξ . Since S is positive, the parameter E appearing in equation (6.1) is naturally positive. G is usually greater than $30/7 (=4.286)$ when the Reynolds number of turbulence is not small [6], so that C is also positive. ζ' can not be negative from the physical point of view. Consequently, we may consider the following three cases according to the values of D and C . Now we express the value of D corresponding to ξ_0 as D_0 .

I) Case when $D_0 = C$.

In this case $\zeta''_0 = 0$ and the linear decay law holds. The slope of the decay curve

in this case becomes

$$\zeta' = \frac{5}{2} \cdot \frac{S^2 R_M}{\left(G - \frac{30}{7}\right)^2} = 10 \frac{R_M}{R_i^2}. \quad (6.3)$$

This result has been already given by Batchelor and Townsend* [6].

II) Case when $D_0 > C$.

In this case $\zeta_0'' > 0$ and the decay curve, U^2/\bar{u}^2 vs. x/M , is convex downwards. As x/M increases and decay proceeds, ζ' becomes greater and $(D-C)$ smaller, and ζ'' decreases monotonously. As the decay proceeds further, the value of D approaches the value of C and ζ'' tends to 0, and the decay curve approaches asymptotically a straight line with a slope,

$$\zeta'_0 = \frac{5}{2} \cdot \frac{S^2 R_M}{\left(G - \frac{30}{7}\right)^2}. \quad (6.4)$$

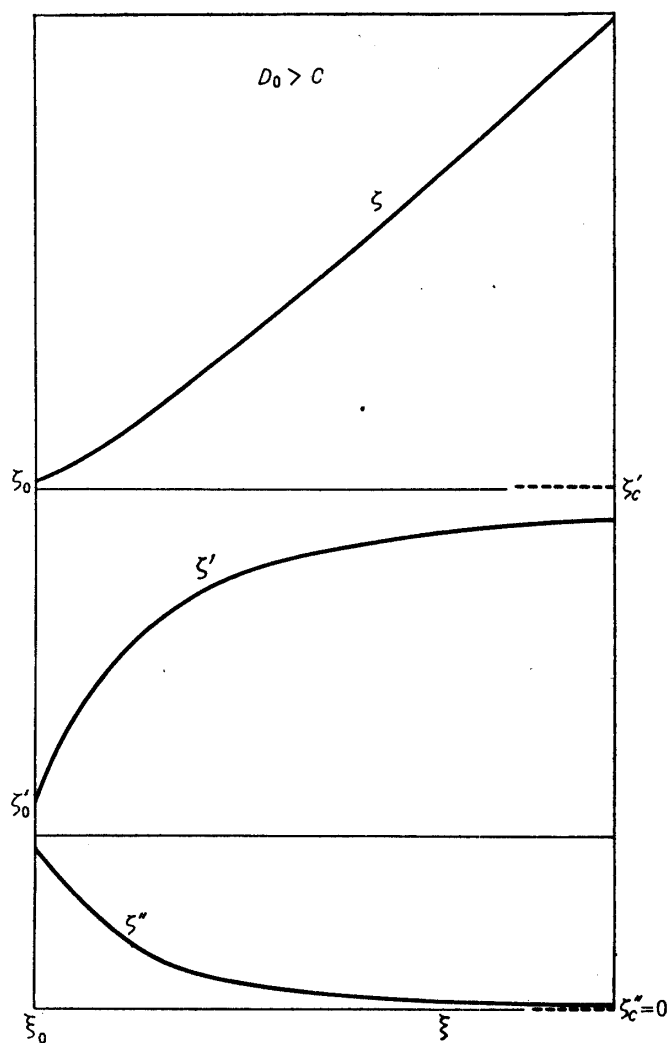


FIGURE 7. A solution of Eq. (5.7) when $D_0 > C$.

* When the linear decay law holds, the relation, $G = (30/7) + (1/2)R_i S$, is obtained from equation (3.4).

The circumstance is represented qualitatively in Fig. 7. The curve, λ^2 vs. x/M , is also convex downwards, and as x/M increases, λ^2 increases monotonously and approaches the straight line with the slope, $10M^2/R_M$. On the other hand, R_λ decreases monotonously*, and as x/M increases, it approaches the critical value

$$R_{\lambda c} = \frac{2}{S} \left(G - \frac{30}{7} \right), \quad (6.5)$$

asymptotically. Such a tendency that appears in this case is observed usually on the turbulence behind a single grid.

III) Case when $D_0 < C$.

In this case, $\zeta''_0 < 0$ and the decay curve, U^2/\bar{u}^2 vs. x/M , is convex upwards. As x/M increases and the decay proceeds, the absolute value of $(D-C)$ becomes smaller and also the absolute value of ζ'' decreases. As decay proceeds further, the value of D approaches the value of C and ζ'' tends to 0, and the decay curve approaches asymptotically a straight line with a slope,

$$\zeta'_G = \frac{5}{2} \cdot \frac{S^2 R_M}{\left(G - \frac{30}{7} \right)^2}. \quad (6.6)$$

The circumstance is represented qualitatively in Fig. 8. The curve, λ^2 vs. x/M , is also convex upwards, and as x/M increases, λ^2 increases monotonously and approaches asymptotically the straight line with the slope, $10M^2/R_M$. On the other hand, R_λ increases monotonously and as x/M increases, R_λ approaches the critical value,

$$R_{\lambda c} = \frac{2}{S} \left(G - \frac{30}{7} \right), \quad (6.7)$$

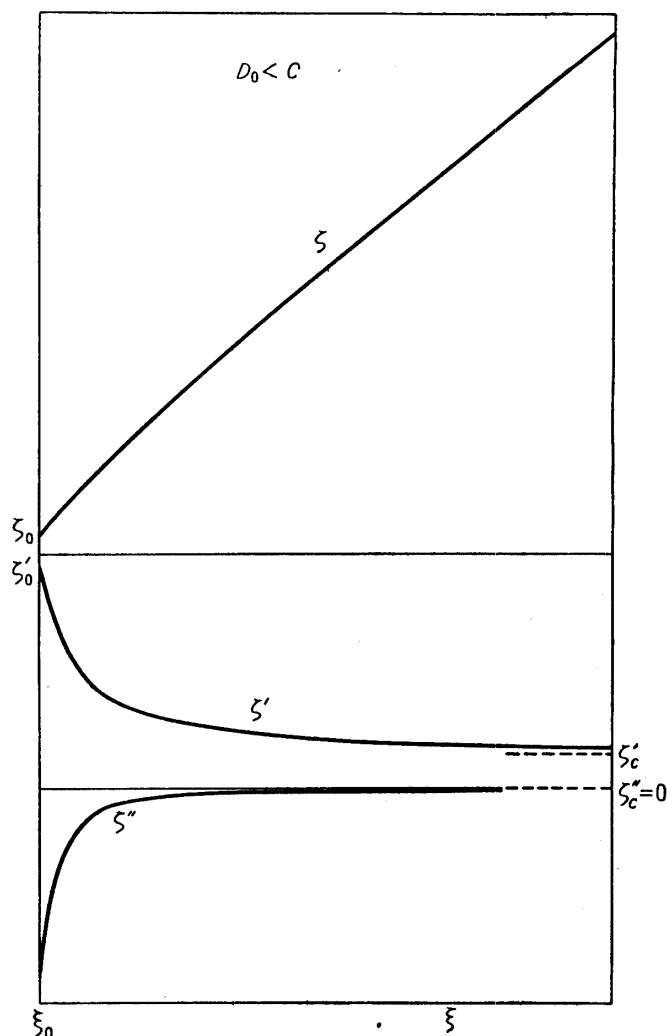
asymptotically. Such a tendency that appears in this case is observed usually on the turbulence with superposed disturbances of low wave numbers behind two grids.

The important conclusions derived from the general characteristics of the solution of the energy decay equation (5.7) for the turbulence in the initial period of decay are: (1) in the initial period of decay when the partial self-preservation of the correlation functions holds and S and G are constant, the linear decay law does not hold in general and also R_λ is not constant, (2) the energy decay of turbulence varies according to the initial conditions of turbulence generation. Namely, even when the mesh Reynolds number, R_M , of turbulence produced by a grid is the same, if the initial conditions are altered and the values of S and G vary, the decay curve may be convex upwards or downwards, or may be nearly a straight line or not.

* According to equation (5.9), $R_\lambda^2 = 10R_M/\zeta'$. Therefore,

$$\frac{dR_\lambda^2}{d\xi} = \frac{-10R_M\zeta''}{(\zeta')^2} = \frac{10R_M(C-D)}{\zeta}.$$

Hence $dR_\lambda^2/d\xi < 0$ when $D > C$ and $dR_\lambda^2/d\xi > 0$ when $C > D$. In both cases, as x/M increases, the value of $dR_\lambda^2/d\xi$ tends to 0. In Lin's decay law, on the other hand, R_λ is a linear function of x/M as indicated by equation (1.4), so that the critical value does not exist.

FIGURE 8. A solution of Eq. (5.7) when $D_0 < C$.

The theory on the decay law which has been developed in this paper has the important significance in view of the fact that the effect of the initial conditions upon the energy decay could be examined quantitatively through such statistical parameters as S , G and R_M . But it must be noticed that, since this theory is developed under the assumption of $S = \text{const.}$ and $G = \text{const.}$, the range, over which this theory can be possibly applied, is limited to the range,

$$x/M < (150 \sim 200)$$

for which the conditions of $S = \text{const.}$ and $G = \text{const.}$ are confirmed by the experiments.

Finally, let us examine λ^2 -curve. λ^2 may be calculated by equation (5.8),

$$\lambda^2 = \frac{10M^2}{R_M} \cdot \frac{\xi}{\xi'}. \quad (5.8)$$

Therefore,

$$\frac{d\lambda^2}{d\xi} = \frac{10M^2}{R_M} \left[\frac{(\xi')^2 - \xi\xi''}{(\xi')^2} \right] = \frac{10M^2}{R_M} [1 + C - D]. \quad (6.8)$$

λ^2 must increase as the decay proceeds from the physical point of view. Namely,

the relation such as

$$(1+C-D) > 0 \quad (6.9)$$

must be satisfied always in the decay process. If this condition is not satisfied, we must have the unreasonable result that λ^2 decreases as the decay time proceeds. If we denote the value of ζ' which satisfies the condition,

$$1+C-D \equiv 1+C-\frac{E}{\sqrt{\zeta'}}=0, \quad (6.10)$$

as ζ'_i, ζ'_o becomes

$$\zeta'_i = \frac{5}{2} \cdot \frac{S^2 R_M}{\left(G - \frac{15}{7}\right)^2}. \quad (6.11)$$

Namely, this theory can be applied for a limited range,

$$\zeta' > \zeta'_i. \quad (6.12)$$

Hence, the possible range for ζ' is

$$\left. \begin{array}{l} \zeta'_o \geq \zeta' > \zeta'_i \text{ for the case where } D > C, \\ \zeta' \geq \zeta'_o > \zeta'_i \text{ for the case where } D < C. \end{array} \right\} \quad (6.13)$$

Therefore, the case where $D < C$ is out of question. However, when $D > C$ and if the Reynolds number of turbulence becomes larger and the value of G increases, the ratio

$$\frac{\zeta'_o}{\zeta'_i} = \frac{\left(G - \frac{15}{7}\right)^2}{\left(G - \frac{30}{7}\right)^2} \quad (6.14)$$

approaches 1 and the possible range for ζ' decreases, so that the decay curve becomes nearly a straight line. Considering these characteristics, we conclude that if there exists a value of ζ' which lies outside the range given by equation (6.13) (such circumstance may come about actually in the very early period of decay in the case when $D > C$ and the Reynolds number of turbulence is large and G is also large—See the calculation of an example presented in § 7), both S and G become, or at least one of them becomes not constant in this stage of decay.

7. EXAMPLES OF CALCULATION OF THE ENERGY DECAY EQUATION

Considering the general characteristics of the solution of the energy decay equation (5.7), we may expect that the decay curve determined by solving this equation will show a good coincidence with the decay curve obtained in the experiments. Then we shall try to solve this equation actually for the cases of our experiments [11] [12] and of the experiments by Batchelor and Townsend [6] [7], and to compare these calculated results with the experimental results.

The calculation was made with four cases. (I) Experiment by the present author on the turbulence behind a single grid of $M=1 \text{ cm}^*$. (II) Experiment by

* See Fig. 1.

the present author on the turbulence behind a single grid of $M=5$ cm*. (III) Experiment by the present author behind two grids ($M_1=5$ cm and $M_2=1$ cm) placed 450 cm apart*. (IV) Experiment by Batchelor and Townsend on the turbulence behind a single grid of $M=1.27$ cm†. The values of the parameters employed actually in the calculation are presented in the following table (in Table 1).

TABLE 1.

Calculation No.	I	II	III	IV
Experimenter	Tsuji	Tsuji	Tsuji	{ Batchelor Townsend
Mesh length of the grid, cm	1	5	$\left\{ \begin{array}{l} M_1=5 \\ M_2=1 \\ X=450 \end{array} \right\}$	1.27
U m/s.	10	10	10	12.86
R_M	6.317×10^3	3.191×10^4	6.916×10^3	1.125×10^4
$(x/M)_0$	20	30	20	20
(U^2/\bar{u}^2)	1.170×10^3	1.015×10^3	9.517×10^2	1.10×10^3
$[d(U^2/\bar{u}^2)/d(x/M)]_0$	66.8	43.2	102	84
S^*	0.420	0.300	0.390	0.390
G^*	8.71	15.06	11.02	9.32
C	2.065	5.028	3.143	2.349
E	24.63	39.54	23.93	30.52
D_0	3.014	6.016	2.370	3.330
ζ'_c	142.3	61.85	57.99	168.8
ζ'_i	64.60	43.01	33.38	83.04
Figure No.	9	10	11	12

* As already mentioned in footnote on p. 98, S and G were not measured directly in our experiments. Therefore, we estimate the values of S and G by referring to the experimental results by Batchelor and Townsend (Refer to Figs. 8, 9, 10 and 11 of Reference [6]) and by considering the measured values of R_λ and R_M obtained in our experiments.

We solved the equation (5.7) by numerical integrations. The results of the theoretical calculation and the experimental values for examples (I), (II), (III) and (IV) are presented in Figs. 9, 10, 11 and 12, respectively. In the examples (I), (III) and (IV), the calculation proceeded starting from $(x/M)_0=20$. In the example (II), on the other hand, when we advance the numerical integration starting from $(x/M)_0=20$, it becomes $(1+C-D_0)<0$ and $\zeta'_0<\zeta'_i$, so that the value of ζ' gets out of the limited range as discussed in the preceding chapter. Therefore, we start the calculation from $(x/M)_0=30$ in this case. As will be seen in these figures, it was confirmed that the results of the theoretical calculation of the energy decay show good agreement with the measured values. On the other hand, the results of the theoretical calculation about R_λ seem to agree not so well with the measured

* See Fig. 1.

† We employ the experimental values presented in Fig. 5 of Reference [6].

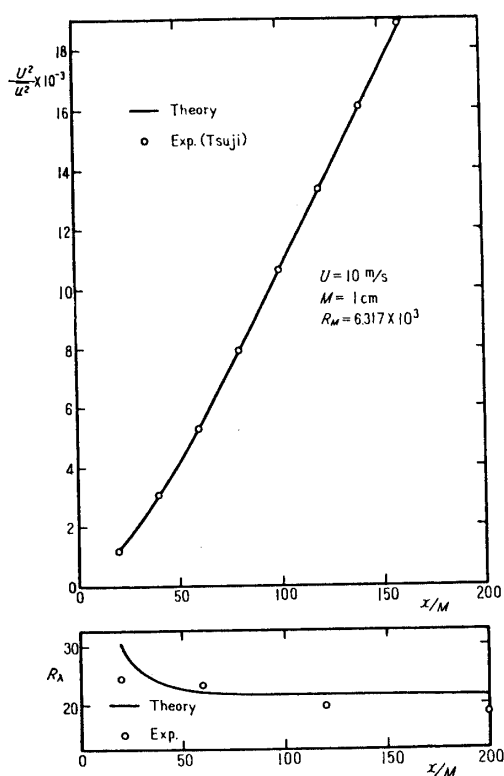


FIGURE 9. Decay of turbulence and variation of R_λ in the initial period (after Tsuji, 1955, Ref. [12]).

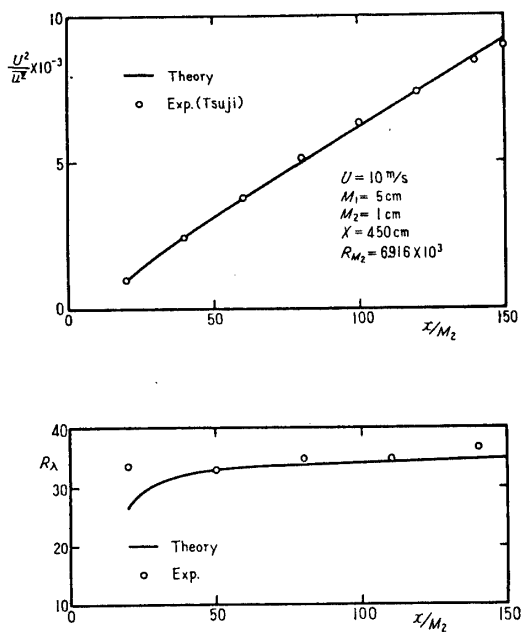


FIGURE 11. Decay of turbulence and variation of R_λ in the initial period (after Tsuji, 1955, Ref. [12]).

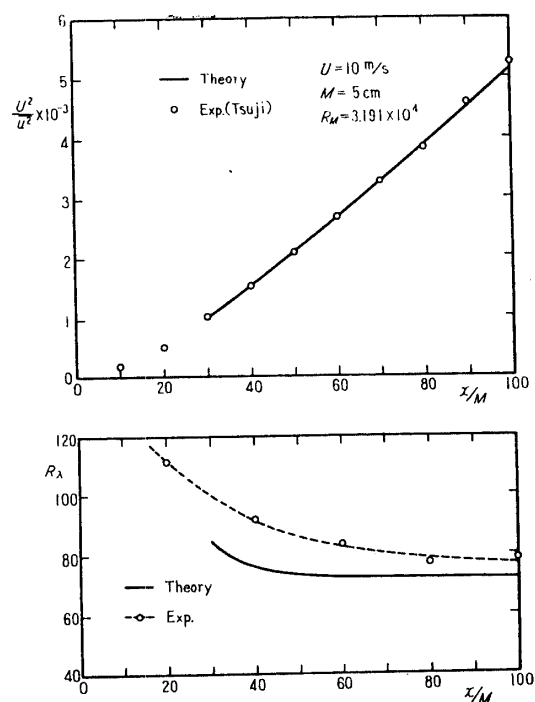


FIGURE 10. Decay of turbulence and variation of R_λ in the initial period (after Tsuji, 1955, Ref. [12]).

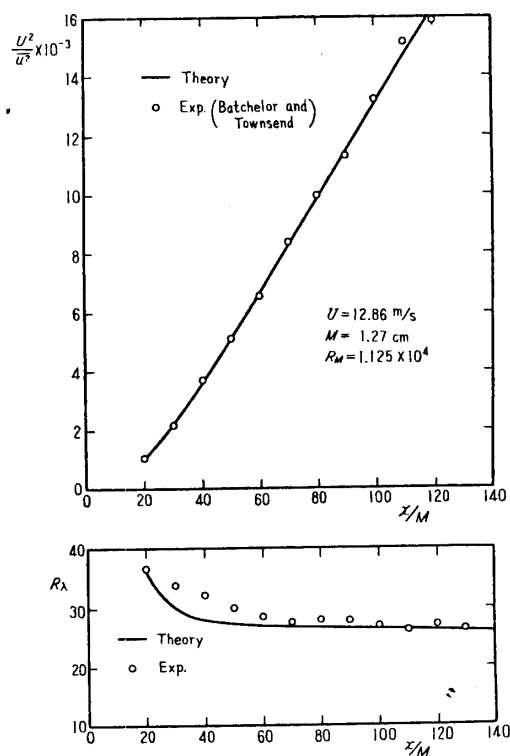


FIGURE 12. Decay of turbulence and variation of R_λ in the initial period (after Batchelor and Townsend, 1947, Ref. [6]).

values quantitatively as the case of energy decay, but satisfactory agreement is observed qualitatively. Since the accuracy of the measurements of λ is not so good as measurements of \overline{u}^2 , discrepancies to this extent may be thought to be unavoidable.

8. CONCLUSIONS

In this paper, we have discussed the self-contradiction involved in the theory about the linear decay law and Lin's decay law, which are well known energy decay laws of isotropic turbulence in the initial period of decay. Then eliminating these self-contradictions and using the simple assumption that the partial self-preservation of the correlation functions near $r=0$ holds and both S and G are constant during decay as confirmed by experiments, we have tried to determine the decay curve by solving the energy decay equation (5.7) which has been derived from the energy equation (2.10) and the vorticity equation (3.1). The conclusions drawn from these theoretical studies are follows:

1. The linear decay law can not be derived by assuming the partial self-preservation of the correlation functions, but can be derived only on the assumption of the complete self-preservation of the correlation functions.
2. The linear decay law does not hold actually nor R_λ is ever constant in the initial period of decay.
3. It is necessary that both S and G are, or at least either one of them is, not constant during decay in order that Lin's decay law should hold, therefore Lin's decay law contains a self-contradiction that the law holds only by neglecting the experimentally confirmed fact of the self-preservation of the correlation functions near $r=0$.
4. The new method to determine the decay curve by solving the energy decay equation (5.7) under the condition of $S=\text{const.}$ and $G=\text{const.}$ during decay, is reasonable, because the assumption used is confirmed by experiments and it contains no self-contradiction such as those contained in the linear decay law and Lin's decay law. In fact, it was confirmed that the results of the theoretical calculation show good agreement with the experimental results.
5. The decay curve varies with the initial conditions of turbulence generation and we can examine the effect of the initial conditions of turbulence generation upon the energy decay quantitatively through such statistical parameters as S , G and R_M .

The author would like to express his sincere thanks to Professor Itiro Tani for his stimulating interest and valuable discussions, to Mrs. Chiyoko Asano for carrying out the numerical integration of the energy decay equation, and to Mr. Susumu Ekida for his help in numerical calculations. This work is supported by the grant-in-aid for scientific research of the Ministry of Education.

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April 11, 1959*

REFERENCES

- [1] T. von Kármán and L. Howarth: On the statistical theory of isotropic turbulence, Proc. Roy. Soc., London, A, Vol. 164 (1938), pp. 192/215.
- [2] E. Reissner: Note on the statistical theory of turbulence, Proc. 5th Inter. Congr. Appl. Mech., Boston (1938), pp. 359/361.
- [3] G. K. Batchelor and A. A. Townsend: Decay of turbulence in the final period, Proc. Roy. Soc., London, A, Vol. 194 (1948), pp. 527/543.
- [4] G. K. Batchelor: Energy decay and self-preserving correlation functions in isotropic turbulence, Quart. Appl. Math., Vol. 6 (1948), pp. 97/116.
- [5] G. I. Taylor: Statistical theory of turbulence, Parts 1-4, Proc. Roy. Soc., London, A, Vol. 151 (1935), pp. 421/478.
- [6] G. K. Batchelor and A. A. Townsend: Decay of vorticity in isotropic turbulence, Proc. Roy. Soc., London, A, Vol. 190 (1947), pp. 534/550.
- [7] G. K. Batchelor and A. A. Townsend: Decay of isotropic turbulence in the initial period, Proc. Roy. Soc., London, A, Vol. 193 (1948), pp. 539/558.
- [8] C. C. Lin: Note on the law of decay of isotropic turbulence, Proc. Nat. Acad. Sci., Wash., Vol. 34 (1948), pp. 540/543.
- [9] S. Goldstein: On the law of decay of homogeneous isotropic turbulence and the theories of the equilibrium and similarity spectra, Proc. Camb. Phil. Soc., Vol. 47 (1951), pp. 554/574.
- [10] C. C. Lin: A critical discussion of similarity concepts in isotropic turbulence, Fluid Dynamics, Vol. IV (1953), pp. 19/27.
- [11] H. Tsuji and F. R. Hama: Experiment on the decay of turbulence behind two grids, Journ. Aero. Sci., Vol. 20 (1953), pp. 848/849.
- [12] H. Tsuji: Experimental studies on the characteristics of isotropic turbulence behind two grids, Journ. Phys. Soc. Japan, Vol. 10 (1955), pp. 578/586.
- [13] H. Tsuji: Experimental studies on the spectrum of isotropic turbulence behind two grids, Journ. Phys. Soc. Japan, Vol. 11 (1956), pp. 1096/1104.
- [14] H. W. Liepmann, J. Laufer and K. Liepmann: On the spectrum of isotropic turbulence, NACA TN 2473 (1951).
- [15] G. K. Batchelor and A. A. Townsend: The nature of turbulent motion at large wave numbers, Proc. Roy. Soc., London, A, Vol. 199 (1949), pp. 238/256.